NONLINEAR ELLIPTIC PROBLEM RELATED TO THE HARDY INEQUALITY WITH SINGULAR TERM AT THE BOUNDARY

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain of $\mathbb{R}^N$ and $1 < p < \infty$. The paper is divided in two main parts. In the first part we prove the following *improved Hardy Inequality* for convex domains. Namely, for all $\phi \in W_0^{1,p}(\Omega)$, we have

$$\int\limits_\Omega |\nabla \phi|^p dx - \left(\frac{p-1}{p}\right)^p \int\limits_\Omega \frac{|\phi|^p}{d^p} dx \geq C \int\limits_\Omega |\nabla \phi|^p \left(\log\left(\frac{D}{d}\right)\right)^{-p} dx,$$

where $d(x) = dist(x, \partial\Omega)$, $D > \sup\limits_{x \in \bar{\Omega}} d(x)$ and C is a positive constant depending only on p, N and Ω. The optimality of the exponent of the logarithmic term is also proved. In the second part we consider the following class of elliptic problem

$$\begin{cases} -\Delta u = \dfrac{u^q}{d^2} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < q \leq 2^* - 1$. We investigate the question of existence and nonexistence of positive solutions depending on the range of the exponent q.

1. Introduction

The starting point of this work is the following Hardy inequality stating that given a smooth bounded domain Ω of $\mathbb{R}^N$ and $1 < p < N$, then

$$\Lambda_p \int\limits_\Omega \frac{|\phi|^p}{d^p} dx \leq \int\limits_\Omega |\nabla \phi|^p dx \quad \text{for all } \phi \in W_0^{1,p}(\Omega), \tag{1.1}$$

where

$$d(x) = dist(x, \partial\Omega)$$

and $0 < \Lambda_p \leq \left(\frac{p-1}{p}\right)^p$. In the case where the domain Ω is convex, then $\Lambda_p = \left(\frac{p-1}{p}\right)^p$ and it is never achieved, see for instance [5], [16] and [17]. We refer also to [12] for details and more general Hardy type inequalities.

Many improvements of (1.1) have been found. In [9], the authors obtain a remainder term for the Hardy inequality, namely they show that for any $1 < p < N$ and $p \leq q < p^* \equiv \frac{Np}{N-p}$, there exists a positive constant $C \equiv C(p, q, N, \Omega)$ such that

$$\int\limits_\Omega |\nabla \phi|^p dx - \left(\frac{p-1}{p}\right)^p \int\limits_\Omega \frac{|\phi|^p}{d^p} dx \geq C \left(\int\limits_\Omega |\phi|^q dx\right)^{\frac{p}{q}} \qquad \forall\, \phi \in W_0^{1,p}(\Omega).$$

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In the case where $q = p^*$, then

$$\int\limits_\Omega |\nabla\phi|^p dx - \left(\frac{p-1}{p}\right)^p \int\limits_\Omega \frac{|\phi|^p}{d^p} dx \geq C D_{int}^{-\beta} \left(\int\limits_\Omega d^\alpha |\phi|^q dx\right)^{\frac{p}{q}} \qquad \forall\, \phi \in W_0^{1,p}(\Omega)$$

where $D_{int} = \sup\limits_{x\in\Omega} d(x, \partial\Omega)$, $\alpha > 0$ is any positive constant and $c = c(p, q, N, \alpha) > 0$.

Another approach was elaborated in [2] with $d(x)$ replaced by $d_K(x) = dist(x, K)$ where K is a piecewise smooth surface of codimension k, $1 \leq k \leq N$. In [2] it is proved that, for any $D > \sup\limits_{x\in\Omega} d(x, K)$ and for all $u \in W_0^{1,p}(\Omega)$,

$$\int\limits_\Omega |\nabla\phi|^p dx - \left|\frac{k-p}{p}\right|^p \int\limits_\Omega \frac{|\phi|^p}{d_K^p} dx \geq \frac{p-1}{2p} \left|\frac{k-p}{p}\right|^{p-2} \int\limits_\Omega \frac{|\phi|^p}{d_K^p} \left(\log\left(\frac{D}{d_K}\right)\right)^{-2} dx \tag{1.2}$$

for all $\phi \in W_0^{1,p}(\Omega)$. In our setting, we are interested in the case $K = \partial\Omega$ and so $k = 1$. Also in [2] the authors proved that for $1 \leq q < p$ and $\beta > 1 + \frac{p}{q}$, the following inequality holds true

$$\int\limits_\Omega |\nabla\phi|^p dx - \left(\frac{p-1}{p}\right)^p \int\limits_\Omega \frac{|\phi|^p}{d^p} dx \geq c \left(\int\limits_\Omega |\nabla\phi|^q d^{\frac{p}{q}-1} \left(\log\left(\frac{D}{d}\right)\right)^{-\beta} dx\right)^{\frac{p}{q}}, \tag{1.3}$$

for all $\phi \in W_0^{1,p}(\Omega)$, where $c > 0$ is a universal constant. The exponent of the *logarithm* term in this inequality is optimal.

The first goal of this paper is to improve the above inequality (1.3). In fact we prove the following result.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a convex bounded domain. Suppose that $1 < p < \infty$ and let $D > \sup\limits_{x\in\bar\Omega} d(x)$. Then for all $\phi \in C_0^\infty(\Omega)$:*

1) *if $p < 2$, there exists a constant positive C such that*

$$\int\limits_\Omega |\nabla\phi|^p dx - \left(\frac{p-1}{p}\right)^p \int\limits_\Omega \frac{|\phi|^p}{d^p} dx \geq C \int\limits_\Omega |\nabla\phi|^p \left(\log\left(\frac{D}{d}\right)\right)^{-p} dx, \tag{1.4}$$

2) *if $p \geq 2$, then there exists a constant positive C such that*

$$\int\limits_\Omega |\nabla\phi|^p dx - \left(\frac{p-1}{p}\right)^p \int\limits_\Omega \frac{|\phi|^p}{d^p} dx \geq C \int\limits_\Omega |\nabla\phi|^p \left(\log\left(\frac{D}{d}\right)\right)^{-2} dx. \tag{1.5}$$

Estimates (1.4), (1.5) *are sharp in the sense that the exponents of the term* $\log(\frac{D}{d})$ *in right hand sides cannot be bigger than p and 2 respectively.*

The aim of the second part of this paper is to study a class of nonlinear elliptic equations with a singular potential, more precisely we consider the following problem:

$$\begin{cases} -\Delta u = \dfrac{u^q}{d^2} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $0 < q \le 2^* - 1$, where $2^* = \frac{2N}{N-2}$ for $N \ge 3$.

The case $q = 1$ is widely studied in the literature and it is strongly related to the Hardy inequality (1.1) and the geometry of the domain Ω. If Ω is a regular bounded domain with $-\Delta d \ge 0$ in the sense of distribution, then $\Lambda_2 = \dfrac{1}{4}$ and it never is achieved [2]. Hence the problem (1.6) has no positive solution. Notice that if Ω is a convex bounded domain, then the above condition is satisfied, see for instance [2].

If $\Lambda_2 < \dfrac{1}{4}$, then Λ_2 is achieved and the problem (1.6) with $q = 1$, up to a positive constant in the right hand side, has a positive bounded solution $u \in W_0^{1,2}(\Omega)$ such that

$$C_1 d^\alpha(x) \le u(x) \le c_2 d^\alpha(x) \text{ for all } x \in \Omega$$

where $\alpha = \dfrac{1+\sqrt{1-4\Lambda_2}}{2}$. We refer to [16] for more details and for an example for explicit domains where the Hardy constant is attainted.

For $q \neq 1$, the situation is totally different and it is, in some ways, surprising.

Let us describe some previous results when we replace $d^2(x)$ by the weight $|x|^2$. If $0 \in \Omega$ then we have existence of positive solutions only if $q < 1$. If $q > 1$, then the equation has no weak (distributional) solution, see [3]. In the case where $0 \in \partial\Omega$, the situation is different. Indeed, for $q < 1$, the problem has bounded solutions with finite energy. For $q > 1$, in [7] it is shown that the existence of solutions depends on the geometry of the domain. In fact, if the domain is starshaped with respect to the origin, there are no finite energy solutions. However, in dumbbell domains they proved, using truncation arguments, that the equation has positive bounded solutions.

For the problem (1.6) instead, the situation is quiet different. Indeed, for $q < 1$ we prove a complete blow-up for a natural approximation scheme.

Theorem 1.2. *Assume that $q < 1$ and let u_n be the unique positive solution to the problem*

$$\begin{cases} -\Delta u_n = \dfrac{u_n^q}{(d(x)+\frac{1}{n})^2} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Then $u_n(x) \to \infty$ for all $x \in \Omega$.

As a consequence we show that the problem (1.6) has no very weak solution, in a suitable sense that we describe next.

Definition 1.3. *Let $h(x,u)$ be a Caratheodory function in $\Omega \times \mathbb{R}$. We say that $u \in L^1(\Omega)$ is a very weak solution to the equation*

$$\begin{cases} -\Delta u = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

if $h(x,u) \in L^1(d,\Omega)$ and for all $\psi \in \mathcal{C}^2(\overline{\Omega})$ with $\psi = 0$ on $\partial\Omega$, we have

$$\int_\Omega u(-\Delta\psi)dx = \int_\Omega f\psi dx.$$

As a consequence of the blow-up result in Theorem 1.2, we have the following non-existence result.

Theorem 1.4. *Assume that* $0 < q < 1$*. Then the equation* (1.6) *has no very weak positive solution in the sense of Definition 1.3.*

For $q < 0$, we know from the result of [8] that the problem (1.6) has no regular solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, however Theorem 1.4 provides a strong non existence result.

If we replace the weight d^2 by d^s for some s positive, we can prove the existence of a very weak solution in the sense of Definition 1.3. More precisely we have the next existence result.

Theorem 1.5. *Assume that* $0 < q < 1$*. Then for all* $s < 2$*, the problem*

$$\begin{cases} -\Delta u = \dfrac{u^q}{d^s} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a positive solution u *in the sense of Definition 1.3.*

Going back to equation (1.6) in the range $1 < q < 2^* - 1$ and using blow-up arguments, we are able to show the existence of a solution as a limit of mountain pass solutions of approximated problems.

Theorem 1.6. *Assume that* $1 < q < 2^* - 1$*, then the problem* (1.6) *has a bounded positive solution* $u \in W_0^{1,2}(\Omega)$*.*

For the critical case $q = 2^* - 1$ and if $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^N$, we prove existence of a bounded radial positive solution.

Theorem 1.7. *Let* $\Omega = B_R(0)$ *Assume that* $N \geq 3$ *and* $q = 2^* - 1$ *or* $N = 1, 2$ *and* $q > 1$*. Then problem* (1.6) *has a positive radial solution* u*.*

The paper is organized as follows. In the next section we give some preliminary tools that will be used systematically in the rest of the paper. In particular inequality (2.2) which can be seen as an extension of the Hardy inequality.

Section 3 will be devoted to the "improved Hardy inequality". We first prove (1.4) and (1.5) see Theorem 1.1. In the last part of the proof we show the optimality of the exponent of the logarithmic term in (1.4) and (1.5).

Problem (1.6) with $q < 1$ will be studied in Section 4. We begin by proving a complete blow up for solutions of the approximated problems. As a consequence, we get the non-existence result. Then, we show that this nonexistence result is strongly related to the weight d^2 in the sense that if we replace d^2 by d^s for some $s < 2$, then the problem has at least a distributional solution. Some estimates on the behavior of the solution near the boundary are also obtained.

The case $1 < q < 2^* - 1$ is considered in Section 5. Then using the mountain pass theorem, we get the existence of a solution to a family of approximated problem. Hence, to get the desired existence result, we pass to the limit using Blow-up technics and the nonexistence results obtained by Gidas-Spruck in [10].

In Section 6 we analyze the critical case $q = 2^* - 1$, then if $\Omega = B_R(0)$, using the concentration-compactness argument, we are able to show the existence of a radial positive solution.

In the last Section we collect some open problems.

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2. Preliminaries and previous results

In this section, we collect some preliminaries and useful known results. We begin by the following vectorial inequalities that will be used systematically in first part of the paper. We first recall the following lemma (see [18] and [13] for complete proofs)

Lemma 2.1. *Assume that $1 < p < \infty$, then there exists a positive constant $c \equiv c(p) > 0$ such that for all $a, b \in \mathbb{R}^N$ we have*

1) *If $p < 2$, then*

$$|a-b|^p - |a|^p \geq c\frac{|b|^2}{(|a|+|b|)^{2-p}} - p|a|^{p-2}a.b. \tag{2.1}$$

2) *If $p \geq 2$, then*

$$|a-b|^p - |a|^p \geq c|a|^{p-2}|b|^2 - p|a|^{p-2}a.b,$$

$$|a-b|^p - |a|^p \geq c|b|^p - p|a|^{p-2}a.b. \tag{2.2}$$

Then, we recall the following extension of Hardy inequality obtained in [12].

Theorem 2.2. *Let Ω be bounded domain in $\mathbb{R}^N$ and suppose that $D > \sup\limits_{x\in\Omega} d(x)$. Then there exists a positive constant $C_0 = C(N,p)$ such that for all $u \in C_0^\infty(\Omega)$,*

$$\int\limits_\Omega \frac{|u|^p}{d}\left(\log(\frac{D}{d})\right)^{-p} dx \leq C_0 \int\limits_\Omega |\nabla u|^p d^{p-1} dx.$$

When dealing with the problem (1.6), the next comparison principle will be of great utility, see [4] for the proof.

Lemma 2.3. *(Comparison principle) Let f be a continuous function such that $\dfrac{f(.,u)}{u}$ is decreasing. Assume that $u, v \in W_0^{1,2}(\Omega)$ satisfy*

$$\begin{array}{ll} -\Delta u \geq f(x,u), & u > 0, \quad \text{in } \Omega, \\ -\Delta v \leq f(x,v), & v > 0, \quad \text{in } \Omega. \end{array}$$

Then $u \geq v$ in Ω.

The following weak version of the Harnack inequality is obtained in [3].

Lemma 2.4. *Let $h \in L^\infty(\Omega)$ be a nonnegative function and assume that v solves*

$$\begin{cases} -\Delta v = h(x) \text{ in } \Omega, \\ \quad v = 0 \text{ on } \partial\Omega. \end{cases}$$

Then

$$\frac{v(x)}{d(x)} \geq c(\Omega) \int_\Omega h(x) d(x)\, dx, \qquad \text{for all } x \in \Omega.$$

In the following C will denote a constant which may vary from line to line. Sometimes, when needed, we will explicit the dependence of the constant C on some of the parameters.

3. An improved Hardy inequality

Proof of Theorem 1.1. We divide the proof into four steps.

(1) **The case $p = 2$.**
Let $\phi \in \mathcal{C}_0^\infty(\Omega)$, by a direct computation we get

$$\frac{1}{2}\nabla\left(\frac{\phi^2}{d}\right)\nabla d = \frac{\phi\nabla\phi\nabla d}{d} - \frac{1}{2}\frac{\phi^2}{d^2}|\nabla d|^2,$$

thus

$$|\nabla\phi|^2 - \frac{1}{4}\frac{\phi^2}{d^2} = \left|\nabla\phi - \frac{1}{2}\frac{\phi}{d}\nabla d\right|^2 + \frac{1}{2}\nabla\left(\frac{\phi^2}{d}\right)\nabla d.$$

Since $-\Delta d \geq 0$ in $\mathcal{D}'(\Omega)$, then

$$\int_\Omega \left(|\nabla\phi|^2 - \frac{1}{4}\frac{\phi^2}{d^2}\right) dx \geq \int_\Omega \left|\nabla\phi - \frac{1}{2}\frac{\phi}{d}\nabla d\right|^2 dx. \tag{3.1}$$

Recall that $D > \sup_{x\in\bar\Omega} d(x)$, thus $\left(\log(\frac{D}{d})\right)^{-\alpha} \in L^\infty(\Omega)$ for all $\alpha > 0$. Hence we get the existence of a positive constant $C >$ such that

$$\left|\nabla\phi - \frac{1}{2}\frac{\phi}{d}\nabla d\right|^2 \geq C\left(\log(\frac{D}{d})\right)^{-2}\left|\nabla\phi - \frac{1}{2}\frac{\phi}{d}\nabla d\right|^2.$$

Therefore

$$\left|\nabla\phi - \frac{1}{2}\frac{\phi}{d}\nabla d\right|^2 \geq C\left(\log(\frac{D}{d})\right)^{-2}\left(|\nabla\phi|^2 + \frac{1}{4}|\phi\frac{\nabla d}{d}|^2 - \frac{\phi}{d}\nabla d\nabla\phi\right).$$

By integration and using Young's inequality, it follows that

$$\begin{aligned} &\int_\Omega \left|\nabla\phi - \frac{1}{2}\frac{\phi}{d}\nabla d\right|^2 dx \\ &\geq C\left\{(1-\varepsilon)\int_\Omega |\nabla\phi|^2\left(\log(\frac{D}{d})\right)^{-2} dx - C_\varepsilon\int_\Omega \frac{\phi^2}{d^2}\left(\log(\frac{D}{d})\right)^{-2} dx\right\}. \end{aligned} \tag{3.2}$$

Using inequality (1.2) with $p = 2$ and taking in consideration (3.1) and (3.2), the result follows in this case.

(2) **The case $p > 2$.**

Let $\phi \in \mathcal{C}_0^\infty(\Omega)$ and define $u = \dfrac{\phi}{d^{\frac{p-1}{p}}}$.

From [9], we get the existence of a positive constant $C_1 \equiv C_1(p, N)$ such that

$$\int_\Omega |\nabla \phi|^p dx - \left(\frac{p-1)}{p}\right)^p \int_\Omega \frac{|\phi|^p}{d^p} dx \geq C_1 \int_\Omega d^{p-1} |\nabla u|^p dx.$$

Since $\nabla u = d^{-(\frac{p-1}{p})}(\nabla \phi - \frac{p-1}{p}\frac{\phi}{d}\nabla d)$, then the last inequality became

$$\int_\Omega |\nabla \phi|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\phi|^p}{d^p} dx \geq C_1 \int_\Omega \left|\nabla \phi - \frac{p-1}{p}\frac{\phi}{d}\nabla d\right|^p dx. \tag{3.3}$$

Using the fact that $p > 2$, following the arguments of the first case, we get the existence of positives constants which are independent of ϕ such that

$$\left|\nabla \phi - \frac{p-1}{p}\frac{\phi}{d}\nabla d\right|^p \geq C \left(\log(\frac{D}{d})\right)^{-2} \left|\nabla \phi - \frac{p-1}{p}\frac{\phi}{d^p}\nabla d\right|^p.$$

By (2.2), hence

$$\left|\nabla \phi - \frac{p-1}{p}\frac{\phi}{d}\nabla d\right|^p \geq C \left(\log(\frac{D}{d})\right)^{-2} \Bigg\{ |\nabla \phi|^p + c(p) \left(\frac{p-1}{p}\right)^p \left|\phi \frac{\nabla d}{d}\right|^p$$
$$- p\left|\left(\frac{p-1}{p}\right)\frac{\phi}{d}\nabla d\right|^{p-1} |\nabla \phi| \Bigg\},$$

where $c(p) > 0$. Thus by integration and using Young inequality, we get

$$\begin{aligned} &\int_\Omega \left|\nabla \phi - \left(\frac{p-1}{p}\right)\frac{\phi}{d}\nabla d\right|^p dx \\ &\geq C \left\{ (1-\varepsilon) \int_\Omega |\nabla \phi|^p \left(\log(\frac{D}{d})\right)^{-p} dx - C_\varepsilon \int_\Omega \frac{\phi^p}{d^p}\left(\log(\frac{D}{d})\right)^{-p} dx \right\}. \end{aligned} \tag{3.4}$$

Using again (1.2), combining estimates (3.3) and (3.4), we reach (1.5) and then we conclude.

(3) **The case $1 < p < 2$.** From [2], we know that

$$\int_\Omega |\nabla \phi|^p dx - \left(\frac{p-1)}{p}\right)^p \int_\Omega \frac{|\phi|^p}{d^p} dx \geq C_1 \int_\Omega X^{2-p} \left|\nabla \phi - \left(\frac{p-1)}{p}\right)\frac{\phi}{d}\nabla d\right|^p dx. \tag{3.5}$$

where $X \equiv X(\frac{d(x)}{R})$ with $X(t) = (1 - \log t)^{-1}$ and $R = \sup_{x \in \Omega} d(x)$.

Since $D > R$, we can find $\beta > 0$ such that

$$X^{2-p} \geq \beta \left(\log(\frac{D}{d})\right)^{-p}. \tag{3.6}$$

Thus combining (3.5) and (3.6), we obtain that

$$\int_\Omega |\nabla \phi|^p dx - \left(\frac{p-1)}{p}\right)^p \int_\Omega \frac{|\phi|^p}{d^p} dx \geq C_3 \int_\Omega \left(\log(\frac{D}{d})\right)^{-p} \left|\nabla \phi - \left(\frac{p-1}{p}\right)\frac{\phi}{d}\nabla d\right|^p dx,$$

For a constant $C_3 >$ independent of ϕ.
Using (2.1), we obtain

$$\left(\log(\frac{D}{d})\right)^{-p}\left|\nabla\phi-\frac{p-1}{p}\frac{\phi}{d}\nabla d\right|^p \geq$$
$$C\left(\log(\frac{D}{d})\right)^{-p}\left(|\nabla\phi|^p - p\left|\left(\frac{p-1}{p}\right)\frac{\phi}{d}\nabla d\right|^{p-1}|\nabla\phi|\right).$$

Therefore, by Young's inequality,

$$\int\limits_\Omega \left(\log(\frac{D}{d})\right)^{-p}\left|\nabla\phi-\left(\frac{p-1}{p}\right)\frac{\phi}{d}\nabla d\right|^p dx$$
$$\geq C\left\{(1-\varepsilon)\int\limits_\Omega |\nabla\phi|^p\left(\log(\frac{D}{d})\right)^{-p}dx - C_\varepsilon\int\limits_\Omega \frac{\phi^p}{d^p}\left(\log(\frac{D}{d})\right)^{-p}dx\right\},$$

which implies,

$$\begin{aligned}&\int\limits_\Omega \left(\log(\frac{D}{d})\right)^{-p}\left|\nabla\phi-\left(\frac{p-1}{p}\right)\frac{\phi}{d}\nabla d\right|^p dx + \int\limits_\Omega \frac{\phi^p}{d^p}\left(\log(\frac{D}{d})\right)^{-p}dx\\ &\geq \overline{C}\int\limits_\Omega |\nabla\phi|^p\left(\log(\frac{D}{d})\right)^{-p}dx.\end{aligned} \tag{3.7}$$

Now, using Theorem 2.2 with $u=\frac{\phi}{d^{\frac{p-1}{p}}}$, we get

$$\int\limits_\Omega \frac{\phi^p}{d^p}\left(\log(\frac{D}{d})\right)^{-p}dx \leq C\int\limits_\Omega \left|\nabla\phi-\frac{p-1}{p}\frac{\phi}{d}\nabla d\right|^p dx. \tag{3.8}$$

Thus, by (3.7) and (3.8), it follows that

$$\begin{aligned}&\int\limits_\Omega |\nabla\phi|^p dx - \left(\frac{p-1)}{p}\right)^p\int\limits_\Omega \frac{|\phi|^p}{d^p}dx\\ &\geq C_1\int\limits_\Omega \left|\nabla\phi-\left(\frac{p-1}{p}\right)\frac{\phi}{d}\nabla d\right|^p dx\\ &+C_2\int\limits_\Omega \left(\left(\log(\frac{D}{d})\right)^{-p}\left|\nabla\phi-\left(\frac{p-1}{p}\right)\frac{\phi}{d}\nabla d\right|^p\right)dx\\ &\geq C\int\limits_\Omega |\nabla\phi|^p\left(\log(\frac{D}{d})\right)^{-p}dx.\end{aligned}$$

Hence the result follows at once.

Optimality of exponents.

To prove the optimality of exponents of $\log(\frac{D}{d})$ in the right hand side of inequalities (1.4) and (1.5), we use closely the arguments introduced in [9]
.

Without loss of generality assume that $0\in\partial\Omega$ and we consider $B_\delta(0)$, the ball centered at the origin with δ sufficiently small.

For $\varepsilon > 0$, we set $w_\varepsilon = d^{\frac{p-1}{p}+\varepsilon}\left(\log\left(\frac{D}{d}\right)\right)^\theta$, where $\theta > 0$, to be chosen later.

Let $\phi \in \mathcal{C}_0^2(\Omega)$, be such that $0 \le \phi \le 1$, $supp(\phi) \subset B_\delta(0)$ and $\phi = 1$ in $B_{\frac{\delta}{2}}(0)$.

Define $U_\varepsilon(x) \equiv \phi(x) w_\varepsilon(x)$, then $supp\left(U_\varepsilon\right) \subset B_\delta(0)$.

Let us begin by proving the optimality in the case $p \ge 2$. We argue by contradiction. Suppose the existence of positive constants C and γ such that

$$\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx \ge C \int_\Omega |\nabla u|^p \left(\log\left(\frac{D}{d}\right)\right)^{-2-\gamma} dx$$

holds for all $u \in W_0^{1,p}(\Omega)$. Since $U_\varepsilon \in W_0^{1,p}(\Omega)$ for all $\varepsilon > 0$, it follows that

$$\int_\Omega |\nabla U_\varepsilon|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|U_\varepsilon|^p}{d^p} dx \ge C \int_\Omega |\nabla U_\varepsilon|^p \left(\log\left(\frac{D}{d}\right)\right)^{-2-\gamma} dx. \tag{3.9}$$

Let analyze each term in the above inequality.

If $\theta < \frac{1}{p}$, then following closely the arguments in [9], their results that

$$\int_\Omega |\nabla U_\varepsilon|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|U_\varepsilon|^p}{d^p} dx \le c\varepsilon^{1-p\theta}. \tag{3.10}$$

Now we estimate the second member of right hand in (3.9).

Notice that $\nabla U_\varepsilon = w_\varepsilon \nabla \phi + \phi \nabla w_\varepsilon$, then

$$\begin{aligned}
\int_\Omega |\nabla U_\varepsilon|^p \left(\log\left(\frac{D}{d}\right)\right)^{-2-\gamma} dx &\ge \int_{B_{\frac{\delta}{2}}(0)} |\nabla U_\varepsilon|^p \left(\log\left(\frac{D}{d}\right)\right)^{-2-\gamma} dx \\
&\ge \int_{B_{\frac{\delta}{2}}(0)} |\nabla w_\varepsilon|^p \left(\log\left(\frac{D}{d}\right)\right)^{-2-\gamma} dx \\
&\ge \int_{B_{\frac{\delta}{2}}(0)} d^{-1+p\varepsilon}(\log(\frac{D}{d}))^{p(\theta-1)-2-\gamma} |(\frac{p-1}{p})\log(\frac{D}{d}) - \theta|^p dx.
\end{aligned}$$

Using (2.2), there results that

$$\Big|(\frac{p-1}{p})\log(\frac{D}{d}) - \theta\Big|^p \ \ge\ c(p)\Big(\log(\frac{D}{d})\Big)^p - p\theta\Big(\log(\frac{D}{d})\Big)^{p-1}.$$

Hence

$$\begin{aligned}
&\int_\Omega |\nabla U_\varepsilon|^p \left(\log\left(\frac{D}{d}\right)\right)^{-2-\gamma} dx \\
\ge\ & c(p)\int_{B_{\frac{\delta}{2}}(0)} d^{-1+p\varepsilon}\left(\log\left(\frac{D}{d}\right)\right)^{p\theta-2-\gamma} dx - c(p,\theta)\int_{B_{\frac{\delta}{2}}(0)} d^{-1+p\varepsilon}\left(\log\left(\frac{D}{d}\right)\right)^{p\theta-3-\gamma} dx \\
=:\ & I_1 - I_2.
\end{aligned}$$

By using the change of variables $r = Ds^{\frac{1}{\varepsilon}}$ in I_1 and I_2, we obtain

$$I_1 - I_2 = \varepsilon^{-p\theta+\gamma+1} D^{p\varepsilon}\Bigg[c(p)\int_0^{(\frac{\delta}{2D})^\varepsilon} s^{p-1}\left(\log\left(\frac{1}{s}\right)\right)^{p\theta-2-\gamma} ds - c(p,\theta)\varepsilon\int_0^{(\frac{\delta}{2D})^\varepsilon} s^{p-1}\left(\log\left(\tfrac{1}{s}\right)\right)^{p\theta-3-\gamma} ds\Bigg]. \tag{3.11}$$

Combining (3.10) and (3.11), we reach that

$$D^{p\varepsilon}\Bigg[c(p)\int_0^{(\frac{\delta}{2D})^\varepsilon} s^{p-1}\left(\log\left(\frac{1}{s}\right)\right)^{p\theta-2-\gamma} ds - c(p,\theta)\varepsilon\int_0^{(\frac{\delta}{2D})^\varepsilon} s^{p-1}\left(\log\left(\frac{1}{s}\right)\right)^{p\theta-3-\gamma} ds\Bigg] \geq C\varepsilon^{-\gamma}. \tag{3.12}$$

Since $p > 1$, then, for all $\gamma > 0$, as $\varepsilon \to 0$, we have

$$c(p)\int_0^1 s^{p-1}\left(\log\left(\frac{1}{s}\right)\right)^{p\theta-2-\gamma} ds + c(p,\theta)\int_0^1 s^{p-1}\left(\log\left(\frac{1}{s}\right)\right)^{p\theta-3-\gamma} ds < \infty,$$

hence we reach a contradiction with (3.12) and the result follows in this case.

(4) The case $p < 2$ follows using the same arguments. ∎

Remark 1. *In the case where $p = 2$, then we can define a new space H as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm*

$$||\phi||_H^2 = \int_\Omega \left(|\nabla\phi|^2 - \frac{1}{4}\frac{\phi^2}{d^2}\right)dx.$$

It is clear that H is a Hilbert space. By Theorem 1.1, it follows that

$$W_0^{1,2}(\Omega) \subsetneq H \subsetneq W_0^{1,q}(\Omega)\,\forall\, q < 2.$$

4. The problem (1.6) with $q < 1$

First we give the proof of Theorem 1.2 about the blow-up for the approximated problem.

Proof of Theorem 1.2 in the case $0 < q < 1$**.** Notice that the existence and the uniqueness of u_n follow using classical minimizing arguments and the comparison principle Lemma 2.3. It is clear that $\{u_n\}_n$ is an increasing sequence in n.

We argue by contradiction. We assume that there exist some $x_0 \in \Omega$ such that $u_n(x_0) \leq C$ for all n. Then, by Lemma 2.4 it follows that

$$\frac{u_n(x_0)}{d(x_0)} \geq C\int_\Omega \frac{u_n^q}{(d(y)+\frac{1}{n})^2}d(y)dy.$$

Hence we conclude that

$$\int_\Omega \frac{u_n^q}{(d(y)+\frac{1}{n})^2}d(y)dy \leq C.$$

Since $\{u_n\}_n$ is an increasing sequence in n, we get the existence of a measurable function u such that $u_n \uparrow u$ a. e. in Ω and

$$\frac{u_n^q}{(d(y)+\frac{1}{n})^2}d(y) \to \frac{u^q}{d(y)} \qquad \text{strongly in} \quad L^1(\Omega).$$

Let ρ be the unique solution to the problem

$$-\Delta\rho = 1, \qquad \rho \in W_0^{1,2}(\Omega). \tag{4.1}$$

It is clear that $\rho \in \mathcal{C}^1(\overline{\Omega})$ and $\rho \simeq d$. Using ρ as a test function in (1.7) we reach that

$$\int_\Omega u_n dx = \int_\Omega \frac{u_n^q}{(d(x)+\frac{1}{n})^2}\rho dx \le C\int_\Omega \frac{u_n^q}{(d(x)+\frac{1}{n})^2}d(x)dx \le C.$$

Hence $\|u_n\|_{L^1(\Omega)} \le C$ and then $u_n \to u$ strongly in $L^1(\Omega)$. In the same way and by an approximation argument we can take $\dfrac{\rho}{u_n^s}$, $0<s<1$, as a test function in (1.7). We obtain that

$$\frac{1}{1-s}\int_\Omega u_n^{1-s}dx = s\int_\Omega \frac{|\nabla u_n|^2}{u_n^{s+1}}\rho\, dx + \int_\Omega \frac{u_n^{q-s}}{(d(x)+\frac{1}{n})^2}\rho\, dx.$$

Since $u_n^{1-s} \to u^{1-s}$ strongly in $L^1(\Omega)$, then

$$\int_\Omega \frac{u_n^{q-s}}{(d(x)+\frac{1}{n})^2}\rho\, dx \le C.$$

Therefore using Fatou's lemma we obtain that

$$\int_\Omega \frac{u^{q-s}}{d(x)}dx \le C \qquad \text{for all } 0<s<1.$$

As a conclusion we have proved that

$$\frac{u^q}{d} \in L^1(\Omega) \quad \text{and} \quad \frac{u^{q-s}}{d} \in L^1(\Omega) \quad \text{for all} \quad 0<s<1.$$

Fix s such that $q<s<1$, then since $\frac{s-q}{s}+\frac{q}{s}=1$,

$$F \equiv \Big(\frac{u^q}{d}\Big)^{\frac{s-q}{s}} \in L^{\frac{s}{s-q}}(\Omega) \quad \text{and} \quad G \equiv \Big(\frac{u^{q-s}}{d}\Big)^{\frac{q}{s}} \in L^{\frac{s}{q}}(\Omega).$$

Therefore, using Hölder's inequality we reach that $FG \in L^1(\Omega)$. On the other hand notice that $FG = \frac{1}{d} \notin L^1(\Omega)$, a contradiction. We then conclude that $u_n(x) \to \infty$ for all $x \in \Omega$. ∎

Remark 2. *In the case where $q=0$, if we consider the problem*

$$-\Delta w = \frac{1}{d^s} \tag{4.2}$$

where $s \le 2$, we can prove the following assertions:

1. *If $s<2$, then the problem (4.2) has a unique positive bounded solution $w \in W_0^{1,2}(\Omega)$.*
2. *If $s=2$, then there is non positive bounded solution.*

Notice that, if $s>1$, then $\frac{1}{d^s} \notin L^1(\Omega) \cup \mathcal{M}(\Omega)$ where $\mathcal{M}(\Omega)$ is the space of bounded Radon measures.

For simplicity of writing, we set $\sigma = -q$.

Proof of Theorem 1.2 when $q < 0$**.** Let $u_n \in L^\infty(\Omega)$ be the unique positive solution to the problem

$$
\left\{
\begin{array}{rcl}
-\Delta u_n = \dfrac{1}{u_n^\sigma (d(x)+\frac{1}{n})^2} & & \text{in } \Omega,\\
u_n > 0 & & \text{in } \Omega,\\
u_n = 0 & & \text{on } \partial\Omega.
\end{array}
\right.
\tag{4.3}
$$

We claim that $u_n(x_0) \to \infty$ for all $x_0 \in \Omega$.

The main idea is to construct a suitable subsolution blowing-up at each point of Ω.

For $s \geq 0$, we set

$$H(s) = \Big(\log(s+1) - \frac{s}{s+1}\Big)^{\frac{1}{1+\sigma}},$$

then

$$H'(s) = \frac{1}{\sigma+1}\frac{s}{(s+1)^2} H^{-\sigma}(s)$$

and

$$H''(s) = -\frac{\sigma}{\sigma+1}\Big(\frac{s}{(s+1)^2}\Big)^2 H^{-2\sigma-1}(s) + \frac{1}{\sigma+1}\frac{s-1}{(s+1)^3} H^{-\sigma}(s).$$

Define $v_n = H(C_0 n\phi_1)$ where ϕ_1 is first eigenfunction of the laplacian and C_0 is a positive constant that we will chose later.

In what follows, C will denote a constant which can vary from line to line and that is independent of n.

By a direct computations, we reach that

$$
\begin{array}{rcl}
-\Delta v_n & = & C_0 n H'(C_0 n\phi_1)(-\Delta\phi_1) - C_0^2 n^2 H''(Cn\phi_1)|\nabla\phi_1|^2\\
& \leq & C_0\lambda_1 n\phi_1 H'(C_0 n\phi) + C_0^2 n^2 |H''(C_0 n\phi_1)||\nabla\phi_1|^2.
\end{array}
$$

Notice that

$$
\begin{array}{rcl}
C_0 n\phi_1 H'(C_0 n\phi_1) & = & \dfrac{1}{\sigma+1}\dfrac{(C_0 n\phi_1)^2}{((C_0 n\phi_1)+1)^2} H^{-\sigma}(C_0 n\phi_1) \leq \dfrac{1}{H^\sigma(C_0 n\phi_1)}\\
& \leq & \dfrac{C}{((C_0\phi_1)+\frac{1}{n})^2 H^\sigma(C_0 n\phi_1)}.
\end{array}
$$

On the other hand we have

$$
\begin{array}{rcl}
|C^2 n^2 H''(C_0 n\phi_1)|\nabla\phi_1|^2| & \leq & \dfrac{\sigma C}{\sigma+1}\dfrac{C_0^2}{((C_0\phi_1)+\frac{1}{n})^2 H^\sigma(C_0 n\phi_1)}\Big(\dfrac{C_0\phi_1}{C\phi_1+\frac{1}{n}}\Big)^2 H^{-\sigma-1}(C_0 n\phi_1) +\\
& + & \dfrac{CC_0^2}{\sigma+1}\dfrac{1}{((C_0\phi_1)+\frac{1}{n})^2 H^\sigma(C_0 n\phi_1)}.
\end{array}
$$

Using the fact that $(\dfrac{s}{s+1})^2 H^{-q-1}(s) \leq C$, it follows that

$$|C_0^2 n^2 H''(C_0 n\phi_1)|\nabla\phi_1|^2| \leq \frac{CC_0^2}{\sigma+1}\frac{1}{((C_0\phi_1)+\frac{1}{n})^2 H^\sigma(C_0 n\phi_1)}.$$

Going back to the problem of v_n, we reach that

$$-\Delta v_n \leq \frac{C}{((C_0\phi_1)+\frac{1}{n})^2 v_n^\sigma}.$$

Since $\phi_1(x) \ge C_1 d(x)$, then choosing C_0 such that $C_0\phi_1(x) \ge d(x)$, it follows that

$$-\Delta v_n \le \frac{C}{(d(x)+\frac{1}{n})^2 v_n^\sigma}.$$

We set $\tilde{v}_n = \frac{1}{C^{\frac{1}{q+1}}} v_n$, then $\tilde{v}_n$ satisfies

$$-\Delta \tilde{v}_n \le \frac{1}{(d(x)+\frac{1}{n})^2 \tilde{v}_n^\sigma}.$$

Thus $\tilde{v}_n$ is a subsolution to problem (4.3) and then by the comparison principle in Lemma 2.3, we conclude that $\tilde{v}_n \le u_n$. It is clear that $\tilde{v}_n(x_0) \to \infty$ for all $x_0 \in \Omega$. Hence we conclude. ■

Proof of Theorem 1.4. We argue by contradiction. We assume that the problem (1.6) has a non-negative solution u in the sense of Definition 1.3. By the strong maximum principle $u > 0$ in Ω. Then, we consider the unique solution u_n to the approximated problem (1.7). It is clear that u is a super-solution to problem (1.7). Hence using a variation of the comparison principle Lemma 2.3 we obtain that

$$u_n \le u_{n+1} \le u \qquad \text{for all} \quad n.$$

Hence we get the existence of $\overline{u} \in L^1(\Omega)$ such that $u_n \to \overline{u}$ strongly in $L^1(\Omega)$. This is a contradiction with the result of Theorem 1.2. Thus we conclude. ■

Remark 3. *Notice that the existence of u_n follows by minimizing the functional*

$$J_n(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \frac{1}{q+1}\int_\Omega \frac{|v|^{q+1}}{(d(x)+\frac{1}{n})^2} dx$$

in $W_0^{1,2}(\Omega)$. It is clear that

$$J_n(u_n) = \min_{\{v \in W_0^{1,2}(\Omega)\setminus\{0\}\}} J_n(v) = -\frac{1-q}{1+q}\int_\Omega \frac{u_n^{q+1}}{(d(x)+\frac{1}{n})^2} dx < 0.$$

We claim that $J_n(u_n) \to -\infty$ as $n \to \infty$. Indeed, define $w = \phi_1^\alpha$ where ϕ_1 is the first eigenfunction and $\frac{1}{2} < \alpha < \frac{1}{q+1}$, thus

$$\nabla w = \alpha \phi_1^{\alpha-1} \nabla \phi_1.$$

Recall that $\phi_1 \backsimeq d(x)$, then since $2(\alpha-1) > -1$

$$|\nabla w|^2 = \alpha^2 \phi_1^{2(\alpha-1)} |\nabla \phi_1|^2 \in L^1(\Omega).$$

Hence we conclude that

$$\begin{aligned} J_n(u_n) \quad &\le \quad J_n(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 dx - \frac{1}{q+1}\int_\Omega \frac{w^{q+1}}{(d(x)+\frac{1}{n})^2} dx \\ &\le \quad C - \frac{1}{q+1}\int_\Omega \frac{\phi_1^{\alpha(q+1)}}{(d(x)+\frac{1}{n})^2} dx. \end{aligned}$$

On the other hand it is clear that

$$\frac{\phi_1^{\alpha(q+1)}}{(d(x)+\frac{1}{n})^2} \backsimeq \frac{d^{\alpha(q+1)}}{(d(x)+\frac{1}{n})^2}.$$

Then by the monotone convergence Theorem we reach that

$$\frac{d^{\alpha(q+1)}}{(d(x)+\frac{1}{n})^2}\uparrow d^{\alpha(q+1)-2}.$$

Since $\alpha<\frac{1}{q+1}$, *we conclude that*

$$\int\limits_\Omega d^{\alpha(q+1)-2}=\infty.$$

To prove Theorem 1.5 we need the following result.

Proposition 4.1. *Assume that* $0<r<1$, *then the problem*

$$\begin{cases} -\Delta w=\dfrac{1}{w^r} & \text{in } \Omega,\\ w>0 & \text{in } \Omega,\\ w=0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

has a unique positive solution w *such that* $w\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, *moreover, there exist two positive constants* C_1,C_2 *such that*

$$C_1 d(x)\le w\le C_2 d(x). \tag{4.5}$$

The proof of Proposition 4.1 follows using sub-supersolution arguments.

Proof of Theorem 1.5. We follow by approximation. Let u_n be the unique positive solution to the problem

$$\begin{cases} -\Delta u_n=\dfrac{u_n^q}{(d(x)+\frac{1}{n})^s} & \text{in } \Omega,\\ u_n>0 & \text{in } \Omega,\\ u_n=0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Let w be the solution of problem (4.4) with $r=s-1<1$ if $1<s<2$ and $r\in(0,1)$ is arbitrary if $0<s\le 1$. Using w as a test function in (4.6), we reach that

$$\int\limits_\Omega \frac{u_n}{w^r}dx=\int\limits_\Omega \frac{u_n^q w}{(d(x)+\frac{1}{n})^s}dx.$$

Using estimate (4.5), the definition of r and the Hölder inequality, we obtain that

$$\int\limits_\Omega \frac{u_n}{d^r}dx\le C\int\limits_\Omega \frac{u_n^q}{d^r}dx\le C\Big(\int\limits_\Omega \frac{u_n}{d^r}dx\Big)^q\Big(\int\limits_\Omega \frac{1}{d^r}dx\Big)^{1-q}.$$

Since, in any case, $r<1$, then $\frac{1}{d^r}\in L^1(\Omega)$, thus $\displaystyle\int\limits_\Omega \frac{u_n}{d^r}dx\le C$.

Using the fact that the sequence $\{u_n\}_n$ is monotone in n, we get the existence of a measurable function u such that $\dfrac{u_n}{d^r}\to\dfrac{u}{d^r}$ strongly in $L^1(\Omega)$. It is clear that

$$\frac{u_n^q}{(d(x)+\frac{1}{n})^s}\uparrow\frac{u^q}{d^s}\quad\text{strongly in}\quad L^1(d(x),\Omega)$$

thus u is a distributional solution to problem (4.6). It is not difficult to prove that u is a solution to (4.6) in the sense of Definition 1.3. Notice that if $s<\frac{q+3}{4}$, we can prove that $u \in W_0^{1,2}(\Omega)$, moveover, using elliptic regularity we reach that $u \in L^\infty(\Omega)$. ■

Remark 4.

(1) *Using the fact that* $-\Delta u^\sigma \geq \dfrac{\sigma}{u^{1-\sigma-q}d^s}$ *for any* $0 < \sigma < 1-q$, *we obtain that* $u \geq Cd^{\frac{2}{1-q}}$.

(2) *Notice that if* $q+1 < s < 2$, *then* $\dfrac{u}{d^s} \notin L^1(\Omega)$, *hence by Lemma* (2.4), *it follows that*

$$\frac{u_n(x)}{d(x)} \geq C \int\limits_\Omega \frac{u_n^q}{(d(x)+\frac{1}{n})^s} d(y)dy \text{ for all } x \in \Omega$$

which implies that $u(x) \geq Cd(x)$ *for all* $x \in \Omega$. *Thus*

$$\frac{u^q}{d^s} \geq \frac{C}{d^{s-q}} \notin L^1(\Omega)$$

since $s - q > 1$.

(3) *If* $1+q < s < 2$, *then for all* $\dfrac{2-s}{1-q} < \theta < 1$, *there exists* $C(\theta) > 0$ *such that*

$$u \geq C(\theta)d^\theta \qquad in \quad \Omega.$$

This follows using the fact that if $\dfrac{2-s}{1-q} < \theta < 1$, *then*

$$-\Delta\phi_1^\theta \leq C\,\frac{\phi_1^{q\,\theta}}{d^s}$$

where ϕ_1 *is the first eigenfunction of the laplacian. Thus by the comparison principle Lemma 2.3 and up a constant we reach the desired estimate.*

Remark 5. *If we consider the problem*

$$(4.7) \qquad \begin{cases} -\Delta u = \dfrac{1}{u_n^\sigma d^s(x)} & in\ \Omega, \\ u > 0 & in\ \Omega, \\ u = 0 & on\ \partial\Omega \end{cases}$$

where $s < 2$, *then using sub-supersolution arguments and apriori estimates, we can prove that, for all* $\sigma > 0$, *the problem* (4.7) *has a unique bounded positive solution. We refer to* [8] *for more details and extensions.*

5. The problem (1.6) with $1 < q < 2^* - 1$

Proof of Theorem 1.6. As in the previous section, we argue by approximation. Let $u_n \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ be the “mountain pass solution ”to the approximated problem

$$(5.1) \qquad \begin{cases} -\Delta u_n = \dfrac{u_n^q}{(d(x)+\frac{1}{n})^2} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Notice that u_n is a critical point of the functional

$$J_n(v) = \frac{1}{2}\int\limits_{\Omega} |\nabla v|^2 dx - \frac{1}{q+1}\int\limits_{\Omega} \frac{|v|^{q+1}}{(d(x)+\frac{1}{n})^2} dx.$$

Using [1], we obtain that $J_n(u_n) = c_n$ where

$$c_n = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$$

and

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], \mathbb{R} \text{ with } \gamma(0) = 0 \text{ and } \gamma(1) = v_1 \in W^{1,2}_0(\Omega), J_n(v_1) < 0\}.$$

It is not difficult to prove that there exist $v_1 \in \mathcal{C}^\infty_0(\Omega)$ such that $J_n(v_1) << 0$ uniformly in n.

Since $c_n = \dfrac{p-1}{p+1}\int\limits_{\Omega} |\nabla u_n|^2 dx$, then using the fact that $0 \le c_n \le \max_{t\in[0,\infty)} J(tv_1) \le C$ for all n, we reach that the sequence $\{u_n\}_n$ is bounded in $W^{1,2}_0(\Omega)$.

We claim that

$$||u_n||_{L^\infty(\Omega)} \le C \qquad \text{for all } n.$$

To prove the claim we use blow-up technique as in [6] and [10]. Let $\{x_n\}_n \subset \Omega$ be such that $||u_n||_{L^\infty(\Omega)} = u_n(x_n)$ and suppose by contradiction that $u_n(x_n) \to \infty$ as $n \to \infty$. Since $\{x_n\}_n$ is a bounded sequence, we get the existence of $\overline{x} \in \overline{\Omega}$ such that (up to a subsequence) $x_n \to \overline{x}$.

We divide the proof in two cases:

(1) **The first case:** $\overline{x} \in \Omega$**.** We set $v_n(z) = \dfrac{u_n(\mu_n z + x_n)}{M_n}$ where $M_n = u_n(x_n)$ and $\mu_n = M_n^{\frac{1-q}{2}}$, then v_n solves

$$\begin{cases} -\Delta v_n &=& \dfrac{v_n^q}{(d(\mu_n z + x_n) + \frac{1}{n})^2} \text{ in } \Omega_n, \\ v_n &>& 0 \text{ in } \Omega_n, \\ v_n &=& 0 \text{ on } \partial\Omega_n, \end{cases}$$

where $\Omega_n = \dfrac{1}{\mu_n}(\Omega - x_n)$ is given by the transformation $x \mapsto z = \dfrac{x - x_n}{\mu_n}$.

It is clear that, for z fixed, $d(\mu_n z + x_n) + \frac{1}{n} \to d(\overline{x}) = C$ as $n \to \infty$.

By elliptic regularity, see [11], we have that $v_n \in \mathcal{C}^{0,\nu}$ for some $0 < \nu < 1/2$, moreover, $\|v_n\|_{\mu C^{0,\nu}} \le C$ uniformly in n.

Passing to the limit as $n \to \infty$, we get the existence of $v \in \mathcal{C}^{0,\nu}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $v(z) \le v(0) = 1$ and v solves

$$-\Delta v = Cv^q, \qquad v \ge 0 \quad \text{in } \mathbb{R}^N.$$

Since $q < 2^* - 1$, we get a contradiction with the non-existence result in [10].

(2) **The second case: $\overline{x} \in \partial\Omega$.** In this case we set $\mu_n = M_n^{\frac{1-q}{2}}(d(x_n) + \frac{1}{n})$, then v_n solves

$$\begin{cases} -\Delta v_n &=& v_n^q \Big(\dfrac{d(x_n) + \frac{1}{n}}{d(\mu_n z + x_n) + \frac{1}{n}}\Big)^2 \text{ in } \Omega_n, \\ v_n &>& 0 \text{ in } \Omega_n, \\ v_n &=& 0 \text{ on } \partial\Omega_n, \end{cases}$$

Fix $z \in \mathbb{R}^N$, then $\dfrac{d(x_n) + \frac{1}{n}}{d(\mu_n z + x_n) + \frac{1}{n}} \to 1$ as $n \to \infty$. Thus passing to the limit as $n \to \infty$, we get the existence of v such that either, $v \in \mathcal{C}^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $v(z) \le v(0) = 1$ and v solves

$$-\Delta v = Cv^q, \qquad v \ge 0 \quad \text{in } \mathbb{R}^N,$$

or, up to a translation, $v \in \mathcal{C}^2(\mathbb{R}^N_+) \cap \mathcal{C}^0(\{z \in \mathbb{R}^N, z_N \ge 0\})$ such that v solves

$$-\Delta v = Cv^q, \qquad v \ge 0 \quad \text{in } \mathbb{R}^N_+, \qquad v = 0 \quad \text{on} \quad z_N = 0.$$

Since $q < 2^* - 1$, we again get a contradiction with the non-existence result in [10]. Hence the claim follows at once.

On the other hand it is clear that

$$||u_n||_{L^\infty} \ge \overline{C} \text{ for all } n. \tag{5.2}$$

Otherwise, for some subsequence, we have $||u_n||_{L^\infty} \to 0$, then u_n solves

$$-\Delta u_n \le ||u_n||^{q-1}_{L^\infty} \frac{u_n}{d^2 + \frac{1}{n}}, \qquad u_n \in W_0^{1,2}(\Omega).$$

Choosing n large, we reach that $||u_n||^{q-1}_{L^\infty} << \Lambda_2$, a contradiction with the Hardy inequality (1.1). Hence we conclude that $||u_n||_{L^\infty} \ge \overline{C}$ for all n.

Recall that $u(x_n) = ||u_n||_{L^\infty}$, we claim that $d(x_n) > C_1 > 0$ for all n. We argue by contradiction, if, for some subsequence, $x_n \to \overline{x} \in \partial\Omega$ and $||u_n||_{L^\infty} \to C_2 \ge \overline{C}$. Then as in the proof of the previous uniform estimate, we set

$$v_n(z) = \frac{u_n(\mu_n z + x_n)}{M_n}$$

where

$$\mu_n = M_n^{\frac{1-q}{2}}(d^2(x_n) + \frac{1}{n})^{\frac{1}{2}}.$$

It is clear that $\mu_n \to 0$ as $n \to \infty$. As above we reach that $v_n \to v$ strongly in $\mathcal{C}(\mathbb{R}^N)$ where v solves

$$-\Delta v = Cv^q \text{ in } \mathbb{R}^N,$$

a contradiction with the result of [10]. Hence the claim follows.

We then conclude that $\{u_n\}_n$ is bounded in $L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ and hence there exists $u \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,2}(\Omega) \qquad \text{and} \quad u_n \to u \text{ strongly in } L^p(\Omega)$$

for all $p \ge 1$.

To finish we have just to prove that $u \not\equiv 0$. We argue by contradiction, if $u \equiv 0$, then $u_n \to 0$ strongly in $L^p(\Omega)$ for all $p \geq 1$. We claim that

$$\int\limits_{\Omega} |\nabla u_n|^2 \phi_1 \to 0 \qquad \text{as} \quad n \to \infty,$$

where ϕ_1 is the first eigenfunction of the laplacian.

To prove the claim we use $u_n(\phi_1 + \frac{c}{n})$ as a test function in (5.1) for $c \geq \sup\limits_{\bar{\Omega}} \dfrac{\phi_1(x)}{d(x)}$. Therefore we obtain that

$$\int\limits_{\Omega} |\nabla u_n|^2 (\phi_1 + \frac{c}{n}) + \int\limits_{\Omega} u_n \nabla u_n \nabla \phi_1 \leq c \int\limits_{\Omega} \frac{u_n^{q+1}}{d + \frac{1}{n}}.$$

Hence

$$\begin{aligned} \int\limits_{\Omega} |\nabla u_n|^2 (\phi_1 + \frac{c}{n}) + \frac{\lambda_1}{2} \int\limits_{\Omega} u_n^2 \phi_1 \quad &\leq \quad c \int\limits_{\Omega} \frac{u_n^{q+1}}{d + \frac{1}{n}} \\ &\leq \quad \Big(\int\limits_{\Omega} \frac{u_n^{q+1}}{(d + \frac{1}{n})^2} \Big)^{\frac{1}{2}} \Big(\int\limits_{\Omega} u_n^{q+1} \Big)^{\frac{1}{2}} \\ &\leq \quad C \Big(\int\limits_{\Omega} u_n^{q+1} \Big)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $\int\limits_{\Omega} |\nabla u_n|^2 \phi_1 \to 0$ and the claim follows.

By elliptic regularity we conclude that $u_n \to 0$ strongly in $\mathcal{C}_{loc}(\Omega)$. Since $d(x_n) \geq C > 0$ for all n, then up to a subsequence, $u_n(x_n) \to 0$ as $n \to \infty$, a contradiction with (5.2). Hence $u \gneq 0$ and then the existence result follows. ∎

6. The Problem (1.6) with the critical power $q = 2^* - 1$

In this section we will consider (1.6) in the case $q = 2^* - 1$ if $N \geq 3$ and $q > 1$ if $N = 1, 2$. We will assume that $\Omega = B_R(0)$ is the ball of radius R centered at the origin and we will work in the space $W^{1,2}_{ra}(B_R(0))$ defined as the subspace of $W^{1,2}_0(B_R(0))$ of radial function.

For $N \geq 3$, we define

$$S(R) \equiv \inf_{\phi \in W^{1,2}_{ra}(B_R(0))} \frac{\displaystyle\int_{B_R(0)} |\nabla \phi|^2 dx}{\Big(\displaystyle\int_{B_R(0)} \frac{|\phi|^{2^*}}{d^2(x)} dx \Big)^{\frac{2}{2^*}}}. \tag{6.1}$$

Since ϕ is a radial function, then

$$\frac{\displaystyle\int\limits_{\Omega} |\nabla \phi|^2 dx}{\Big(\displaystyle\int\limits_{\Omega} \frac{|\phi|^{2^*}}{d^2(x)} dx \Big)^{\frac{2}{2^*}}} = \frac{\displaystyle\int_0^R |\phi'(r)|^2 r^{N-1} dr}{\Big(\displaystyle\int_0^R \frac{|\phi|^{2^*}}{(R-r)^2} r^{N-1} dr \Big)^{\frac{2}{2^*}}}.$$

Let us begin by proving the following Proposition.

Proposition 6.1. *Assume that $S(R)$ is defined as in (6.1), then*

(1) $S(R) > 0$ *for all* $R > 0$,
(2) $S(R) = R^{\frac{4}{2^*}} S(1)$.

Proof. We begin with the first point. Let $0 < R_1 < R$, then

$$\begin{aligned}\int_0^R \frac{|\phi|^{2^*}}{(R-r)^2} r^{N-1} dr &= \int_0^{R_1} \frac{|\phi|^{2^*}}{(R-r)^2} r^{N-1} dr + \int_{R_1}^R \frac{|\phi|^{2^*}}{(R-r)^2} r^{N-1} dr \\ &= I(R_1) + J(R_1).\end{aligned}$$

It is clear that

$$I(R_1) \leq \frac{1}{(R-R_1)^2} \int_0^R |\phi|^{2^*} r^{N-1} dr \leq C(R, R_1, N) \, ||\phi||^{2^*}_{W^{1,2}_{ra}(B_R(0))}.$$

We deal now with $J(R_1)$. For $0 < R_1 < r < R$, we have

$$\begin{aligned}|\phi(r)| &\leq \int_r^R |\phi'(s)| ds \leq \int_r^R |\phi'(s)| s^{N-1} s^{1-N} ds \\ &\leq \Big(\int_r^R |\phi'(s)|^2 s^{N-1} ds\Big)^{\frac{1}{2}} \Big(\int_r^R s^{1-N} ds\Big)^{\frac{1}{2}} \\ &\leq C(N) ||\phi||_{W^{1,2}_0} \Big(\frac{C(R)(R-r)}{(rR)^{N-2}}\Big)^{\frac{1}{2}}\end{aligned}$$

where

$$C(R) = \begin{cases} 1 & \text{if} \quad N = 3, \\ R^{N-3} & \text{if} \quad N \geq 4. \end{cases}$$

Hence

$$\int_{R_1}^R \frac{|\phi|^{2^*}}{(R-r)^2} r^{N-1} \leq C(N, R, R_1) ||\phi||^{2^*}_{W^{1,2}_0} \int_{R_1}^R (R-r)^{\frac{2^*}{2}-2} dr \leq C(N, R, R_1) ||\phi||^{2^*}_{W^{1,2}_{ra}(B_R(0))}.$$

Therefore

$$J(R_1) \leq C(N, R, R_1) ||\phi||^{2^*}_{W^{1,2}_{ra}(B_R(0))}$$

and then

$$S(R) \geq \frac{1}{C(N, R, R_1)} > 0.$$

This complete the proof of the point (1).

To prove the second estimate (2) we consider $\phi \in W^{1,2}_{ra}(B_1(0))$ and we define for $0 < r < R$, the function $\psi(r) = \phi(\frac{r}{R})$. It is clear that $\psi \in W^{1,2}_{ra}(B_R(0))$ and a direct computation yields

$$\frac{\displaystyle\int_0^R |\psi'(r)|^2 r^{N-1} dr}{\Big(\displaystyle\int_0^R \frac{|\psi|^{2^*}}{(R-r)^2} r^{N-1} dr\Big)^{\frac{2}{2^*}}} = R^{\frac{4}{2^*}} \frac{\displaystyle\int_0^1 |\phi'(r)|^2 r^{N-1} dr}{\Big(\displaystyle\int_0^1 \frac{|\phi|^{2^*}}{(1-r)^2} r^{N-1dr}\Big)^{\frac{2}{2^*}}}.$$

Thus, taking the infimum on the above identity, we get $S(R) = R^{\frac{4}{2^*}} S(1)$.

∎

We are now in position to prove Theorem 1.7.

Proof of Theorem 1.7 when $N \geq 3$**.** It is clear that if u is a solution to (1.6) in $B_1(0)$, then $v(r) = u(\frac{r}{R})$ is a solution to (1.6) in $B_R(0)$. Hence we have just to show that problem (1.6) has a solution in some ball $B_R(0)$.

Notice that $S(1) \le S$, the Sobolev constant. Hence fix $R<1$ such that $S(R) < S$. To get the desired result we have just to show that $S(R)$ is achieved. Let $\{u_n\}_n \subset W^{1,2}_{ra}(B_R(0))$, be a minimizing sequence of $S(R)$ with

$$\int_0^R \frac{|u_n|^{2^*}}{(R-r)^2} r^{N-1} dr = 1.$$

Without loss of generality we can assume that $u_n \ge 0$.

Hence we obtain that $||u_n||_{W^{1,2}_{ra}(B_R(0))} \le C$ and then we get the existence of $u \in W^{1,2}_{ra}(B_R(0))$ such that

$$u_n \rightharpoonup u \text{ weakly in } W^{1,2}_{ra}(B_R(0)), \qquad u_n \to u \text{ strongly in } L^s(B_R(0))\, \forall\, s < 2^*$$

and $u_n \to u$ strongly in $L^\sigma(B_R(0)\backslash B_\varepsilon(0))$ for all $\sigma > 1$ and for all $\varepsilon > 0$. If $u \ne 0$, then we get easily that u solves (1.6) with $q = 2^* - 1$.

Assume that $u \equiv 0$, then $u_n \to 0$ strongly in $L^\sigma(B_R(0)\backslash B_\varepsilon(0))$ for all $\sigma > 1$ and for all $\varepsilon > 0$. Fix $0 < R_1 < R$, then

$$\frac{|u_n|^{2^*}}{(R-r)^2} r \le C(N,R,R_1)||u_n||^{2^*}_{W^{1,2}_{ra}(B_R(0))}(R-r)^{\frac{2^*}{2}-2}.$$

Since $\frac{2^*}{2} - 2 > -1$, then by the dominated convergence theorem, it follows that

$$\int_{R_1}^R \frac{|u_n|^{2^*}}{(R-r)^2} r^{N-1} dr \to 0 \text{ as } n \to \infty.$$

Thus, for all $1 < R_1 < R$, we have

$$\int_{B_{R_1}(0)} \frac{|u_n|^{2^*}}{(R-|x|)^2} dx \to 1 \text{ as } n \to \infty.$$

Using the Ekeland variational principle, we obtain that, up to a subsequence,

$$-\Delta u_n = S(R)\frac{u_n^{2^*-1}}{(R-|x|)^2} + o(1). \tag{6.2}$$

Now, by the concentration compactness principle, see [14] and [15], it follows that

(1) $|\nabla u_n|^2 \rightharpoonup d\mu \ge \mu_0\delta_0$, $\quad |u_n|^{2^*} \rightharpoonup d\nu = \nu_0\delta_0$,
(2) $\mu_0 \ge S^{\frac{2}{2^*}}\nu_0$

weakly in the sense of measure, where δ_0 is the dirac measure centered at the origin.

Let now $\phi \in \mathcal{C}_0^\infty(B_R(0)) \cap W^{1,2}_{ra}(B_R(0))$ be such that

$$0 \le \phi \le 1, \quad \phi \equiv 1 \quad \text{in } B_\varepsilon(0) \qquad \text{and} \quad \phi \equiv 0 \quad \text{in } B_R(0)\backslash B_\varepsilon(0),$$

then using $u_n\phi$ as a test function in (6.2) and letting $\varepsilon \to 0$, we reach that

$$\mu_0 \le S(R)\nu_0.$$

Since $\mu_0 \ge S^{\frac{2}{2^*}}\nu_0$, then $\mu_0 \le \dfrac{S(R)}{S^{\frac{2}{2^*}}}\nu_0$.

If $\mu_0 = 0$, then $\nu_0 = 0$. Hence

$$\int_{B_R(0)} \frac{|u_n|^{2^*}}{(R-|x|)^2} dx \to \int_{B_R(0)} \frac{|u|^{2^*}}{(R-|x|)^2} dx = 1$$

a contradiction with the fact that $u \equiv 0$.

Now, if $\nu_0 > 0$, then $S^{\frac{2}{2^*}} \le S(R)$. Recall that $S(R) = R^{\frac{4}{2^*}} S(1)$, since $S(1) \le S$, we conclude that $S \ge R^{-\frac{4}{2^*-2}}$. Notice that the Sobolev constant S in independent of the domain, and in particular it is independent of R. Hence, letting $R \to 0$, we reach a contradiction. Thus $u \neq 0$ and solves (1.6) with $q = 2^* - 1$. The strong maximum principle allows us to get that $u > 0$ in $B_R(0)$.

Notice that, from the above computation, we can conclude that

$$\int_{B_R(0)} \frac{|u_n|^{2^*}}{(R-|x|)^2}dx \to \int_{B_R(0)} \frac{|u|^{2^*}}{(R-|x|)^2}dx = 1$$

and then u is a minimizer of $S(R)$.

∎

For the case $N = 1, 2$ we need the next proposition.

Proposition 6.2. *Define*

$$S_q(R) \equiv \inf_{\phi \in W^{1,2}_{ra}(B_R(0))} \frac{\displaystyle\int_{B_R(0)} |\nabla\phi|^2 dx}{\Big(\displaystyle\int_{B_R(0)} \frac{|\phi|^{q+1}}{d^2(x)}dx\Big)^{\frac{2}{q+1}}}.$$

Then

(1) $S_q(R) > 0$ *for all* $R > 0$,

(2) $S_q(R) = R^{\frac{4}{q+1}} S(1)$.

Proof. We begin by proving that $S_q(R) > 0$.

If $N = 1$, then $W^{1,2}_{ra}(B_R(0)) \subset L^\infty(\Omega)$ with a compact inclusion. Hence using Hardy inequality we obtain that

$$\int_{B_R(0)} \frac{|\phi|^{q+1}}{d^2(x)}dx \le ||\phi||^{q-2}_\infty \int_{B_R(0)} \frac{|\phi|^2}{d^2(x)}dx \le C_1 ||\phi||^{q+1}_{W^{1,2}_{ra}(B_R(0))}.$$

Thus

$$\frac{\displaystyle\int_{B_R(0)} |\nabla\phi|^2 dx}{\Big(\displaystyle\int_{B_R(0)} \frac{|\phi|^{q+1}}{d^2(x)}dx\Big)^{\frac{2}{q+1}}} \ge \frac{1}{C_1^{\frac{2}{q+1}}} > 0.$$

As a consequence $S_q(R) \ge \dfrac{1}{C_1^{\frac{2}{q}}}$ and the result follows in this case.

Assume that $N = 2$. We follow closely the computation of Proposition 6.1. Given $0 < R_1 < R$, then

$$\begin{aligned}\int_0^R \frac{|\phi|^{q+1}}{(R-r)^2}rdr &= \int_0^{R_1} \frac{|\phi|^{q+1}}{(R-r)^2}rdr + \int_{R_1}^R \frac{|\phi|^{q+1}}{(R-r)^2}rdr \\ &= I(R_1) + J(R_1).\end{aligned}$$

It is clear that

$$I(R_1) \le \frac{1}{(R-R_1)^2}\int_0^R |\phi|^{q+1} rdr \le C(R, R_1)||\phi||^{q+1}_{W^{1,2}_{ra}(B_R(0))}.$$

We deal now with $J(R_1)$. It is clear that for $R_1 < r < R$, we have

$$|\phi(r)| \le ||\phi||_{W^{1,2}_{ra}(B_R(0))}\Big(\frac{R-r}{R_1})\Big)^{\frac{1}{2}}.$$

Thus

$$\int_{R_1}^{R} \frac{|\phi|^{q+1}}{(R-r)^2} r \leq C(R_1,R)||\phi||^{q+1}_{W^{1,2}_{ra}(B_R(0))} \int_{R_1}^{R} (R-r)^{\frac{q+1}{2}-2} dr.$$

Since $q>1$, then $\int_{R_1}^{R}(R-r)^{\frac{q+1}{2}-2}dr<\infty$. Therefore,

$$J(R_1) \leq C(N,R,R_1)||\phi||^{q+1}_{W^{1,2}_{ra}(B_R(0))}.$$

Combining the above estimates, we reach the desired result.

The point (2) follows exactly as the point (2) in Proposition 6.1. Hence we omit it here. This conclude the proof of the desired result. ∎

Proof of Theorem 1.7 when $N=1,2$**.** We have just to show that $S_q(R)$ is achieved.

Let $\{u_n\}_n \subset W^{1,2}_{ra}(B_R(0))$ be a minimizing sequence of $S_q(R)$ with

$$\int_0^R \frac{|u_n|^{q+1}}{(R-r)^2} r^{N-1} dr = 1.$$

It is clear that the sequence $\{u_n\}_n$ is bounded in $W^{1,2}_{ra}(B_R(0))$ and then $u_n \rightharpoonup u$ weakly in $W^{1,2}_{ra}(B_R(0))$.

If $N=1$, then, up to a subsequence, $u_n \to u$ strongly in $\mathcal{C}(\overline{\Omega})$.

Since $|u_n(r)| \leq ||u_n||_{W^{1,2}_{ra}(B_R(0))}(R-r)^{\frac{1}{2}}$, then we conclude that

$$\frac{|u_n|^{q+1}}{(R-r)^2} \leq C(R-r)^{\frac{q-3}{2}}.$$

Since $q>1$, then $(R-r)^{\frac{q-3}{2}} \in L^1(0,R)$ and then by the dominated convergence Theorem we reach that

$$\frac{|u_n|^{q+1}}{(R-r)^2} \to \frac{|u|^{q+1}}{(R-r)^2} \text{ strongly in } L^1(0,R).$$

Thus $\int_0^R \dfrac{|u|^{q+1}}{(R-r)^2} dr = 1$ and then u solves (1.6). It is not difficult to prove that $u_n \to u$ strongly in $W^{1,2}_{ra}(B_R(0))$.

Consider now the case $N=2$. It is clear that, for $R_1<R$ fixed we have

$$\frac{|u_n|^{q+1}}{(R-r)^2} \to \frac{|u|^{q+1}}{(R-r)^2} \text{ strongly in } L^1(0,R_1).$$

To deal with the set (R_1,R), we use the estimate

$$|u_n(r)| \leq ||u_n||_{W^{1,2}_0}\Big(\frac{R-r}{R_1})\Big)^{\frac{1}{2}}.$$

The existence result now follows using the dominated convergence Theorem.

∎

7. Further results and Open problems

Assume that $0 < q < 1 < p$ and consider the following concave-convex problem

$$(7.1)\qquad \begin{cases} -\Delta u = \lambda u^q + \dfrac{u^p}{d^2} & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where $\lambda > 0$. Using a sub-supersolution arguments we can prove that, for λ small, problem (7.1) has a positive bounded solution for all $p > 1$. To see that we have just to build a suitable supersolution.

Let $\psi \in W_0^{1,2}(\Omega)$ be the positive solution of the problem

$$(7.2)\qquad \begin{cases} -\Delta \psi = \dfrac{1}{\psi^\beta} & \text{in } \Omega,\\ \psi = 0 & \text{on } \partial\Omega,\end{cases}$$

with $0 < \beta < 1$. It is clear that $C_1 d(x) \le \psi \le C_2 d(x)$ for some $C_1, C_2 > 0$. Since $p > 1$, then we can choose $\beta < 1$ such that $p > 2 - \beta$. Hence we can choose $A > 0$ such that $A\psi$ is a supersolution to the problem (7.1) at least for λ small. It is clear that if w, the unique positive solution to

$$\begin{cases} -\Delta w = \lambda w^q & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega,\end{cases}$$

is a subsolution to (7.1) with $w \le A\psi$ (that follows using the comparison principle in Lemma 2.3). Thus an iteration argument allows us to conclude.

For problem (7.1), we can summarize the main results in the following Theorem.

Theorem 7.1. *Define*

$$M = \sup\{\lambda > 0 : \text{ the problem (7.1) has a positive solution }\}$$

then $M < \infty$ and

1. *For all $\lambda < M$, then problem (7.1) has a minimal positive bounded solution.*
2. *If $\lambda > M$, there is no positive solutions.*
3. *If $p < 2^* - 1$, there exits a second positive solution at least for λ small.*

7.1. **Open problems.** In this subsection we collect some open problems.

1. In Theorem 1.7, we have considered the case $\Omega = B_R(0)$ and we have proved the existence of a positive radial solution. The behavior of the minimizing sequence near the boundary of Ω was of great utility to get the compactness of the minimizing sequence. However the arguments used are not applicable for a general domain Ω. It seems to be interesting to develop new arguments in order to analyze the critical problem in general domains.
2. The case $q > 2^* - 1$, is also interesting including for radial domain (when $N \ge 3$). Notice if we set

$$S_q(R) \equiv \inf_{\phi \in W_{ra}^{1,2}} \frac{\displaystyle\int_{B_R(0)} |\nabla \phi|^2 dx}{\Big(\displaystyle\int_{B_R(0)} \frac{|\phi|^q}{d^2} dx\Big)^{\frac{2}{q}}}.$$

then $S_q(R) = 0$ for all $R > 0$. However it is not clear how to prove that the unique "bounded" solution is 0.

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