Snaking on Networks:

from Local Solutions to Turing Patterns

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Background

- Patterns are found throughout nature and identifying them and understanding their origin forms the basis of much of our scientific understanding.
- Alan Turing laid down the basic mechanisms for spatial pattern formation, based the loss of stability of an equilibrium state to a patterned state [1].
- Recent attention has been focussed on non-regular *complex network* topologies, which are a model for many phenomena in a variety of fields including biology and sociology [2].

Outline of this Work

- The emergence of patterns of activity on complex networks with reaction-diffusion dynamics on the nodes has been numerically studied.
- The transition between previously analysed single-node solutions [3] and fully developed global activation states (so called *Turing states*) has been investigated.
- We reveal *snaking bifurcations* connecting different localised solutions, as found on regular lattices, shedding light on the origin of previously reported multistable *"Turing patterns"* [4].

The Mimura-Murray model

- Reaction-diffusion system:

$$\dot{u} = f(u,v) - DLu$$
$$\dot{v} = g(u,v) - \sigma DLv$$

with

$$f(u,v) = \frac{au + bu^2 - u^3}{c} - uv, \qquad g(u,v) = uv - v - dv^2$$

at the parameter values $a = 35$, $b = 16$, $c = 9$, $d = 2/5$.

- u: activator, v: inhibitor; $L = (L_{(i,j)})$ network Laplacian;
- the model has a stable equilibrium at $(\bar{u}, \bar{v}) = (5, 10)$,
- Turing instability: equilibrium destabilized on increase of σ,
  - undergoes a *supercritical* bifurcation at $\sigma = \sigma_T \approx 15.5$.

Scale-Free "Complex" Networks

- Preferential-attachment networks such as the Barabási–Albert model are found throughout nature and have a *scale-free* power-law degree distribution [5].

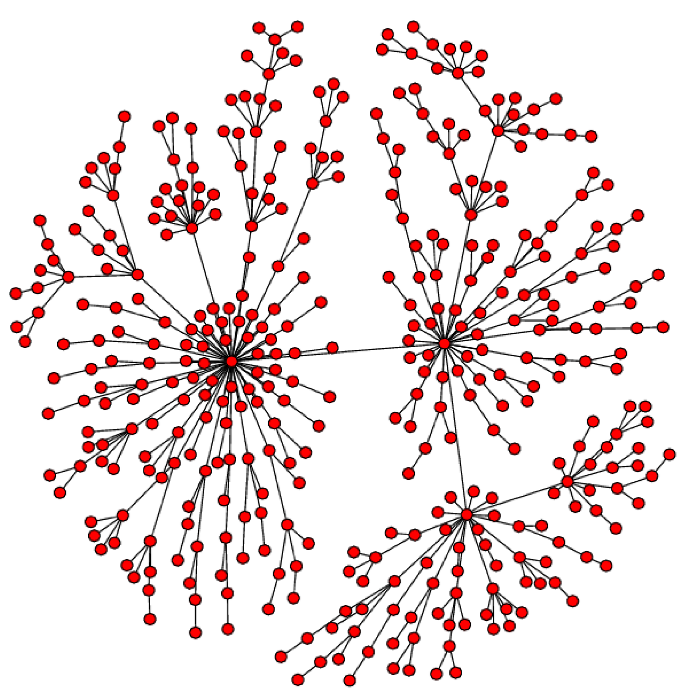
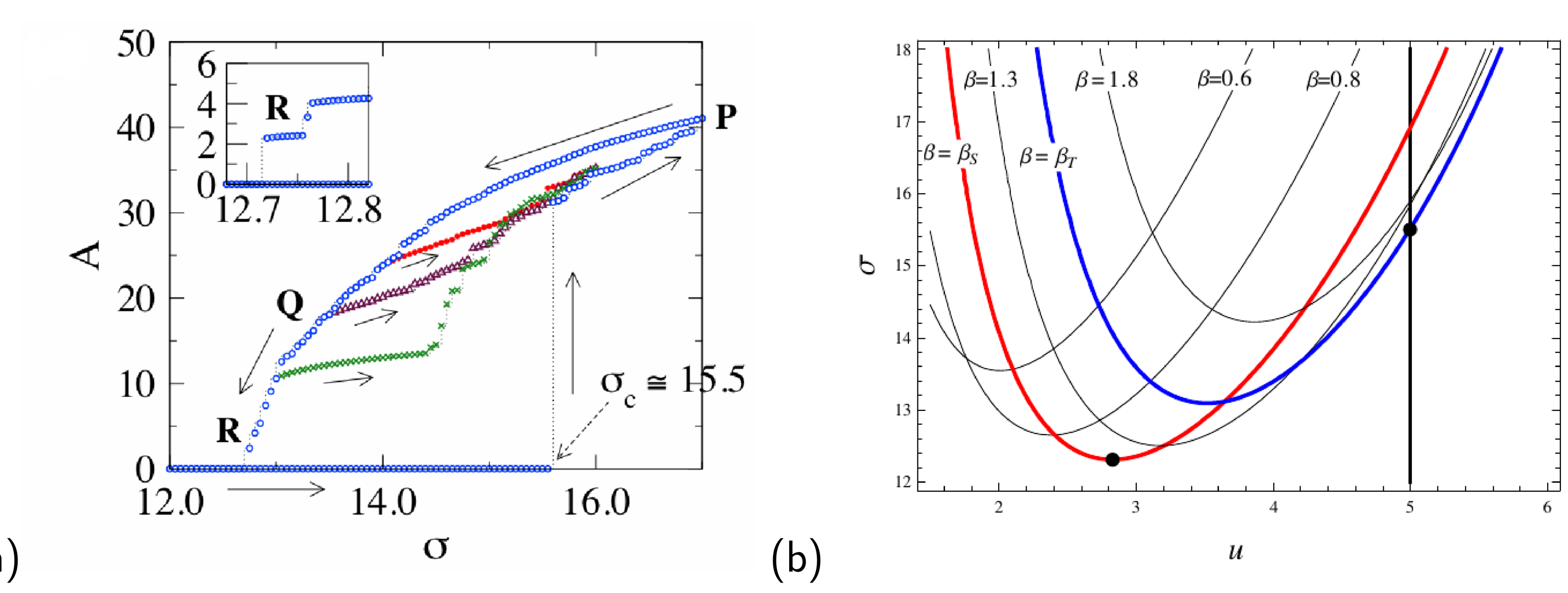


Figure 1: A "Complex" Network

Mimura-Murray on Complex Networks

- Coexistence and multi-stability of a huge variety of patterns was previously found [4].



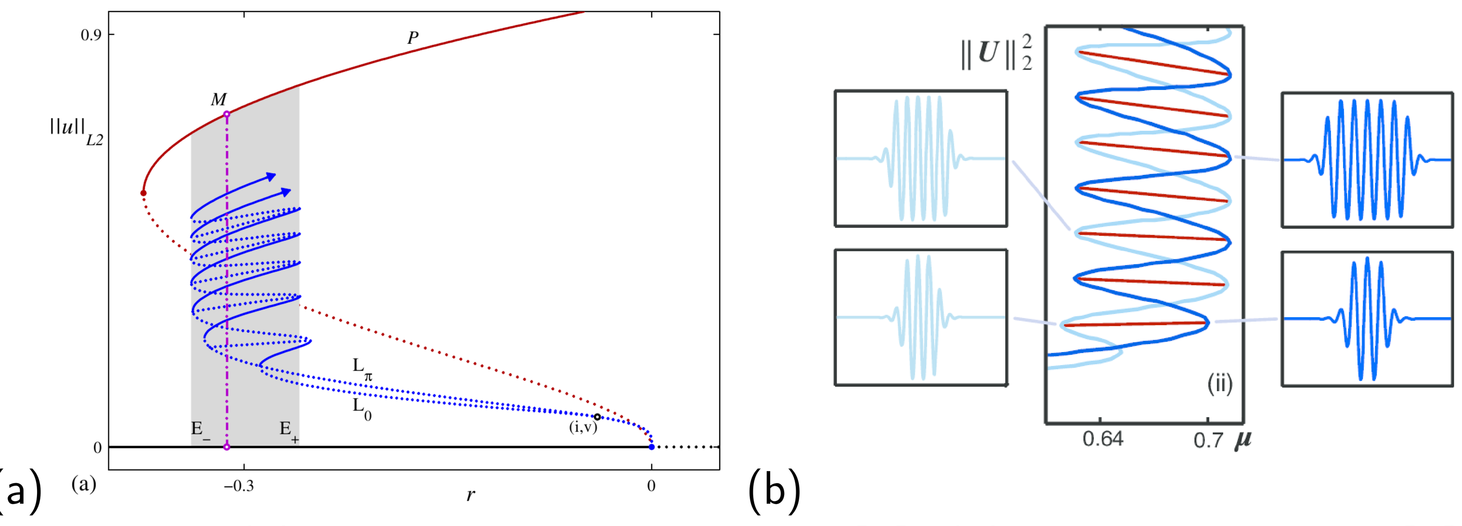
(a) (b)

Figure 2: (a) from Nakao & Mikhailov (2010) [4]; (b) from Wolfrum [3].

- Explained analytically by considering single differentiated nodes (SDN) with others fixed at equilibrium [3], multiple bifurcation curves found – one for each node.

Localised Patterns and Snaking Bifurcations

- Theory exists for development of patterns on reaction-diffusion systems on regular lattices:



(a) (b)

Figure 3: (a) from Burke & Knobloch [6]; (b) from Beck et al. [7]

- localised patterns develop via homoclinic "snaking bifurcations" from a ground-state;
- each level of the "snake" is associated with increasingly more *activated nodes*.

Snaking Bifurcations on Complex Mimura-Murray

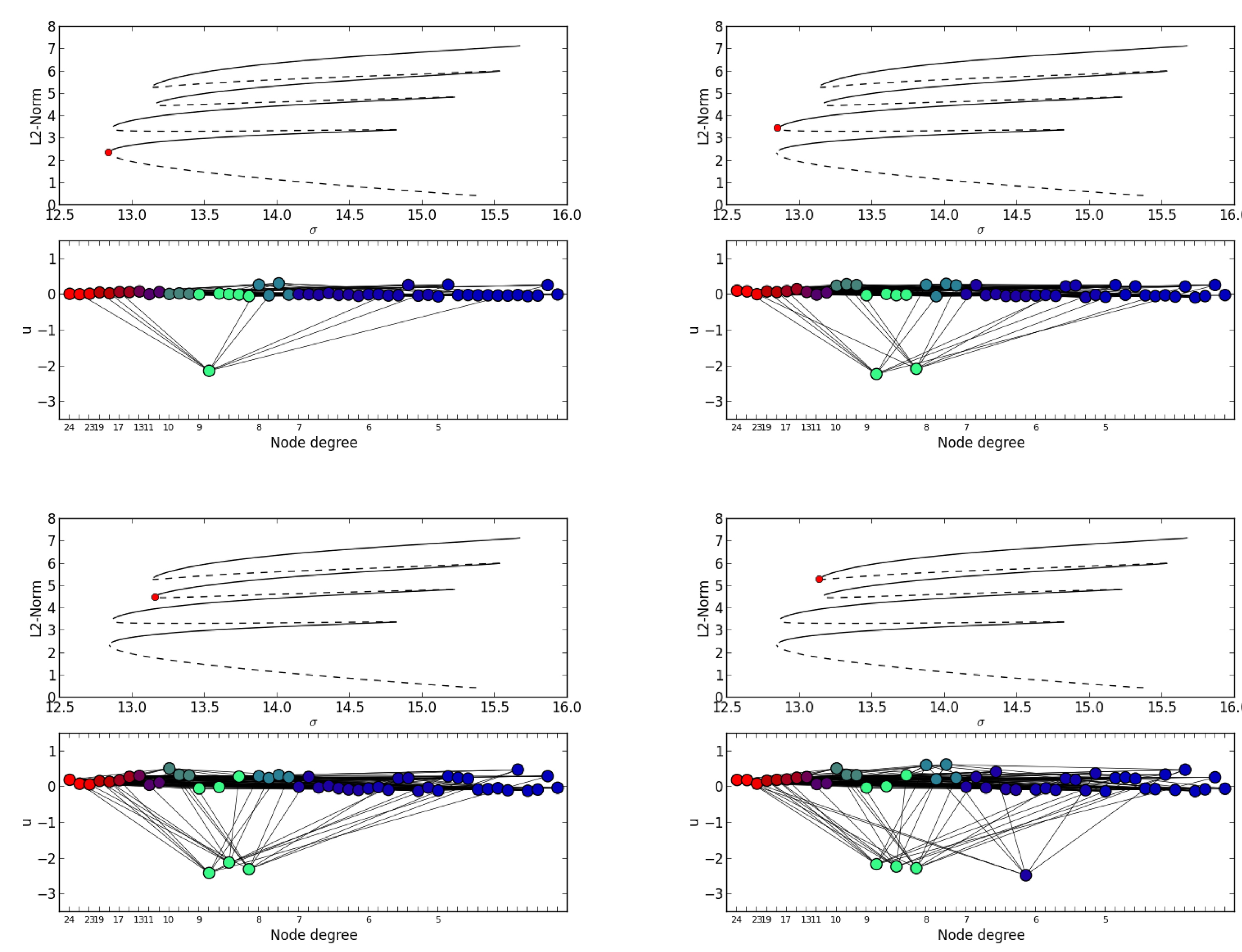
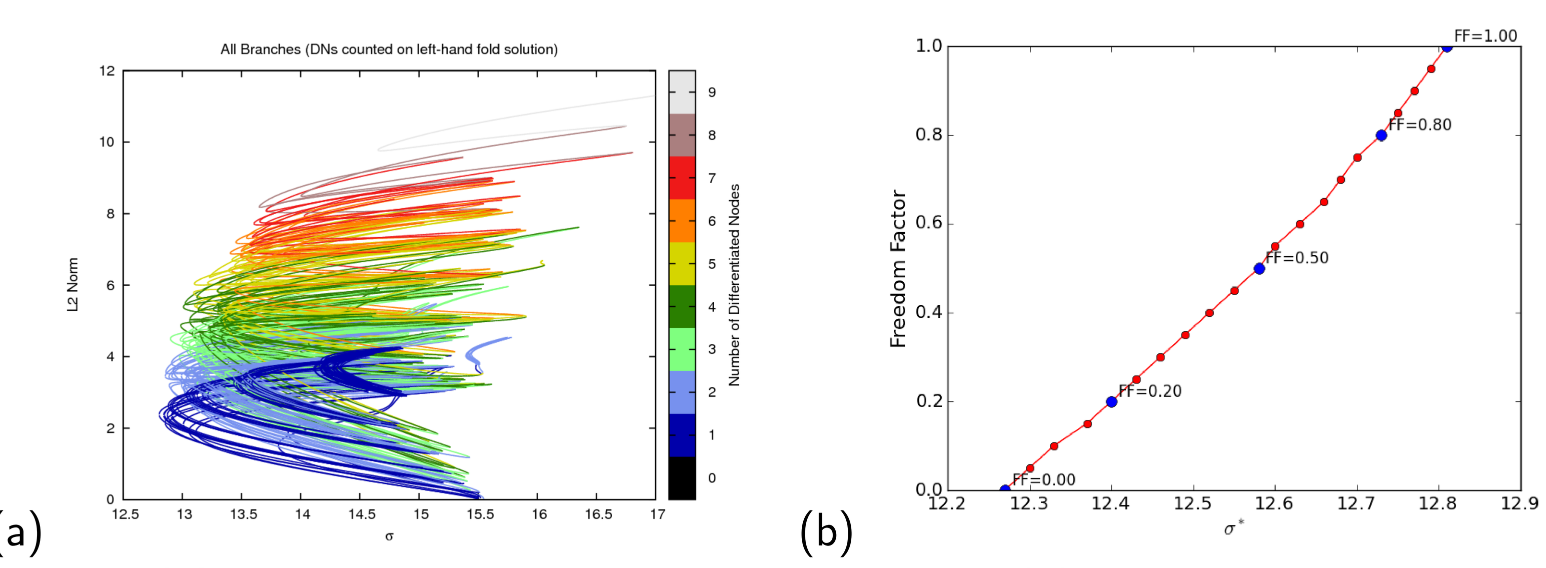


Figure 4: We find that snaking is also the route to multiply differentiated patterns on Mimura-Murray systems on complex scale-free networks.

The Bigger Picture



(a) (b)

Figure 5: (a) The "zoo" of solutions is immense; (b) Attenuating the influence of neighbouring nodes reveals the deviation of the minimum (σ^*) of the region of existence of the differentiated states from the single-node (SDN) theory.

Conclusions

- Reaction-diffusion systems on complex scale-free networks display a rich set of coexisting solutions, from single differentiated nodes to full-scale *Turing patterns*.
- The patterns of activity, originating in singly-differentiated-node solutions, develop into full-scale *patterns* of activity via *snaking bifurcations*, as found in other related systems.
- Simple analysis can begin to reveal the origins of the full complexity of these systems.

References

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