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A block hybrid integrator for numerically solving fourth-order Initial Value Problems

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ABSTRACT

A Linear Multistep Hybrid Block method with four intra-step grid points is presented for approximating directly the solution of fourth order Initial Value Problems (IVPs). Multiple Finite Difference formulas are derived and combined in a block formulation to form a numerical integrator that provides direct solution to fourth order IVPs over sub-intervals. The properties and convergence of the proposed method are discussed. The superiority of this method over existing methods is established numerically on different test problems.

1. Introduction

In this work, fourth-order problems of the type:

$$y^{(iv)} = f(x, y, y', y'', y'''), \quad a \le x \le b, \tag{1}$$

with appropriate initial conditions

$$y(a) = \alpha_0, y'(a) = \alpha_1, y''(a) = \alpha_2, y'''(a) = \alpha_3, \tag{2}$$

are solved numerically, where a, b, α_i, $i = 0, 1, 2, 3$ are real numbers. We will see later that the developed method may also be used for solving system of differential equations.

Problems arising from engineering and other sciences, just to mention a few, have been modeled using high order linear and nonlinear IVPs. One of the applications of fourth order problems arises in the static deflection of a uniform beam or a cantilever beam (with left end embedded and right end free gives rise to fourth order IVPs (see [4,12]).

Several numerical methods exist in the literature, but just a few are specially designed to solve fourth order ordinary differential equations. For instance, Yap and Ismail [2] developed a Block Hybrid Collocation Method (BHCM) and applied it to solve fourth order IVPs. They considered three off-grid points by means of collocation. Abdelrahim and Omar [3] developed a four-step implicit block method with three generalized off-step points and applied it to solve fourth order IVPs directly. In the work by Awoyemi [13], a multiderivative collocation method was developed to obtain the approximation of fourth order IVPs. In [10], Hussain et al. presented a Runge–Kutta type method for directly solving this kind of problems.

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In this work, we consider linear multistep hybrid formulas with four mid intra-step grid points and fifth derivatives at the two endpoints. These formulas are constructed using a collocation approach, and put together to form a Block Hybrid Method (see [1,5,6,8,9]).

2. Derivation of the method

This section describes the derivation of a continuous implicit four intra-step hybrid block method for the solution of the IVP (1). Consider the grid points on the interval of integration $[a, b]$, $\pi_N \equiv \{a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b\}$, with h the constant step size, $h = x_j - x_{j-1},\ j = 1, 2, \ldots, N$. The method relies on the approximation of the exact solution $y(x)$ of (1) at the grid points of four adjacent subintervals and the corresponding intermediate points, by the polynomial $p(x)$ given by

$$y(x) \simeq p(x) = \sum_{i=0}^{14} \rho_i x^i. \tag{3}$$

This yields

$$y^{(iv)}(x) \simeq p^{(4)}(x) = \sum_{i=4}^{14} \rho_i i(i-1)(i-2)(i-3)x^{i-4}, \tag{4}$$

$$y^{(v)}(x) \simeq p^{(5)}(x) = \sum_{i=5}^{14} \rho_i i(i-1)(i-2)(i-3)(i-4)x^{i-5}. \tag{5}$$

Here, ρ_i are real unknown coefficients to be determined. In order to simplify the derivation of the method we will consider the first block, $[x_0, x_4]$, as the formulas obtained may be easily shifted to successive blocks. Thus, we consider the points $x_{\frac{j}{2}} = \frac{j}{2}h,\ j = 0, 1, \ldots, 8$ in the four-step scheme for approximating the solution on $[x_0, x_4]$. After considering the approximations in (3) up to the third derivative applied to the point x_0, the fourth derivative in (4) applied to the points $x_{\frac{j}{2}}$, $j = 0, 1, \ldots, 8$ and the fifth derivative in (5) applied to the points x_j, $j = 0, 4$, we obtain a system of 15 equations with 15 unknowns (the ρ_i, $i = 0, 1, \ldots, 14$). This system may be written in matrix form as

$$\begin{pmatrix}
1 & x_0 & x_0^2 & x_0^3 & x_0^4 & x_0^5 & x_0^6 & \ldots & x_0^{13} & x_0^{14} \\
0 & 1 & 2x_0 & 3x_0^2 & 4x_0^3 & 5x_0^4 & 6x_0^5 & \ldots & 13x_0^{12} & 14x_0^{13} \\
0 & 0 & 2 & 6x_0 & 12x_0^2 & 20x_0^3 & 30x_0^4 & \ldots & 156x_0^{11} & 182x_0^{12} \\
0 & 0 & 0 & 6 & 24x_0 & 60x_0^2 & 120x_0^3 & \ldots & 1716x_0^{10} & 2184x_0^{11} \\
0 & 0 & 0 & 0 & 24 & 120x_0 & 360x_0^2 & \ldots & 17160x_0^9 & 24024x_0^{10} \\
0 & 0 & 0 & 0 & 24 & 120x_{\frac{1}{2}} & 360x_{\frac{1}{2}}^2 & \ldots & 17160x_{\frac{1}{2}}^9 & 24024x_{\frac{1}{2}}^{10} \\
0 & 0 & 0 & 0 & 24 & 120x_1 & 360x_1^2 & \ldots & 17160x_1^9 & 24024x_1^{10} \\
0 & 0 & 0 & 0 & 24 & 120x_{\frac{3}{2}} & 360x_{\frac{3}{2}}^2 & \ldots & 17160x_{\frac{3}{2}}^9 & 24024x_{\frac{3}{2}}^{10} \\
0 & 0 & 0 & 0 & 24 & 120x_2 & 360x_2^2 & \ldots & 17160x_2^9 & 24024x_2^{10} \\
0 & 0 & 0 & 0 & 24 & 120x_{\frac{5}{2}} & 360x_{\frac{5}{2}}^2 & \ldots & 17160x_{\frac{5}{2}}^9 & 24024x_{\frac{5}{2}}^{10} \\
0 & 0 & 0 & 0 & 24 & 120x_3 & 360x_3^2 & \ldots & 17160x_3^9 & 24024x_3^{10} \\
0 & 0 & 0 & 0 & 24 & 120x_{\frac{7}{2}} & 360x_{\frac{7}{2}}^2 & \ldots & 17160x_{\frac{7}{2}}^9 & 24024x_{\frac{7}{2}}^{10} \\
0 & 0 & 0 & 0 & 24 & 120x_4 & 360x_4^2 & \ldots & 17160x_4^9 & 24024x_4^{10} \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_0 & \ldots & 154440x_0^8 & 240240x_0^9 \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_4 & \ldots & 154440x_4^8 & 240240x_4^9
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \\ \rho_7 \\ \rho_8 \\ \rho_9 \\ \rho_{10} \\ \rho_{11} \\ \rho_{12} \\ \rho_{13} \\ \rho_{14}
\end{pmatrix}
=
\begin{pmatrix}
y_0 \\ y_0' \\ y_0'' \\ y_0''' \\ f_0 \\ f_{\frac{1}{2}} \\ f_1 \\ f_{\frac{3}{2}} \\ f_2 \\ f_{\frac{5}{2}} \\ f_3 \\ f_{\frac{7}{2}} \\ f_4 \\ g_0 \\ g_4
\end{pmatrix}$$

where the approximate values are given by $y_i^{(j)} \simeq y^{(j)}(x_i)$, $f_i = f(x_i, y_i, y_i', y_i'', y_i''')$, and $g_i \simeq \frac{df(x,y,y',y'',y''')}{dx}\Big|_{\{x=x_i, y=y_i, y^{(j)}=y_i^{(j)}\}, j=1,2,3}$.

Solving the system above using a CAS like Mathematica we can obtain the values of the coefficients ρ_i, $i = 0, 1, \ldots, 14$, which are not included here. After simplification, the approximating polynomial in (3) adopts the form

$$p(x) = \sum_{i=0}^{3} \alpha_i(x) y_0^{(i)} h^i + h^4 \sum_{i=0}^{8} \beta_i(x) f_{i/2} + h^5(\gamma_0(x) g_0 + \gamma_4(x) g_4) \tag{6}$$

where the $\alpha's$, $\beta's$ and $\gamma's$ are continuous coefficients (which are cumbersome expression that are not included here, but can be easily obtained with the help of a CAS).

Now, we evaluate $p(x)$ in (6) at the points $x = x_{\frac{i}{2}}$, $i = 1, 2, \ldots, 8$, and after some simplifications, we obtain the following main methods:

$$\left.\begin{aligned}
y_{\frac{1}{2}} = {} & y_0 + \frac{hy_0'}{2} + \frac{h^2 y_0''}{8} + \frac{h^3 y_0'''}{48} + h^4\Bigg(\frac{337092419021 f_0}{156223497830400} + \frac{292918007 f_{\frac{1}{2}}}{406832025600} - \frac{37197037 f_1}{66421555200} + \frac{9931411 f_{\frac{3}{2}}}{17791488000} \\
& - \frac{50529077 f_2}{106274488320} + \frac{721757 f_{\frac{5}{2}}}{2264371200} - \frac{1210801 f_3}{7380172800} + \frac{27789161 f_{\frac{7}{2}}}{406832025600} - \frac{15088644019 f_4}{781117489152000}\Bigg) + h^5\left(\frac{2821493 g_0}{27051687936} + \frac{1858909 g_4}{572247244800}\right) \\
y_1 = {} & y_0 + hy_0' + \frac{h^2 y_0''}{2} + \frac{h^3 y_0'''}{6} + h^4\Bigg(\frac{1495257787 f_0}{58677696000} + \frac{6383912 f_{\frac{1}{2}}}{297972675} - \frac{37451 f_1}{3474900} + \frac{9346 f_{\frac{3}{2}}}{868725} \\
& - \frac{1419283 f_2}{155675520} + \frac{185188 f_{\frac{5}{2}}}{30405375} - \frac{43469 f_3}{13899600} + \frac{387302 f_{\frac{7}{2}}}{297972675} - \frac{6225119 f_4}{16951334400}\Bigg) + h^5\left(\frac{913259 g_0}{605404800} + \frac{2803 g_4}{45405360}\right) \\
y_{\frac{3}{2}} = {} & y_0 + \frac{3\ hy_0'}{2} + \frac{9h^2 y_0''}{8} + \frac{9h^3 y_0'''}{16} + h^4\Bigg(\frac{315524886687 f_0}{3214475264000} + \frac{46825209 f_{\frac{1}{2}}}{386355200} - \frac{24065289 f_1}{820019200} + \frac{21137157 f_{\frac{3}{2}}}{512512000} \\
& - \frac{9357849 f_2}{262406144} + \frac{1120689 f_{\frac{5}{2}}}{46592000} - \frac{10181709 f_3}{820019200} + \frac{26006427 f_{\frac{7}{2}}}{5022617600} - \frac{4710304629 f_4}{3214475264000}\Bigg) + h^5\left(\frac{560084463 g_0}{91842150400} + \frac{22637961 g_4}{91842150400}\right) \\
y_2 = {} & y_0 + 2\ hy_0' + 2h^2 y_0'' + \frac{4h^3 y_0'''}{3} + h^4\Bigg(\frac{147826319 f_0}{595945350} + \frac{36780032 f_{\frac{1}{2}}}{99324225} - \frac{28352 f_1}{2027025} + \frac{3438592 f_{\frac{3}{2}}}{30405375} \\
& - \frac{253 f_2}{2835} + \frac{366592 f_{\frac{5}{2}}}{6081075} - \frac{3008 f_3}{96525} + \frac{1292288 f_{\frac{7}{2}}}{99324225} - \frac{10978741 f_4}{2979726750}\Bigg) + h^5\left(\frac{89147 g_0}{5675670} + \frac{17593 g_4}{28378350}\right) \\
y_{\frac{5}{2}} = {} & y_0 + \frac{5\ hy_0'}{2} + \frac{25h^2 y_0''}{8} + \frac{125h^3 y_0'''}{48} + h^4\Bigg(\frac{3144064437125 f_0}{6248939913216} + \frac{40940215625 f_{\frac{1}{2}}}{48819843072} + \frac{708940625 f_1}{7970586624} + \frac{26069375 f_{\frac{3}{2}}}{90574848} \\
& - \frac{106945546875 f_2}{63764692992} + \frac{120144875 f_{\frac{5}{2}}}{996323328} - \frac{71153125 f_3}{1138655232} + \frac{98121875 f_{\frac{7}{2}}}{3755372544} - \frac{5140964375 f_4}{694326657024}\Bigg) + h^5\left(\frac{3192063125 g_0}{99189522432} + \frac{370815625 g_4}{297568567296}\right) \\
y_3 = {} & y_0 + 3\ hy_0' + \frac{9h^2 y_0''}{2} + \frac{9h^3 y_0'''}{2} + h^4\Bigg(\frac{559610739 f_0}{627827200} + \frac{1957122 f_{\frac{1}{2}}}{1226225} + \frac{133191 f_1}{400400} + \frac{11412 f_{\frac{3}{2}}}{17875} \\
& - \frac{146529 f_2}{640640} + \frac{5562 f_{\frac{5}{2}}}{25025} - \frac{11007 f_3}{100100} + \frac{56376 f_{\frac{7}{2}}}{1226225} - \frac{3145959 f_4}{241472000}\Bigg) + h^5\left(\frac{32157 h^5 g_0}{560560} + \frac{49167 h^5 g_4}{22422400}\right) \\
y_{\frac{7}{2}} = {} & y_0 + \frac{7\ hy_0'}{2} + \frac{49h^2 y_0''}{8} + \frac{343h^3 y_0'''}{48} + h^4\Bigg(\frac{3280726311163 f_0}{2277310464000} + \frac{3217491263 f_{\frac{1}{2}}}{1186099200} + \frac{7309414721 f_1}{9488793600} + \frac{400359547 f_{\frac{3}{2}}}{323481600} \\
& - \frac{3318995939 f_2}{15182069760} + \frac{7526124179 f_{\frac{5}{2}}}{17791488000} - \frac{524361593 f_3}{3162931200} + \frac{87283553 f_{\frac{7}{2}}}{1186099200} - \frac{9503592581 f_4}{455462092800}\Bigg) + h^5\left(\frac{1087295251 g_0}{11678515200} + \frac{9697639 g_4}{2760376320}\right) \\
y_4 = {} & y_0 + 4\ hy_0' + 8h^2 y_0'' + \frac{32h^3 y_0'''}{3} + h^4\Bigg(\frac{3246302912 f_0}{1489863375} + \frac{1268842496 f_{\frac{1}{2}}}{297972675} + \frac{8851456 f_1}{6081075} + \frac{65536 f_{\frac{3}{2}}}{30375} \\
& - \frac{20224 f_2}{243243} + \frac{24051712 f_{\frac{5}{2}}}{30405375} - \frac{1101824 f_3}{6081075} + \frac{36110336 f_{\frac{7}{2}}}{297972675} - \frac{5220896 f_4}{165540375}\Bigg) + h^5\left(\frac{668032 h^5 g_0}{4729725} + \frac{75392 h^5 g_4}{14189175}\right)
\end{aligned}\right\} \tag{7}$$

Then, evaluating $p'(x)$ in (6), at the points $x = x_{\frac{i}{2}}$, $i = 1, 2, \ldots, 8$, we obtain the following formulas for approximating the first derivatives:

$$
\left.
\begin{aligned}
y'_{\frac{1}{2}} =\ & y'_0 + \frac{hy''_0}{2} + \frac{h^2y'''_0}{8} + h^3\left(\frac{130299259219f_0}{8136640512000} + \frac{509169931f_{\frac{1}{2}}}{67805337600} - \frac{11332931f_1}{2075673600} + \frac{22339813f_{\frac{3}{2}}}{4151347200} - \frac{1154233f_2}{253034496}\right.\\
&\left. + \frac{7033927f_{\frac{5}{2}}}{2306304000} - \frac{13011319f_3}{8302694400} + \frac{44188141f_{\frac{7}{2}}}{67805337600} - \frac{1198899799f_4}{6509312409600}\right) + h^4\left(\frac{178138097g_0}{206644838400} + \frac{3839317g_4}{123986903040}\right)\\
y'_1 =\ & y'_0 + hy''_0 + \frac{h^2y'''_0}{2} + h^3\left(\frac{22061331283f_0}{254270016000} + \frac{1924388f_{\frac{1}{2}}}{19864845} - \frac{2391523f_1}{64864800} + \frac{56344f_{\frac{3}{2}}}{1447875} - \frac{1726601f_2}{51891840}\right.\\
&\left. + \frac{226084f_{\frac{5}{2}}}{10135125} - \frac{2363f_3}{205920} + \frac{474256f_{\frac{7}{2}}}{99324225} - \frac{114404293f_4}{84756672000}\right) + h^4\left(\frac{3264091g_0}{605404800} + \frac{68699g_4}{302702400}\right)\\
y'_{\frac{3}{2}} =\ & y'_0 + \frac{3hy''_0}{2} + \frac{9h^2y'''_0}{8} + h^3\left(\frac{85673001969f_0}{401809408000} + \frac{817974369f_{\frac{1}{2}}}{2511308800} - \frac{2196099f_1}{102502400} + \frac{22589577f_{\frac{3}{2}}}{256256000} - \frac{24933339f_2}{328007680}\right.\\
&\left. + \frac{13214367f_{\frac{5}{2}}}{256256000} - \frac{195633f_3}{7321600} + \frac{28053783f_{\frac{7}{2}}}{2511308800} - \frac{635955543f_4}{200904704000}\right) + h^4\left(\frac{311747967g_0}{22960537600} + \frac{12232053g_4}{22960537600}\right)\\
y'_2 =\ & y'_0 + 2hy''_0 + 2h^2y'''_0 + h^3\left(\frac{786030943f_0}{1986484500} + \frac{22976384f_{\frac{1}{2}}}{33108075} + \frac{204376f_1}{2027025} + \frac{320896f_{\frac{3}{2}}}{1447875} - \frac{18848f_2}{135135}\right.\\
&\left. + \frac{45952f_{\frac{5}{2}}}{482625} - \frac{99928f_3}{2027025} + \frac{682112f_{\frac{7}{2}}}{33108075} - \frac{11595887f_4}{1986484500}\right) + h^4\left(\frac{26773g_0}{1051050} + \frac{9293g_4}{9459450}\right)\\
y'_{\frac{5}{2}} =\ & y'_0 + \frac{5hy''_0}{2} + \frac{25h^2y'''_0}{8} + h^3\left(\frac{82531925075f_0}{130186248192} + \frac{9784510625f_{\frac{1}{2}}}{8136640512} + \frac{7795625f_1}{23721984} + \frac{83309125f_{\frac{3}{2}}}{166053888} - \frac{841840625f_2}{5313724416}\right.\\
&\left. + \frac{24889075f_{\frac{5}{2}}}{166053888} - \frac{8554375f_3}{110702592} + \frac{263294375f_{\frac{7}{2}}}{8136640512} - \frac{796570375f_4}{86790832128}\right) + h^4\left(\frac{1017691375g_0}{24797380608} + \frac{38318125g_4}{24797380608}\right)\\
y'_3 =\ & y'_0 + 3hy''_0 + \frac{9y'''_0}{2} + h^3\left(\frac{2913731487f_0}{3139136000} + \frac{2269296f_{\frac{1}{2}}}{1226225} + \frac{75951f_1}{114400} + \frac{23148f_{\frac{3}{2}}}{25025} - \frac{8667f_2}{128128}\right.\\
&\left. + \frac{4968f_{\frac{5}{2}}}{17875} - \frac{89613f_3}{800800} + \frac{57996f_{\frac{7}{2}}}{1226225} - \frac{8420679f_4}{627827200}\right) + h^4\left(\frac{676161g_0}{11211200} + \frac{2025g_4}{896896}\right)\\
y'_{\frac{7}{2}} =\ & y'_0 + \frac{7hy''_0}{2} + \frac{49h^2y'''_0}{8} + h^3\left(\frac{121286820613f_0}{94887936000} + \frac{104340257f_{\frac{1}{2}}}{39536640} + \frac{1310256913f_1}{1186099200} + \frac{4435720247f_{\frac{3}{2}}}{2965248000} + \frac{159649693f_2}{1265172480}\right.\\
&\left. + \frac{541538347f_{\frac{5}{2}}}{988416000} - \frac{5529503f_3}{59304960} + \frac{12920467f_{\frac{7}{2}}}{197683200} - \frac{432422501f_4}{23721984000}\right) + h^4\left(\frac{351091027g_0}{4217241600} + \frac{38807363g_4}{12651724800}\right)\\
y'_4 =\ & y'_0 + 4hy''_0 + 8h^2y'''_0 + h^3\left(\frac{836474624f_0}{496621125} + \frac{354254848f_{\frac{1}{2}}}{99324225} + \frac{3352576f_1}{2027025} + \frac{22380544f_{\frac{3}{2}}}{10135125} + \frac{25088f_2}{57915}\right.\\
&\left. + \frac{9601024f_{\frac{5}{2}}}{10135125} + \frac{34816f_3}{675675} + \frac{14319616f_{\frac{7}{2}}}{99324225} - \frac{3750752f_4}{165540375}\right) + h^4\left(\frac{519872g_0}{4729725} + \frac{18848g_4}{4729725}\right)
\end{aligned}
\right\} \qquad (8)
$$

Similarly, evaluating $p''(x)$ in (6), at the points $x = x_{\frac{i}{2}},\ i = 1, 2, \ldots, 8,$ we obtain the formulas for approximating the second derivatives:

$$
\left.\begin{aligned}
y''_{\frac{1}{2}} = {}& y''_0 + \frac{hy'''_0}{2} + h^2\left(\frac{29891673451 f_0}{357654528000} + \frac{21119 f_{\frac{1}{2}}}{345600} - \frac{5043739 f_1}{127733760} + \frac{15287341 f_{\frac{3}{2}}}{399168000} - \frac{3661313 f_2}{113541120} + \frac{8574319 f_{\frac{5}{2}}}{399168000}\right.\\
&\left. - \frac{2344229 f_3}{212889600} + \frac{170323 f_{\frac{7}{2}}}{37255680} - \frac{9422989 f_4}{7299072000}\right) + h^3\left(\frac{17340643 g_0}{3406233600} + \frac{11731 g_4}{54067200}\right)\\
y''_1 = {}& y''_0 + hy'''_0 + h^2\left(\frac{39420877 f_0}{199584000} + \frac{344101 f_{\frac{1}{2}}}{1091475} - \frac{21 f_1}{400} + \frac{20869 f_{\frac{3}{2}}}{259875} - \frac{286259 f_2}{3991680} + \frac{4237 f_{\frac{5}{2}}}{86625}\right.\\
&\left. - \frac{63353 f_3}{2494800} + \frac{1657 f_{\frac{7}{2}}}{155925} - \frac{467641 f_4}{155232000}\right) + h^3\left(\frac{24251 g_0}{1900800} + \frac{2249 g_4}{4435200}\right)\\
y''_{\frac{3}{2}} = {}& y''_0 + \frac{3hy'''_0}{2} + h^2\left(\frac{272752743 f_0}{883097600} + \frac{4122009 f_{\frac{1}{2}}}{6899200} + \frac{1061937 f_1}{7884800} + \frac{75 f_{\frac{3}{2}}}{512} - \frac{1377081 f_2}{12615680} + \frac{72117 f_{\frac{5}{2}}}{985600}\right.\\
&\left. - \frac{298311 f_3}{7884800} + \frac{109107 f_{\frac{7}{2}}}{6899200} - \frac{791613 f_4}{176619520}\right) + h^3\left(\frac{506421 g_0}{25231360} + \frac{3807 g_4}{5046272}\right)\\
y''_2 = {}& y'''_0 + 2hy'''_0 + h^2\left(\frac{41749 f_0}{99225} + \frac{45568 f_{\frac{1}{2}}}{51975} + \frac{10928 f_1}{31185} + \frac{2560 f_{\frac{3}{2}}}{6237} - \frac{16 f_2}{135} + \frac{14848 f_{\frac{5}{2}}}{155925}\right.\\
&\left. - \frac{2608 f_3}{51975} + \frac{512 f_{\frac{7}{2}}}{24255} - \frac{17 f_4}{2835}\right) + h^3\left(\frac{571 g_0}{20790} + \frac{g_4}{990}\right)\\
y''_{\frac{5}{2}} = {}& y''_0 + \frac{5hy'''_0}{2} + h^2\left(\frac{217663025 f_0}{408748032} + \frac{25854875 f_{\frac{1}{2}}}{22353408} + \frac{4794875 f_1}{8515584} + \frac{748975 f_{\frac{3}{2}}}{1064448} + \frac{14076875 f_2}{204374016} + \frac{75 f_{\frac{5}{2}}}{512}\right.\\
&\left. - \frac{1676375 f_3}{25546752} + \frac{86375 f_{\frac{7}{2}}}{3193344} - \frac{7288475 f_4}{953745408}\right) + h^3\left(\frac{678125 g_0}{19464192} + \frac{58375 g_4}{45416448}\right)\\
y''_3 = {}& y''_0 + 3hy'''_0 + h^2\left(\frac{11111631 f_0}{17248000} + \frac{19359 f_{\frac{1}{2}}}{13475} + \frac{4779 f_1}{6160} + \frac{9573 f_{\frac{3}{2}}}{9625} + \frac{14013 f_2}{49280} + \frac{3807 f_{\frac{5}{2}}}{9625}\right.\\
&\left. - \frac{21 f_3}{400} + \frac{81 f_{\frac{7}{2}}}{2695} - \frac{150741 f_4}{17248000}\right) + h^3\left(\frac{20799 g_0}{492800} + \frac{729 g_4}{492800}\right)\\
y''_{\frac{7}{2}} = {}& y''_0 + \frac{7hy'''_0}{2} + h^2\left(\frac{5519521651 f_0}{7299072000} + \frac{724759 f_{\frac{1}{2}}}{422400} + \frac{90371239 f_1}{91238400} + \frac{73127257 f_{\frac{3}{2}}}{57024000} + \frac{24420571 f_2}{48660480} + \frac{38135083 f_{\frac{5}{2}}}{57024000}\right.\\
&\left. + \frac{1630279 f_3}{10137600} + \frac{21119 f_{\frac{7}{2}}}{345600} - \frac{90739621 f_4}{7299072000}\right) + h^3\left(\frac{24142741 g_0}{486604800} + \frac{328937 g_4}{162201600}\right)\\
y''_4 = {}& y''_0 + 4hy'''_0 + h^2\left(\frac{947108 f_0}{1091475} + \frac{311296 f_{\frac{1}{2}}}{155925} + \frac{62464 f_1}{51975} + \frac{16384 f_{\frac{3}{2}}}{10395} + \frac{22208 f_2}{31185} + \frac{16384 f_{\frac{5}{2}}}{17325}\right.\\
&\left. + \frac{62464 f_3}{155925} + \frac{311296 f_{\frac{7}{2}}}{1091475} + \frac{148 f_4}{10395}\right) + \frac{592 h^3 g_0}{10395}
\end{aligned}\right\} \quad (9)
$$

Finally, evaluating $p'''(x)$ in (6), at the points $x = x_{\frac{i}{2}},\ i = 1, 2, \ldots, 8,$ we obtain the formulas for approximating the third derivatives given by:

$$
\left.\begin{aligned}
y'''_{\frac{1}{2}} &= y'''_0 + h\Bigg(\frac{10679850817 f_0}{44706816000} + \frac{19430681 f_{\frac{1}{2}}}{55883520} - \frac{13615823 f_1}{79833600} + \frac{10732441 f_{\frac{3}{2}}}{66528000} - \frac{1073407 f_2}{7983360} + \frac{1975213 f_{\frac{5}{2}}}{22176000} \\
&\quad - \frac{726493 f_3}{15966720} + \frac{5265763 f_{\frac{7}{2}}}{279417600} - \frac{237618977 f_4}{44706816000}\Bigg) + h^2\left(\frac{3590473 g_0}{212889600} + \frac{190073 g_4}{212889600}\right) \\
y'''_{1} &= y'''_0 + h\Bigg(\frac{21921839 f_0}{99792000} + \frac{632069 f_{\frac{1}{2}}}{1091475} + \frac{128761 f_1}{623700} + \frac{1921 f_{\frac{3}{2}}}{259875} - \frac{6667 f_2}{249480} + \frac{6079 f_{\frac{5}{2}}}{259875} \\
&\quad - \frac{8453 f_3}{623700} + \frac{941 f_{\frac{7}{2}}}{155925} - \frac{1230073 f_4}{698544000}\Bigg) + h^2\left(\frac{6731 g_0}{475200} + \frac{997 g_4}{3326400}\right) \\
y'''_{\frac{3}{2}} &= y'''_0 + h\Bigg(\frac{24860509 f_0}{110387200} + \frac{272331 f_{\frac{1}{2}}}{492800} + \frac{445953 f_1}{985600} + \frac{170879 f_{\frac{3}{2}}}{492800} - \frac{12321 f_2}{98560} + \frac{36129 f_{\frac{5}{2}}}{492800} \\
&\quad - \frac{35297 f_3}{985600} + \frac{50067 f_{\frac{7}{2}}}{3449600} - \frac{63963 f_4}{15769600}\Bigg) + h^2\left(\frac{23679 g_0}{1576960} + \frac{153 g_4}{225280}\right) \\
y'''_{2} &= y'''_0 + h\Bigg(\frac{1387597 f_0}{6237000} + \frac{615424 f_{\frac{1}{2}}}{1091475} + \frac{64496 f_1}{155925} + \frac{160768 f_{\frac{3}{2}}}{259875} + \frac{5552 f_2}{31185} + \frac{1024 f_{\frac{5}{2}}}{86625} \\
&\quad - \frac{2032 f_3}{155925} + \frac{1024 f_{\frac{7}{2}}}{155925} - \frac{86699 f_4}{43659000}\Bigg) + h^2\left(\frac{433 g_0}{29700} + \frac{71 g_4}{207900}\right) \\
y'''_{\frac{5}{2}} &= y'''_0 + h\Bigg(\frac{80310805 f_0}{357654528} + \frac{6213125 f_{\frac{1}{2}}}{11176704} + \frac{1393625 f_1}{3193344} + \frac{296525 f_{\frac{3}{2}}}{532224} + \frac{768125 f_2}{1596672} + \frac{150995 f_{\frac{5}{2}}}{532224} \\
&\quad - \frac{165625 f_3}{3193344} + \frac{198875 f_{\frac{7}{2}}}{11176704} - \frac{1687925 f_4}{357654528}\Bigg) + h^2\left(\frac{127025 g_0}{8515584} + \frac{6625 g_4}{8515584}\right) \\
y'''_{3} &= y'''_0 + h\Bigg(\frac{1916713 f_0}{8624000} + \frac{1521 f_{\frac{1}{2}}}{2695} + \frac{3189 f_1}{7700} + \frac{5843 f_{\frac{3}{2}}}{9625} + \frac{1179 f_2}{3080} + \frac{5997 f_{\frac{5}{2}}}{9625} \\
&\quad + \frac{299 f_3}{1540} - \frac{117 f_{\frac{7}{2}}}{13475} + \frac{7047 f_4}{8624000}\Bigg) + h^2\left(\frac{1791 g_0}{123200} - \frac{9 g_4}{123200}\right) \\
y'''_{\frac{7}{2}} &= y'''_0 + h\Bigg(\frac{206023153 f_0}{912384000} + \frac{3145261 f_{\frac{1}{2}}}{5702400} + \frac{5087719 f_1}{11404800} + \frac{5145343 f_{\frac{3}{2}}}{9504000} + \frac{559433 f_2}{1140480} + \frac{1486219 f_{\frac{5}{2}}}{3168000} \\
&\quad + \frac{6513913 f_3}{11404800} + \frac{1270003 f_{\frac{7}{2}}}{5702400} - \frac{16782353 f_4}{912384000}\Bigg) + h^2\left(\frac{460159 g_0}{30412800} + \frac{79919 g_4}{30412800}\right) \\
y'''_{4} &= y'''_0 + h\Bigg(\frac{240662 f_0}{1091475} + \frac{622592 f_{\frac{1}{2}}}{1091475} + \frac{62464 f_1}{155925} + \frac{32768 f_{\frac{3}{2}}}{51975} + \frac{11104 f_2}{31185} + \frac{32768 f_{\frac{5}{2}}}{51975} \\
&\quad + \frac{62464 f_3}{155925} + \frac{622592 f_{\frac{7}{2}}}{1091475} + \frac{240662 f_4}{1091475}\Bigg) + h^2\left(\frac{148 g_0}{10395} - \frac{148 g_4}{10395}\right)
\end{aligned}\right\} \tag{10}
$$

All the formulas in (7)–(10) considered together form the block method, that should be applied sequentially on block of intervals of the form $[x_n, x_{n+4}]$, $n = 0, 4, N-4$, where N, the number of subintervals, must be a multiple of 4 in order to reach after an integer number of blocks the final point $x_N = b$.

3. Analysis of the method

3.1. Local truncation error and order

Let us consider the linear difference operators associated with the formulas in (7), which could be written as

$$
\begin{aligned}
\mathcal{L}_{i/2}[z(x);\ h] \equiv z\left(x + \frac{i}{2}h\right) - \Bigg[\sum_{k=0}^{3} \alpha_k(x) z^{(k)}(x) h^k + h^4 \sum_{k=0}^{8} \beta_k(x) z^{(4)}\left(x + \frac{k}{2}h\right) \\
+ h^5\left(\gamma_0(x) z^{(5)}(x) + \gamma_4(x) z^{(5)}(x + 4h)\right)\Bigg]
\end{aligned} \tag{11}
$$

for $i = 1(1)8$. The local truncation error of each of the formulas in (7) is the amount by which the exact solution of the ODE fails to satisfy the corresponding difference operator. Thus, if we consider the exact solution $y(x)$ in (11), after expanding in Taylor series around x we get that each of the local truncation errors is of the form

$$
\mathcal{L}[y(x);\ h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_q h^q y^{(q)}(x) + O(h^{(q+1)}). \tag{12}
$$

where the C_i are constants. If we have that the first $p+4$ constants vanish, that is,

$$C_0 = C_1 = C_2 = \cdots = C_{p+3} = 0, \text{ and } C_{p+4} \neq 0$$

then, it is

$$\mathcal{L}[y(x);\ h] = C_{p+4}h^{p+4}y^{(p+4)}(x) + O(h^{p+5}) \tag{13}$$

and p is called the order of the formula and C_{p+4} is known as the principal error constant. In this the case for all the formulas, the method is said to be consistent of order p (see [7]). Taking $x = x_n$, the local truncation errors and principal error constants of the main formulas in (7) are given by

$$\begin{aligned}
\mathcal{L}_{\frac{1}{2}}[y(x_n);\ h] &= -\frac{111253613y^{(15)}(x_n)h^{15}}{565618332716236800} + O(h^{16}); & \mathcal{L}_1[y(x_n);\ h] &= -\frac{2557283y^{(15)}(x_n)h^{15}}{690452066304000} + O(h^{16});\\
\mathcal{L}_{\frac{3}{2}}[y(x_n);\ h] &= -\frac{19360377y^{(15)}(x_n)h^{15}}{1293137477632000} + O(h^{16}); & \mathcal{L}_2[y(x_n);\ h] &= -\frac{1597y^{(15)}(x_n)h^{15}}{42141849750} + O(h^{16});\\
\mathcal{L}_{\frac{5}{2}}[y(x_n);\ h] &= -\frac{1731780625y^{(15)}(x_n)h^{15}}{22624733308649472} + O(h^{16}); & \mathcal{L}_3[y(x_n);\ h] &= -\frac{8523y^{(15)}(x_n)h^{15}}{63141478400} + O(h^{16});\\
\mathcal{L}_{\frac{7}{2}}[y(x_n);\ h] &= \frac{12530643727y^{(15)}(x_n)h^{15}}{57716156399616000} + O(h^{16}); & \mathcal{L}_4[y(x_n);\ h] &= -\frac{6898y^{(15)}(x_n)h^{15}}{21070924875} + O(h^{16}).
\end{aligned} \tag{14}$$

For the formulas in (8)–(10) the local truncation errors may be obtained similarly. From the above results, the order of the block method is $p = 11$.

3.2. Zero-stability and convergence

A numerical method is zero-stable if the solutions remain bounded as $h \to 0$, which means that the method does not provide solutions that grow unbounded as the number of steps increases. To show the zero-stability of the above block method (7)–(10) after taking $h \to 0$ the method may be rewritten in matrix form as

$$A_0 Y_\mu = A_1 Y_{\mu-1} \tag{15}$$

where

$$Y_\mu = \left(Y_\mu^0, Y_\mu^1, Y_\mu^2, Y_\mu^3\right)^T, \quad Y_{\mu-1} = \left(Y_{\mu-1}^0, Y_{\mu-1}^1, Y_{\mu-1}^2, Y_{\mu-1}^3\right)^T$$

$$\begin{aligned}
Y_\mu^0 &= (y_{\frac{1}{2}}, y_1, y_{\frac{3}{2}}, y_2, y_{\frac{5}{2}}, y_3, y_{\frac{7}{2}}, y_4)\\
&\ \vdots\\
Y_\mu^3 &= (y'''_{\frac{1}{2}}, y'''_1, y'''_{\frac{3}{2}}, y'''_2, y'''_{\frac{5}{2}}, y'''_3, y'''_{\frac{7}{2}}, y'''_4)\\
Y_{\mu-1}^0 &= (y_{\frac{1}{2}-1}, y_0, y_{\frac{3}{2}-1}, y_1, y_{\frac{5}{2}-1}, y_2, y_{\frac{7}{2}-1}, y_3)\\
&\ \vdots\\
Y_{\mu-1}^3 &= (y'''_{\frac{1}{2}-1}, y'''_0, y'''_{\frac{3}{2}-1}, y'''_1, y'''_{\frac{5}{2}-1}, y'''_2, y'''_{\frac{7}{2}-1}, y'''_3)
\end{aligned}$$

A_0 is the identity matrix of order 32, $A_0 = I_{32}$, and A_1 is a 32×32 matrix given by

$$\begin{pmatrix} A_{11} & & & \\ & A_{22} & & \\ & & A_{33} & \\ & & & A_{44} \end{pmatrix}$$

with the A_{ii} identical 8×8 matrices given by

$$A_{ii} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

The characteristic polynomial of the matrix A_{11} is given as $|A_{11} - \lambda I_8| = 0$, that is, $\lambda^7(\lambda - 1) = 0$. The roots of the characteristic polynomial are $\lambda_q = 0$, for $q = 1, \ldots, 7$ and $\lambda_8 = 1$. For matrices A_{22}, A_{33} and A_{44} corresponding to the approximate values for the derivatives up to the third order the results are similar. Consequently, the method is zero-stable, since the roots of the characteristic polynomial are all zero except one, whose modulus is one (see [7,16]). For convergence, we state the following theorem.

Theorem 3.1. *Henrici [15]. A linear multistep method is said to be convergent if it is consistent (with order $p \geq 1$) and it is zero-stable.*

By the above analysis, the method has order $p = 11$, and is zero-stable. Thus, by the above theorem, the method is convergent.

4. Computational procedure

The proposed method is implemented in a block mode without the need of predictors. On each block interval of the form $[x_n, x_{n+4}], n = 0, 4, N-4$, being N a multiple of 4 in order to have an integer number of blocks, we solve the system given by all the formulas (7)–(10). To do that we use the Newton's method taking as starting values (denoted with a bar over them) the approximations provided by the Taylor formulas up to the order of the differential equation, that is,

$$\bar{y}_{n+\frac{i}{2}} = y_n + i\frac{h}{2}y'_n + \frac{1}{2}\left(i\frac{h}{2}\right)^2 y''_n + \frac{1}{6}\left(i\frac{h}{2}\right)^3 y'''_n + \frac{1}{24}\left(i\frac{h}{2}\right)^4 f_n$$

$$\bar{y}'_{n+\frac{i}{2}} = y'_n + i\frac{h}{2}y''_n + \frac{1}{2}\left(i\frac{h}{2}\right)^2 y'''_n + \frac{1}{6}\left(i\frac{h}{2}\right)^3 f_n$$

$$\bar{y}''_{n+\frac{i}{2}} = y''_n + i\frac{h}{2}y'''_n + \frac{1}{2}\left(i\frac{h}{2}\right)^2 f_n$$

$$\bar{y}'''_{n+\frac{i}{2}} = y'''_n + i\frac{h}{2}f_n\,,$$

for $i = 1, \ldots, 8$. The presence of $g_n, n = 0(4)N$, in the formulas of the block method, which approximates the fifth derivative at x_n, that is, $g_n \simeq y^{(v)}(x_n)$, requires the calculation of

$$y^{(v)}(x) = \frac{df(x, y(x), y'(x), y''(x), y'''(x))}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y'(x) + \frac{\partial f}{\partial y'}y''(x) + \frac{\partial f}{\partial y''}y'''(x) + \frac{\partial f}{\partial y'''}f(x, y(x), y'(x), y'''(x))\,,$$

which can be obtained by hand in simple cases, or with the use of a computer algebra system in the more complicated ones.

The block method has been implemented using the system *Mathematica*, enhanced by the feature ***NSolve[]*** for linear problems while nonlinear problems were solved by Newton's method enhanced by the feature ***FindRoot[]***, as summarized in the code below.

Algorithm: .

Data: a, b (integration interval), N (number of steps), $y_{00}, y_{10}, y_{20}, y_{30}$ (initial values), f, $\frac{df}{dx}$
Result: sol, discrete approximate solution of the IVP (1) and (2)

Let $n = 0, h = \frac{b-a}{N}$;
Let $x_n = a, y_n = y_{00}, y'_n = y_{10}, y''_n = y_{20}, y'''_n = y_{30}$. Let sol=$\{(x_n, y_n)\}$.
Solve (7)-(10) to get $y_{n+i}, y'_{n+i}, y''_{n+i}, y'''_{n+i}$, $i = 1(1)4$
Let $sol = sol \cup \{(x_{n+i}, y_{n+i})\}_{i=1(1)4}$.
Let $x_n = x_n + 4h, y_n = y_{n+4}, y'_n = y'_{n+4}, y''_n = y''_{n+4}, y'''_n = y'''_{n+4}$
Let $n = n + 4$
if $n = N$ **then**
  go to 12
else
  go to 3;
end
End

Remark 4.1. Note that the proposed method may also be used for solving systems of fourth-order differential equations, by considering a component-wise implementation. For a system of m equations, given in vector form as

$$\mathbf{y}^{(iv)} = \mathbf{f}(x, \mathbf{y}^T, \mathbf{y}'^T, \mathbf{y}''^T, \mathbf{y}'''^T), \qquad a = x_0 \leq x \leq b = x_N$$

with initial values

$$\mathbf{y}(a) = \mathbf{y}_0, \quad \mathbf{y}'(a) = \dot{\mathbf{y}}_0, \quad \mathbf{y}''(a) = \ddot{\mathbf{y}}_0, \quad \mathbf{y}'''(a) = \dddot{\mathbf{y}}_0,$$

where $\mathbf{y} = (y_1, \ldots, y_m)^T$, $\mathbf{y}^{(i)} = (y_1^{(i)}, \ldots, y_m^{(i)})^T$, $i = 1, 2, 3$,

$$\mathbf{f}(x, \mathbf{y}^T, \mathbf{y}'^T, \mathbf{y}''^T, \mathbf{y}'''^T) = (f_1(x, \mathbf{y}^T, \mathbf{y}'^T, \mathbf{y}''^T, \mathbf{y}'''^T), \ldots, f_m(x, \mathbf{y}^T, \mathbf{y}'^T, \mathbf{y}''^T, \mathbf{y}'''^T))^T,$$

and $\mathbf{y}_0 = (y_{1,0}, \ldots, y_{m,0})^T$, $\mathbf{y}_0' = (\dot{y}_{1,0}, \ldots, \dot{y}_{m,0})^T$, $\mathbf{y}_0'' = (\ddot{y}_{1,0}, \ldots, \ddot{y}_{m,0})^T$, $\mathbf{y}_0''' = (\dddot{y}_{1,0}, \ldots, \dddot{y}_{m,0})^T$, we apply the method to each of the scalar equations in the differential system. In the general case this would result in an algebraic system of $32 \times m$ equations, that may be solved using again the Newton's method, as in the scalar case. To get the approximate values of the fifth derivative of each component at $x_n, n = 0(4)N$, denoted by $g_{i,n} \simeq y_i^{(5)}(x_n)$, $i = 1, \ldots, m$, we use the formula

$$y_i^{(5)}(x) = \frac{df_i(x, y_1, \ldots, y_m, y_1', \ldots, y_m', y_1'', \ldots, y_m'', y_1''', \ldots, y_m''')}{dx}$$

$$= \frac{\partial f_i}{\partial x} + \sum_{j=1}^{m} \frac{\partial f_i}{\partial y_j} y_j' + \sum_{j=1}^{m} \frac{\partial f_i}{\partial y_j'} y_j'' + \sum_{j=1}^{m} \frac{\partial f_i}{\partial y_j''} y_j''' + \sum_{j=1}^{m} \frac{\partial f_i}{\partial y_j'''} f_j.$$

5. Numerical examples

Here, some numerical examples are presented to show the accuracy of the developed Block Hybrid Integrator (BHI). In the examples considered the absolute errors were obtained as $Err = \|y_i - y(x_i)\|_\infty$, where y_i is the approximate solution obtained using BHI or any other numerical approach used for comparisons, and $y(x_i)$ is the exact solution of the problem considered at the grid points $\{x_i\}_{i=1}^N$. The approximate solutions with the BHI are compared with the approximate solutions obtained with different methods in the literature. We have considered the block hybrid method of eighth order in [2], named as BHCM4, and the block hybrid method of eighth order in [3] named as BH8, which have been developed for directly solving fourth-order problems. We have also included here, the block 10-step Generalized Adams–Moulton Method which we named AM. This method is also a self-starting block method of order 11 and is used for solving first-order problems. We have implemented this method by reducing all the scalar fourth-order problems into systems of four first-order problems. The overall maximum error was thus compared with maximum errors obtained using the BHI and other methods mentioned earlier. Furthermore, we have plotted the efficiency curves which show the maximum absolute errors (in logarithmic scale) versus CPU times and number of function evaluations (FEVAL) for the BHI and the AM. The resulting plots of efficiency curves give in all cases a line for the BHI data below the line for the AM results, indicating that the performance of BHI is better than that of the AM. We have not found any other higher order methods in the literature for directly solving the fourth-order problems, in order to make other comparisons. Anyway, the results in the tables and the efficiency plots give an idea of the good performance of the proposed method. The obtained results clearly display the efficiency of the BHI for the following examples.

Problem 1. Consider the linear IVP;

$$y^{(iv)} - y''' - y'' - y' - 2y = 0; \quad 0 \leq x \leq 2,$$

$$y(0) = -1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 30,$$

whose exact solution is $y(x) = 2e^{2x} - 5e^{-x} + 2\cos x - 9\sin x$. This problem has been also used in [2,3] to check the behavior of the methods developed there. In Table 1 we can see that the best performance corresponds to the BHI method. Table 2 shows the absolute errors (AE) at different points of the integration interval when $h = 0.1$. in Fig. 1 we can see the efficiency curves of the BHI and AM methods.

Problem 2. Consider the nonlinear IVP, (see [2]).

$$y^{(iv)} - (y')^2 + yy''' = -4x^2 + e^x(1 + x^2 - 4x), \quad 0 \leq x \leq 1,$$

$$y(0) = y'(0) = 1, \quad y''(0) = 3, \quad y'''(0) = 1,$$

whose exact solution is $y(x) = x^2 + e^x$. Table 3 shows the errors for different methods, where the best performance corresponds to the BHI. The efficiency curves for the BHI and AM methods are shown in Fig. 2.

Problem 3: Application to ship dynamics. Consider the linear IVP;

$$y^{(iv)} + 3y'' + (2 + \epsilon\cos(\Omega t))y = 0; \quad 0 \leq t \leq 15,$$

Table 1
Comparison of the maximum absolute errors obtained for Problem 1.

h	Method	Err
0.1	BHI	2.06E−14
	AM	8.33E−8
	BH8	8.07E−10
	BHCM4	1.74E−8
0.05	BHI	4.88E−18
	AM	1.92E−11
	BH8	3.22E−12
	BHCM4	8.45E−11
0.025	BHI	1.18E−21
	AM	2.27E−13
	BH8	7.30E−16
	BHCM4	3.69E−13
0.02	BHI	8.11E−23
	AM	5.68E−14
	BHCM4	7.11E−14

Table 2
Absolute errors at different points, obtained for Problem 1, with $h = 0.1$.

x	AE with BHI	AE with BH8	AE with BHCM4
0.2	3.84639E−18	3.512997 E−13	2.318519 E−13
0.4	3.52637E−17	4.183300 E−12	2.260324 E−12
0.6	1.33530E−16	1.430233 E−11	1.965140 E−11
0.8	3.72718E−16	3.592435 E−11	9.914494 E−11
1.0	8.66967E−16	7.276201 E−11	3.311345 E−10
1.2	1.82155E−15	1.336016 E−10	9.000018 E−10
1.4	3.55154E−15	2.234540 E−10	2.117600 E−9
1.6	6.60586E−15	3.579606 E−10	4.550582 E−9
1.8	1.18243E−14	5.433495 E−10	9.117964 E−9
2.0	2.06365E−14	8.079574 E−10	1.740907 E−8

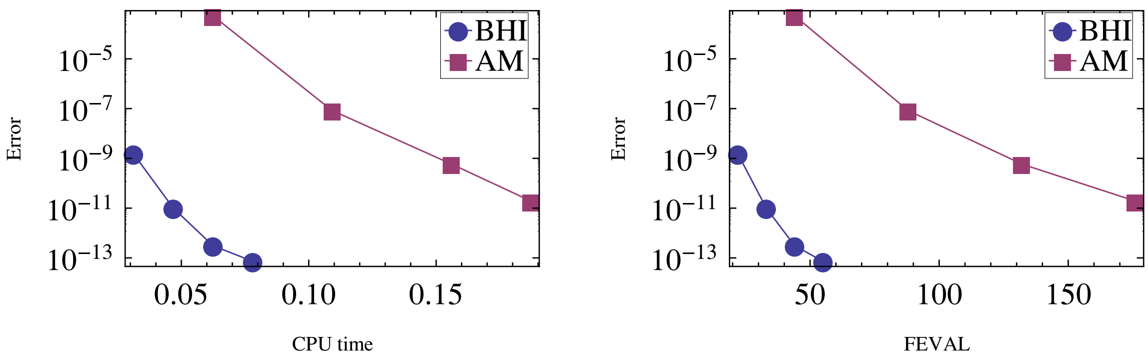

Fig. 1. Efficiency curves showing errors versus CPU times and number of function evaluations (FEVAL) for Problem 1.

$$y(0) = 1, \quad y'(0) = y''(0) = y'''(0) = 0,$$

taken from [2], which corresponds to a physical problem from ship dynamics. When a sinusoidal wave of frequency Ω passes along a ship or offshore structure, the resultant fluid actions vary with time t, see [14]. For $\epsilon = 0$ the theoretical solution is known, $y(t) = 2\cos(t) - \cos(t\sqrt{2})$, so we will take this value for the numerical experiments. Table 4 shows the errors obtained with different block hybrid methods, where the best performance corresponds to the BHI. The efficiency curves are shown in Fig. 3.

Problem 4: Application to beam problem. Consider the linear IVP which illustrates an ill-posed problem of a Beam on Elastic Foundation, which has appeared in [10].

$$y^{(iv)} + y = 1, \quad 0 \le x \le 1,$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0,$$

Table 3
Comparison of the maximum absolute errors obtained for Problem 2:

h	Method	Err
0.2	BHI	1.21E−17
	AM	5.59E−10
	BHCM4	2.38E−12
0.1	BHI	5.20E−21
	AM	2.84E−14
	BHCM4	1.95E−14

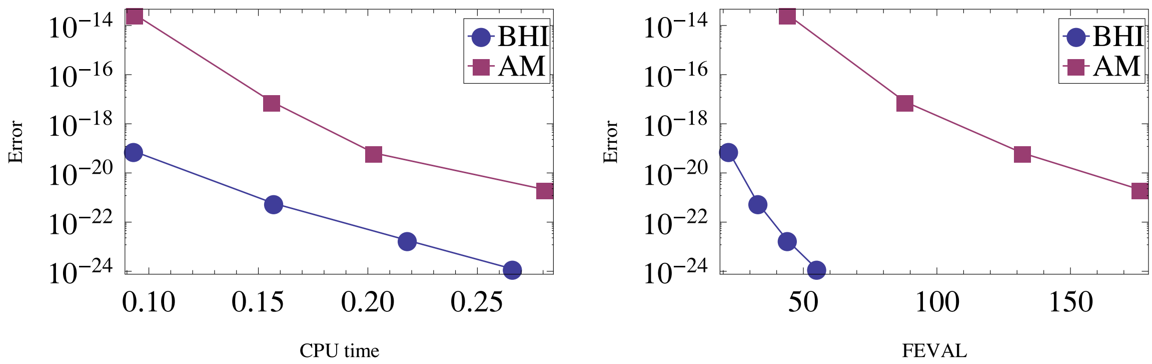

Fig. 2. Efficiency curves showing errors versus CPU times and number of function evaluations (FEVAL) for Problem 2.

Table 4
Comparison of the maximum absolute errors obtained for Problem 3.

h	Method	*Err*
0.25	BHI	1.55E−12
	AM	4.72E−7
	BH8	8.15E−8
	BHCM4	5.2E−7
0.1	BHI	1.93E−15
	AM	1.34E−11
	BH8	3.6E−11
	BHCM4	2.8E−10

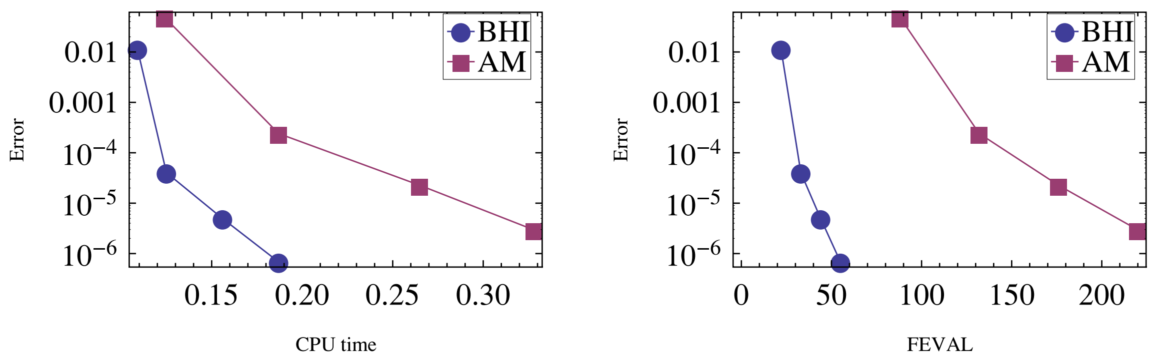

Fig. 3. Efficiency curves showing errors versus CPU times and number of function evaluations (FEVAL) for Problem 3.

Here, y is the normalized vertical displacement (deviation), and 1 is the normalized distributed load, with the above initial conditions, see [11].

The exact solution is

$$y(x) = 1 - \frac{1}{2} e^{-\frac{x}{\sqrt{2}}} (1 + e^{\sqrt{2}x}) \cos\left(\frac{x}{\sqrt{2}}\right).$$

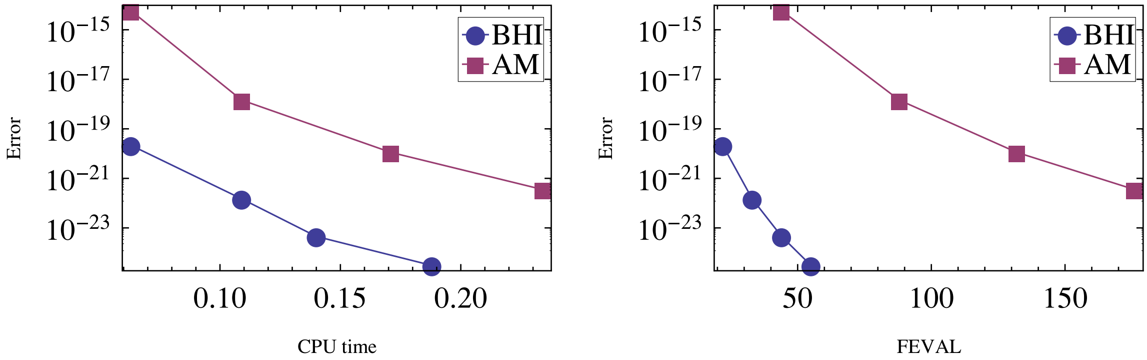

Fig. 4. Efficiency curves showing errors versus CPU times and number of function evaluations (FEVAL) for Problem 4.

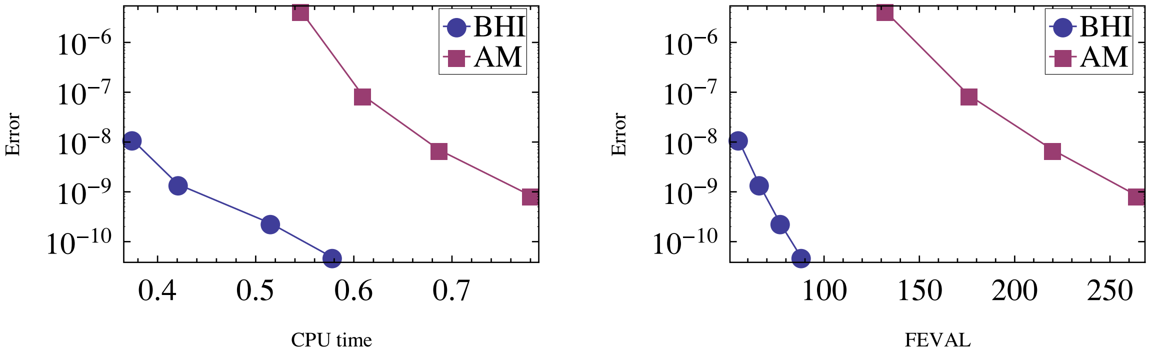

Fig. 5. Efficiency curves showing errors versus CPU times and number of function evaluations (FEVAL) for Problem 5.

We solved this problem with the BHI method taking $N = 20, 40, 80, 160$. The efficiency curves in Fig. 4 show the maximum absolute errors versus the CPU times and the number of function evaluations. The results with the method BHI are compared with the block form of the generalized Adams–Moulton method denoted as AM. The graphs clearly indicate that the BHI method presents the best performance.

Problem 5. Consider the nonlinear IVP, which has appeared in [10], given by

$$y^{(iv)} - y^2 = \cos^2(x) + \sin(x) - 1, \quad 0 \le x \le 10\,,$$

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = -1\,,$$

whose exact solution is $y(x) = \sin(x)$. We have solved this problem with the BHI method taking $h = 1/2^i, i = 1, 2, 3, 4$. The efficiency plot in Fig. 5 shows the proposed method performs better than the generalized Adams–Moulton method AM.

Problem 6. Consider the nonlinear IVP (see [10])

$$y^{(iv)} = \frac{3\sin(y)(3 + 2\sin^2(y))}{\cos^7(y)}, \quad 0 \le x \le \pi/4\,,$$

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1\,,$$

whose exact solution is $y(x) = \arcsin(x)$. We have solved this problem with the BHI method taking $N = 20, 40, 80, 160$, and the results have been compared with the block form of the generalized Adams–Moulton method (AM). The efficiency curves in Fig. 6 show the good performance of the BHI method.

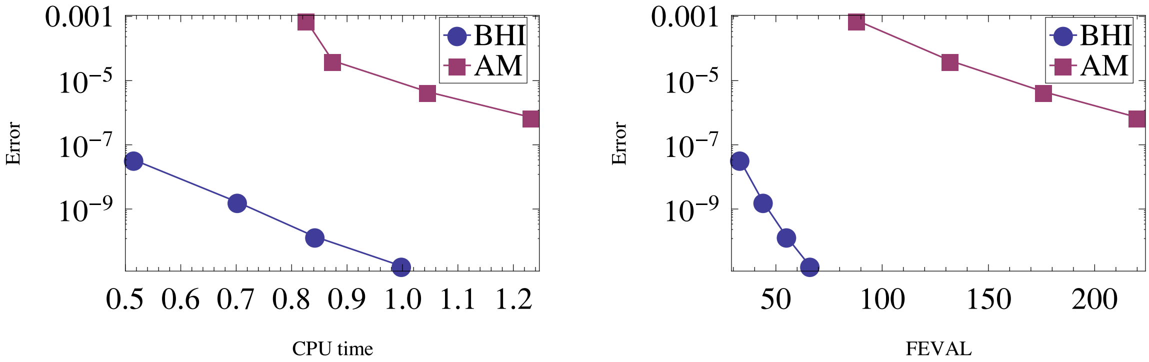

Fig. 6. Efficiency curves showing errors versus CPU times and number of function evaluations (FEVAL) for Problem 6.

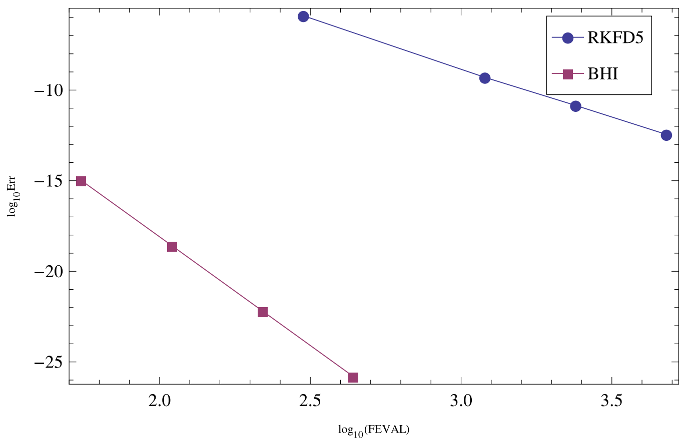

Fig. 7. Efficiency curves for Problem 7.

Problem 7. Consider the nonlinear system as follows, (see [10])

$$\left.\begin{array}{rcl} y^{(iv)} &=& y+\frac{1}{\sqrt{y^2+z^2}}-\frac{1}{\sqrt{w^2+u^2}} \\ && y(0)=1,\quad y'(0)=0,\quad y''(0)=-1,\quad y'''(0)=0 \\ z^{(iv)} &=& z+\frac{1}{\sqrt{y^2+z^2}}-\frac{1}{\sqrt{w^2+u^2}} \\ && z(0)=0,\quad z'(0)=1,\quad z''(0)=0,\quad z'''(0)=-1 \\ w^{(iv)} &=& 16w+\frac{1}{\sqrt{y^2+z^2}}-\frac{1}{\sqrt{w^2+u^2}} \\ && w(0)=1,\quad w'(0)=0,\quad w''(0)=-4,\quad w'''(0)=0 \\ u^{(iv)} &=& 16u+\frac{1}{\sqrt{y^2+z^2}}-\frac{1}{\sqrt{w^2+u^2}} \\ && u(0)=0,\quad u'(0)=2,\quad u''(0)=0,\quad u'''(0)=-8 \end{array}\right\} \tag{16}$$

where the exact solution is given by

$$\left.\begin{array}{l} y=\cos(x) \\ z=\sin(x) \\ w=\cos(2x) \\ u=\sin(2x) \end{array}\right\} \tag{17}$$

The considered integration interval is [0, 2]. We have solved this problem with the BHI method for $N = 20, 40, 80, 160$. Fig. 7 shows the good performance of the proposed method compared with the RKFD5 method in [10].

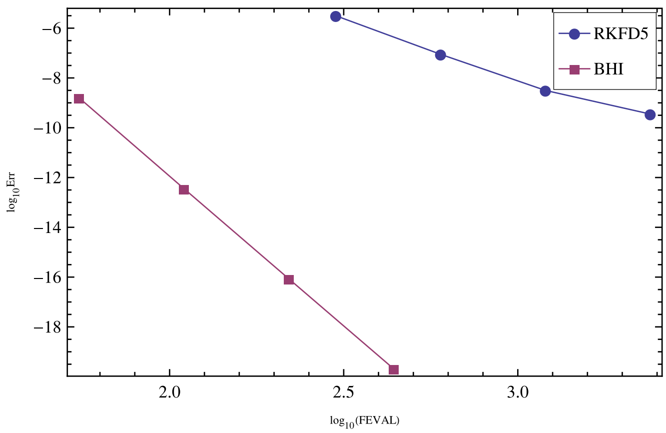

Fig. 8. Efficiency curves for Problem 8.

Problem 8. Finally, consider the nonlinear system (see [10]) given by

$$\left.\begin{array}{l} y^{(iv)} = \frac{z^2}{w} \\ y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 1, \quad y'''(0) = 1 \\ z^{(iv)} = 16\frac{w^2}{u} \\ z(0) = 1, \quad z'(0) = 2, \quad z''(0) = 4, \quad z'''(0) = 8 \\ w^{(iv)} = 81\frac{u^2}{y^5} \\ w(0) = 1, \quad w'(0) = 3, \quad w''(0) = 9, \quad w'''(0) = 27 \\ u^{(iv)} = 256y^4 \\ u(0) = 1, \quad u'(0) = 4, \quad u''(0) = 16, \quad u'''(0) = 64 \end{array}\right\} \tag{18}$$

where the exact solution is

$$\left.\begin{array}{l} y = e^x \\ z = e^{2x} \\ w = e^{3x} \\ u = e^{4x} \end{array}\right\}. \tag{19}$$

The problem has been integrated in the interval [0, 2] with the BHI method taking again $N = 20, 40, 80, 160$. The efficiency curves in Fig. 8 show the outperformance of the BHI method.

6. Conclusion

A block hybrid integrator (BHI) obtained combining different linear multistep formulas is proposed and applied to solve directly fourth-order IVPs in ordinary differential equations. It was shown that the method is very flexible, easy to derive and may be applied to solve diverse kinds of fourth-order IVPs as can be seen in the numerical examples. The method shows a very high accuracy when compared the numerical results to the exact solution and a very good performance compared with existing methods in the literature.

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