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On the convergence of a regularization scheme for approximating cavitation solutions with prescribed cavity volume size

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Abstract

Let $\Omega \in \mathbb{R}^n$, $n = 2, 3$, be the region occupied by a hyperelastic body in its reference configuration. Let $E(\cdot)$ be the stored energy functional and let $\mathbf{x}_0$ be a flaw point in Ω (i.e., a point of possible discontinuity for admissible deformations of the body). For $V > 0$ fixed, let $\mathbf{u}_V$ be a minimizer of $E(\cdot)$ among the set of discontinuous deformations $\mathbf{u}$ constrained to form a hole of prescribed volume V at $\mathbf{x}_0$ and satisfying the homogeneous boundary data $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for $\mathbf{x} \in \partial\Omega$. In this paper we describe a regularization scheme for the computation of both $\mathbf{u}_V$ and $E(\mathbf{u}_V)$ and study its convergence properties. In particular, we show that as the regularization parameter goes to zero, (a subsequence) of the regularized constrained minimizers converge weakly in $W^{1,p}(\Omega \setminus \overline{\mathcal{B}_\delta(\mathbf{x}_0)})$ to a minimizer $\mathbf{u}_V$ for any $\delta > 0$. We obtain various sensitivity results for the dependence of the energies and Lagrange multipliers of the regularized constrained minimizers on the boundary data $\mathbf{A}$ and on the volume parameter V. We show that both the regularized constrained minimizers and $\mathbf{u}_V$ satisfy suitable weak versions of the corresponding Euler–Lagrange equations. In addition we describe the main features of a numerical scheme for approximating $\mathbf{u}_V$ and $E(\mathbf{u}_V)$ and give numerical examples for the case of a stored energy function of an elastic fluid and in the case of the incompressible limit.

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1 Introduction

In this paper we consider the problem of numerically computing a special type of minimizers within the context of a variational theory of nonlinear elasticity that allows for cavitation. The particular problem we study is that in which the minimizer of the stored energy functional belongs to a set of discontinuous deformations that satisfy homogeneous boundary data and that produce a (not necessarily spherical) hole within the deformed body of prescribed volume V. The proposed numerical scheme essentially consists of approximating the original constrained problem by a sequence of *regularized constrained problems* over punctured domains, where the punctures are taken around possible flaw points within the body. If ε represents the diameter of the punctures in the regularized domains, we show the convergence of the numerical approximations to that of the original problem as $\varepsilon \searrow 0$.

The problem considered in this paper, though related, differs significantly from that of "standard" cavitation (see [1], [15]) in which just the homogeneous boundary data $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for $\mathbf{x} \in \partial\Omega$, is specified. Depending on the matrix $\mathbf{A}$, the global minimizer may be discontinuous producing a hole or cavitation inside the body of volume that depends among other things on the norm of $\mathbf{A}$. The numerical aspects of cavitation have been studied among others by [10], [11], [16], [8], [12], and [13]. A fundamental problem in studies of cavitation is to analytically or computationally characterize the boundary data $\mathbf{A}$ for which cavitation occurs. In [18] the authors introduced the concept of the *volume derivative* (cf. (3.16)) as a tool for characterizing these boundary displacements. The problem considered in this paper is central for the definition and computation of the volume derivative.

Each of the regularized constrained problems over punctured domains mentioned previously is approximated by a sequence of regularized (unconstrained) but penalized problems. For a quadratic penalty, the convergence of the minimizers in this inner iteration can be easily established (cf. [18]). In this paper, instead of just using a penalty parameter to deal with the volume constraint in the regularized problems, we employ a penalty–multiplier technique, also called augmented Lagrangians (cf.[14]), that leads to a more stable numerical scheme for computing such minimizers. Under the standard assumption that the sequence of the generated multipliers remains bounded (cf. [3]), we then show that the minimizers of the regularized (unconstrained) penalized prob-

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lems with its corresponding sequence of multipliers, have subsequences converging to a minimizer and to a multiplier respectively, of the corresponding regularized constrained problem.

To introduce the results in the paper, consider a nonlinear hyperelastic body occupying the bounded region $\Omega \subset \mathbb{R}^n$ in its reference state. A deformation of the body is a mapping $\mathbf{u} : \Omega \to \mathbb{R}^n$ satisfying the local invertibility condition

$$\det \nabla \mathbf{u}(\mathbf{x}) > 0 \quad \text{a.e. } \mathbf{x} \in \Omega.$$

The energy stored in the deformed body under a deformation $\mathbf{u}$ is given by

$$E(\mathbf{u}) = \int_\Omega W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}, \tag{1.1}$$

where $W : M_+^{n \times n} \to \mathbb{R}$ is the stored energy function of the material and $M_+^{n \times n}$ denotes the set of $n \times n$ matrices with positive determinant. For a fixed matrix $\mathbf{A} \in M_+^{n \times n}$, we consider deformations satisfying the *displacement boundary* condition:

$$\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \text{ for } \mathbf{x} \in \partial\Omega. \tag{1.2}$$

We fix a "flaw" point $\mathbf{x}_0 \in \Omega$ and take the *admissible set* of deformations to be

$$\mathcal{A}_\mathbf{A} = \left\{ \mathbf{u} \in W^{1,p}(\Omega) \mid \exists \, \alpha \geq 0 \text{ such that } \mathrm{Det}\nabla\mathbf{u} = \det\nabla\mathbf{u} \, \mathcal{L}^n + \alpha\delta_{\mathbf{x}_0}, \right. \tag{1.3}$$
$$\left. \det \nabla\mathbf{u} > 0 \text{ a.e., } \mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \text{ on } \partial\Omega, \ \mathbf{u} \text{ satisfies INV on } \Omega \right\}.$$

Here $\mathrm{Det}\nabla\mathbf{u}$ denotes the distributional determinant of $\mathbf{u}$, defined by

$$< \mathrm{Det}\nabla\mathbf{u}, \phi > = -\frac{1}{n} \int_\Omega \nabla\phi \cdot (\mathrm{adj}\,\nabla\mathbf{u})\mathbf{u} \, d\mathbf{x}, \quad \forall \ \phi \in C_0^\infty(\Omega), \tag{1.4}$$

$\mathcal{L}^n$ denotes n-dimensional Lebesgue measure, $p > n - 1$, $\delta_{\mathbf{x}_0}$ denotes the Dirac measure supported at $\mathbf{x}_0 \in \Omega$, and (INV) denotes the condition[1] relating to invertibility introduced in Definition 3.2 of [15]. Results in [21] give conditions on the stored energy function W under which a minimiser for (1.1) exists on the set $\mathcal{A}_\mathbf{A}$. The results of Henao and Mora-Corral [6] give conditions under which a minimiser also exists in the case $p = n - 1$ and their work in [7] includes justification of the interpretation of α in (1.3) as the volume of

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[1]For technical reasons, the deformation $\mathbf{u}$ has to be extended to a larger domain, whilst still satisfying (INV) on the extended domain, for example by setting it equal to $A\mathbf{x}$ outside Ω (see [21] for further details). Henceforth we shall assume that all deformations have been extended accordingly without introducing any extra notation.

the hole formed by the deformation. Hence if $\mathbf{u} \in \mathcal{A}_{\mathbf{A}}$ and $\alpha > 0$, then the deformation $\mathbf{u}$ produces a hole of volume α in the deformed body.

The requirement that deformations produce a hole of volume V in the deformed body is equivalent to the *integral constraint*:

$$c(\mathbf{u}) \equiv \int_{\Omega} \det \nabla \mathbf{u} \, d\mathbf{x} - (\det \mathbf{A}) \, |\Omega| + V = 0. \tag{1.5}$$

(Here $|\Omega|$ is the volume of Ω and $0 < V < (\det \mathbf{A}) \, |\Omega|$.) Thus the *volume constrained problem* that we consider in this paper is given by:

$$\begin{cases} \min_{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}} & E(\mathbf{u}), \\ \text{subject to} & c(\mathbf{u}) = 0. \end{cases} \tag{1.6}$$

For any $\varepsilon > 0$ sufficiently small, let

$$\Omega_{\varepsilon} = \Omega \setminus \overline{\mathcal{B}_{\varepsilon}(\mathbf{x}_0)}.$$

(Here and henceforth, we use the notation $\mathcal{B}_{\varepsilon}(\mathbf{x}_0)$ for the open ball of radius ε centered at $\mathbf{x}_0$.) The *regularized constrained* minimization problem is given by:

$$\begin{cases} \min_{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^{\varepsilon}} & E_{\varepsilon}(\mathbf{u}), \\ \text{subject to} & c_{\varepsilon}(\mathbf{u}) = 0, \end{cases} \tag{1.7}$$

where

$$E_{\varepsilon}(\mathbf{u}) = \int_{\Omega_{\varepsilon}} W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}, \quad c_{\varepsilon}(\mathbf{u}) = \int_{\Omega_{\varepsilon}} \det \nabla \mathbf{u} \, d\mathbf{x} - (\det \mathbf{A}) \, |\Omega| + V,$$

and

$$\begin{aligned} \mathcal{A}_{\mathbf{A}}^{\varepsilon} \;=\; & \{\mathbf{u} \in W^{1,p}(\Omega_{\varepsilon}) \,|\, \mathrm{Det} \nabla \mathbf{u} = (\det \nabla \mathbf{u}) \, \mathcal{L}^n, \; \det \nabla \mathbf{u} > 0 \text{ a.e.,} \\ & \mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \text{ on } \partial\Omega, \; \mathbf{u} \text{ satisfies INV}\}, \end{aligned}$$

Note that deformations $\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^{\varepsilon}$ are "regular" in the sense that the singular part (with respect to Lebesgue measure) of the distribution $\mathrm{Det} \nabla \mathbf{u}$ is zero. We set $\mathcal{A}_{\mathbf{A}}^{0} = \mathcal{A}_{\mathbf{A}}$ and $c_0 = c$, and define the sets

$$\mathcal{C}_{\mathbf{A}}^{\varepsilon} \equiv \{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^{\varepsilon} \,|\, c_{\varepsilon}(\mathbf{u}) = 0\}.$$

Thus (1.7) is now equivalent to $\min_{\mathbf{u} \in \mathcal{C}_{\mathbf{A}}^{\varepsilon}} E_{\varepsilon}(\mathbf{u})$.

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Remark 1.1. The hypotheses and results of [21] are easily adapted to prove that a (not necessarily unique) minimiser $\mathbf{u}_{V,\varepsilon}$ of E_ε on $\mathcal{C}_{\mathbf{A}}^\varepsilon$ exists for each $\varepsilon \geq 0$ and $V > 0$ small enough.

To compute approximations of the constrained problem (1.7), we use a penalty–multiplier method in which the energy functional in (1.7) is replaced by:

$$E_{\varepsilon,\mu,\eta}(\mathbf{u}) = E_\varepsilon(\mathbf{u}) + \mu\, c_\varepsilon(\mathbf{u}) + \frac{1}{2}\eta\, c_\varepsilon(\mathbf{u})^2. \tag{1.8}$$

Here η is a "large" positive parameter and $\mu \in \mathbb{R}$. Thus we replace the regularized constrained problem (1.7) with the regularized "unconstrained" problem:

$$\inf_{\mathbf{u}\in\mathcal{A}_{\mathbf{A}}^\varepsilon} E_{\varepsilon,\mu,\eta}(\mathbf{u}). \tag{1.9}$$

In Proposition 2.1 we prove the existence of a minimizer $\mathbf{u}_{V,\varepsilon,\mu,\eta}$ for (1.9). Then in Theorem 2.2 and for $\varepsilon, V > 0$ fixed, we show how to construct sequences $\{\mu_j\}$, $\{\eta_j\}$ and give conditions under which $\{\mathbf{u}_{V,\varepsilon,\mu_j,\eta_j}\}$ converges weakly in $W^{1,p}(\Omega_\varepsilon)$ to a solution $\mathbf{u}_{V,\varepsilon}$ of (1.7), and with $c_\varepsilon(\mathbf{u}_{V,\varepsilon,\mu_j,\eta_j}) \to 0$. In Theorem 2.5 we establish a result on the weak form of the Euler–Lagrange equations for the minimizer $\mathbf{u}_{V,\varepsilon}$. This result is then used to study the sensitivity of the minimum energy $E_\varepsilon(\mathbf{u}_{V,\varepsilon})$ and its corresponding Lagrange multiplier, with respect to variations in the boundary data $\mathbf{A}$ and the volume V (Theorem 2.7).

In Section 3 we prove several key results that will be used as the basis for a numerical scheme for computing a minimizer $\mathbf{u}_V$ of (1.6). First in Theorem 3.3 we show that for a sequence $\{\varepsilon_j\}$ converging to zero, a subsequence of the corresponding regularized constrained minimizers $\{\mathbf{u}_{V,\varepsilon_j}\}$, converges weakly in $W^{1,p}(\Omega_\delta)$ to a solution $\mathbf{u}_V$ of (1.6), for any $\delta > 0$. The main difficulty in this proof is to show that the limiting function $\mathbf{u}_V$ is a solution of (1.6), in particular that it satisfies the integral volume constraint in (1.6). Two other important results in Section 3 are, firstly, on the weak form of the Euler–Lagrange equations satisfied by the minimizer $\mathbf{u}_V$ (Theorem 3.5) and, secondly, a result on the convergence as $V \searrow 0$ of the Lagrange multiplier μ_V corresponding to the volume constraint on $\mathbf{u}_V$, to the volume derivative (see Theorem 3.6). We should mention that apart from the extra complications of dealing with the volume constraint, a major technical difficulty in this section is due to the fact that the domains of the functions in the sequences appearing in most of the calculations, are changing around the possible flaw point $\mathbf{x}_0$ with the sequential index. This adds a considerable level of complication to obtain the various estimates needed to establish certain limits of weakly converging sequences.

A simple class of polyconvex isotropic stored energy functions to which the results in this paper can be applied is given by

$$W(\mathbf{F}) = \frac{\kappa}{p} |\mathbf{F}|^p + h(\det \mathbf{F}), \tag{1.10}$$

where $|\mathbf{F}| = \sqrt{\mathbf{F} \cdot \mathbf{F}}$, $\kappa > 0$, $p \in (n-1, n)$ and $h : (0, \infty) \to (0, \infty)$ is such that

$$h \text{ is a } C^2 \text{ convex function and} \tag{1.11a}$$

$$h(\delta) \to \infty \text{ and } \frac{h(\delta)}{\delta} \to \infty \text{ as } \delta \to 0, \infty \text{ respectively.} \tag{1.11b}$$

However, we note that the results of this paper are readily extended to apply to more general polyconvex stored energy functions under varied hypotheses.

In Section 4 we describe the main features of a numerical scheme for approximating solutions of the problem (1.6). Then we use this scheme in a numerical example for the case of an elastic fluid (corresponding to $\kappa = 0$ in (1.10)). For this class of materials and for a spherical domain, an exact solution of (1.6) is known and we can thus check the various convergence results in the paper in this case. We also report some simulations for the so called *incompressible limit* case. Here, we add a term of the form $k(\det \mathbf{F} - 1)^2$ to (1.10) (with $\kappa > 0$), where $k > 0$ is a given constant and, although the solutions of the intermediate problems with k given are not known explicitly, the limiting case ($k \to \infty$) corresponds to an incompressible material for which the solution is known explicitly. Thus, in this case, we can test the robustness of the scheme by computing the solutions of several intermediate volume constrained problems (with V fixed but k varying), and test for convergence of the computed solutions to the limiting incompressible solution as k gets large.

2 A penalty–multiplier method for the solution of the regularized constrained problems

In this section we study the approximation of minimizers of the constrained minimization problem (1.7) by minimizers of the unconstrained minimization problems (1.9) for some sequences $\{\mu_j\}$, $\{\eta_j\}$. We assume that the stored energy function $W(\mathbf{F})$ satisfies the following:

H1: (Polyconvexity) There exists $G : (M_+^{n \times n})^{n-1} \times (0, \infty) \to \mathbb{R}$ continuous and convex such that

$$W(\mathbf{F}) = \begin{cases} G(\mathbf{F}, \det \mathbf{F}) & , \quad n = 2, \\ G(\mathbf{F}, \operatorname{adj} \mathbf{F}, \det \mathbf{F}) & , \quad n = 3. \end{cases}$$

H2: (Growth) There exists $p \in (n-1, n)$, $c_1 > 0$, and a C^2 function h such that

$$W(\mathbf{F}) \geq c_1 |\mathbf{F}|^p + h(\det \mathbf{F}) \quad \text{for} \quad \mathbf{F} \in M_+^{n \times n},$$

where the function h satisfies conditions (1.11).

We begin by showing the existence of minimizers for problem (1.9). Note that because of the boundary and INV conditions for functions $\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^\varepsilon$, we have that

$$-(\det \mathbf{A}) |\Omega| + V \leq c_\epsilon(\mathbf{u}) \leq \int_{\Omega_\varepsilon} \det \nabla \mathbf{u}(\mathbf{x}) \, d\mathbf{x} \leq (\det \mathbf{A}) |\Omega|, \qquad (2.1)$$

by the non–negativity of the determinant and so $c_\varepsilon(\mathbf{u})$ is (uniformly) bounded on $\mathcal{A}_{\mathbf{A}}^\varepsilon$.

Proposition 2.1. *For any $\mu \in \mathbb{R}$, $\eta > 0$, there exists a minimizer $\mathbf{u}_{V,\varepsilon,\mu,\eta} \in \mathcal{A}_{\mathbf{A}}^\varepsilon$ of $E_{\varepsilon,\mu,\eta}(\mathbf{u})$ on $\mathcal{A}_{\mathbf{A}}^\varepsilon$. Moreover, for any $\delta > 0$, the parameter η can be chosen sufficiently large such that the minimizer $\mathbf{u}_{V,\varepsilon,\mu,\eta}$ satisfies that $|c_\varepsilon(\mathbf{u}_{V,\varepsilon,\mu,\eta})| < \delta$.*

Proof. Since the homogeneous deformation $\mathbf{u} = A\mathbf{x}$ lies in $\mathcal{A}_{\mathbf{A}}^\varepsilon$,

$$g^* := \inf_{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^\varepsilon} E_{\varepsilon,\mu,\eta}(\mathbf{u}) < \infty.$$

By the non–negativity of W and since $\eta > 0$, we obtain $E_{\varepsilon,\mu,\eta}(\mathbf{u}) \geq \mu c_\varepsilon(\mathbf{u})$ for all $\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^\varepsilon$. By the uniform boundedness of $c_\varepsilon(\mathbf{u})$ mentioned above, it follows that $g^* \neq -\infty$.

Let now $\{\mathbf{u}_k\}$ in $\mathcal{A}_{\mathbf{A}}^\varepsilon$ be an infimizing sequence, i.e., $E_{\varepsilon,\mu,\eta}(\mathbf{u}_k) \to g^*$. Since $\{\mu c_\varepsilon(\mathbf{u}_k)\}$ is bounded, we can find an $L > 0$ such that $\mu c_\varepsilon(\mathbf{u}_k) \geq -L$ for all k. Thus, for k sufficiently large, we obtain

$$\int_{\Omega_\varepsilon} W(\nabla \mathbf{u}_k(\mathbf{x})) \, d\mathbf{x} - L \leq g^* + 1.$$

It follows now from the growth hypotheses (H1)–(H2) that there exists a subsequence $\{\mathbf{u}_{k_j}\}$ which converges weakly in $W^{1,p}(\Omega_\varepsilon)$ to a function $\mathbf{u}^*$, and that $\{\det \nabla \mathbf{u}_{k_j}\}$ converges weakly in $L^1(\Omega_\varepsilon)$ to a function θ. Since $p \in (n-1, n)$, it follows from [15, Theorem 4.2], that $\mathbf{u}^*$ satisfies condition INV, $\theta = \det \nabla \mathbf{u}^*$, and $\det \nabla \mathbf{u}^* > 0$ almost everywhere. Thus $\mathbf{u}^* \in \mathcal{A}_{\mathbf{A}}^\varepsilon$.

Upon adapting the lower semi–continuity results in [21], it follows that $E_{\varepsilon,\mu,\eta}$ is sequentially weakly lower semi–continuous. Thus we have that

$$E_{\varepsilon,\mu,\eta}(\mathbf{u}^*) \leq \liminf_{j \to \infty} E_{\varepsilon,\mu,\eta}(\mathbf{u}_{k_j}) = g^*,$$

i.e., that $\mathbf{u}_{V,\varepsilon,\mu,\eta} \equiv \mathbf{u}^* \in \mathcal{A}_{\mathbf{A}}^\varepsilon$ is a minimizer.

For the last part of the proposition, we argue by contradiction. Suppose that for some δ_0 there exists a sequence $\eta_j \to \infty$ such that the corresponding minimizers $\{\mathbf{u}_j\}$ satisfy $|c_\varepsilon(\mathbf{u}_j)| \geq \delta_0$ for all j. Note that for all j,

$$E_{\varepsilon,\mu,\eta_j}(\mathbf{u}_j) \leq f_\varepsilon^*, \tag{2.2}$$

where f_ε^* is the minimum value in (1.7) (cf. Remark 1.1). Since $\{\mu c_\varepsilon(\mathbf{u}_j)\}$ is bounded, we can find $L > 0$ such that, $\mu c_\varepsilon(\mathbf{u}_j) \geq -L$ for all j. Hence

$$f_\varepsilon^* \geq \mu c_\varepsilon(\mathbf{u}_j) + \frac{1}{2}\eta_j c_\varepsilon(\mathbf{u}_j)^2 \geq -L + \frac{1}{2}\eta_j \delta_0^2 \to \infty,$$

which leads to a contradiction, completing the proof. $\qquad\square$

We now show how to construct sequences $\{\mu_j\}$ and $\{\eta_j\}$ and give hypotheses under which, the computed minimizers in (1.9), converge to a solution of (1.7).

Theorem 2.2. *Let the stored energy function W satisfy the conditions H1–H2. Let $\gamma \in (0,1)$, $\beta > 1$, $\eta_1 > 0$, $\mu_1 \in \mathbb{R}$, and $\mathbf{u}_0 \in \mathcal{A}_{\mathbf{A}}^\varepsilon$ be given. Let the sequences $\{\mu_j\}$, $\{\eta_j\}$, and $\{\mathbf{u}_j\}$ be given by:*

$$E_{\varepsilon,\mu_j,\eta_j}(\mathbf{u}_j) = \min_{\mathbf{u}\in\mathcal{A}_{\mathbf{A}}^\varepsilon} E_{\varepsilon,\mu_j,\eta_j}(\mathbf{u}), \tag{2.3a}$$

$$\mu_{j+1} = \mu_j + \eta_j c_\varepsilon(\mathbf{u}_j), \tag{2.3b}$$

$$\eta_{j+1} = \begin{cases} \eta_j & , \quad if \ |c_\varepsilon(\mathbf{u}_j)| \leq \gamma |c_\varepsilon(\mathbf{u}_{j-1})|, \\ \beta\eta_j & , \quad otherwise. \end{cases} \tag{2.3c}$$

Assume that $\{\mu_j\}$ is bounded. Then $c_\varepsilon(\mathbf{u}_j) \to 0$, and $\{\mathbf{u}_j\}$ has a subsequence $\{\mathbf{u}_{j_k}\}$ that converges weakly in $W^{1,p}(\Omega_\varepsilon)$ to a minimizer $\mathbf{u}_\varepsilon$ of problem (1.7) and with

$$E_\varepsilon(\mathbf{u}_\varepsilon) = \liminf_k E_{\varepsilon,\mu_{j_k},\eta_{j_k}}(\mathbf{u}_{j_k}). \tag{2.4}$$

Proof. By Proposition 2.1, a function $\mathbf{u}_j \in \mathcal{A}_{\mathbf{A}}^\varepsilon$ satisfying (2.3a) exists for each j. From (2.2) we get that

$$E_{\varepsilon,\mu_j,\eta_j}(\mathbf{u}_j) \leq f_\varepsilon^*, \quad \forall j.$$

From this inequality and using that W is nonnegative, we get that

$$\mu_j c_\varepsilon(\mathbf{u}_j) + \frac{1}{2}\eta_j c_\varepsilon(\mathbf{u}_j)^2 \leq f_\varepsilon^*, \quad \forall j. \tag{2.5}$$

Note that the sequence $\{\eta_j\}$ is increasing. Thus in (2.3c) we have two possibilities:

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1. the sequence $\{\eta_j\}$ remains bounded, in which case, $|c_\varepsilon(\mathbf{u}_j)| \leq \gamma \, |c_\varepsilon(\mathbf{u}_{j-1})|$ is satisfied for all but finitely many indexes j. Clearly $c_\varepsilon(\mathbf{u}_j) \to 0$ in this case.

2. Otherwise (for a subsequence) $\eta_j \to \infty$, in which case (2.5) and the boundedness of $\{\mu_j\}$ would imply that $c_\varepsilon(\mathbf{u}_j) \to 0$.

Thus, in both cases, we have that $c_\varepsilon(\mathbf{u}_j) \to 0$.

If $\mu_j c_\varepsilon(\mathbf{u}_j) \geq -L$ for all j, where $L > 0$, then from (2.2) we get that

$$\int_{\Omega_\varepsilon} W(\nabla \mathbf{u}_j(\mathbf{x})) \, \mathrm{d}\mathbf{x} - L \leq f_\varepsilon^*.$$

By the arguments in the proof of Proposition 2.1, there exists a subsequence $\{\mathbf{u}_{j_k}\}$ which converges weakly in $W^{1,p}(\Omega_\varepsilon)$ to a function $\mathbf{u}_\varepsilon$, and such that $\{\det \nabla \mathbf{u}_{j_k}\}$ converges weakly in $L^1(\Omega_\varepsilon)$ to $\det \nabla \mathbf{u}_\varepsilon$, where $\mathbf{u}_\varepsilon$ satisfies condition INV and $\det \nabla \mathbf{u}_\varepsilon > 0$ almost everywhere. Thus $\mathbf{u}_\varepsilon \in \mathcal{A}_{\mathbf{A}}^\varepsilon$ and $c_\varepsilon(\mathbf{u}_\varepsilon) = 0$. Moreover, since $\mu_j c_\varepsilon(\mathbf{u}_j) \to 0$ by the assumed boundedness of $\{\mu_j\}$, we have that

$$f_\varepsilon^* \leq E_\varepsilon(\mathbf{u}_\varepsilon) \leq \liminf_k E_{\varepsilon, \mu_{j_k}, \eta_{j_k}}(\mathbf{u}_{j_k}) \leq f_\varepsilon^*.$$

It follows that $\mathbf{u}_\varepsilon$ is a minimizer of (1.7) and that (2.4) holds. $\qquad \square$

Remark 2.3. The multiplier iteration (2.3b) is the most common type of iteration used in the augmented Lagrangian scheme. The motivation for this formula comes from the observation that the multiplier for the problem $\inf_{\mathbf{u} \in \mathcal{C}_{\mathbf{A}}^\varepsilon} E_{\varepsilon, \mu_j, 0}(\mathbf{u})$ is $\mu_\varepsilon - \mu_j$ where μ_ε is the Lagrange multiplier corresponding to the problem (1.7). On the other hand, since $E_{\varepsilon, \mu_j, \eta_j}(\cdot)$ is a quadratic penalty function for this problem, one expects $\eta_j c_\varepsilon(\mathbf{u}_j)$ to be close to $\mu_\varepsilon - \mu_j$ for η_j sufficiently large. Hence $\mu_\varepsilon - \mu_j \approx \eta_j c_\varepsilon(\mathbf{u}_j)$ from which (2.3b) follows. (See [9], [14, Pages 451–452].)

Remark 2.4. The assumption of boundedness on the multiplier sequence $\{\mu_j\}$ is typical of local convergence results for the augmented Lagrangian scheme (cf. [3, Proposition 2]). One could in practice enforce this condition by requiring that the iterates remain on a prescribed bounded interval. However if this interval does not contain the actual multiplier μ_ε of the problem (1.7), then this would impede the convergence of $\{\mu_j\}$ to μ_ε. A better practice is just to monitor the growth of the μ_j to detect some possible tendency to unboundedness.

Our next results give conditions under which the minimizer $\mathbf{u}_\varepsilon$ in (1.7), satisfies a weak form of the Euler-Lagrange equations for this problem. We use the following modified version of hypothesis H2 for the term $\tilde{W}$ in the stored energy function in the next theorem:

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H3: (Growth) There exists a C^2 function h such that

$$\tilde{W}(\mathbf{F}) \geq h(\det \mathbf{F}) \quad \text{for} \quad \mathbf{F} \in M_+^{n \times n},$$

where the function h satisfies conditions (1.11).

In the following we relabelled the subsequence $\{\mathbf{u}_{j_k}\}$ in Theorem 2.2 to $\{\mathbf{u}_j\}$.

Theorem 2.5. *Let $\{\mathbf{u}_j\}$ be the sequence in Theorem 2.2 that converges weakly in $W^{1,p}(\Omega_\varepsilon)$ to a solution $\mathbf{u}_\varepsilon$ of (1.7) and with $\{\mu_j\}$ bounded. Assume that the stored energy function W is uniformly quasiconvex of the form $\gamma|\mathbf{F}|^p + \tilde{W}(\mathbf{F})$ where $\gamma > 0$ and $\tilde{W}$ satisfy H1 and H3. Furthermore, assume there exist constants $K, \varepsilon_0 > 0$ such that[2]:*

$$\left| \frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\mathbf{CF})\mathbf{F}^T \right| \leq K\left[W(\mathbf{F}) + 1 \right] \quad \text{for all } \mathbf{F} \in \mathrm{M}_+^{n \times n}, \tag{2.6}$$

whenever $|\mathbf{C} - \mathbf{I}| < \varepsilon_0$. Then $\{\mu_j\}$ has a subsequence converging to μ_ε, where[3]

$$\int_{\Omega_\varepsilon} \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon \left(\mathrm{adj}\, \nabla \mathbf{u}_\varepsilon \right)^T \right] \cdot \nabla[\mathbf{v}(\mathbf{u}_\varepsilon)]\, \mathrm{d}\mathbf{x} = 0, \tag{2.7}$$

for all $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$, where $\mathcal{E} = \{\mathbf{Ax} : \mathbf{x} \in \Omega\}$. Moreover if $\mathbf{u}_\varepsilon \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega}_\varepsilon)$ with $\det \nabla \mathbf{u}_\varepsilon > 0$ in Ω_ε, then

$$\mathrm{div}\left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon \left(\mathrm{adj}\, \nabla \mathbf{u}_\varepsilon \right)^T \right] = \mathbf{0}, \quad in \;\; \Omega_\varepsilon, \tag{2.8a}$$

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{Ax} \quad on \;\; \partial\Omega, \tag{2.8b}$$

$$\left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon \left(\mathrm{adj}\, \nabla \mathbf{u}_\varepsilon \right)^T \right] \mathbf{n} = \mathbf{0} \quad on \;\; \partial\mathcal{B}_\varepsilon(\mathbf{x}_0), \tag{2.8c}$$

$$\int_{\Omega_\varepsilon} \det \nabla \mathbf{u}_\varepsilon \, \mathrm{d}\mathbf{x} = (\det \mathbf{A}) \, |\Omega| - V. \tag{2.8d}$$

Proof. To show (2.7), we first derive the corresponding equilibrium equation for each $\mathbf{u}_j$. We use variations of $\mathbf{u}_j$ of the form $\mathbf{u}_s = \mathbf{u}_j + s\mathbf{v}(\mathbf{u}_j)$ where $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$. From [21, Corollary 6.4] it follows that for s sufficiently small, the function

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[2]We refer to [2] for a discussion of (2.6) an other related constitutive hypotheses.
[3]μ_ε is the Lagrange multiplier corresponding to the volume constraint in (1.7) and is a measure of the Cauchy stress acting on the deformed inner cavity.

$\mathbf{u}_s \in \mathcal{A}_{\mathbf{A}}^\varepsilon$. (Note that the variation $\mathbf{u}_s$ is not required to satisfy the constraint $c_\varepsilon(\mathbf{u}) = 0$ as $\mathbf{u}_j$ is a solution of an unconstrained problem!) To show (2.7) for $\mathbf{u}_j$, first note that

$$\int_{\Omega_\varepsilon} [W(\nabla \mathbf{u}_s) - W(\nabla \mathbf{u}_j)] \mathrm{d}\mathbf{x} = s \int_{\Omega_\varepsilon} \left[\int_0^1 \frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}} ([\mathbf{I} + st\nabla\mathbf{v}(\mathbf{u}_j)]\nabla\mathbf{u}_j)\nabla\mathbf{u}_j^T \, \mathrm{d}t \right] \cdot \nabla\mathbf{v}(\mathbf{u}_j) \mathrm{d}\mathbf{x}$$

It follows now from (2.6) that for s small enough,

$$\left| \int_0^1 \frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}} ([\mathbf{I} + st\nabla\mathbf{v}(\mathbf{u}_j))]\nabla\mathbf{u}_j)\nabla\mathbf{u}_j^T \, \mathrm{d}t \right| \leq K[W(\nabla\mathbf{u}_j) + 1] \in L^1(\Omega_\varepsilon).$$

Upon invoking the Dominated Convergence Theorem, we obtain

$$\lim_{s \to 0} \frac{1}{s} \int_{\Omega_\varepsilon} [W(\nabla\mathbf{u}_s) - W(\nabla\mathbf{u}_j)] \, \mathrm{d}\mathbf{x} = \int_{\Omega_\varepsilon} \frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}} (\nabla\mathbf{u}_j)\nabla\mathbf{u}_j^T \cdot \nabla\mathbf{v}(\mathbf{u}_j) \, \mathrm{d}\mathbf{x} \qquad (2.9)$$

Also

$$\mu_j[c_\varepsilon(\mathbf{u}_s) - c_\varepsilon(\mathbf{u}_j)] + \frac{1}{2}\eta_j[c_\varepsilon^2(\mathbf{u}_s) - c_\varepsilon^2(\mathbf{u}_j)]$$
$$= [\mu_j + \frac{1}{2}\eta_j(c_\varepsilon(\mathbf{u}_s) + c_\varepsilon(\mathbf{u}_j))][c_\varepsilon(\mathbf{u}_s) - c_\varepsilon(\mathbf{u}_j)].$$

Now

$$c_\varepsilon(\mathbf{u}_s) - c_\varepsilon(\mathbf{u}_j) = s \int_{\Omega_\varepsilon} \left[\int_0^1 [\mathrm{adj}\,(\mathbf{I} + st\nabla\mathbf{v}(\mathbf{u}_j))]^T \, \mathrm{d}t \right] \cdot \nabla\mathbf{v}(\mathbf{u}_j) \det \nabla\mathbf{u}_j \, \mathrm{d}\mathbf{x}.$$

It follows now since $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$, that

$$\lim_{s \to 0} \frac{1}{s}[c_\varepsilon(\mathbf{u}_s) - c_\varepsilon(\mathbf{u}_j)] = \int_{\Omega_\varepsilon} [\mathbf{I} \cdot \nabla\mathbf{v}(\mathbf{u}_j)] \det \nabla\mathbf{u}_j \, \mathrm{d}\mathbf{x}.$$

Combining this with (2.9) and using that $c_\varepsilon(\mathbf{u}_s) \to c_\varepsilon(\mathbf{u}_j)$ as $s \to 0$, we get that

$$\left. \frac{\mathrm{d}}{\mathrm{d}s} E_{\varepsilon,\mu_j,\eta_j}(\mathbf{u}_s) \right|_{s=0} = \int_{\Omega_\varepsilon} \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}} (\nabla\mathbf{u}_j)\nabla\mathbf{u}_j^T \right.$$
$$\left. + (\mu_j + \eta_j c_\varepsilon(\mathbf{u}_j))(\det \nabla\mathbf{u}_j)\mathbf{I} \right] \cdot \nabla\mathbf{v}(\mathbf{u}_j) \, \mathrm{d}\mathbf{x}.$$

Since $\mathbf{u}_j$ is a minimizer, we must have that

$$\int_{\Omega_\varepsilon} \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}} (\nabla\mathbf{u}_j)\nabla\mathbf{u}_j^T + (\mu_j + \eta_j c_\varepsilon(\mathbf{u}_j))(\det \nabla\mathbf{u}_j)\mathbf{I} \right] \cdot \nabla\mathbf{v}(\mathbf{u}_j) \, \mathrm{d}\mathbf{x} = 0, \qquad (2.10)$$

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for all such $\mathbf{v}$'s. Recall the subsequence $\{\mathbf{u}_j\}$ converges weakly in $W^{1,p}(\Omega_\varepsilon)$ to $\mathbf{u}_\varepsilon$, with $\det \nabla \mathbf{u}_j \rightharpoonup \det \nabla \mathbf{u}_\varepsilon$ in $L^1(\Omega_\varepsilon)$. Furthermore, because of (2.4), we may assume that the sequence is such that

$$E_\varepsilon(\mathbf{u}_\varepsilon) = \lim_j E_{\varepsilon,\mu_j,\eta_j}(\mathbf{u}_j),$$

and, by the boundedness of $\{\mu_j\}$, that $\mu_j \to \mu_\varepsilon$ for some μ_ε. Thus

$$\lim_j \left[\mu_j c_\varepsilon(\mathbf{u}_j) + \frac{1}{2}\eta_j c_\varepsilon(\mathbf{u}_j)^2 \right] = 0.$$

It follows now that

$$
\begin{aligned}
E_\varepsilon(\mathbf{u}_\varepsilon) \;-\; \gamma \int_{\Omega_\varepsilon} |\nabla \mathbf{u}_\varepsilon|^p \, d\mathbf{x} &= \int_{\Omega_\varepsilon} \tilde{W}(\nabla \mathbf{u}_\varepsilon) \, d\mathbf{x}, \\
&\leq \liminf_j \left[\int_{\Omega_\varepsilon} \tilde{W}(\nabla \mathbf{u}_j) \, d\mathbf{x} + \mu_j c_\varepsilon(\mathbf{u}_j) + \frac{1}{2}\eta_j c_\varepsilon(\mathbf{u}_j)^2 \right], \\
&= \lim_j E_{\varepsilon,\mu_j,\eta_j}(\mathbf{u}_j) - \gamma \limsup_j \int_{\Omega_\varepsilon} |\nabla \mathbf{u}_j|^p \, d\mathbf{x},
\end{aligned}
$$

from which we obtain

$$\limsup_k \int_{\Omega_\varepsilon} |\nabla \mathbf{u}_j|^p \, d\mathbf{x} \leq \int_{\Omega_\varepsilon} |\nabla \mathbf{u}_\varepsilon|^p \, d\mathbf{x}.$$

This together with the weak convergence of $\{\mathbf{u}_j\}$ to $\mathbf{u}_\varepsilon$ in $W^{1,p}(\Omega_\varepsilon)$, implies the strong convergence (of a subsequence not relabelled) $\mathbf{u}_j \to \mathbf{u}_\varepsilon$ in $W^{1,p}(\Omega_\varepsilon)$. Thus, we may assume that $\{\mathbf{u}_j\}$ and $\{\nabla \mathbf{u}_j\}$ converge almost everywhere to $\mathbf{u}_\varepsilon$ and $\nabla \mathbf{u}_\varepsilon$ respectively. Thus using (2.6) and the Dominated Convergence Theorem in (2.10) (dropping to the subsequence $\{\mathbf{u}_j\}$), we obtain

$$\int_{\Omega_\varepsilon} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon)\nabla \mathbf{u}_\varepsilon^T + \mu_\varepsilon(\det \nabla \mathbf{u}_\varepsilon)\mathbf{I} \right] \cdot \nabla \mathbf{v}(\mathbf{u}_\varepsilon) \, d\mathbf{x} = 0.$$

Since $(\det \nabla \mathbf{u}_\varepsilon)\mathbf{I} = (\operatorname{adj} \nabla \mathbf{u}_\varepsilon)^T \nabla \mathbf{u}_\varepsilon^T$ and $\nabla[\mathbf{v}(\mathbf{u}_\varepsilon)] = \nabla \mathbf{v}(\mathbf{u}_\varepsilon)\nabla \mathbf{u}_\varepsilon$, it follows that the above equation is equivalent to (2.7).

Now assume that $\mathbf{u}_\varepsilon \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega}_\varepsilon)$ with $\det \nabla \mathbf{u}_\varepsilon > 0$ in Ω_ε. Note that (2.8b) and (2.8d) follow from the fact that $\mathbf{u}_\varepsilon$ is a solution of (1.7). The proof that (2.8a) holds is similar to the one given in [21, Theorem 5.1] and thus we omit it. Now multiply (2.8a) by $\mathbf{v}(\mathbf{u}_\varepsilon)$ where $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$, and integrate by parts using (2.7) to get that

$$\int_{\partial\Omega_\varepsilon} \mathbf{v}(\mathbf{u}_\varepsilon) \cdot \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon(\operatorname{adj} \nabla \mathbf{u}_\varepsilon)^T \right] \mathbf{n} \, ds(\mathbf{x}) = 0.$$

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Since the normal $\mathbf{n}$ to $\partial\Omega_\varepsilon$ is mapped by $\mathbf{u}_\varepsilon$ to

$$\tilde{\mathbf{n}}(\mathbf{u}_\varepsilon) = (\det \nabla \mathbf{u}_\varepsilon)(\nabla \mathbf{u}_\varepsilon)^{-T}\mathbf{n},$$

upon setting $\mathbf{y} = \mathbf{u}_\varepsilon(\mathbf{x})$, the previous equation is equivalent to:

$$\int_{\mathbf{u}_\varepsilon(\partial\Omega_\varepsilon)} \mathbf{v}(\mathbf{y}) \cdot [\mathbf{T}(\mathbf{y}) + \mu_\varepsilon \mathbf{I}]\,\tilde{\mathbf{n}}(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) = 0, \tag{2.11}$$

where the Cauchy stress tensor $\mathbf{T}(\mathbf{u}_\varepsilon)$ is given by

$$\mathbf{T}(\mathbf{u}_\varepsilon) = (\det \nabla \mathbf{u}_\varepsilon)^{-1}\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_\varepsilon)(\nabla \mathbf{u}_\varepsilon)^T.$$

From (2.11) and the arbitrariness of $\mathbf{v}$, we get that

$$[\mathbf{T}(\mathbf{y}) + \mu_\varepsilon \mathbf{I}]\,\tilde{\mathbf{n}}(\mathbf{y}) = \mathbf{0}, \quad \forall\, \mathbf{y} \in \mathbf{u}_\varepsilon(\partial\mathcal{B}_\varepsilon(\mathbf{x}_0)).$$

which after changing variables back to Ω_ε yields (2.8c). $\qquad\square$

Remark 2.6. The hypotheses on W in Theorem 2.5 are satisfied by the model stored energy function (1.10). The argument used in the proof of this theorem to get the strong convergence of the sequence $\{\mathbf{u}_j\}$ to $\mathbf{u}_\varepsilon$ in $W^{1,p}(\Omega_\varepsilon)$ from its weak convergence, is a slight variation of the one due to Evans [4].

We now study the sensitivity of the attained minimum value in (1.7) with respect to changes in the matrix $\mathbf{A}$ and the volume parameter V. In the usual sensitivity theorems of optimization theory, the parameters that change are on the right hand sides of the constraints which is the case for V in our problem. As for the matrix $\mathbf{A}$, it appears both in the right hand side of the volume constraint and in the displacement boundary condition on $\partial\Omega$. Thus our calculation for the sensitivity with respect to $\mathbf{A}$ picks up an additional term from $\partial\Omega$. We use the notation $\mathbf{u}_\varepsilon(\cdot, \mathbf{A}, V)$ to emphasize the dependence of the minimizer on both $\mathbf{A}$ and V. With the aid of Theorem 2.5 it is not difficult to show now that the following result holds.

Theorem 2.7. *Let $\mathbf{u}_\varepsilon(\cdot, \mathbf{A}, V)$ be a minimizer in (1.6) and assume that $\mathbf{u}_\varepsilon(\cdot, \mathbf{A}, V)$ $\in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega}_\varepsilon)$ and that $\mathbf{u}_\varepsilon \in C^2(\Omega_\varepsilon \times M_+^{n\times n} \times (0, V_0))$ for some $V_0 > 0$. Then for $\mathbf{A} = diag(\lambda_1, \ldots, \lambda_n)$, $\lambda_i > 0$ for all i, we have that*

$$\frac{\partial}{\partial\lambda_i}E_\varepsilon(\mathbf{u}_\varepsilon(\cdot, \mathbf{A}, V)) = \int_{\partial\Omega} x_i\mathbf{e}_i \cdot \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon\,(\mathrm{adj}\,\nabla \mathbf{u}_\varepsilon)^T\right]\mathbf{n}\,\mathrm{d}s \tag{2.12a}$$

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$$- \mu_\varepsilon \, |\Omega| \, \frac{\det \mathbf{A}}{\lambda_i}, \quad i = 1, \dots, n,$$

$$\frac{\partial}{\partial V} E_\varepsilon(\mathbf{u}_\varepsilon(\cdot, \mathbf{A}, V)) = \mu_\varepsilon. \tag{2.12b}$$

where $\{\mathbf{e}_k\}$ is the standard basis of $\mathbb{R}^n$. Moreover with the "dots" denoting derivatives with respect to V, we have:

$$\frac{\partial \mu_\varepsilon}{\partial V} = \int_{\Omega_\varepsilon} \nabla \dot{\mathbf{u}}_\varepsilon \cdot \mathbf{C}(\nabla \mathbf{u}_\varepsilon)[\nabla \dot{\mathbf{u}}_\varepsilon] \, \mathrm{d}\mathbf{x} - \mu_\varepsilon \int_{\Omega_\varepsilon} (\mathrm{adj}\, \nabla \mathbf{u}_\varepsilon)^T \cdot \nabla \ddot{\mathbf{u}}_\varepsilon \, \mathrm{d}\mathbf{x}, \tag{2.13a}$$

$$\frac{\partial \mu_\varepsilon}{\partial \lambda_i} = \int_{\partial \Omega} x_i \mathbf{e}_i \cdot \left[\mathbf{C}(\nabla \mathbf{u}_\varepsilon)[\nabla \dot{\mathbf{u}}_\varepsilon] + \frac{\partial \mu_\varepsilon}{\partial V} (\mathrm{adj}\, \nabla \mathbf{u}_\varepsilon)^T \right] \mathbf{n} \, \mathrm{d}s \tag{2.13b}$$

$$+ \mu_\varepsilon \int_{\partial \Omega} x_i \mathbf{e}_i \cdot \Big[((\mathrm{adj}\, \nabla \mathbf{u}_\varepsilon)^T \cdot \nabla \dot{\mathbf{u}}_\varepsilon) \mathbf{I}$$

$$- (\mathrm{adj}\, \nabla \mathbf{u}_\varepsilon)^T \, \nabla \dot{\mathbf{u}}_\varepsilon^T \Big] (\nabla \mathbf{u}_\varepsilon)^{-T} \mathbf{n} \, \mathrm{d}s - |\Omega| \, \frac{\det \mathbf{A}}{\lambda_i} \frac{\partial \mu_\varepsilon}{\partial V},$$

where $\mathbf{C}(\mathbf{F})$ denotes the elasticity tensor (fourth order) at $\mathbf{F}$.

3 Convergence of the regularized constrained minimizers

We now show that the regularized constrained minimizers given by Theorem 2.2, converge as $\varepsilon \searrow 0$ to a solution of the "non–regular" constrained problem (1.6). The first part of the proof of this result, dealing with the convergence and the existence of the limit, is very similar to that in [23, Theorem 4.1] and consequently we sketch most of it. The second part in which we show that the limiting function is actually a solution of (1.6) is more subtle due to the treatment of the integral volume constraint in (1.6) and the varying domains. For the proofs of the main results in this section we make use of the following two lemmas which we state without proofs[4].

Lemma 3.1. *There exists $V_0 > 0$ such that for any $V \in (0, V_0)$, there exists $\varepsilon_0(V) > 0$ such that*

$$\mathcal{C}_\mathbf{A}^\varepsilon \equiv \{ \mathbf{u} \in \mathcal{A}_\mathbf{A}^\varepsilon \, | \, c_\varepsilon(\mathbf{u}) = 0 \} \neq \emptyset,$$

---

[4]The construction in the proof of Lemma 3.1 is based on translations and dilations of functions like (4.6) adjusted to satisfy the volume constraint over Ω_ε, while that for Lemma 3.2 makes use of the Implicit Function Theorem on variations of any $\mathbf{u} \in \mathcal{C}_\mathbf{A}^0$ similar to those used in the proof of Theorem 2.5.

for all $\varepsilon \in [0, \varepsilon_0(V))$. Moreover, if W is nonnegative and for any $0 < \gamma < \delta$ there exists a constant $K > 0$ such that

$$W(\mathbf{F}) \leq K(\|\mathbf{F}\|^p + 1), \quad \text{whenever} \ \det \mathbf{F} \in [\gamma, \delta], \tag{3.1}$$

then for any nonnegative sequence $\varepsilon_j \to 0$, there exists a sequence $\mathbf{z}_{\varepsilon_j} \in \mathcal{C}_\mathbf{A}^{\varepsilon_j}$ such that

$$E_{\varepsilon_j}(\mathbf{z}_{\varepsilon_j}) \leq C, \quad \forall\, j,$$

for some constant $C > 0$.

Lemma 3.2. *Let Ω be a bounded, open set, and let the stored energy function W satisfy conditions H1–H2 and (2.6). Let $\mathbf{u} \in \mathcal{C}_\mathbf{A}^0$ and $V \in (0, |\Omega| \det \mathbf{A})$. Then for any for any sequence $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$, there exists a sequence of functions $\{\hat{\mathbf{u}}_j\}$ in $W^{1,p}(\Omega)$ with $\hat{\mathbf{u}}_j|_{\Omega_{\varepsilon_j}} \in \mathcal{C}_\mathbf{A}^{\varepsilon_j}$ for each j, and such that*

$$\lim_{j \to \infty} \int_\Omega W(\nabla \hat{\mathbf{u}}_j(\mathbf{x})) \, \mathrm{d}\mathbf{x} = \int_\Omega W(\nabla \mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x}.$$

We now have one of the main results of this paper.

Theorem 3.3. *Let the hypotheses in Lemma 3.2 hold. For $V \in (0, |\Omega| \det \mathbf{A})$, let $\{\varepsilon_j\}$ be a sequence of positive numbers converging to zero, and for each ε_j, let $\mathbf{u}_j$ be a minimizer for (1.7). Then $\{\mathbf{u}_j\}$ has a subsequence $\{\mathbf{u}_{j_k}\}$ such that for any $\delta > 0$,*

$$\mathbf{u}_{j_k} \rightharpoonup \mathbf{u}_V \quad in \quad W^{1,p}(\Omega_\delta),$$

where the function $\mathbf{u}_V$ is a solution of (1.6), and with

$$E(\mathbf{u}_V) = \lim_k E_{\varepsilon_{j_k}}(\mathbf{u}_{j_k}). \tag{3.2}$$

Proof. We let

$$\mathcal{C}_\mathbf{A}^\varepsilon \equiv \{\mathbf{u} \in \mathcal{A}_\mathbf{A}^\varepsilon \,|\, c_\varepsilon(\mathbf{u}) = 0\}, \quad \varepsilon \geq 0.$$

It follows from Lemma 3.1 that these sets are non empty for ε small enough. Thus each $\mathbf{u}_j$ satisfies:

$$E_{\varepsilon_j}(\mathbf{u}_j) = \min_{\mathbf{u} \in \mathcal{C}_\mathbf{A}^{\varepsilon_j}} \int_{\Omega_{\varepsilon_j}} W(\nabla \mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x} = \min_{\mathbf{u} \in \mathcal{C}_\mathbf{A}^{\varepsilon_j}} E_{\varepsilon_j}(\mathbf{u}).$$

Now we fix an index $J \in \mathbb{N}$ and take $j > J$. It follows from hypothesis (H2) on W and Poincaré's inequality, that for some constant $K > 0$:

$$E_{\varepsilon_J}(\mathbf{u}_j) \geq K \|\mathbf{u}_j\|_{W^{1,p}(\Omega_{\varepsilon_J})}^p, \quad j > J.$$

Again, it follows from (H2) that we may assume that W is non negative. Hence

$$E_{\varepsilon_J}(\mathbf{u}_j) \leq E_{\varepsilon_j}(\mathbf{u}_j) \leq C, \quad j > J,$$

where the constant C is given by Lemma 3.1. Combining this with the previous inequality we get that (for a subsequence) $\{\mathbf{u}_j\}$ converges weakly in $W^{1,p}(\Omega_{\varepsilon_J})$ to a function $\mathbf{u}^J$, and that $\{\det \nabla \mathbf{u}_j\}$ converges weakly in $L^1(\Omega_{\varepsilon_J})$ to a function θ^J. Since $p \in (n-1, n)$, it follows from [15, Theorem 4.2], that $\mathbf{u}^J$ satisfies condition INV, $\theta^J = \det \nabla \mathbf{u}^J$, and $\det \nabla \mathbf{u}^J > 0$ almost everywhere. By choosing an appropriate diagonal sequence, it is shown in [23] that there exists a subsequence $\{\mathbf{u}_{j_k}\}$ and a function $\mathbf{u}_V \in W^{1,p}(\Omega)$ such that

$$\mathbf{u}_{j_k} \rightharpoonup \mathbf{u}_V, \quad \text{in} \quad W^{1,p}(\Omega_{\varepsilon_J}).$$

The results in [23, Section 4.2] show that $\mathbf{u}_V \in \mathcal{A}_{\mathbf{A}}$.

It remains to show that $\mathbf{u}_V$ is a solution of (1.6). By the results quoted in the previous paragraph, we get that the subsequence $\{\mathbf{u}_{j_k}\}$ has the property that

$$\det \nabla \mathbf{u}_{j_k} \rightharpoonup \det \nabla \mathbf{u}_V, \quad \text{in} \quad L^1(\Omega_{\varepsilon_J}).$$

Since $\mathbf{u}_{j_k} \in \mathcal{C}_{\mathbf{A}}^{\varepsilon_{j_k}}$, we also have that

$$\int_{\Omega_{\varepsilon_{j_k}}} \det \nabla \mathbf{u}_{j_k} \, \mathrm{d}\mathbf{x} = (\det \mathbf{A}) \, |\Omega| - V, \quad \forall k.$$

Now we extend $\det \nabla \mathbf{u}_{j_k}$ to Ω as follows:

$$g_k(\mathbf{x}) = \begin{cases} \det \nabla \mathbf{u}_{j_k}(\mathbf{x}), & \mathbf{x} \in \Omega_{\varepsilon_{j_k}}, \\ 0, & \mathbf{x} \in \Omega \setminus \Omega_{\varepsilon_{j_k}}. \end{cases}$$

Clearly $g_k \in L^1(\Omega)$ and

$$\int_{\Omega} g_k \, \mathrm{d}\mathbf{x} = \int_{\Omega_{\varepsilon_{j_k}}} \det \nabla \mathbf{u}_{j_k} \, \mathrm{d}\mathbf{x} = (\det \mathbf{A}) \, |\Omega| - V, \quad \forall k.$$

Writing

$$\int_{\Omega} (\det \nabla \mathbf{u}_V - g_k) \, \mathrm{d}\mathbf{x} = \int_{\Omega_{\varepsilon_J}} (\det \nabla \mathbf{u}_V - g_k) \, \mathrm{d}\mathbf{x} + \int_{\mathcal{B}_{\varepsilon_J}(\mathbf{x}_0)} (\det \nabla \mathbf{u}_V - g_k) \, \mathrm{d}\mathbf{x}, \quad (3.3)$$

we note that the second term above can be made arbitrarily small by taking J sufficiently large. To see this we first observe that

$$\int_{\mathcal{B}_{\varepsilon_J}(\mathbf{x}_0)} g_k \, \mathrm{d}\mathbf{x} = \int_{\mathcal{D}_k} \det \nabla \mathbf{u}_{j_k} \, \mathrm{d}\mathbf{x},$$

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where $\mathcal{D}_k = \mathcal{B}_{\varepsilon_J}(\mathbf{x}_0) \setminus \mathcal{B}_{\varepsilon_{j_k}}(\mathbf{x}_0)$. Now using Jensen's inequality and the convexity of $h(\cdot)$, we get that

$$|\mathcal{D}_k|\, h\left(\frac{1}{|\mathcal{D}_k|}\int_{\mathcal{D}_k} \det \nabla \mathbf{u}_{j_k}\, \mathrm{d}\mathbf{x}\right) \leq \int_{\mathcal{D}_k} h(\det \nabla \mathbf{u}_{j_k})\, \mathrm{d}\mathbf{x}.$$

By Lemma 3.1 the right hand side of this inequality is uniformly bounded. Thus our statement about the second term in (3.3) now follows from (1.11b) and arguing by contradiction. Now once J is fixed, the first term in (3.3) can be made arbitrarily small as g_k equals $\det \nabla \mathbf{u}_{j_k}$ over Ω_{ε_J} for k sufficiently large and by the weak convergence of $\{\det \nabla \mathbf{u}_{j_k}\}$ to $\det \nabla \mathbf{u}_V$ in $L^1(\Omega_{\varepsilon_J})$. This shows that

$$\int_\Omega \det \nabla \mathbf{u}_V\, \mathrm{d}\mathbf{x} = \lim_{k\to\infty} \int_\Omega g_k\, \mathrm{d}\mathbf{x} = (\det \mathbf{A})\, |\Omega| - V, \tag{3.4}$$

Hence $\mathbf{u}_V \in \mathcal{C}_{\mathbf{A}}^0$. We now show that $\mathbf{u}_V$ is a minimizer over $\mathcal{C}_{\mathbf{A}}^0$.

For any $\mathbf{u} \in \mathcal{C}_{\mathbf{A}}^0$ and for the subsequence $\{\varepsilon_{j_k}\}$ above, let $\{\hat{\mathbf{u}}_{j_k}\}$ be the corresponding sequence given by Lemma 3.2 with the property that

$$\lim_{k\to\infty} \int_\Omega W(\nabla \hat{\mathbf{u}}_{j_k}(\mathbf{x}))\, \mathrm{d}\mathbf{x} = \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}. \tag{3.5}$$

As a function over $\Omega_{\varepsilon_{j_k}}$, we have that $\hat{\mathbf{u}}_{j_k} \in \mathcal{C}_{\mathbf{A}}^{\varepsilon_{j_k}}$. Since $\mathbf{u}_{j_k}$ is the minimizer over $\mathcal{C}_{\mathbf{A}}^{\varepsilon_{j_k}}$, we have that

$$\int_{\Omega_{\varepsilon_{j_k}}} W(\nabla \mathbf{u}_{j_k}(\mathbf{x}))\, \mathrm{d}\mathbf{x} \leq \int_{\Omega_{\varepsilon_{j_k}}} W(\nabla \hat{\mathbf{u}}_{j_k}(\mathbf{x}))\, \mathrm{d}\mathbf{x}. \tag{3.6}$$

Let $N > 0$ be given. For $k > N$ and the nonnegativity of W we get that

$$\int_{\Omega_{\varepsilon_{j_N}}} W(\nabla \mathbf{u}_{j_k}(\mathbf{x}))\, \mathrm{d}\mathbf{x} \leq \int_{\Omega_{\varepsilon_{j_k}}} W(\nabla \mathbf{u}_{j_k}(\mathbf{x}))\, \mathrm{d}\mathbf{x}. \tag{3.7}$$

By the results in [1], the functional $E_{\varepsilon_{j_N}}(\cdot)$ (cf. (1.7)) is weakly lower semi–continuous over $\mathcal{A}_{\mathbf{A}}^{\varepsilon_{j_N}}$. Using this and since $\mathbf{u}_{j_k} \rightharpoonup \mathbf{u}_V$ in $W^{1,p}(\Omega_{\varepsilon_{j_N}})$, we conclude that

$$\int_{\Omega_{\varepsilon_{j_N}}} W(\nabla \mathbf{u}_V(\mathbf{x}))\, \mathrm{d}\mathbf{x} \leq \liminf_k \int_{\Omega_{\varepsilon_{j_N}}} W(\nabla \mathbf{u}_{j_k}(\mathbf{x}))\, \mathrm{d}\mathbf{x}. \tag{3.8}$$

From the nonnegativity of W, it follows from (3.5)–(3.8) that

$$\int_{\Omega_{\varepsilon_{j_N}}} W(\nabla \mathbf{u}_V(\mathbf{x}))\, \mathrm{d}\mathbf{x} \leq \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}.$$

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Since N is arbitrary, we can conclude that

$$\int_\Omega W(\nabla \mathbf{u}_V(\mathbf{x})) \, \mathrm{d}\mathbf{x} \le \int_\Omega W(\nabla \mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x}.$$

Since $\mathbf{u} \in \mathcal{C}_{\mathbf{A}}^0$ is arbitrary, we get that $\mathbf{u}_V$ is a minimizer over $\mathcal{C}_{\mathbf{A}}^0$. If we set $\mathbf{u} = \mathbf{u}_V$ in (3.5), then we get as well that

$$\int_\Omega W(\nabla \mathbf{u}_V(\mathbf{x})) \, \mathrm{d}\mathbf{x} = \liminf_k \int_{\Omega_{\varepsilon_{j_k}}} W(\nabla \mathbf{u}_{j_k}(\mathbf{x})) \, \mathrm{d}\mathbf{x},$$

from which the result about the energies follows upon taking another subsequence. $\quad\square$

Remark 3.4. We should point out that since the minimizers in (1.6) are not necessarily unique, the results of Theorem 3.3 are true for one such minimizer and the convergence is for a subsequence.

We now derive an expression for a weak form of the equilibrium equations for the minimizer $\mathbf{u}_V$ in Theorem 3.3. Although the next result seems similar to that of Theorem 2.5, the proof is somewhat more technical due to the fact that the domains of the sequence of approximating functions are changing with the sequential index. We use the notation $\mu_j = \mu_{\varepsilon_j}$ for the Lagrange multiplier (cf. (2.7)) corresponding to the minimizer $\mathbf{u}_j$ of (1.7) in the statement of Theorem 3.3.

Theorem 3.5. *Assume that* (2.6) *and the hypotheses in Theorem 3.3 hold, and that the stored energy function W is of the form $\gamma|\mathbf{F}|^p + \tilde{W}(\mathbf{F})$ where $\gamma > 0$ and $\tilde{W}$ satisfy H1 and H3. Let $\mathbf{u}_V$ be the minimizer in Theorem 3.3. Then there exists $\mu_V \in \mathbb{R}$, a limit point of $\{\mu_j\}$, such that*

$$\int_\Omega \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_V) + \mu_V \, (\mathrm{adj} \, \nabla \mathbf{u}_V)^T \right] \cdot \nabla[\mathbf{v}(\mathbf{u}_V)] \, \mathrm{d}\mathbf{x} = 0, \qquad (3.9)$$

for all $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$ where $\mathcal{E} = \{\mathbf{A}\mathbf{x} : \mathbf{x} \in \Omega\}$. Moreover, if $\mathbf{u}_V \in C^2(\Omega \setminus \{\mathbf{x}_0\}) \cap C^1(\overline{\Omega} \setminus \{\mathbf{x}_0\})$ with $\det \nabla \mathbf{u}_V > 0$ in $\Omega \setminus \{\mathbf{x}_0\}$, then

$$\mathrm{div} \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_V) + \mu_V \, (\mathrm{adj} \, \nabla \mathbf{u}_V)^T \right] = \mathbf{0}, \quad in \ \Omega \setminus \{\mathbf{x}_0\}, \qquad (3.10)$$

and

$$\lim_{\delta \to 0} \int_{\partial \mathcal{B}_\delta(\mathbf{x}_0)} \mathbf{v}(\mathbf{u}_V) \cdot \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_V) + \mu_V (\mathrm{adj} \, \nabla \mathbf{u}_V)^T \right] \mathbf{n} \, \mathrm{d}s(\mathbf{x}) = 0. \qquad (3.11)$$

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Proof. Let $\{\mathbf{u}_{j_k}\}$ be the subsequence given by Theorem 3.3 such that (3.2) holds and for any $\delta > 0$,

$$\mathbf{u}_{j_k} \rightharpoonup \mathbf{u}_V \quad \text{in} \quad W^{1,p}(\Omega_\delta), \tag{3.12a}$$

$$\det \nabla \mathbf{u}_{j_k} \rightharpoonup \det \nabla \mathbf{u}_V \quad \text{in} \quad L^1(\Omega_\delta). \tag{3.12b}$$

Here $\mathbf{u}_{j_k}$ is the minimizer given by Theorem 2.2 corresponding to ε_{j_k}. The proof is divided into several steps.

Step 1: We first show that

$$\int_\Omega |\nabla \mathbf{u}_V|^p \, \mathrm{d}\mathbf{x} = \lim_k \int_{\Omega_{\varepsilon_{j_k}}} |\nabla \mathbf{u}_{j_k}|^p \, \mathrm{d}\mathbf{x}. \tag{3.13}$$

Note that

$$\int_{\Omega_\delta} \tilde{W}(\nabla \mathbf{u}_V) \, \mathrm{d}\mathbf{x} \le \liminf_k \int_{\Omega_\delta} \tilde{W}(\nabla \mathbf{u}_{j_k}) \, \mathrm{d}\mathbf{x} \le \liminf_k \int_{\Omega_{\varepsilon_{j_k}}} \tilde{W}(\nabla \mathbf{u}_{j_k}) \, \mathrm{d}\mathbf{x},$$

from which it follows that

$$\int_\Omega \tilde{W}(\nabla \mathbf{u}_V) \, \mathrm{d}\mathbf{x} \le \liminf_k \int_{\Omega_{\varepsilon_{j_k}}} \tilde{W}(\nabla \mathbf{u}_{j_k}) \, \mathrm{d}\mathbf{x}.$$

Using this we have now:

$$\begin{aligned}
\int_\Omega W(\nabla \mathbf{u}_V) \, \mathrm{d}\mathbf{x} \;-\; \gamma \int_\Omega |\nabla \mathbf{u}_V|^p \, \mathrm{d}\mathbf{x} &= \int_\Omega \tilde{W}(\nabla \mathbf{u}_V) \, \mathrm{d}\mathbf{x} \\
&\le \liminf_k \int_{\Omega_{\varepsilon_{j_k}}} \tilde{W}(\nabla \mathbf{u}_{j_k}) \, \mathrm{d}\mathbf{x} \\
&\le \lim_k \int_{\Omega_{\varepsilon_{j_k}}} W(\nabla \mathbf{u}_{j_k}) \, \mathrm{d}\mathbf{x} - \gamma \limsup_k \int_{\Omega_{\varepsilon_{j_k}}} |\nabla \mathbf{u}_{j_k}|^p \, \mathrm{d}\mathbf{x},
\end{aligned}$$

which upon invoking (3.2) yields that

$$\limsup_k \int_{\Omega_{\varepsilon_{j_k}}} |\nabla \mathbf{u}_{j_k}|^p \, \mathrm{d}\mathbf{x} \le \int_\Omega |\nabla \mathbf{u}_V|^p \, \mathrm{d}\mathbf{x}.$$

Since $\mathbf{u}_{j_k} \rightharpoonup \mathbf{u}_V$ in $W^{1,p}(\Omega_\delta)$, we get that

$$\int_{\Omega_\delta} |\nabla \mathbf{u}_V|^p \, \mathrm{d}\mathbf{x} \le \liminf_k \int_{\Omega_\delta} |\nabla \mathbf{u}_{j_k}|^p \, \mathrm{d}\mathbf{x} \le \liminf_k \int_{\Omega_{\varepsilon_{j_k}}} |\nabla \mathbf{u}_{j_k}|^p \, \mathrm{d}\mathbf{x},$$

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which by the arbitrariness of δ leads to

$$\int_\Omega |\nabla \mathbf{u}_V|^p \, \mathrm{d}\mathbf{x} \leq \liminf_k \int_{\Omega_{\varepsilon_{j_k}}} |\nabla \mathbf{u}_{j_k}|^p \, \mathrm{d}\mathbf{x}.$$

This combined with our previous result yields (3.13).

Step 2: We now show that

$$\nabla \mathbf{u}_{j_k} \to \nabla \mathbf{u}_V \text{ a.e. in } \Omega_\delta,$$

for any $\delta > 0$. Let $\psi : \mathbb{R} \to [0, \infty)$ be a smooth function such that $\psi(t) = 0$ for $t \leq 1$ and $\psi(t) = 1$ for $t \geq \frac{4}{3}$. For each k let $\hat{\mathbf{u}}_{j_k} : \Omega \to \mathbb{R}^n$ be given by:

$$\hat{\mathbf{u}}_{j_k}(\mathbf{x}) = \begin{cases} \psi\left(\frac{2|\mathbf{x}-\mathbf{x}_0|}{|\mathbf{x}-\mathbf{x}_0|+\varepsilon_{j_k}}\right) \mathbf{u}_{j_k}(\mathbf{x}) & , \quad \mathbf{x} \in \Omega_{\varepsilon_{j_k}}, \\ \mathbf{0} & , \quad \mathbf{x} \in \Omega \setminus \Omega_{\varepsilon_{j_k}}. \end{cases}$$

Using (3.12a) one can show now that $\hat{\mathbf{u}}_{j_k} \rightharpoonup \mathbf{u}_V$ in $W^{1,p}(\Omega)$ and that

$$\int_\Omega |\nabla \mathbf{u}_V|^p \, \mathrm{d}\mathbf{x} = \lim_k \int_\Omega |\nabla \hat{\mathbf{u}}_{j_k}|^p \, \mathrm{d}\mathbf{x}. \tag{3.14}$$

Since $\hat{\mathbf{u}}_{j_k} \rightharpoonup \mathbf{u}_V$ in $W^{1,p}(\Omega)$, it follows now that (for a subsequence) $\nabla \hat{\mathbf{u}}_{j_k} \to \nabla \mathbf{u}_V$ a.e. in Ω. Since $\hat{\mathbf{u}}_{j_k}(\mathbf{x}) = \mathbf{u}_{j_k}(\mathbf{x})$ for all $\mathbf{x} \in \Omega_{2\varepsilon_{j_k}}$, it follows that $\nabla \mathbf{u}_{j_k} \to \nabla \mathbf{u}_V$ a.e. in Ω_δ for any $\delta > 0$.

Step 3: Finally we show now that the weak form (3.9) of the equilibrium equations for the minimizer $\mathbf{u}_V$ holds. Once again the varying domains in the sequence $\{\mathbf{u}_{j_k}\}$ complicates the analysis. The convergence a.e. established in Step 2 is an essential ingredient for the following arguments.

Let $N > 0$ be such that $\varepsilon_{j_k} < \delta$ for $k > N$. Using Theorem 2.5 we get now that

$$\begin{aligned} 0 &= \int_{\Omega_{\varepsilon_{j_k}}} \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_{j_k})\nabla \mathbf{u}_{j_k}^T + \mu_{j_k}(\det \nabla \mathbf{u}_{j_k})\mathbf{I}\right] \cdot \nabla \mathbf{v}(\mathbf{u}_{j_k}) \, \mathrm{d}\mathbf{x} \\ &= \int_{\mathcal{B}_\delta(\mathbf{x}_0) \setminus \mathcal{B}_{\varepsilon_{j_k}}(\mathbf{x}_0)} \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_{j_k})\nabla \mathbf{u}_{j_k}^T + \mu_{j_k}(\det \nabla \mathbf{u}_{j_k})\mathbf{I}\right] \cdot \nabla \mathbf{v}(\mathbf{u}_{j_k}) \, \mathrm{d}\mathbf{x} \\ &\quad + \int_{\Omega_\delta} \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_{j_k})\nabla \mathbf{u}_{j_k}^T + \mu_{j_k}(\det \nabla \mathbf{u}_{j_k})\mathbf{I}\right] \cdot \nabla \mathbf{v}(\mathbf{u}_{j_k}) \, \mathrm{d}\mathbf{x}, \quad k > N. \quad (3.15) \end{aligned}$$

for all $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$. It follows from Step 2, hypothesis (2.6) and the generalized Dominated Convergence Theorem (see [20]), that

$$\lim_k \int_{\Omega_\delta} \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_{j_k})\nabla \mathbf{u}_{j_k}^T\right] \cdot \nabla \mathbf{v}(\mathbf{u}_{j_k}) \, \mathrm{d}\mathbf{x} = \int_{\Omega_\delta} \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_V)\nabla \mathbf{u}_V^T\right] \cdot \nabla \mathbf{v}(\mathbf{u}_V) \, \mathrm{d}\mathbf{x}.$$

Also from (3.12b) and [21, Lemma 6.7] we get that

$$\lim_k \int_{\Omega_\delta} (\det \nabla \mathbf{u}_{j_k})\mathbf{I} \cdot \nabla \mathbf{v}(\mathbf{u}_{j_k}) \, \mathrm{d}\mathbf{x} = \int_{\Omega_\delta} (\det \nabla \mathbf{u}_V)\mathbf{I} \cdot \nabla \mathbf{v}(\mathbf{u}_V) \, \mathrm{d}\mathbf{x}.$$

By an argument similar to the one within the proof of Theorem 3.3 (cf. (3.3)), we get that the integrals

$$\int_{\mathcal{B}_\delta(\mathbf{x}_0) \setminus \mathcal{B}_{\varepsilon_{j_k}}(\mathbf{x}_0)} (\det \nabla \mathbf{u}_{j_k})\mathbf{I} \cdot \nabla \mathbf{v}(\mathbf{u}_{j_k}) \, \mathrm{d}\mathbf{x},$$

can be made arbitrarily small as $\delta \searrow 0$ and $k \to \infty$. Now let

$$w_k(\mathbf{x}) = \begin{cases} W(\nabla \mathbf{u}_{j_k}(\mathbf{x})) & , \quad \mathbf{x} \in \Omega_{\varepsilon_{j_k}}, \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

Since $\nabla \mathbf{u}_{j_k} \to \nabla \mathbf{u}_V$ a.e. in Ω_δ for any $\delta > 0$, we have that $w_k \to W(\nabla \mathbf{u}_V)$ a.e. in Ω. Also

$$\|w_k\|_{L^1(\Omega)} = \int_{\Omega_{\varepsilon_{j_k}}} W(\nabla \mathbf{u}_{j_k}(\mathbf{x})) \, \mathrm{d}\mathbf{x} \to \int_\Omega W(\nabla \mathbf{u}_V(\mathbf{x})) \, \mathrm{d}\mathbf{x} = \|W(\nabla \mathbf{u}_V)\|_{L^1(\Omega)},$$

by (3.2). It follows now that $w_k \to W(\nabla \mathbf{u}_V)$ in $L^1(\Omega)$, which implies that $\{w_k\}$ is equi-integrable. This property of the w_k's together with (2.6) can be used now to show that the integrals

$$\int_{\mathcal{B}_\delta(\mathbf{x}_0) \setminus \mathcal{B}_{\varepsilon_{j_k}}(\mathbf{x}_0)} \frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_{j_k})\nabla \mathbf{u}_{j_k}^T \cdot \nabla \mathbf{v}(\mathbf{u}_{j_k}) \, \mathrm{d}\mathbf{x},$$

can be made arbitrarily small as $\delta \searrow 0$ and $k \to \infty$. Thus letting first $k \to \infty$ in (3.15), and then letting $\delta \searrow 0$, we get that for some $\mu_V \in \mathbb{R}$,

$$\int_\Omega \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_V)\nabla \mathbf{u}_V^T + \mu_V(\det \nabla \mathbf{u}_V)\mathbf{I}\right] \cdot \nabla \mathbf{v}(\mathbf{u}_V) \, \mathrm{d}\mathbf{x} = 0,$$

for all $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$.

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Now assume that $\mathbf{u}_V \in C^2(\Omega \setminus \{\mathbf{x}_0\}) \cap C^1(\overline{\Omega} \setminus \{\mathbf{x}_0\})$ with $\det \nabla \mathbf{u}_V > 0$ in $\Omega \setminus \{\mathbf{x}_0\}$. The proof of (3.10) is similar to the one given in [21, Theorem 5.1] and thus we omit it. Let $\delta > 0$ be given. If we multiply (3.10) by $\mathbf{v}(\mathbf{u}_V)$, where $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$ and constant over $\mathcal{K}$, and integrate by parts over Ω_δ, we get that

$$
\int_{\Omega_\delta} \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_V) + \mu_V \left(\mathrm{adj}\, \nabla \mathbf{u}_V\right)^T \right] \cdot \nabla[\mathbf{v}(\mathbf{u}_V)] \, \mathrm{d}\mathbf{x}
$$
$$
= \int_{\partial \mathcal{B}_\delta(\mathbf{x}_0)} \mathbf{v}(\mathbf{u}_V) \cdot \left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_V) + \mu_V(\mathrm{adj}\, \nabla \mathbf{u}_V)^T \right] \mathbf{n} \, \mathrm{d}s(\mathbf{x}).
$$

Taking the limit as $\delta \searrow 0$ and using (3.9) we get that (3.11) holds. $\qquad \square$

We now establish a very nice connection between the Lagrange multipliers μ_V and the volume derivative (cf [18]). The volume derivative of the stored energy function W at the boundary displacement $\mathbf{A}$ is given by:

$$
G(\mathbf{A}) = \lim_{V \to 0^+} \inf_{\mathbf{u} \in \mathcal{A}_{\mathbf{A},V}} \frac{E(\mathbf{u}) - E(\mathbf{Ax})}{V} = \lim_{V \to 0^+} \frac{E(\mathbf{u}_V) - E(\mathbf{Ax})}{V}, \tag{3.16}
$$

where $E(\cdot)$ is as in (1.1) and $\mathbf{u}_V$ is the minimizer from Theorems 3.3 and 3.5. In the following, we write $\mathbf{u}(\mathbf{x}, V)$ instead of $\mathbf{u}_V(\mathbf{x})$.

Theorem 3.6. *Assume that* (2.6) *and the hypotheses in Theorems 3.3 and 3.5 hold and that:*

1. $\mathbf{u}(\cdot, V) \in C^2(\Omega \setminus \{\mathbf{x}_0\}) \cap C^1(\overline{\Omega} \setminus \{\mathbf{x}_0\})$ *with* $\det \nabla \mathbf{u}_V > 0$ *in* $\Omega \setminus \{\mathbf{x}_0\}$;

2. $\mathbf{u}(\cdot, \cdot) \in C^3((\overline{\Omega} \setminus \{\mathbf{x}_0\}) \times (0, V_0))$ *for some* $V_0 > 0$;

3. *for some* $\delta > 0$, $\mathbf{u}(\cdot, V) \to \mathbf{u}_h(\cdot)$ *in* $C^2(\overline{\Omega} \setminus \mathcal{B}_\delta(\mathbf{x}_0))$ *as* $V \searrow 0$, *where* $\mathbf{u}_h(\mathbf{x}) = \mathbf{Ax}$ *for all* $\mathbf{x} \in \overline{\Omega}$.

Then

$$
G(\mathbf{A}) = \lim_{V \to 0^+} \mu_V,
$$

where $\{\mu_V\}$ *are the Lagrange multipliers from Theorem 3.5.*

Proof. Under the stated hypotheses it is shown in [18, Prop. 5.4] that

$$
G(\mathbf{A}) = -\frac{1}{n} \lim_{V \to 0^+} \frac{1}{V} \left[\lim_{\delta \to 0^+} \int_{\partial \mathcal{B}_\delta(\mathbf{x}_0)} (\mathbf{u}_V - \mathbf{Ax}_0) \cdot \frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u}_V) \mathbf{n} \, \mathrm{d}s(\mathbf{x}) \right],
$$

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where the normal $\mathbf{n}$ to $\partial\mathcal{B}_\delta(\mathbf{x}_0)$ points in the outward direction. Let $K \subset \Omega$ be a compact set such that $\mathbf{u}_V(\mathcal{B}_\delta(\mathbf{x}_0)) \subset K$ for all δ and V sufficiently small. Then taking $\mathbf{v}$ in (3.11) such that $\mathbf{v}(\mathbf{x}) = \mathbf{x} - \mathbf{A}\mathbf{x}_0$ for $\mathbf{x} \in K$, we get that

$$\lim_{\delta \to 0^+} \int_{\partial\mathcal{B}_\delta(\mathbf{x}_0)} (\mathbf{u}_V - \mathbf{A}\mathbf{x}_0) \cdot \frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla\mathbf{u}_V)\mathbf{n}\,\mathrm{d}s(\mathbf{x})$$
$$= -\mu_V \lim_{\delta \to 0^+} \int_{\partial\mathcal{B}_\delta(\mathbf{x}_0)} (\mathbf{u}_V - \mathbf{A}\mathbf{x}_0) \cdot (\mathrm{adj}\,\nabla\mathbf{u}_V)^T\mathbf{n}\,\mathrm{d}s(\mathbf{x}).$$

But

$$\int_{\partial\mathcal{B}_\delta(\mathbf{x}_0)} (\mathbf{u}_V - \mathbf{A}\mathbf{x}_0) \cdot (\mathrm{adj}\,\nabla\mathbf{u}_V)^T\mathbf{n}\,\mathrm{d}s(\mathbf{x})$$
$$= \int_{\partial\Omega} (\mathbf{u}_V - \mathbf{A}\mathbf{x}_0) \cdot (\mathrm{adj}\,\nabla\mathbf{u}_V)^T\mathbf{n}\,\mathrm{d}s(\mathbf{x}) - n\int_{\Omega\backslash\overline{\mathcal{B}_\delta(\mathbf{x}_0)}} \det\nabla\mathbf{u}_V\,\mathrm{d}\mathbf{x}.$$

Thus

$$\lim_{\delta \to 0^+} \int_{\partial\mathcal{B}_\delta(\mathbf{x}_0)} (\mathbf{u}_V - \mathbf{A}\mathbf{x}_0) \cdot (\mathrm{adj}\,\nabla\mathbf{u}_V)^T\mathbf{n}\,\mathrm{d}s(\mathbf{x})$$
$$= \int_{\partial\Omega} (\mathbf{u}_V - \mathbf{A}\mathbf{x}_0) \cdot (\mathrm{adj}\,\nabla\mathbf{u}_V)^T\mathbf{n}\,\mathrm{d}s(\mathbf{x}) - n\int_{\Omega} \det\nabla\mathbf{u}_V\,\mathrm{d}\mathbf{x}$$
$$= n(\det\mathbf{A})\,|\Omega| - n[(\det\mathbf{A})\,|\Omega| - V] = nV.$$

It follows now that

$$G(\mathbf{A}) = -\frac{1}{n}\lim_{V \to 0^+}\frac{1}{V}(-\mu_V\,nV) = \lim_{V \to 0^+}\mu_V.$$

$\square$

4   Numerical results

In this section we describe some of the elements of a numerical procedure based on the results of the previous sections, to compute a minimizer of (1.6). In addition we work a numerical example in which we check the convergence as $\varepsilon \searrow 0$ predicted by Theorem 3.3 and another example in which we test the robustness of the method in the so called incompressible limit.

For given values of ε, V, we use the method outlined in Theorem 2.2 to compute the minimizer $\mathbf{u}_\varepsilon$ in (1.7). The minimizers in (2.3a) of Theorem 2.2 (dropping the subscript "j") are computed using the *gradient flow equation*[5]:

$$\Delta \mathbf{u}_t = -\mathrm{div}\left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}) + (\mu + \eta c_\varepsilon(\mathbf{u}))(\mathrm{adj}\,\nabla \mathbf{u})^t\right], \quad \text{in}\ \ \Omega_\varepsilon, \tag{4.1}$$

where for all $t \geq 0$, $\mathbf{u}(\mathbf{x}, t) = \mathbf{A}\mathbf{x}$ over $\partial\Omega$ and

$$\left[\nabla \mathbf{u}_t + \frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}) + (\mu + \eta c_\varepsilon(\mathbf{u}))(\mathrm{adj}\,\nabla \mathbf{u})^t\right]\mathbf{n} = \mathbf{0}, \quad \text{on}\ \partial\mathcal{B}_\varepsilon(\mathbf{x}_0). \tag{4.2}$$

The gradient flow equation leads to a descent method for the solution of (i) of Theorem 2.2. (For more details about gradient flow methods (also called Sobolev gradient methods) and there properties, we refer to [19]. For further applications of this technique in other problems leading to cavitation see [8].) After discretization of the partial derivative with respect to "t", one can use a finite element method to solve the resulting flow equation. In particular, if we let $\Delta t > 0$ be given, and set $t_{i+1} = t_i + \Delta t$ where $t_0 = 0$, we can approximate $\mathbf{u}_t(\mathbf{x}, t_i)$ with:

$$\mathbf{z}_i(\mathbf{x}) = \frac{\mathbf{u}_{i+1}(\mathbf{x}) - \mathbf{u}_i(\mathbf{x})}{\Delta t},$$

where $\mathbf{u}_i(\mathbf{x}) = \mathbf{u}(\mathbf{x}, t_i)$, etc.. (We take $\mathbf{u}_0(\mathbf{x})$ to be some initial deformation satisfying the boundary condition on $\partial\Omega$, e.g., $\mathbf{A}\mathbf{x}$.) Inserting this approximation into the weak form of (4.1), (4.2), and evaluating the right hand side of (4.1) at $\mathbf{u} = \mathbf{u}_i$, we arrive at the following iterative formula:

$$\int_{\Omega_\varepsilon} \nabla \mathbf{z}_i \cdot \nabla \mathbf{v}\,\mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon}\left[\frac{\mathrm{d}W}{\mathrm{d}\mathbf{F}}(\nabla \mathbf{u}_i) + (\mu + \eta c_\varepsilon(\mathbf{u}_i))(\mathrm{adj}\,\nabla \mathbf{u}_i)^t\right] \cdot \nabla \mathbf{v}\,\mathrm{d}\mathbf{x} = 0, \tag{4.3}$$

for all $\mathbf{v}$ vanishing on $\partial\Omega$ and sufficiently smooth so that the integrals above are well defined. Given $\mathbf{u}_i$, one can solve the above equation for $\mathbf{z}_i$ via some finite element scheme, and then set $\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta t\,\mathbf{z}_i$. This process is repeated for $i = 0, 1, \ldots$, until $\mathbf{u}_{i+1} - \mathbf{u}_i$ is "small" enough (10^{-3} in the calculations below), or some maximum value of "t" is reached, declaring the last $\mathbf{u}_i$ as an approximation of $\mathbf{u}_\varepsilon$. This whole process is repeated for smaller values of of ε, to obtain as a result an approximation of the minimizer $\mathbf{u}_V$ in (1.6).

---

[5]It follows from (2.10) that the Euler–Lagrange equations for the minimizer in (2.3a) of Theorem 2.2 are formally given by equating to zero the right hand side of (4.1).

For the computations we used the stored energy function (1.10) in which:

$$h(d) = c_1 d^{e_1} + c_2 d^{-e_2}, \tag{4.4}$$

where $c_1, c_2 \geq 0$ and $e_1, e_2 > 0$. The reference configuration is stress free provided[6]:

$$c_2 = \frac{\kappa(\sqrt{n})^{q-2} + c_1 e_1}{e_2}. \tag{4.5}$$

The case $\kappa = 0$ in (1.10) is called an *elastic fluid.*

For an elastic fluid in which $\Omega = \mathcal{B} \equiv \mathcal{B}_1(\mathbf{0})$ and $\mathbf{x}_0 = \mathbf{0}$, the minimizer $\mathbf{u}_V$ in (1.6) is given[7] by (see [18]):

$$\mathbf{u}_V(\mathbf{x}) = [dR^n + (1-d)]^{1/n} \frac{\mathbf{Ax}}{R}, \quad R = \|\mathbf{x}\|, \tag{4.6}$$

where d is given by

$$d = 1 - \frac{nV}{\omega_n \det \mathbf{A}}.$$

(V is assumed to be sufficiently small as to guarantee that $d > 0$.) It follows that $\det \nabla \mathbf{u}_V = d \det \mathbf{A}$. Thus we have that

$$E(\mathbf{u}_V) = \int_{\mathcal{B}} h(\det \nabla \mathbf{u}_V) \, d\mathbf{x} = \frac{\omega_n}{n} h(d \det \mathbf{A}),$$

where ω_n denotes the area of the unit sphere in $\mathbb{R}^n$. We now consider the particular case in which

$$n = 2, \quad c_1 = 1, \quad e_1 = 2, \quad e_2 = 1, \quad V = \pi(0.15)^2, \quad \mathbf{A} = \mathrm{diag}(1.1, 1.4). \tag{4.7}$$

Using the formulas above, we get that

$$E(\mathbf{u}_V) = \pi \, h((1.1)(1.4) - 0.15^2) = 11.3750.$$

For the parameters in Theorem 2.2 we used $\gamma = 0.25$, $\beta = 2$, with the stopping criteria in (2.3b) of Theorem 2.2 given by $|\mu_{j+1} - \mu_j| < 10^{-3}|\mu_j|$. The solution of the sub–problems (4.3) was done using the package freefem++ (see [5]) with first-order Crouzeix–Raviart finite elements. We show in Table 1 the results in this case for the method described

---

[6]For the stored energy function (1.10), we have that $\frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) = \kappa |\mathbf{F}|^{q-2} \mathbf{F} + h'(\det \mathbf{F})(\mathrm{adj} \, \mathbf{F})^T$. Thus in the case (4.4), $\frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}) = \mathbf{0}$ for (4.4) if and only if (4.5) holds.

[7]The minimizer in (1.7) is given by a similar expression but replacing d with $d_\varepsilon = d/(1 - \varepsilon^n)$.

at the beginning of this section and for the data (4.7). Each line in this table shows, for a given ε, the last computed step of the method outlined in Theorem 2.2. Note that the penalty parameter η (fifth column) does not become too large, thus avoiding the ill–conditioning associated with large values of these parameters. Also the computed energy values are approaching the exact energy 11.3750 in accordance with the result in Theorem 3.3, to within the convergence tolerances in the gradient flow and penalty multiplier iterations, and finite element approximations. In Figure 1 we show the initial finite element mesh and final computed deformation corresponding to $\varepsilon = 0.00625$. The hole (which is not circular) inside the computed deformation satisfies the constraint of having area $V = \pi(0.15)^2$ with an error of the order of $O(10^{-7})$.

Table 1: Convergence of the regularized minimizers for the case of a two dimensional elastic fluid and data (4.7).

ε	$c_\varepsilon(\mathbf{u}_\varepsilon)$	$E_{\varepsilon,\mu,\eta}(\mathbf{u}_\varepsilon)$	μ	η
0.1	-2.92883e-05	11.3636	-2.22599	40
0.05	-4.85216e-06	11.3699	-2.19213	160
0.025	1.26299e-06	11.3717	-2.18469	160
0.0125	-2.08708e-06	11.3721	-2.18249	320
0.00625	4.99878e-06	11.3723	-2.17622	160

For the stored energy corresponding to an elastic fluid, it is shown in [18] that the volume derivative at the matrix $\mathbf{A}$ is given by $-h'(\det \mathbf{A})$. For the data (4.7) we get a value of -2.2367 for the volume derivative. If we repeat the calculation in Table 1 corresponding to $\varepsilon = 0.00625$ but with prescribed area $V = \pi(0.01)^2$, we get a multiplier value upon convergence of -2.2375, which approximates quite well the exact volume derivative to within the convergence tolerances.

The incompressible case of our problem corresponds to the case in which $\det \nabla \mathbf{u}$ is set to one in the h term of (1.10), and we minimize in (1.6) subject to the additional constraint of $\det \nabla \mathbf{u} = 1$. In this case, for $\Omega = \mathcal{B}_1(\mathbf{0})$ and $\mathbf{A} = \lambda \mathbf{I}$, assuming that the minimizer $\mathbf{u}_V$ is radial[8], then $\mathbf{u}_V(\mathbf{x}) = r(\|\mathbf{x}\|)\frac{\mathbf{x}}{\|\mathbf{x}\|}$ where

$$r(R) = \sqrt[n]{R^n + \lambda^n - 1}, \quad \lambda^n = 1 + c_V^n, \quad V = \frac{\omega_n}{n}c_V^n. \tag{4.8}$$

(Note that the boundary displacement λ is completely determined by the volume constraint parameter V.) Our next simulation is for what is called the *incompressible limit*.

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[8]See [22] for conditions under which a minimizer is radial.

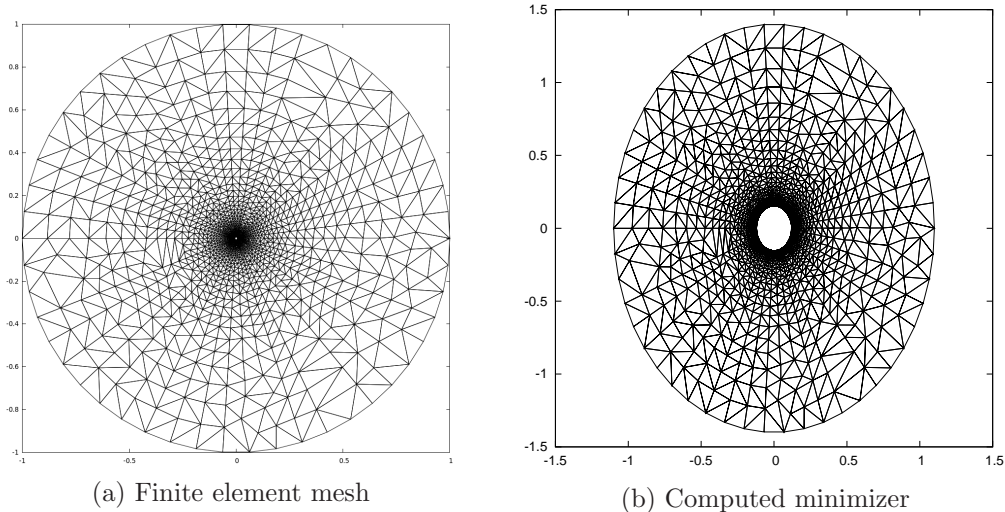(a) Finite element mesh

(b) Computed minimizer

Figure 1: Finite element mesh and computed minimizer for the case of an elastic fluid and a spherical domain for the data (4.7).

In particular, we consider the stored energy function (1.10) in which

$$h(d) = c_1 d^{e_1} + c_2 d^{-e_2} + k(d-1)^2, \tag{4.9}$$

where $c_1, c_2 \geq 0$, $e_1, e_2 > 0$, and $k \geq 0$ is a "large" parameter. The parameters for the simulations (not including k) were taken to be:

$$n = 2, \quad \kappa = 1, \quad q = 1.5, \quad c_1 = 1, \quad e_1 = 2, \quad e_2 = 1, \quad c_V = 0.5.$$

The energy of the discrete version of (4.8) is given approximately by 16.6089. In Table 2 we show the results obtained by solving the nearly incompressible problem (1.7) (using (4.9) in (1.10)) via the regularized method (with $\varepsilon = 0.05$) for increasing values of k. The results in columns three and four in Table 2 show that both, the energy of the incompressible exact solution (4.8) and the incompressibility condition, are approximated quite well to within the discretization and convergence tolerances. In Figure 2 we show a graph of the determinant of the computed approximate minimizer corresponding to the last line in Table 2.

Table 2: Results obtained in the incompressible limit case.

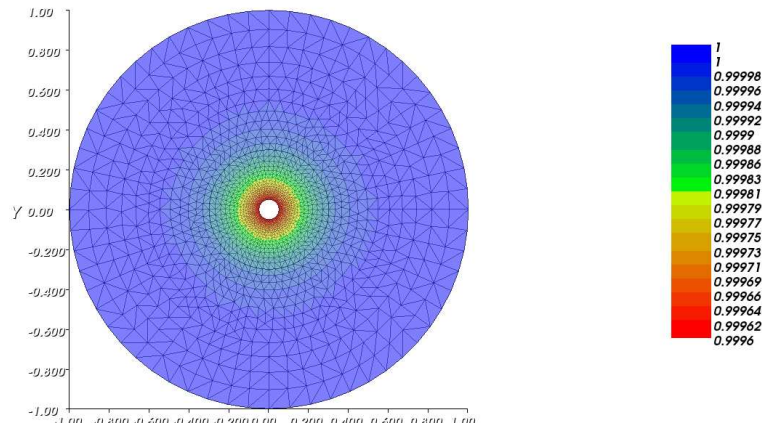| k	$c_\varepsilon(\mathbf{u}_\varepsilon)$	$E_\varepsilon(\mathbf{u}_\varepsilon)$	$\|\det \nabla \mathbf{u}_\varepsilon - 1\|_1$
10	9.71729e-05	16.6068	6.8198e-03
100	1.53748e-05	16.6074	9.20077e-04
1000	6.51531e-07	16.6075	9.47614e-05

Figure 2: Graph of the determinant of the last approximate minimizer in the incompressible limit case.

5 Final Comments

In [18, Proposition 6.1] the authors introduced a numerical scheme for the solution of (1.6) based on approximating the original constrained problem by a sequence of regularized constrained problems over punctured domains. They anticipated without proof, the convergence of the corresponding regularised minimizers to a solution of (1.6) as $\varepsilon \searrow 0$. The result in Theorem 3.3 fills that gap. Moreover, the regularized constrained problems over punctured domains are solved numerically via a penalty–multiplier technique that leads to a more stable numerical scheme for this internal iteration, as compared to the standard quadratic penalty method. This is the case as in general one achieves convergence in the penalty–multiplier method without having to make the penalty parameter excessively large, which could lead to numerical ill conditioning.

Let $\mu_\varepsilon(\mathbf{A}, V)$ be the multiplier in Theorems 2.5 and 2.7. By combining Theorems 2.5 and 3.6 we get that (cf. (3.16))

$$G(\mathbf{A}) = \lim_{V \to 0} \lim_{\varepsilon \to 0} \mu_\varepsilon(\mathbf{A}, V). \tag{5.1}$$

Thus the computation of $\mu_\varepsilon(\mathbf{A}, V)$ for progressively smaller values of V, ε leads to a numerical scheme for approximating the volume derivative $G(\mathbf{A})$, perhaps more robust than the one employed in [18] based on difference quotients.

The set

$$\mathcal{F} = \left\{ \mathbf{A} \, : \, G(\mathbf{A}) = 0, \, \mathbf{A} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \, \lambda_i > 0, \, 1 \leq i \leq n \right\}.$$

is called the *fracture surface* associated to the stored energy function W. In [18] the authors give justifications for the interpretation of $\mathcal{F}$ as the boundary of the set of boundary displacement matrices leading or inducing to cavitation as defined in [1]. Let

$$\mathcal{F}_V^\varepsilon = \left\{ \mathbf{A} \, : \, \mu_\varepsilon(\mathbf{A}, V) = 0, \, \mathbf{A} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \, \lambda_i > 0, \, 1 \leq i \leq n \right\}. \tag{5.2}$$

Note that for the matrices in $\mathcal{F}_V^\varepsilon$, the corresponding minimizer $\mathbf{u}_\varepsilon$ produces a stress–free inner cavity of volume V (cf. 2.8c). It follows from (5.1) that the computation of the sets $\mathcal{F}_V^\varepsilon$ in (5.2) for progressively smaller values of V, ε, leads to a numerical scheme for approximating the fracture surface $\mathcal{F}$.

This method for approximating $\mathcal{F}$ in a certain sense generalizes to the nonradial case the inverse method proposed in [17] for computing the critical λ_c in the radial case. This is so because in the method proposed in [17], when one specifies the cavity radius $r(0)$ of a radial deformation of an ε punctured ball, we are specifying the volume of the cavity which is spherical in that case. Then one determines the boundary displacement λ for

which the radial minimizer has the proposed cavity radius and zero Cauchy stress in the inner cavity. In the present context, computing $\mathcal{F}_V^\varepsilon$ is equivalent to specifying the volume of the cavity, and then determining the boundary deformation matrices $\mathbf{A}$ that lead to solutions of (1.7) that produce a stress–free cavity, not necessarily spherical, with the specified volume. Contrary to the radial case in which there is only one critical value of λ_c, in the nonradial case $\mathcal{F}$ is in general an $n-1$ dimensional surface.

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