Afrika Matematika (2023) 34:36
https://doi.org/10.1007/s13370-023-01075-3

A trigonometrically fitted intra-step block Falkner method for the direct integration of second-order delay differential equations with oscillatory solutions

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Received: 2 July 2022 / Accepted: 1 June 2023

Abstract
An intra-step block Falkner method whose coefficients depend on a parameter ω and the step length h is presented in this study for solving numerically second-order delay differential equations with oscillatory solutions. In the development of the method, the collocation and interpolation techniques were employed. The investigation of the properties of the method has shown that it is zero-stable and consistent, and consequently, convergent. The application of the method to some standard problems from the scientific literature show that it produced very accurate results.

Keywords Delay differential equation · Falkner block method · Intra-step points · Oscillatory solution

Mathematics Subject Classification 65L05 · 65L06

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1 Presentation

This paper is targeted at finding numerical approximations for second-order Delay Differential Equations (DDEs) whose structure is given as

$$\begin{aligned} &y''(t) = f(t, y(t), y(t-\tau)), \quad a \le t \le b, \quad \tau \ge 0, \\ &y(t_0) = y_0, \quad y'(t_0) = y_0', \\ &y(t) = \varphi(t), \ t \le a, \end{aligned} \tag{1}$$

where τ is the delay argument and φ is the primary function. We assume that the first derivative is absent in the differential equation, that the true solution shows an oscillatory or periodic behavior whose frequency can be estimated in advance, and $f : R \times R^{2d} \to R^d$ is a sufficiently smooth function where d is the dimension of the system.

By large, real-life phenomena are modelled using Ordinary Differential Equations (ODEs). However, these equations do not plainly address certain situations where the states of the current system depend on both the current and the previous state of the system. To make these models more reliable, DDEs are utilized to portray these phenomena, providing a good realistic simulation of them. The major difference between ODEs and DDEs is that while the solutions of ODEs are considered at the present state, the solutions of DDEs contain in addition the past state. There are numerous applications that are notably described with DDEs, a list of which can be found in Rasdi et al. [1], Abdulganiy et al. [2] and Ahmad et al. [3], among others.

In general, different approaches have been used to solve DDEs semi analytically and numerically. Such methods include the Variational Iteration Method (Khader [4]), the Adomian Decomposition Method (Ogunfiditimi [5]), Runge–Kutta Methods (Ismail et al. [6]), Block methods that use different interpolation polynomials such as Lagrange, Neville and Hermite to compute the delay term (Ken and Ismail [7] and Majid et al. [8]), Block methods with the delay term calculated with the initial function rather than using interpolation (Akinfenwa et al. [9], and Abdulganiy et al. [2]).

These methods are applied so that Eq. (1) is transformed into an equivalent system of first-order equations. In any case, a more demanding memory storage during execution and even a more complex computer code for the method, particularly the subroutine needed to supply starting values required for such methods, and the large dimension of the emerging first-order system are some of the limitations to reformulate a second-order Differential Equation as a system of first-order Differential Equations.

Consequently, techniques for the direct integration of Eq. (1) have been proposed. Among such methods are the explicit Runge–Kutta–Nyström methods (Papageorgiou and Famelis [10]), one-step block methods (Rasdi et al. [1]), the direct two-point block method of Adams–Moulton type (Seong and Majid [11]), a spline collocation method (El-safty [12]), and the Adomian decomposition method (Evans and Raslan [13]). Most of these methods do not perform well in case of oscillatory solutions according to Ehigie et al. [14], due to the particular nature of the solutions.

Falkner developed a multistep method to solve second-order Initial Value Problems (IVPs) directly. The explicit and implicit forms of Falkner methods are due to Falkner [15] and Collatz [16] respectively. Whereas Ramos et al. [17, 18] presented some modification to the traditional Falkner methods whose basis functions are either polynomials or rational functions, trigonometrically-fitted Falkner-type methods that exploit the fact that the solution of the IVP is periodic were presented by Li and Wu [19], and also by Ehigie and Okunuga [20]. A number of fitted approaches have appeared in the recent literature (see Fang et al. [21], Jator et al. [22, 23], Ramos and Vigo-Aguiar [24], and Abdulganiy et al. [25, 26]). Only a few works in the literature such as those by Ahmad et al. [3, 27], Ismail et al. [28] and Senu et al. [29] have considered the use of an adapted method to numerically integrate Eq. (1), hence the motivation for the present study.

The current work presents a Trigonometrically Fitted Intra-Step Block Falkner (TFIBF) method using the multistep collocation technique for the direct integration of Eq. (1) assuming that the solution presents an oscillatory or periodic behavior, where the frequency can be estimated in advance. We assume that the exact solution can be approximated by a linear combination of polynomials and trigonometric terms. The need to gain more order in the approach while maintaining excellent stability prompted the use of the intra-step strategy. Intra-step formulas were first presented to circumvent the limitation of the Dahlquist barrier in such a way that the customary linear multistep formulas were enhanced by considering intra-step points between some grid points in the formulation process (Gupta [30], Alkasassbeh et al. [31]). Albeit these formulas retain both higher-order and superb stability features, intra-step methods suffer from the need to formulate predictors for the estimation of the corrector at intra-step points, making the methodology more tiresome and inefficient (Lambert [32]). In this paper, a blockwise implementation approach is embraced as a swap for the conventional stepwise execution to bypass the shortage of the predictor-corrector mode.

The rest of the article is arranged as follows: the formulation of the TFIBF is detailed in Sect. 2. The essential elements of the TFIBF are studied in Sect. 3. Whereas some numerical experiments are provided in Sect. 4 to exemplify the superb performance of the method, Sect. 5 contains the conclusions.

2 Formulation of the TFIBF

For the derivation of the method, we will consider $y(t)$ as a scalar function, that is, we take the dimension $d = 1$. This is not a drawback, since the method can be applied in a component-wise mode to solve a system. The primary formulas of the Trigonometrically Fitted Intra-Step Block Falkner (TFIBF) method in this study (with a parameter ω incorporated as $u = \omega h$) are of the form

$$\begin{cases} y_{n+1} = y_n + hy'_n + h^2(\beta_0(u)\, f_n + \beta_\mu(u)\, f_{n+\mu} + \beta_1(u)\, f_{n+1}) \\ hy'_{n+1} = hy'_n + h^2(\bar{\beta}_0(u)\, f_n + \bar{\beta}_\mu(u)\, f_{n+\mu} + \bar{\beta}_1(u)\, f_{n+1}), \end{cases} \tag{2}$$

while the secondary formulas are given as

$$\begin{cases} y_{n+\mu} = y_n + \mu hy'_n + h^2(\beta_0^1(u)\, f_n + \beta_\mu^1(u)\, f_{n+\mu} + \beta_1^1(u)\, f_{n+1}) \\ hy'_{n+\mu} = hy'_n + h^2(\bar{\beta}_0^1(u)\, f_n + \bar{\beta}_\mu^1(u)\, f_{n+\mu} + \bar{\beta}_1^1(u)\, f_{n+1}), \end{cases} \tag{3}$$

where $\mu = \frac{1}{2}$ is the intra-step point and $\beta_j, \bar{\beta}_j, \beta_j^1, \bar{\beta}_j^1$ are coefficients to be distinctively determined, that depend on the parameter ω and on the step-size $h = t_{n+1} - t_n$. Customarily, $y_{n+j}, y'_{n+j}, f_{n+j}$ are approximate values of $y\left(t_{n+j}\right), y'(t_{n+j})$, and $f\left(t_{n+j}, y(t_{n+j}), y(t_{n+j} - \tau)\right)$.

The true solution $y(t)$ is considered to be locally approximated on the interval $\left[t_n, t_{n+1}\right]$ by a solution $\gamma(t)$ of the form

$$\gamma(t) = \xi_0 + \xi_1 t + \xi_2 t^2 + \xi_3 \sin(\omega t) + \xi_4 \cos(\omega t), \tag{4}$$

where the coefficients ξ_i will be obtained demanding that the following system of equations is satisfied

$$\begin{cases} \gamma(t_n) = y_n \\ \gamma'(t_n) = y'_n \\ \gamma''(t_{n+j}) = f_{n+j}, \quad j = 0, \mu, 1. \end{cases} \tag{5}$$

This system can be written in matrix form as

$$\Delta\Theta = \Lambda, \tag{6}$$

where

$$\Delta = \begin{bmatrix} 1 & t_n & {t_n}^2 & \sin(\omega t_n) & \cos(\omega t_n) \\ 0 & 1 & 2t_n & \cos(\omega t_n)\,\omega & -\sin(\omega t_n)\,\omega \\ 0 & 0 & 2 & -\sin(\omega t_n)\,\omega^2 & -\cos(\omega t_n)\,\omega^2 \\ 0 & 0 & 2 & -\sin(\omega t_{n+\mu})\,\omega^2 & -\cos(\omega t_{n+\mu})\,\omega^2 \\ 0 & 0 & 2 & -\sin(\omega t_{n+1})\,\omega^2 & -\cos(\omega t_{n+1})\,\omega^2 \end{bmatrix}, \Theta = \begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}, \Lambda = \begin{bmatrix} y_n \\ y_n' \\ f_n \\ f_{n+\mu} \\ f_{n+1} \end{bmatrix}.$$

Equation (6) is solved for the coefficients $\xi_i,\ i = 0(1)4$, with the aid of Crammer's rule. Each ξ_i is calculated as $\xi_i = \frac{det(\Delta_i)}{det(\Delta)}$, where Δ_i is found after substituting the i-th column of Δ with Λ. This results in

$$\begin{cases} \xi_0 = \frac{\chi}{2u^2(-\sin(u)+2\sin(u/2))} \\ \xi_1 = \frac{\left(\begin{array}{c} u\left(f_n t_n + f_{n+1} t_n - 2y'_n\right)\sin(u/2) - h\left(f_n - f_{n+1}\right)\cos(u/2) \\ +u\left(f_{n+1/2} t_n - y'_n\right)\sin(u) - h\left(\left(-f_n + f_{n+1/2}\right)\cos(u) - f_{n+1/2} + f_{n+1}\right) \end{array} \right)}{u(-\sin(u)+2\sin(u/2))} \\ \xi_2 = \frac{(-f_n - f_{n+1})\sin(u/2) + \sin(u) f_{n+1/2}}{2\sin(u) - 4\sin(u/2)} \\ \xi_3 = -\frac{h^2\left((f_n - f_{n+1})\cos\left(\frac{ut_{n+1/2}}{h}\right) + (-f_n + f_{n+1/2})\cos\left(\frac{ut_{n+1}}{h}\right) + \cos\left(\frac{ut_n}{h}\right)(f_{n+1} - f_{n+1/2})\right)}{u^2(\sin(u) - 2\sin(u/2))} \\ \xi_4 = \frac{h^2\left((f_n - f_{n+1})\sin\left(\frac{ut_{n+1/2}}{h}\right) + (-f_n + f_{n+1/2})\sin\left(\frac{ut_{n+1}}{h}\right) + \sin\left(\frac{ut_n}{h}\right)(f_{n+1} - f_{n+1/2})\right)}{u^2(\sin(u) - 2\sin(u/2))} \end{cases} \tag{7}$$

where

$$\begin{aligned} \chi = & \left(\left((f_n + f_{n+1})\,{t_n}^2 - 4y'_n t_n + 4y_n\right)u^2 + 2h^2(f_n - f_{n+1})\right)\sin(u/2) \\ & +2hut_n(f_n - f_{n+1})\cos(u/2) + \left(-f_{n+1/2}{t_n}^2 + 2y'_n t_n - 2y_n\right)u^2 \\ & +2h^2\left(-f_n + f_{n+1/2}\right)\sin(u) + 2h\left(\left(-f_n + f_{n+1/2}\right)\cos(u) - f_{n+1/2} + f_{n+1}\right)ut_n. \end{aligned}$$

We then substitute the value of each ξ_i specified by Eq. (7) into Eq. (4) to obtain the continuous formula as follows

$$\gamma(t,u) = y_n + hy'_n + h^2\left(\beta_0(t,u)\,f_n + \beta_\mu(t,u)\,f_{n+\mu} + \beta_1(t,u)\,f_{n+1}\right), \tag{8}$$

where we have included explicitly u into $\gamma(t,u)$ to highlight the dependence on this parameter.

2.1 Specific formulation of the TFIBF

The continuous formula in Eq. (8) and its first derivative are evaluated at $t = \{t_{n+1}, t_\mu\}$ to get the two principal formulas and the two complimentary formulas in the form of Eqs. (2) and (3), respectively, to form the block method TFIBF. It is emphasized that when $u \to 0$, the coefficients of the TFIBF may suffer substantial cancellations affecting the calculations. In this situation, the expansion of the coefficients in Taylor's series is usually considered (see Lambert [32]).

The coefficients of the four formulas of the block TFIBF together with their series expansions up to the eight order are as follows:

$$\begin{cases} \beta_0(u) = \frac{\left(-u^2-4\right)\sin(u/2)-2\cos(u)u+2\cos(u/2)u+2\sin(u)}{2u^2(-2\sin(u/2)+\sin(u))} \\ \quad = \frac{1}{6}+\frac{u^2}{480}+\frac{19u^4}{483840}+\frac{17u^6}{19353600}+\frac{29u^8}{1362493440}+O(h^{10}) \\ \beta_\mu(u) = -\frac{u\sin(u)+2\cos(u)-2}{2u(2\sin(u/2)-\sin(u))} \\ \quad = \frac{1}{3}-\frac{u^2}{720}-\frac{u^4}{80640}-\frac{u^6}{9676800}-\frac{u^8}{1226244096}+O(h^{10}) \\ \beta_1(u) = \frac{-\cos(u/2)u^2+2\sin(u/2)u-4(\cos(u/2))^2-u^2+4}{4u^2(\cos(u/2))^2-4u^2} \\ \quad = -\frac{u^2}{1440}-\frac{13u^4}{483840}-\frac{u^6}{1290240}-\frac{251u^8}{12262440960}+O(h^{10}) \end{cases} \tag{9}$$

$$\begin{cases} \bar{\beta}_0(u) = \frac{u\sin(u/2)+\cos(u)-1}{u(2\sin(u/2)-\sin(u))} \\ \quad = \frac{1}{6}+\frac{u^2}{720}+\frac{u^4}{80640}+\frac{u^6}{9676800}+\frac{u^8}{1226244096}+O(h^{10}) \\ \bar{\beta}_\mu(u) = -\frac{u\sin(u)+2\cos(u)-2}{u(2\sin(u/2)-\sin(u))} \\ \quad = \frac{2}{3}-\frac{u^2}{360}-\frac{u^4}{40320}-\frac{u^6}{4838400}-\frac{u^8}{613122048}+O(h^{10}) \\ \bar{\beta}_1(u) = \frac{\cos(u)+u\sin(u/2)-1}{u(2\sin(u/2)-\sin(u))} \\ \quad = \frac{1}{6}+\frac{u^2}{720}+\frac{u^4}{80640}+\frac{u^6}{9676800}+\frac{u^8}{1226244096}+O(h^{10}) \end{cases} \tag{10}$$

$$\begin{cases} \beta_0^1(u) = \frac{\left(-u^2-16\right)\sin(u/2)-4\cos(u)u+4\cos(u/2)u+8\sin(u)}{8u^2(-2\sin(u/2)+\sin(u))} \\ \quad = \frac{7}{96}+\frac{7u^2}{7680}+\frac{71u^4}{3870720}+\frac{53u^6}{123863040}+\frac{23u^8}{2179989504}+O(h^{10}) \\ \beta_\mu^1(u) = -\frac{u^2\sin(u)+4\cos(u)u+16\sin(u/2)-8\sin(u)-4u}{8u^2(2\sin(u/2)-\sin(u))} \\ \quad = \frac{1}{16}-\frac{u^2}{2304}-\frac{u^4}{276480}-\frac{u^6}{34406400}-\frac{u^8}{4459069440}+O(h^{10}) \\ \beta_1^1(u) = \frac{u\sin(u/2)+4\cos(u/2)-4}{8u(2\sin(u/2)-\sin(u))} \\ \quad = -\frac{1}{96}-\frac{11u^2}{23040}-\frac{19u^4}{1290240}-\frac{247u^6}{619315200}-\frac{1013u^8}{98099527680}+O(h^{10}) \end{cases} \tag{11}$$

$$\begin{cases} \bar{\beta}_0^1(u) = -\frac{-u\sin(u/2)+4\cos(u/2)-2\cos(u)-2}{2u(2\sin(u/2)-\sin(u))} \\ \quad = \frac{5}{24} + \frac{19u^2}{5760} + \frac{23u^4}{322560} + \frac{263u^6}{154828800} + \frac{1033u^8}{24524881920} + O(h^{10}) \\ \bar{\beta}_\mu^1(u) = -\frac{u\sin(u)+2\cos(u)-2}{2u(2\sin(u/2)-\sin(u))} \\ \quad = \frac{1}{3} - \frac{u^2}{720} - \frac{u^4}{80640} - \frac{u^6}{9676800} - \frac{u^8}{1226244096} + O(h^{10}) \\ \bar{\beta}_1^1(u) = \frac{u\sin(u/2)+4\cos(u/2)-4}{2u(2\sin(u/2)-\sin(u))} \\ \quad = -\frac{1}{24} - \frac{11u^2}{5760} - \frac{19u^4}{322560} - \frac{247u^6}{154828800} - \frac{1013u^8}{24524881920} + O(h^{10}) \end{cases} \tag{12}$$

Remark 1 It is noted here that taking limit when $u \to 0$ in the coefficients in (9)–(12), the block hybrid Falkner method obtained using a polynomial basis is recovered.

3 Essential elements of the TFIBF

The essential elements of the TFIBF which include the local truncation error, zero- and linear stability, convergence, and region of absolute stability are addressed in this section.

3.1 Local truncation error and consistency of TFIBF

The theory of linear operators in Lambert [32] is employed to establish the Local Truncation Error (LTE) of all the formulas in the TFIBF. Since the formulas in (2) and (3) are of the type of generalized multistep methods, we consider the associated difference operators $\mathcal{L}[y(t_n);h]$, $\mathcal{L}'[y(t_n);h]$ to the principal formulas and $\mathcal{L}_\mu[y(t_n);h]$, $\mathcal{L}'_\mu[y(t_n);h]$, to the complimentary ones, defined respectively as follows

$$\begin{cases} \mathcal{L}[y(t_n);h] = y(t_n+h) - \begin{pmatrix} y(t_n) + hy'(t_n) + h^2\beta_0(u)y''(t_n) \\ +h^2\beta_\mu(u)y''(t_n+\mu h) + h^2\beta_1(u)y''(t_n+h) \end{pmatrix} \\ \mathcal{L}'[y(t_n);h] = hy'(t_n+h) - \begin{pmatrix} hy'(t_n) + h^2\bar{\beta}_0(u)y''(t_n) \\ +h^2\bar{\beta}_\mu(u)y''(t_n+\mu h) + h^2\bar{\beta}_1(u)y''(t_n+h) \end{pmatrix} \\ \mathcal{L}_\mu[y(t_n);h] = y(t_n+\mu h) - \begin{pmatrix} y(t_n) + \mu hy'(t_n) + h^2\beta_0^1(u)y''(t_n) \\ +h^2\beta_\mu^1(u)y''(t_n+\mu h) + h^2\beta_1^1(u)y''(t_n+h) \end{pmatrix} \\ \mathcal{L}'_\mu[y(t_n);h] = hy'(t_n+\mu h) - \begin{pmatrix} hy'(t_n) + h^2\bar{\beta}_0^1(u)y''(t_n) \\ +h^2\bar{\beta}_\mu^1(u)y''(t_n+\mu h) + h^2\bar{\beta}_1^1(u)y''(t_n+h) \end{pmatrix}. \end{cases} \tag{13}$$

We use Taylor's series and expand the above formulas in powers of h, after substituting the coefficients defined in Eqs. (9)–(12) into the corresponding formula above, with the assumption that $y(t)$ is a sufficiently differentiable function. After some simplifications, the local truncation errors for each of the formulas in (13) are given, respectively, by

$$LTE = \begin{cases} \frac{h^5}{720}\left(y^{(5)}(t_n) + \omega^2 y^{(3)}(t_n)\right) + O\left(h^6\right) \\ -\frac{h^6}{2880}\left(y^{(6)}(t_n) + \omega^2 y^{(4)}(t_n)\right) + O\left(h^7\right) \\ \frac{h^5}{1440}\left(y^{(5)}(t_n) + \omega^2 y^{(3)}(t_n)\right) + O\left(h^6\right) \\ \frac{h^5}{384}\left(y^{(5)}(t_n) + \omega^2 y^{(3)}(t_n)\right) + O\left(h^6\right) \end{cases} \tag{14}$$

Remark 2 It is emphasized that the order p of each formula in the TFIBF is at least $p = 3$.

Theorem 1 *If the exact solution of the problem (1) is a linear combination of the basis function* $\left\{1, t, t^2, sin(\omega t), cos(\omega t)\right\}$*, then the local truncation errors of the formulas in the TFIBF vanish.*

Proof Solving the differential equation $y^{(5)}(t) + \omega^2 y^{(3)}(t) = 0$ results in the following fundamental set of solutions $\left\{1, t, t^2, sin(\omega t), cos(\omega t)\right\}$, and thus the required result follows immediately. □

3.1.1 Consistency

Remark 3 Following the definition by Lambert [32], a numerical approach for solving (1) is consistent if it has an order greater than one. Thus, the TFIBF is consistent.

3.2 Stability analysis

The TFIBF given in (2)–(3) can be reformulated as follows

$$(A_1 \otimes I)\Upsilon_{n+1} = (A_0 \otimes I)\Upsilon_n + h^2(B_0 \otimes I)F_n + h^2(B_1 \otimes I)F_{n+1} \tag{15}$$

with $\Upsilon_{n+1} = (y_{n+\mu}, y_{n+1}, hy'_{n+\mu}, hy'_{n+1})^T$, $\Upsilon_n = (y_{n-1+\mu}, y_n, hy'_{n-1+\mu}, hy'_n)^T$, $F_{n+1} = (f_{n+\mu}, f_{n+1}, hf'_{n+\mu}, hf'_{n+1})^T$ and $F_n = (f_{n-1+\mu}, f_n, hf'_{n-1+\mu}, hf'_n)^T$, I is the 4×4 unit matrix, $\otimes$ denotes the Kronecker product of matrices. A_0, A_1, B_0 and B_1 are 4×4 matrices containing the coefficients of the formulas and are given as follows

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B_0 = \begin{bmatrix} 0 & \beta_0^1 & 0 & 0 \\ 0 & \beta_0 & 0 & 0 \\ 0 & \bar{\beta}_0^1 & 0 & 0 \\ 0 & \bar{\beta}_0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} \beta_1^1 & \beta_2^1 & 0 & 0 \\ \beta_1 & \beta_2 & 0 & 0 \\ \bar{\beta}_1^1 & \bar{\beta}_2^1 & 0 & 0 \\ \bar{\beta}_1 & \bar{\beta}_2 & 0 & 0 \end{bmatrix}.$$

3.2.1 Zero-stability

The concept of zero-stability refers to the behavior of the solutions of the system in (15) when h tends to 0. In this case, the system in (15) results in

$$A_1\Upsilon_{n+1} - A_0\Upsilon_n = 0, \tag{16}$$

where A_1 and A_0 are 4×4 constant matrices.

Definition 1 A given numerical integrator is zero-stable provided the modulus of the roots of its first characteristic polynomial, $\Gamma(\Omega) = \det(\Omega A_1 - A_0)$, is less than or equal to one, and for those of modulus one, the multiplicity is at most two (see Fatunla [33]).

Proposition 2 *The TFIBF is zero-stable.*

Proof From the normalized first characteristic polynomial of the TFIBF, we have that

$$\Omega A_1 - A_0 = \begin{bmatrix} \Omega & -1 & 0 & -\frac{1}{2} \\ 0 & \Omega - 1 & 0 & -1 \\ 0 & 0 & \Omega & -1 \\ 0 & 0 & 0 & \Omega - 1 \end{bmatrix},$$

so that the characteristic equation is $\Gamma(\Omega) \equiv \det(\Omega A_1 - A_0) = 0$, that is, $\Omega^2 (\Omega - 1)^2 = 0$. Thus, TFIBF is zero-stable according to Definition 1. □

3.2.2 Linear stability

Apply the TFIBF specified by the formulas in Eqs. (2) and (3) whose coefficients are given in Eqs. (9)–(12) to the test equation $y'' = -\lambda^2 y$ and take $z = \lambda h$ to obtain

$$\Upsilon_{n+1} = \Psi(z, u)\Upsilon_n, \tag{17}$$

where
$\Upsilon_{n+1} = (y_{n+\mu}, y_{n+1}, hy'_{n+\mu}, hy'_{n+1})^T$, $\Upsilon_n = (y_{n-1+\mu}, y_n, hy'_{n-1+\mu}, hy'_n)^T$, and

$$\Psi(z, u) = (A_1 - zB_1)^{-1}(A_0 + zB_0) \tag{18}$$

is the so-called amplification matrix, which determines the behavior of the TFIBF concerning the stability. The amplification matrix $\Psi(z, u)$ for TFIBF has eigenvalues given by $(\theta_1, \theta_2, \theta_3, \theta_4) = (0, 0, 0, \theta_4)$, where $\theta_4 (z, u)$ is the stability function.

Remark 4 The function $\theta_4 (z, u)$ is a rational one specified by

$$\theta_4 (z, u) = \frac{\Pi_4 (z, u)}{\Phi_4 (z, u)}, \tag{19}$$

where

$$\begin{aligned}
\Pi_4(z, u) = {} & 4\sqrt{2}(u^2(((z^2 + 8)u^2 + 8z^2) \cos(u/2) \\
& -8u^2 - 8z^2)(-3/4uz^2(u^2 + z^2)(\cos(u/2))^2 \\
& +((u^4 + (-1/8z^4 + 2z^2)u^2 + z^4) \sin(u/2) \\
& +1/2uz^2(u^2 + z^2)) \cos(u/2) - (u^2 + z^2) \\
& ((u^2 + z^2) \sin(u/2) - 1/4uz^2)(\cos(u/2) + 2)^2 \sin(u/2)z^2)^{\frac{1}{2}} \\
& -6(-1/3((z^2 + 4)u^2 \\
& +4z^2uz^2(\cos(u/2))^2 + (u^2 + z^2)(((z^2 + 8/3)u^2 + 8/3z^2) \\
& \sin(u/2) + 4/3\, uz^2) \cos(u/2) \\
& -2/3((z^2 + 4)u^2 + 4\, z^2)(u^2 + z^2) \\
& \sin(u/2) + 1/6\, u^3 z^4)(\cos(u/2) + 2) \\
\Phi_4(z, u) = {} & 2(\cos(u/2) + 2)((4u^3 z^2 + 4uz^4)(\cos(u/2))^2 \\
& +(((z^2 - 8)u^2 - 8z^2) \sin(u/2) - 4uz^2) \\
& (u^2 + z^2) \cos(u/2) - 2(u^2 + z^2)((z^2 - 4)u^2 - 4z^2) \sin(u/2) \\
& +1/2u^3 z^4)
\end{aligned}$$

Definition 2 (Jator [34]) The region of linear stability of the TFIBF is the domain in the $z - u$ plane in which the spectral radius of the amplification matrix, $\rho\,(\Psi\,(z, u))$, verifies

$$\|\rho\,(\Psi\,(z, u))\| \le 1.$$

The $z - u$ stability region generated for TFIBF is plotted in Fig. 1, where the colored region (blue) is the stability region for the test problem $y'' = -\lambda^2 y$.

4 Numerical examples

The proposed TFIBF in this study is executed in blockwise form without requiring starting values or/and predictors. A written algorithm in Maple 2016.1 is developed for TFIBF. We note that no interpolation is required for the determination of the delay term, which is calculated as explained in [11]. The values of the fitting parameters used in the numerical examples were taken from the referenced problems. However, the strategies for the frequency choice considered by [24] can be explored. In the numerical investigations, we plotted the graphs of the absolute errors between the exact solutions $y(t_n)$ and the numerical solutions $\{y_n\}$ obtained using the TFIBF as a measure of accuracy, whereas the computational efficiency is shown through the plots of the maximum errors versus the computational time (CPU) and the plots of the maximum errors versus the number of function evaluation (NFE) respectively in comparison with the following methods

1. TF-NSIHM (5): An order five trigonometrically fitted hybrid method in [27],
2. TF-NSIHM (4): An order four trigonometrically fitted hybrid method in [28],
3. TF-BEHM (5): An order five trigonometrically fitted hybrid method in [3].
4. TDRK3(5): An order five trigonometrically fitted method in [29].

4.1 Example 1

As a first model, consider the DDE given by

$$\begin{aligned} &y''(t) = -\tfrac{\sin(t)}{2-\sin(t)}\, y(t-\pi), \quad 0 \le t \le 8\pi, \\ &y(0) = 2, \quad y'(0) = 1, \end{aligned} \tag{20}$$

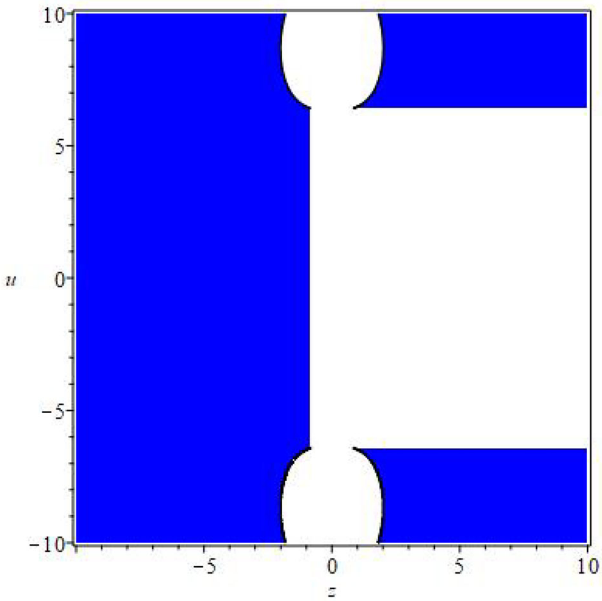

Fig. 1 Linear stability region in the $z-u$-plane for TFIBF

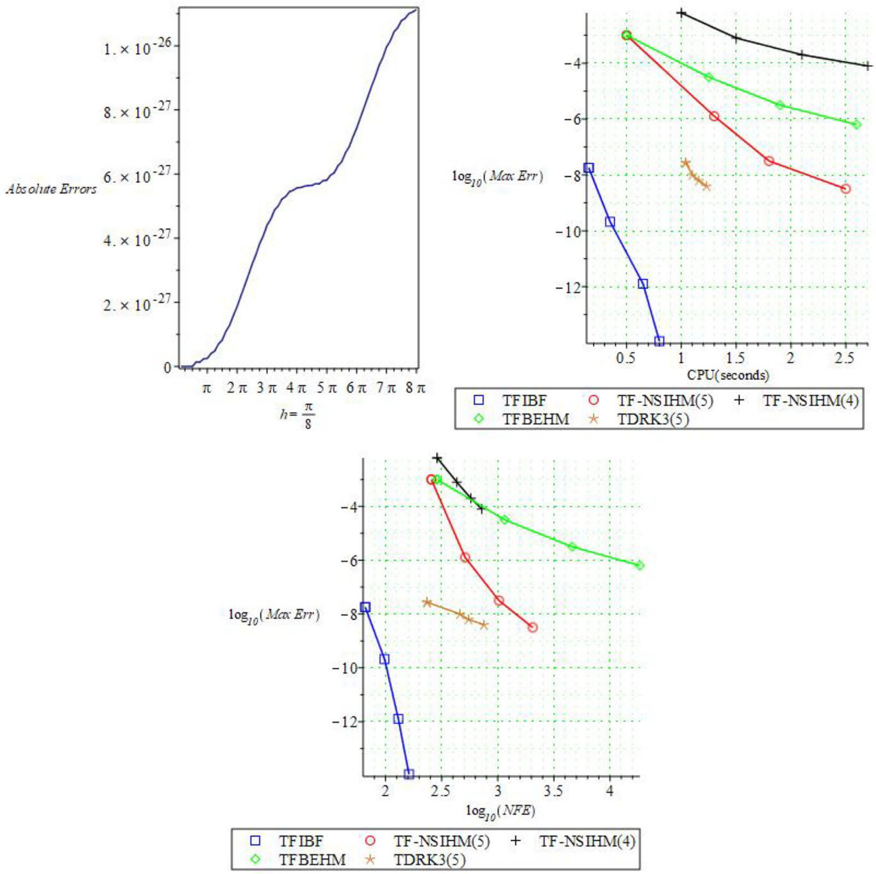

Fig. 2 Absolute errors (top left) and efficiency curves (top right and bottom) for Example 1

whose solution in closed form is given as $y(t) = 2 + \sin(t)$. Equation (20) is solved in [27] and [29] with step sizes $h = \frac{\pi}{2^i}, i = 1, 2, 3, 4$, in [28] the step sizes are $h = \frac{\pi}{2i}, i = 2, 3, 4, 5$, while in [3] they choose $h = \frac{\pi}{4^i}, i = 1, 2, 3, 4$, as the integration step sizes. For the implementation of this problem, the fitting frequency is chosen as $\omega = 1$ (see [3, 27, 28]) and step sizes $h = \frac{\pi}{2i}, i = 1, 2, 3, 4$ were considered. The absolute errors of the numerical solutions provided by the TFIBF with $h = \frac{\pi}{8}$ in comparison with the exact solutions are shown in Fig. 2 (top left). The efficiency curves with the different methods considered are plotted in Fig. 2 (top right and bottom), showing the numerical superiority of the TFIBF.

4.2 Example 2

Consider the DDE given by

$$\begin{aligned} &y''(t) - \tfrac{1}{2}y(t-\pi) + \tfrac{1}{2}y(t) = 0, \quad 0 \le t \le 8\pi, \\ &y(0) = 0, \quad y'(0) = 1, \end{aligned} \tag{21}$$

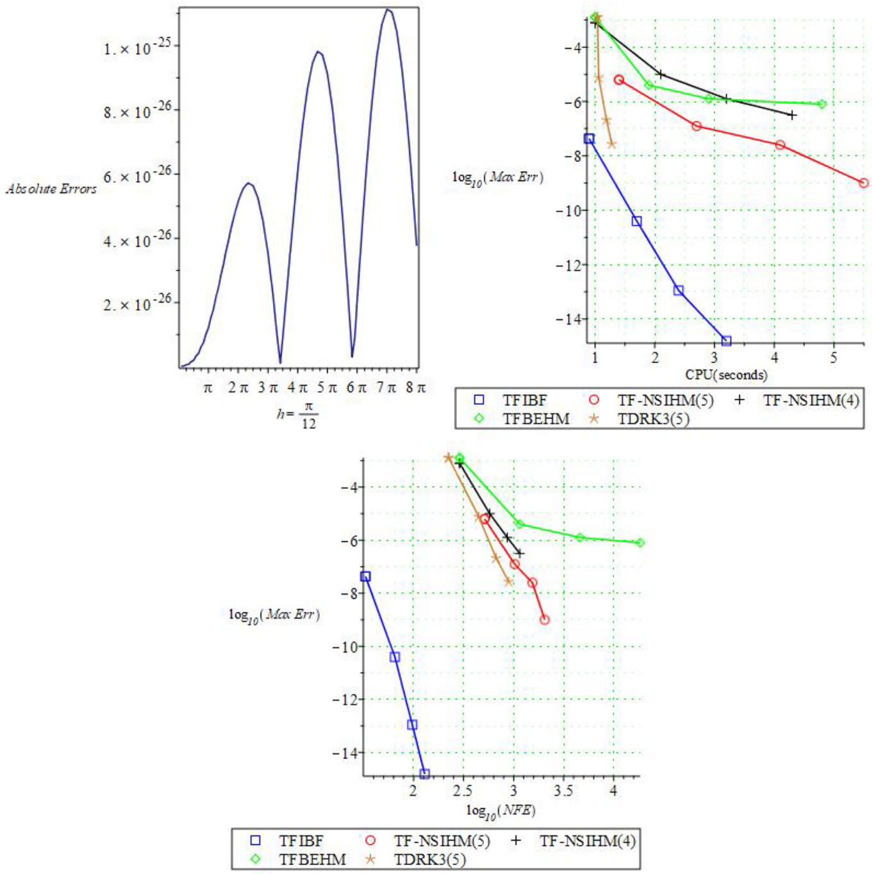

Fig. 3 Absolute errors (top left) and efficiency curves (top right and bottom) for Example 2

whose exact solution is $y(t) = \sin(t)$. The fitting frequency is selected as $\omega = 1$ as in [3, 27, 28], and the same step lengths $h = \frac{\pi}{4i}, i = 1, 2, 3, 4$, as in [27, 28] are used for comparisons. The absolute errors of the discrete solutions provided by the TFIBF with $h = \frac{\pi}{12}$ in comparison with the exact solutions are shown in Fig. 3 (top left). The efficiency curves for different methods are presented in Fig. 3 (top right and bottom), showing the good performance of the TFIBF.

4.3 Example 3

Consider the DDE specified by

$$\begin{aligned} &y''(t) - y(t-\pi) = 0, \quad 0 \le t \le 8\pi, \\ &y(0) = 0, \quad y'(0) = 1, \end{aligned} \tag{22}$$

whose solution in closed form is given as $y(t) = \sin(t)$. Whereas Eq. (22) is integrated with step sizes $h = \frac{\pi}{2i}, i = 1, 2, 3, 4$, the fitting frequency is selected as $\omega = 1$. Figure 4 displays the the absolute errors of the discrete solutions provided by the TFIBF with $h = \frac{\pi}{8}$ as they

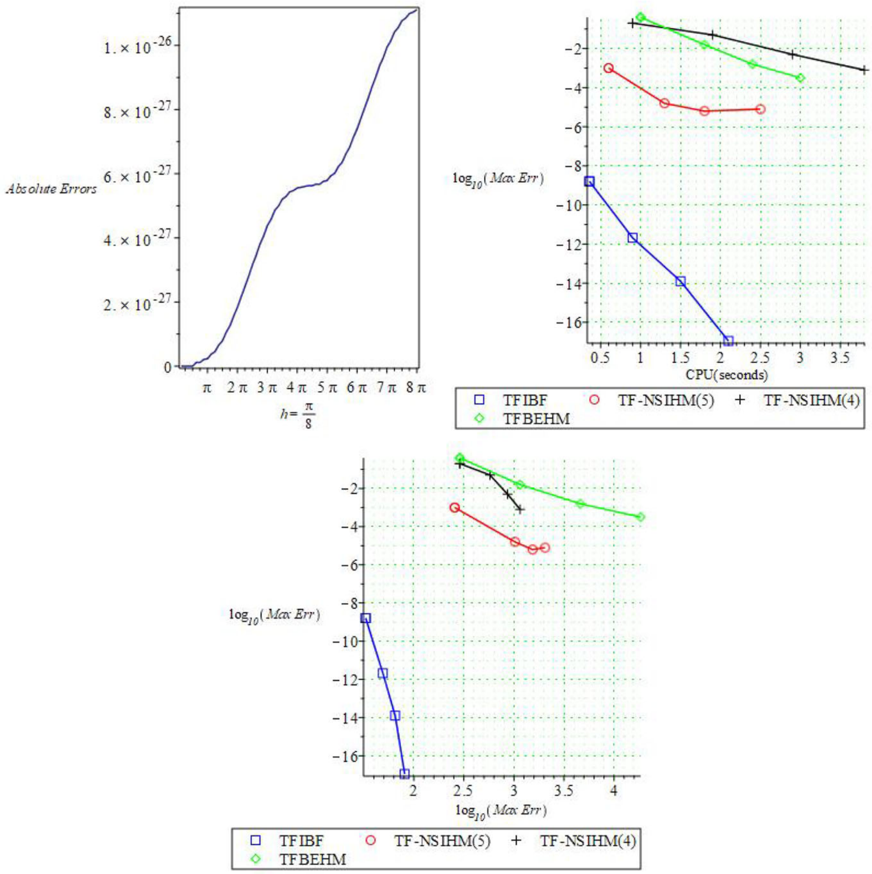

Fig. 4 Absolute errors (top left) and efficiency curves (top right and bottom) for Example 3

compare with the exact solutions (top left). The efficiency curves for different methods are shown in Fig. 4 (top right and bottom) again indicating the good performance of the TFIBF.

4.4 Example 4

Consider the non-homogeneous DDE in [10] specified by

$$\begin{aligned} &y''(t) = -y(t) - y(t - \tfrac{3\pi}{2}) + 3\cos(t) + 5\sin(t), \quad 0 \le t \le 10, \\ &y(0) = -5, \quad y'(0) = 3, \end{aligned} \tag{23}$$

whose exact solution is given as $y(t) = 3\sin(t) - 5\cos(t)$. For the integration of Eq. (23), the fitting frequency is selected as $\omega = 1$ while the step sizes are taken as $h = \frac{1}{2^i}$, $i = 1, 2, 3, 4$, for the implementation of the TFIBF and other methods it compared. The visual representation of the absolute errors of the discrete solutions of TFIBF in comparison with the exact solutions with $h = \frac{1}{8}$ is displayed in Fig. 5 (top left). The efficiency curves for different methods are shown in Fig. 5 (top right and bottom) again indicating the good performance of the TFIBF.

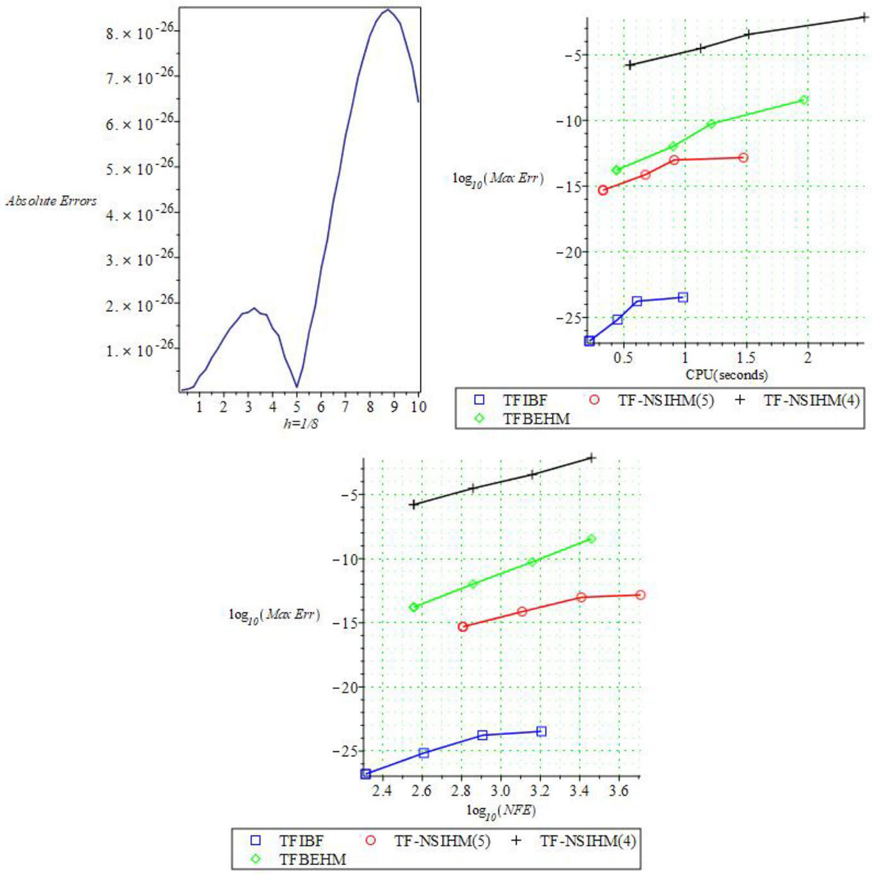

Fig. 5 Absolute errors (top left) and efficiency curves (top right and bottom) for Example 4

4.5 Example 5

Consider the non-linear DDE in [10] specified by

$$\begin{aligned} &y''(t) = -\tfrac{1}{2}y(t) - \tfrac{1}{2} + y(\tfrac{1}{2}t - \tfrac{\pi}{4})^2, \quad 2 \leq t \leq 12, \\ &y(2) = 0.9092974268, \quad y'(2) = -0.4161468365, \end{aligned} \tag{24}$$

whose exact solution is given as $y(t) = \sin(t)$. Whereas Fig. 6 (top right and bottom) shows the advantage in terms of performance of the TFIBF over the methods in [3, 27, 28] with step sizes chosen as $h = \frac{1}{2^i}, i = 1, 2, 3, 4$, and the fitting frequency as $\omega = 1$ for the implementation, Fig. 6 (top left) reveals how the TFBIF fares with the exact solution for $h = \frac{1}{8}$.

Equation (24) is solved in the interval $0 \leq t \leq 10\pi$ with initial conditions given as $y(0) = 0, \quad y'(0) = 1$. The results of the proposed TFIBF in comparison with the TDRK3(5) in [29] are presented in Fig. 7 with step sizes taken as $h = \frac{1}{2^i}, i = 1, 2, 3, 4, 5$ and the fitting frequency as $\omega = 1$. Whereas Fig. 7 (top left) shows how the TFBIF compares to the exact solution for $h = \frac{1}{2}$, Fig. 7 (top right and bottom) shows the TFIBF's significant advantage over TDRK3 (5).

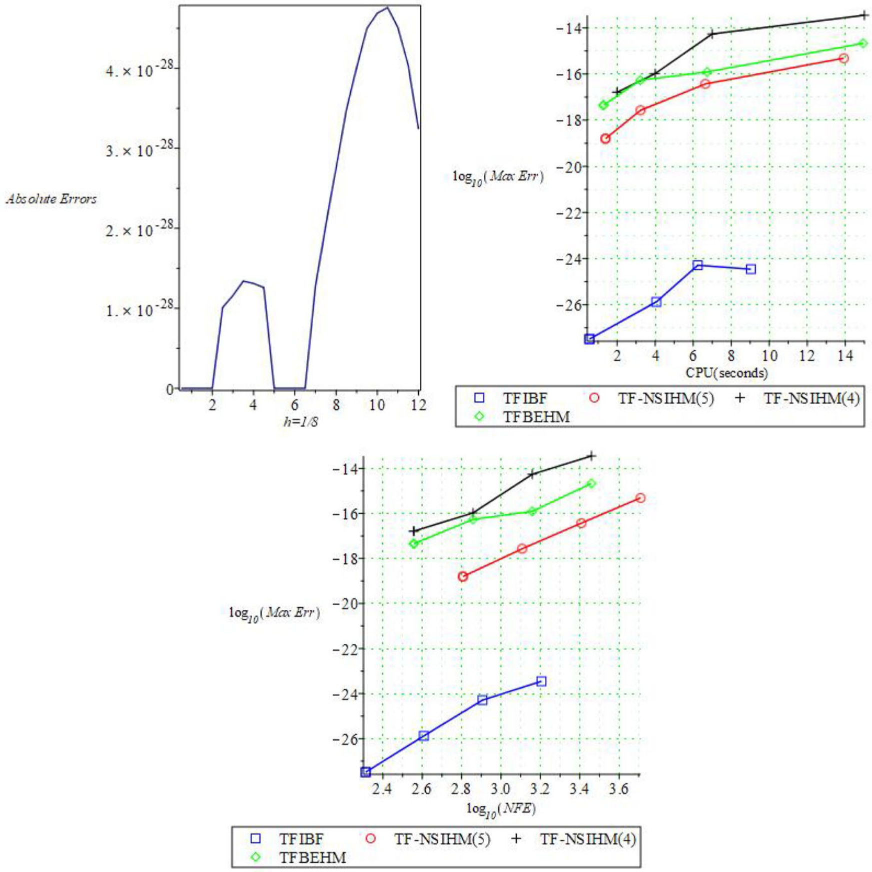

Fig. 6 Absolute errors (top left) and efficiency curves (top right and bottom) for Example 5

4.6 Example 6

We consider a Bessel-type equation involving a state-dependent delay given in [35]

$$
\begin{aligned}
&y''(t) = -\left(100 + \frac{1}{4t^2}\right) y(t) - y\left(t - 1 - y^2(t)\right), \quad 3 \le t \le 10, \\
&y(t) = \sqrt{t} J_0(10t), \quad t \le 3, \\
&y'(t) = \frac{1}{2\sqrt{t}} \left[J_0(10t) - 20t J_1(10t)\right], \quad t \le 3,
\end{aligned} \tag{25}
$$

where J_0 and J_1 are the Bessel functions of first and second kind, respectively. The exact solution of Eq. (25) is given as $y(t) = \sqrt{t} J_0(10t)$ with the initial conditions taken as $y(3) \simeq -0.1495937357$, $y'(3) \simeq 1.982031871$ and the fitting frequency, ω, according to [35] is approximately 10. While Fig. 8 (top) shows the absolute errors between the exact solutions and the approximate results obtained by TFIBF with the step size $h = \frac{1}{40}$, Fig. 8 (bottom) presents the efficiency curves of the comparison between the TFIBF and the third order Intra-

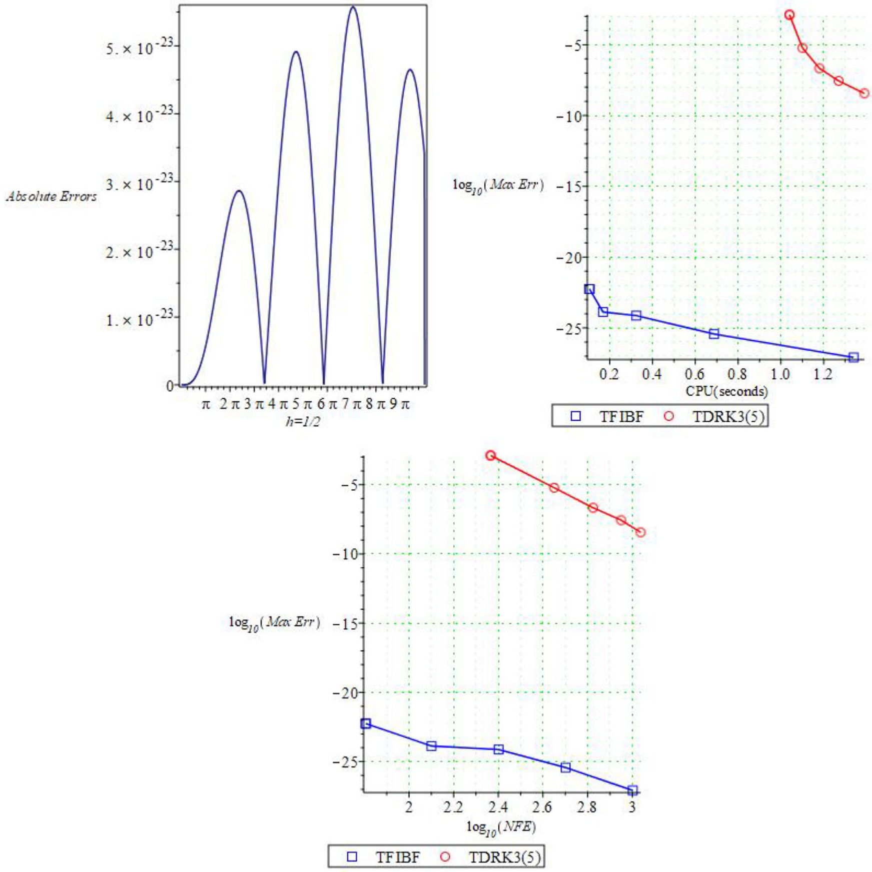

Fig. 7 Absolute errors (top left) and efficiency curves (top right and bottom) for Example 5

step Block Falkner (IBF) method whose basis function is polynomial with the step sizes taken as $h = \dfrac{0.1}{2^i}$, $i = 0, 1, 2, 3$.

4.7 Example 7: an application to Mathieu equation

In this subsection, we apply TFIBF to solve the well-known non-linear delay differential equation called Mathieu equation in engineering. Consider the Mathieu equation with delay given by

$$y''(t) + (\mu + \alpha \cos(t))y(t) + \delta y^3 = \lambda y(t - \tau) \tag{26}$$

where $\mu, \alpha, \lambda, \delta$ and τ are the parameters described as follows:

- μ is the simple harmonic oscillator's frequency squared,
- α is the parametric resonance amplitude,
- λ is the delay amplitude,
- δ is the cubic non-linearity amplitude, and

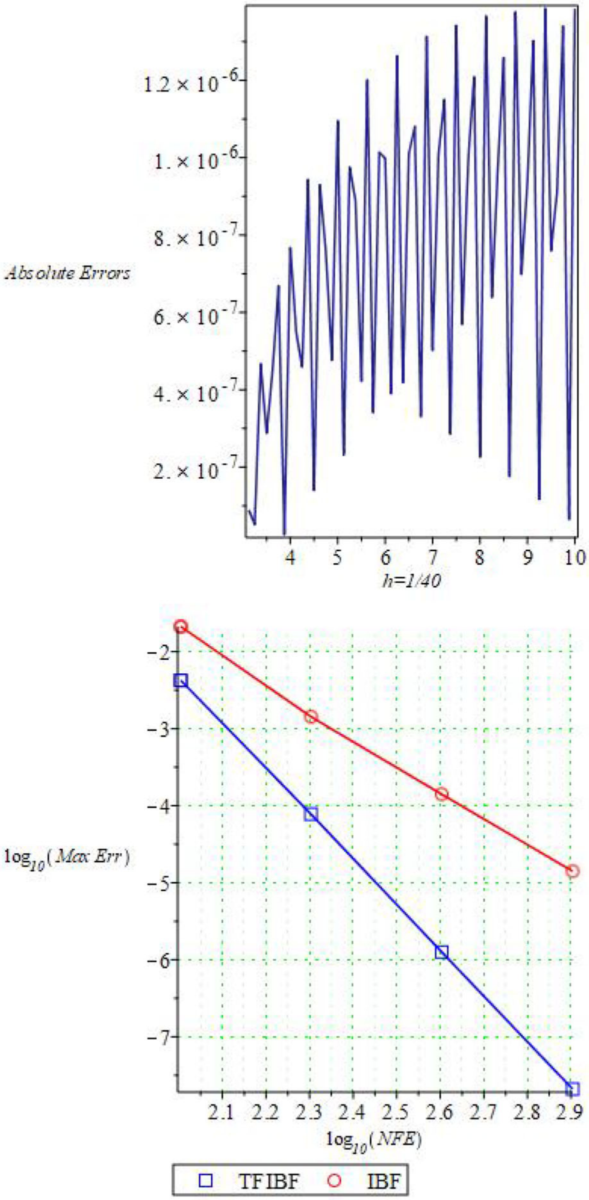

Fig. 8 Absolute errors (top) and efficiency curves (bottom) for Example 6

τ is the time delay.

According to Morrison and Rand [36], various special cases of Eq. (26) exist, depending on which parameters are zero. In this article, we have taken $\tau = 2\pi$, $\delta = 0$, and $\alpha = \mu = \lambda = 1$, and have taken the initial conditions as $y(0) = 0, \quad y'(0) = 1$ and the fitting frequency as $\omega = 1$ (see [29]). The comparison of the TFIBF and one of the ODE solvers in Maple Computer Algebra System ***dverk78*** is shown in Fig. 9.

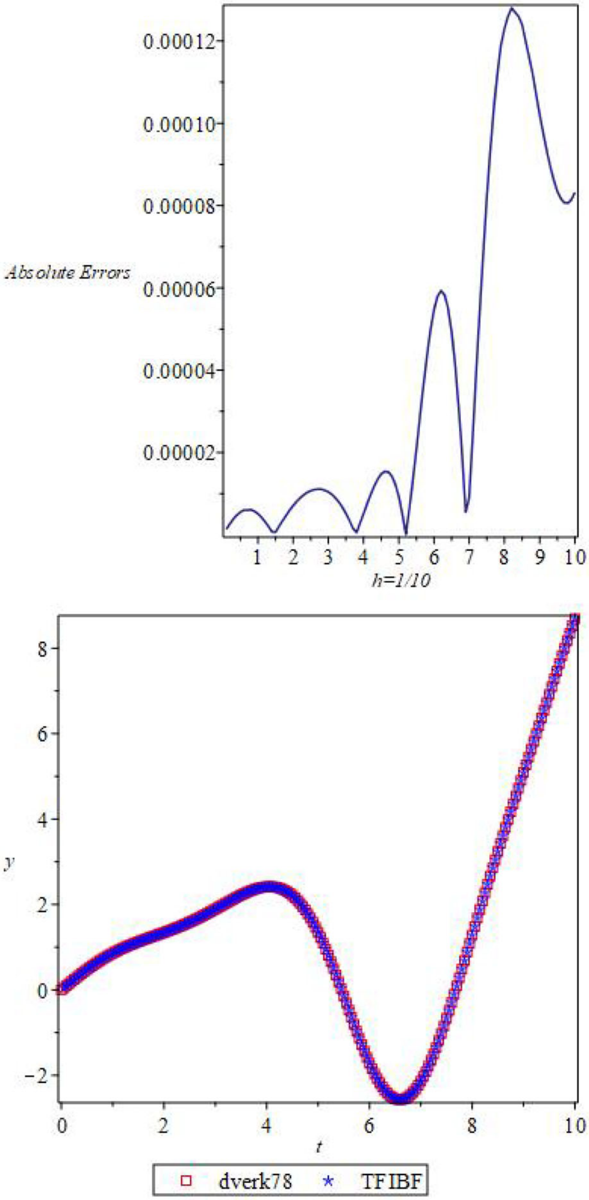

Fig. 9 Absolute errors (top) and discrete solutions (bottom)between dverk78 and TFIBF with $h = 1/10$ for Example 7

We remark that the ***dsolve*** command, with the options ***numeric*** and ***method=dverk78***, finds a numerical solution using a Runge–Kutta–Nyström pair of orders 7(8). The ***dverk78*** method can work in arbitrary precision depending on the Digits setting and can be used to obtain high accuracy solutions for ODE systems. The digit is set as required, so that tol=$O(Float(1, -Digits)^{9/8})$, and the absolute error tolerance ('abserr') and relative error tolerance ('relerr') are set to the default values, abserr=relerr=$1. \times 10^{-8}$.

5 Conclusion

An intra-step block Falkner method whose coefficients are in trigonometric form, is proposed for the direct integration of Eq. (1). It is established that the TFIBF is consistent, zero-stable, and convergent. The computational accuracy and efficiency of the TFIBF are shown in Figs. 2–8 , and in particular, the accuracy of TFIBF for the Mathieu equation is displayed in Fig. 9. Looking at the approximate results provided by the developed method, it is evident that it works better than other methods in the literature. Therefore, it can be considered as an acceptable alternative to solve the type of problem considered.

Author Contributions The authors contributed equally to this work.

Funding This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declarations

Conflict of interest The authors declare no conflict of interest.

Consent to participate The authors consented to the participation.

Consent for publication The authors consented to the publication of the article.

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