A Note on the Incompatibility of Strategy-proofness and Pareto-optimality in Quasi-linear Settings with Public Budgets*

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Abstract

We study the problem of allocating multiple identical items that may be complements to budget-constrained bidders with private values. We show that there does not exist a deterministic mechanism that is individually rational, strategy-proof, Pareto-efficient, and that does not make positive transfers. This is true even if there are only two players, two items, and the budgets are common knowledge. The same impossibility naturally extends to more abstract social choice settings with an arbitrary outcome set, assuming players with quasi-linear utilities and public budget limits. Thus, the case of infinite budgets (in which the VCG mechanism satisfies all these properties) is really the exception.

JEL Classification Numbers: C70, D44, D82

Keywords: Budget constraints, Strategy-proofness, Pareto-optimality

*We wish to thank Ron Holzman for finding an error in a previous version.

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1 Introduction

It is well known that the possibility of designing strategy-proof and Pareto-efficient mechanisms depends on the structure of players' utilities. Most remarkably, when utilities are quasi-linear in money, the VCG mechanism (Groves (1973)) is strategy-proof and Pareto-efficient. In recent years, several works study, in the context of auctions, whether this possibility holds if the quasi-linearity assumption is slightly relaxed, to allow for hard budget constraints. For example, Aggarwal, Muthukrishnan, Pal and Pal (2009) and Ashlagi, Braverman, Hassidim, Lavi and Tennenholtz (2010) construct strategy-proof and Pareto-efficient deterministic auctions for unit-demand bidders with private values and budget constraints. In addition, naturally, these auctions are individually rational, and never make positive transfers. In contrast, with multi-unit demand, Fiat, Leonardi, Saia and Sankowski (2011) show that there does not exist such auctions, even with only two items, two bidders with multi-unit demand, and public (commonly known) budgets.

For *identical* items, the situation with multi-unit demand is slightly better: Dobzinski, Lavi and Nisan (2008) show that Ausubel's clinching auction (uniquely) satisfies all above-mentioned properties, assuming public budgets and additive valuations. In this paper we show that when the identical items are complements, the impossibility returns. Specifically, even with two identical items, two bidders with valuations that are complements, and even if budgets are public (i.e., commonly known), there still does not exist individually rational, strategy-proof, and Pareto-efficient auctions that make no positive transfers.

The no-positive-transfers requirement that we use is quite weak, as we only require the *sum* of players' payments to be non-negative. This is clearly an important design criterion in most auction settings. In fact, with an unlimited amount of positive transfers, one can simply increase the players' budgets to be ineffective, and then use the VCG mechanism (and indeed the Groves mechanism can be tuned to make only positive transfers).

Our proof is quite simple, and does not rely on previous impossibilities. In comparison, the proof of Fiat et al. (2011) for non-identical items relies on the uniqueness result of Dobzinski et al. (2008). Briefly, the proof is composed of two main claims that contradict each other, but must be satisfied by any mechanism with the above four properties. Let b_i denote the budget of player i, and assume $b_2 > b_1$. We show that if $v_2(2) > b_1$ (where $v_i(q)$ denotes player i's value for q items), player 2 must pay zero when she receives one item. On the other hand, if $v_2(2) < b_1$, prices must be regular VCG prices. These two claims contradict, since when player 2 has low values she prefers to falsely declare high values.

There are many classic as well as recent results on impossibilities in mechanism design without the quasi-linearity assumption, starting from Gibbard (1973) and Satterthwaite (1975), where the typical result is that only dictatorship (or sequential dictatorship) is strategy-proof and Pareto-efficient. Interestingly, here, even dictatorship is not a candidate, as dictatorship is simply not efficient when utilities are quasi-linear. Our result implies that there really is *no* mechanism that satisfies the above desired properties, for almost all structures of valuations over some abstract set of outcomes. The two extremes of zero budgets, and of infinite budgets, are just two rare exceptions.

A possible way to advance, given these impossibilities, is to study the "constrained efficiency" problem of maximizing efficiency subject to Bayesian incentive-compatibility constraints, as initiated in Maskin (2000). It is also important to understand the powers and limitations of randomized mechanisms, as initiated by Bhattacharya, Conitzer, Munagala and Xiax (2010) for the case of a divisible good.

In section 2 we formally define our model, and state our result. The proof is given in section 3.

2 Problem Statement and Main Result

We study the following very simple setting. The simplicity of the setting only strengthens our result, since it is an impossibility. There are two identical items and two players $1, 2$. Each player i has a private valuation function $v_i(q)$ that denotes i's non-negative real value for $q = 1, 2$ items, where $v_i(2) \geq v_i(1)$ (free disposal), and for notational simplicity we also use $v_i(0) = 0$. Each player additionally has a commonly known budget constraint $b_i > 0$. The utility of player i from obtaining $q = 1, 2$ items for a price p_i is quasi-linear up to the budget constraint, i.e., it is $v_i(q) - p_i$ if $p_i \leq b_i$, and it is some arbitrary negative number if $p_i > b_i$. An *outcome* is a tuple (q_1, q_2, p_1, p_2) where $q_1, q_2 \in \{0, 1, 2\}$, $q_1 + q_2 \leq 2$, and $p_i \leq b_i$ for $i = 1, 2$. (since budgets are public it simplifies notation and it is without loss of generality to disallow outcomes in which a player pays more than her budget).

Relying on the revelation principle, we consider only direct mechanisms. In a direct deterministic mechanism, each player i reports some type $\tilde{v}_i(\cdot)$, and, given these reports and the knowledge of b_1, b_2, the mechanism decides on an outcome $(q_i(\tilde{v}_1(\cdot), \tilde{v}_2(\cdot)), p_i(\tilde{v}_1(\cdot), \tilde{v}_2(\cdot)))_{i=1,2}$. Having fixed the parameters b_1, b_2, and some direct mechanism, we denote by $u_i((\tilde{v}_1(\cdot), \tilde{v}_2(\cdot)), v_i(\cdot))$ player i's resulting utility from the outcome of the mechanism when her private type is $v_i(\cdot)$ and the declarations are $(\tilde{v}_1(\cdot), \tilde{v}_2(\cdot))$.

We consider the following four standard and desirable properties. The first two properties address strategic issues, and need to be satisfied for any player i, any true valuation $v_i(\cdot)$ of i, any possible declaration $\tilde{v}_i(\cdot)$ of player i, and any possible declaration of the other player (say j), $\tilde{v}_j(\cdot)$.

Individual rationality (IR). A mechanism is ex-post individually rational if player i can always obtain a non-negative utility by truth-telling, i.e., $u_i((v_i(\cdot), \tilde{v}_j(\cdot)), v_i(\cdot)) \geq 0$.

Strategy proofness (SP). A mechanism is strategy-proof if truth-telling is a dominant strategy, i.e., $u_i((v_i(\cdot), \tilde{v}_j(\cdot)), v_i(\cdot)) \geq u_i((\tilde{v}_i(\cdot), \tilde{v}_j)(\cdot), v_i(\cdot))$.

The last two properties address design issues, and need to be satisfied for any tuple of valuations/declarations $(v_1(\cdot), v_2(\cdot))$.

No Positive Transfers (NPT). A mechanism satisfies no positive transfers if $p_1(v_1(\cdot), v_2(\cdot)) + p_2(v_1(\cdot), v_2(\cdot)) \geq 0$.

Pareto optimality (PO). A mechanism is Pareto-optimal if its outcome is Pareto optimal. Formally, there does not exist another outcome $\tilde{o} = (\tilde{q}_1, \tilde{q}_2, \tilde{p}_1, \tilde{p}_2)$ that is preferred by both bidders, i.e., $v_i(\tilde{q}_i) - \tilde{p}_i \geq v_i(q_i(v_1(\cdot), v_2(\cdot))) - p_i(v_1(\cdot), v_2(\cdot))$ for $i = 1, 2$, and by the seller, i.e., $\tilde{p}_1 + \tilde{p}_2 \geq p_1(v_1(\cdot), v_2(\cdot)) + p_2(v_1(\cdot), v_2(\cdot))$, with at least one strict inequality.[1]

If valuations were additive, Ausubel's clinching auction would satisfy all these properties, as Dobzinski et al. (2008) show. Even in our setting, there do exist mechanisms that satisfy any three of these four properties. Without IR, for example, one can charge each player her full budget, and choose some arbitrary allocation independently of the declarations. Without NPT, as remarked earlier, there exists a possible Groves mechanism. This paper shows that combining the four properties together is unfortunately impossible.

Theorem 1. *In our setting, if $b_1 \neq b_2$, there does not exist any mechanism that satisfies IR, SP, NPT, and PO.*

This theorem implies the same impossibility for all settings that generalize ours, since any mechanism for a more general setting can be used to construct a mechanism with the same properties for our setting. In particular, this gives an impossibility for any number of identical or non-identical items and any number of players with general valuations over the set of items. Similarly, it implies the same impossibility for the abstract social choice setting with at least three alternatives and an unrestricted domain of players' valuations.

3 Proof

The proof is composed of four components, that together yield a contradiction to the existence of a mechanism that satisfies the four mentioned properties. Without loss of generality we assume throughout that $b_2 > b_1$.

First component: $q_i = 0$ implies $p_i = 0$. The first component shows that if player i receives no items, her price is exactly zero. A very similar argument appears in Dobzinski et al. (2008) and in Fiat et al. (2011), we include it here mainly for completeness, but also since the exact technical connection is vague (as these papers study a different setting). We should also note that this claim does not immediately follow from IR and NPT – these requirements only imply that $p_i \leq 0$, and a-priori it may well be that $p_i < 0$ (i.e., a positive transfer to i) if the other player who receives both items has a positive payment that can balance the transfer. We start with a simple case.

[1] Note that $\tilde{o}$ must satisfy $\tilde{p}_i \leq b_i$ by definition. The seller must be included in this definition to preclude the trivial improvement of bidders' utilities by reducing prices (e.g., to zero).

Claim 1. *If* $v_i(2) = v_i(1) = 0$ *then* $p_i = 0$.

Proof. Let j denote the other player, let $v_j(\cdot)$ be her declaration, and let (q_1, q_2, p_1, p_2) be the resulting outcome. If $v_j(2) = 0$, IR implies $p_i, p_j \leq 0$ and NPT then implies $p_i = p_j = 0$. Thus, assume $v_j(2) > v_j(1) > 0$. In this case PO implies $q_j = 2$, otherwise the outcome that keeps the same prices and sets $\tilde{q}_j = 2, \tilde{q}_i = 0$ Pareto improves the previous outcome (player j's utility strictly increases since $v_j(2) > v_j(1) > 0$ and player i's utility does not change since $v_i(2) = v_i(1) = 0$). Now, IR implies $p_j \leq v_j(2)$. Since for *any* $v_j(2) > v_j(1) > 0$ (having fixed $v_i(\cdot) = 0$), player j receives both items, strategy-proofness implies that $p_j \leq 0$, otherwise player j can report some positive $\tilde{v}_j(2) < p_j$, still receive both items, and pay strictly less, thus strictly improving her utility and contradicting SP.

Since $p_j \leq 0$, and IR requires $p_i \leq 0$, NPT now implies $p_j = p_i = 0$, and the claim follows. □

Claim 2. *Whenever* $q_i = 0$*,* $p_i = 0$.

Proof. Suppose by contradiction that there exist $v_1(\cdot), v_2(\cdot)$ such that $q_i(v_1(\cdot), v_2(\cdot)) = 0$ but $p_i(v_1(\cdot), v_2(\cdot)) \neq 0$. By IR, $p_i(v_1(\cdot), v_2(\cdot)) < 0$, hence i's utility is strictly positive. Then, in case i's true value is $v_i'(\cdot) = 0$, i can increase her utility by misreporting her type to be $v_i(\cdot)$ (if she reports her true value $v_i'(\cdot)$, claim 1 implies that her utility will be zero). This contradicts SP. Thus, $p_i(v_1(\cdot), v_2(\cdot)) = 0$, as claimed. □

Second component: the case where $v_2(1) = 0$**.** (Recall that we assume $b_2 > b_1$.)

Claim 3. *Suppose* $\min\{b_2, v_2(2)\} > v_1(2) > b_1$*, and* $v_2(1) = 0$*. Then* $q_2 = 2$*, and* $q_1 = p_1 = 0$.

Proof. Suppose that the outcome of the mechanism for this tuple of valuations is q_1, q_2, p_1, p_2. We show that the claim directly follows from PO. Suppose by contradiction that $q_2 < 2$. Then, we argue that the following outcome is a Pareto improvement: $\tilde{q}_1 = 0, \tilde{p}_1 = p_1 - \min\{b_2, v_2(2)\}, \tilde{q}_2 = 2, \tilde{p}_2 = p_2 + \min\{b_2, v_2(2)\}$. To verify this, note that $\tilde{p}_2 \leq b_2$ since $p_2 \leq 0$ (by IR, since $q_2 \leq 1$ and $v_2(1) = 0$). Also, by definition, $\tilde{p}_1 + \tilde{p}_2 = p_1 + p_2$. Player 2's utility does not decrease since the added value is $v_2(2)$ and the added price is at most that. Player 1's utility strictly increases, since the decrease in her value is at most $v_1(2)$ and the decrease in her price is $\min\{b_2, v_2(2)\} > v_1(2)$. This shows that we have indeed constructed a Pareto improvement, which is a contradiction. We conclude that it must be that $q_2 = 2$ and $q_1 = 0$. Claim 2 now implies $p_1 = 0$. □

While the last claim requires $v_2(2) > v_1(2) > b_1$, the next claim allows very large or very small values, without a connection between the values of the two players.

Claim 4. *Suppose* $v_2(1) = 0$ *and* $v_1(2) > v_1(1)$*. Then either* $q_1 = 2$ *or* $q_2 = 2$.

Proof. This again follows directly from PO. Since $v_i(2) > v_i(1)$ for $i = 1, 2$, PO implies $q_1 + q_2 = 2$. Thus, we only need to rule out the case that $q_1 = q_2 = 1$. This outcome is Pareto dominated by the following outcome: $\tilde{q}_1 = 2, \tilde{p}_1 = p_1, \tilde{q}_2 = 0, \tilde{p}_2 = p_2$. (Player 1's utility strictly increases since $v_1(2) > v_1(1)$, player 2's utility is the same in both cases as $v_2(1) = 0$, and prices are the same.) □

Third component: some bounds on prices when $v_2(2) > b_1$. We will show a case where player 2's price for one item is at most zero. This makes the contradiction very close, as clearly an efficient mechanism cannot give an item "for free". We start with 2's price for two items.

Claim 5. *Suppose $v_2(2) > b_1$, $v_2(1) = 0$, and $v_1(2) > v_1(1)$. Then $q_2 = 2$, and $p_2 \leq b_1$.*

Proof. This now follows from SP. By Claim 4, either $q_1 = 2$, or $q_2 = 2$. Suppose by contradiction that $q_1 = 2$, and consider the case where player 1's true valuation is $\tilde{v}_1(\cdot)$, where $\min\{b_2, v_2(2)\} > \tilde{v}_1(2) > b_1$. By Claim 3, if player 1 truthfully reports $\tilde{v}_1(\cdot)$ as her valuation, her resulting utility will be exactly zero. If, however, player 1 misreports her valuation to be $v_1(\cdot)$, she will receive two items, and by IR will pay at most b_1. Since $\tilde{v}_1(2) > b_1$, misreporting her value in this case strictly increases player 1's utility, contradicting SP. Thus, $q_2 = 2$. Since this is true for *every* $v_2(2) > b_1$, IR and SP imply $p_2 \leq b_1$. □

Claim 6. *Suppose $v_2(2) > v_2(1) > b_1$, $v_2(2) - v_2(1) \leq b_1 < v_1(1)$. Then $q_2 = 1$ and $p_2 \leq 0$.*

Proof. We first show that $q_2 = 1$. If $q_2 = 0$, player 2's resulting utility is exactly zero by Claim 2. However, if player 2 will declare a false valuation $\tilde{v}_2(\cdot)$ such that $\tilde{v}_2(2) = v_2(2)$ and $\tilde{v}_2(1) = 0$ she will receive two items and will pay at most b_1 (by Claim 5), hence will obtain a strictly positive utility, contradicting SP. If $q_2 = 2$, the following outcome is a Pareto improvement: $\tilde{q}_1 = \tilde{q}_2 = 1, \tilde{p}_1 = p_1 + \Delta, \tilde{p}_2 = p_2 - \Delta$, where $\Delta = v_2(2) - v_2(1)$, and this contradicts PO. Thus, $q_2 = 1$. Since player 2 can receive two items and pay at most b_1 by declaring $\tilde{v}_2(1) = 0$, SP implies $v_2(1) - p_2 \geq v_2(2) - b_1$, i.e. $p_2 \leq b_1 - (v_2(2) - v_2(1))$. Since this is true also when $v_2(2) - v_2(1) = b_1$, SP implies that $p_2 \leq 0$. □

Forth component: allocation and prices when all values are smaller than b_1. We now study the complementary case where values are relatively small. We show that in this case the mechanism must choose the VCG outcome. In particular, in some cases player 2 receives one item and pays a strictly positive price. This will imply the theorem by contradicting SP, as if player 2 misreports her value to be larger than b_1 she can receive one item "for free". We start with a standard claim.

Claim 7. *Let i, j be two distinct players, and fix some valuation $\tilde{v}_j$ for player j. If $q_i(v_i(\cdot), v_j(\cdot)) = q_i(\tilde{v}_i(\cdot), v_j(\cdot))$ for two valuations $v_i(\cdot), \tilde{v}_i(\cdot)$ of player i, then it must be that $p_i(v_i(\cdot), v_j(\cdot)) = p_i(\tilde{v}_i(\cdot), v_j(\cdot))$ as well.*

Proof. Otherwise, if w.l.o.g. $p_i(v_i(\cdot), v_j(\cdot)) < p_i(\tilde{v}_i(\cdot), v_j(\cdot))$, when i's true valuation is $\tilde{v}_i(\cdot)$ she can increase her utility by declaring $v_i(\cdot)$ (this way, her price will decrease while the allocation remains the same), contradicting SP. □

We continue by analyzing the case where all values are smaller than b_1. One should not be surprised that in this case the mechanism must be VCG, as this case is similar to the case when there are no budgets at all, and for this case it is well known that the unique strategy-proof and efficient mechanism is VCG. In this sense, the proof of the following claim is quite standard, and we provide it mainly for completeness in Appendix A.

Claim 8. *Suppose* $\min_{i=1,2} b_i > \max_{i=1,2} v_i(2)$. *Then the allocation* (q_1, q_2) *maximizes the welfare* $v_1(q_1) + v_2(q_2)$, *if* $q_i = 2$ *then* $p_i = v_j(2)$, *and if* $q_i = 1$ *then* $p_i = v_j(2) - v_j(1)$, *where* j *is the other player.*

Concluding the proof of Theorem 1. We show how all the above implies that, in our setting, there does not exist a mechanism that satisfies IR, SP, NPT, and PO. Assume by contradiction the existence of such a mechanism. Fix valuations $v_1(\cdot)$ and $v_2(\cdot)$ such that $v_1(2) > b_1 + v_1(1)$ and $v_1(1) > b_1 > v_2(2) > v_2(1) > 0$. For these valuations, it cannot be that $q_1 = 0$, otherwise a Pareto improvement is $\tilde{q}_1 = 2, \tilde{q}_2 = 0, \tilde{p}_1 = p_1 + v_2(2), \tilde{p}_2 = p_2 - v_2(2)$ (in fact $p_1 = 0$ by claim 2). We next show $q_1 \neq 1$ and $q_1 \neq 2$, achieving a contradiction.

Suppose $q_1 = 1$. If $b_1 - p_1 \geq v_2(1)$, a Pareto improvement is $\tilde{q}_1 = 2, \tilde{q}_2 = 0, \tilde{p}_1 = p_1 + v_2(1), \tilde{p}_2 = p_2 - v_2(1)$, which is a contradiction. If $b_1 - p_1 < v_2(1)$, player 1 can increase her utility by declaring a valuation $\tilde{v}_1(\cdot)$ such that $v_2(2) - v_2(1) < \tilde{v}_1(2) = \tilde{v}_1(1) < b_1$. By claim 8, in this case player 1 will still receive one item, and will pay $v_2(2) - v_2(1) < b_1 - v_2(1) < p_1$. Thus, player 1 is able to strictly increase her utility, contradicting SP. We conclude that $q_1 \neq 1$.

Finally, suppose $q_1 = 2$. By claim 2 the utility of player 2 in this case is exactly zero. However, by claim 6, if player 2 declares a valuation $\tilde{v}_2(\cdot)$ such that $\tilde{v}_2(2) > \tilde{v}_2(1) > b_1$ and $\tilde{v}_2(2) - \tilde{v}_2(1) \leq b_1$, she will receive one item and will pay a non-positive price, resulting in a positive utility. That is, player 2 can increase her utility to be strictly positive instead of zero by declaring $\tilde{v}_2(\cdot)$, a contradiction to SP. We conclude that $q_1 \neq 2$ as well, and therefore we have contradicted the existence of a mechanism that satisfies IR, SP, NPT, and PO. This concludes the proof of Theorem 1.

A Proof of Claim 8

We need to prove that, if $\min_{i=1,2} b_i > \max_{i=1,2} v_i(2)$, the allocation (q_1, q_2) maximizes the welfare $v_1(q_1) + v_2(q_2)$. Furthermore, if $q_i = 2$ then $p_i = v_j(2)$, and if $q_i = 1$ then $p_i = v_j(2) - v_j(1)$, where j is the other player.

Allocation. First suppose $v_i(2) > \max\{v_j(2), v_1(1) + v_2(1)\}$. In this case, if $q_i < 2$, the following outcome is a Pareto improvement: $\tilde{q}_i = 2, \tilde{p}_i = p_i + (v_i(2) - v_i(q_i)), \tilde{q}_j = 0, \tilde{p}_j = p_j - (v_i(2) - v_i(q_i))$. By IR, $p_i \leq v_i(q_i)$. Thus, $\tilde{p}_i \leq v_i(2) < b_i$. Since $v_i(2) \geq v_i(q_i)$, $\tilde{p}_j \leq p_j \leq b_j$. Clearly, $\tilde{p}_1 + \tilde{p}_2 = p_1 + p_2$, and player i's utility is exactly the same in both outcomes, as the added value $(v_i(2) - v_i(q_i))$ is exactly balanced by the increase in price. Finally, since $v_i(2) - v_i(q_i) > v_j(q_j)$ (whether $q_i \leq 1$

and $q_j \leq 1$, or $q_j = 2$ and $q_i = 0$), player j's utility strictly increases, as her decrease in value is $v_j(q_j)$ and her decrease in price is $v_i(2) - v_i(q_i)$. This shows that we have indeed constructed a Pareto improvement, contradicting PO. We conclude that in this case $q_i = 2$.

Now suppose $v_1(1) + v_2(1) > \max_{i=1,2} v_i(2)$. We need to show that $q_1 = q_2 = 1$. Suppose by contradiction that there exists a player i with $q_i = 0$, and let j be the other player. Then, the following outcome is a Pareto improvement: $\tilde{q}_1 = \tilde{q}_2 = 1, \tilde{p}_i = v_i(1), \tilde{p}_j = p_j - v_i(1)$. We have $\tilde{p}_i < b_i$ since $v_i(1) < b_i$ by assumption, and $\tilde{p}_j < b_j$ since $p_j < b_j$ by IR. By Claim 2 $p_i = 0$, hence $p_1 + p_2 = p_j = \tilde{p}_1 + \tilde{p}_2$. Player i's utility is zero in both outcomes. The utility of player j strictly increases: $v_j(\tilde{q}_j) - \tilde{p}_j - (v_j(q_j) - p_j) = v_j(\tilde{q}_j) - v_j(q_j) + v_i(1) \geq v_j(1) - v_j(2) + v_i(1) > 0$. We have therefore showed a Pareto improvement, contradicting PO. Hence, $q_1 = q_2 = 1$.

Payments. We first show that $q_i = 2$ implies $p_i = v_j(2)$. By the allocation part of the proof $q_i = 2$ for every $\tilde{v}_i(\cdot)$ such that $\min(b_1, b_2) > \tilde{v}_i(2) > v_j(2)$ and $\tilde{v}_i(1) = 0$. Thus, SP and IR imply $p_i \leq v_j(2)$, otherwise player i can declare $p_i > \tilde{v}_i(2) > v_j(2)$ and $\tilde{v}_i(1) = 0$, still win two items, and pay at most $\tilde{v}_i(2)$ which is strictly less than p_i. Similarly, again by the allocation part of the proof, $q_i = 0$ for every $\tilde{v}_i(\cdot)$ such that $v_j(2) > \tilde{v}_i(2)$ and $\tilde{v}_i(1) = 0$. By claim 2, if player i has true value $v_j(2) > \tilde{v}_i(2)$ and $\tilde{v}_i(1) = 0$ her utility is exactly zero. Thus, SP implies that $p_i \geq v_j(2)$, otherwise player i with true value $v_j(2) > \tilde{v}_i(2) > p_i$ and $\tilde{v}_i(1) = 0$ she can falsely declare v_i and obtain strictly positive utility. As a conclusion, $p_i = v_j(2)$.

Second, we argue that $q_i = 1$ implies $p_i = v_j(2) - v_j(1)$. Assume $v_j(1) > 0$, otherwise $v_i(2) \geq v_i(1) + v_j(1)$ and we can assume $q_i = 2$. By the allocation part of the proof $q_i = 1$ for every $\tilde{v}_i(\cdot)$ such that $\min(b_1, b_2) > \tilde{v}_i(2) = \tilde{v}_i(1) > v_j(2) - v_j(1)$, and $q_i = 0$ for every $\tilde{v}_i(\cdot)$ such that $\tilde{v}_i(2) = \tilde{v}_i(1) < v_j(2) - v_j(1)$. Similarly to the previous paragraph, by SP this implies $p_i = v_j(2) - v_j(1)$ (if $p_i > v_j(2) - v_j(1)$, i can increase utility by declaring $p_i > \tilde{v}_i(2) = \tilde{v}_i(1) > v_j(2) - v_j(1)$, and if $p_i < v_j(2) - v_j(1)$, when i's true utility is $p_i < \tilde{v}_i(2) = \tilde{v}_i(1) < v_j(2) - v_j(1)$ she can increase utility by declaring v_i). This concludes the proof of claim 8.

References

Aggarwal, G., S. Muthukrishnan, D. Pal and M. Pal (2009). "General auction mechanism for search advertising." In *Proc. of the 18th International Conference on World Wide Web (WWW'09).*

Ashlagi, I., M. Braverman, A. Hassidim, R. Lavi and M. Tennenholtz (2010). "Position auctions with budgets: Existence and uniqueness." *The B.E. Journal of Theoretical Economics (Advances)*, **10**(1).

Bhattacharya, S., V. Conitzer, K. Munagala and L. Xiax (2010). "Incentive compatible budget elicitation in multi-unit auctions." In *Proc. of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA).*

Dobzinski, S., R. Lavi and N. Nisan (2008). "Multi-unit auctions with budget limits." In *Proc. of the 49th Annual Symposium on Foundations of Computer Science (FOCS).*

Fiat, A., S. Leonardi, J. Saia and P. Sankowski (2011). "Single valued combinatorial auctions with budgets." In *Proceedings of the 12th ACM Conference on Electronic Commerce (ACM-EC)*, pp. 223–232.

Gibbard, A. (1973). "Manipulation of voting schemes: a general result." *Econometrica*, **41**, 587–601.

Groves, T. (1973). "Incentives in teams." *Econometrica*, pp. 617–631.

Maskin, E. S. (2000). "Auctions, development, and privatization: Efficient auctions with liquidity-constrained buyers." *European Economic Review*, **44**(4-6), 667–681.

Satterthwaite, M. (1975). "Strategy-proofness and arrow's condition: Existence and correspondence theorems for voting procedures and social welfare functions." *Journal of Economic Theory*, pp. 187–217.