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Research Article

A 14-Order Hybrid Block Method in Variable Step-Size Mode for Solving Second-Order Initial Value Problems

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The search for efficient higher order methods is a constant goal in numerical analysis. In this paper, a higher order two-step hybrid block method is presented to directly solve second-order initial value problems in ordinary differential equations. In addition to the higher order, the proposed method has been formulated in variable step-size mode to extract its best performance. Comparisons with other methods in the literature show the good accuracy it can provide. Theoretical aspects such as linear stability and convergence analysis are also discussed.

1. Introduction

In literature, the numerical solution of the general second-order initial-value problem (IVP) of the form

$$\begin{aligned} &y'' = f(x, y, y'),\ x \in [a, b],\\ &y(a) = y_0,\ y'(a) = y'_0, \end{aligned} \tag{1}$$

has been on the rise. This is because it models many physical applied problems [1]. Sometimes Equation (1) can be transformed into a system of first-order ordinary differential equations (ODEs) and thus, solving it to obtain its numerical solution. A particular drawback of the latter is the high cost of CPU time due to the greater number of function evaluations [2]. The advantage of solving Equation (1) directly without considering transforming it resides in the fact that it achieves efficiency in terms of accuracy and less CPU time [3].

Few of these numerical approaches that are applied directly to solve Equation (1) include but are not limited to: Runge–Kutta methods, differential transform methods (DTM), linear multistep methods (LLMs), to mention but a few. In recent times, linear multistep block methods, credited to Milne [4], have been widely applied to solve Equation (1) directly. These methods have great advantages since they overcome the intersections of pieces of solutions and do not require any starting values provided by other methods, that is, they are self starting. For recent methods solving Equation (1) and higher order equations using linear multistep block methods, see [1, 3, 5–10]. It is worthy to mention that most of the approaches found used a constant step-size h. This approach may perform poorly, especially if there are rapid and slow changes of the solution over the interval of integration. Efficient codes for solving IVPs is meant to automatically select the suitable step-size to achieve efficiency [11].

The two approaches to achieving this include: applying a scheme such that its coefficients rely on the ratios of the step sizes, or the use of a second technique to provide estimates of the local errors. The aim is to vary the step sizes so that one retains local errors smaller than a given tolerance and concurrently solving the underlying problem as efficiently as possible.

This paper aims to achieve a higher order variable step-size block hybrid integrator for solving Equation (1) directly. Though, a higher order method does not guarantee a better efficiency in terms of smaller global errors, as can be seen in comparing a method of seventh order by Jator [9], which performs better than a method of order eight by Tsitouras

[12]. However, the higher order method derived here has more advantages which include, a small number of integration subintervals, smaller number of function evaluations, less costly in terms of CPU time, and small global error in terms of accuracy and its wide robust application for solving general second order IVPs. It is worth noting that the method implemented in its block form is self-starting. This means that, it does not require any external method to obtain the starting values. The novelty of the derived method is hinged on the advantages aforementioned among others.

2. Fundamentals

Definition 1. A linear k-step method for solving Equation (1) is usually written in the form

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i f_{n+i}, \tag{2}$$

where $y_{n+i} \approx y(x_{n+i}) \approx y(x_n + ih)$, $f_{n+i} = f(x_{n+i}, y_{n+i}) \approx f(x_n + ih, y(x_n + ih))$.

If $\beta_k \neq 0$ the method is of implicit type, otherwise it is explicit. The coefficients are usually normalized assuming that $\alpha_k = 1$.

Remark 1. *The k-step LMM Equation (2) is called linear because it involves only linear combinations of the y_{n+k} and the f_{n+k}.*

Definition 2. The k-step method has first and second characteristic polynomials of the form

$$\begin{aligned} \rho(r) &= r^k + \alpha_{k-1} r^{k-1} + \cdots + \alpha_0 \\ \sigma(r) &= \beta_k r^k + \beta_{k-1} r^{k-1} + \cdots + \beta_0. \end{aligned} \tag{3}$$

Definition 3. Given a continuously differentiable function $z(x)$, we may associate a linear difference operator $\mathscr{L}_h$ to the LMM Equation (2) given by

$$\mathscr{L}_h(z(x);h) \equiv \sum_{j=0}^{k} \left(\alpha_j z(x+jh) - h^2 \beta_j z''(x+jh)\right). \tag{4}$$

Expanding Equation (4) in Taylor series about the point $h = 0$, the following is obtained

$$\mathscr{L}_h(z(x);h) = C_0 z(x) + C_1 h z'(x) + C_2 h^2 z''(x) + \cdots + C_p h^p z^{(p)}(x) + O(h^{p+1}), \tag{5}$$

where $C_0, C_1, C_2, \ldots$ are linear combinations of the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_k (\alpha_k = 1), \beta_0, \beta_1, \ldots, \beta_k$.

Definition 4. The LMM Equation (2) is said to be order p if $C_0 = C_1 = C_2 = \cdots = C_{p+2} = 0$, and $C_{p+3} \neq 0$ in which

$$\mathscr{L}_h[y(x);h] = C_{p+3} h^{p+3} y^{(p+3)}(x) + O(h^{p+4}). \tag{6}$$

In this case, C_{p+3} is known as the principal error constant.

Definition 5. The difference operator $\mathscr{L}_h$ is said to be consistent of order p if it satisfies Equation (6) with $p > 0$ for every sufficiently differentiable function z.

Remark 2. *An LMM whose associated difference operator is consistent of order $p > 0$ is said to be consistent.*

Definition 6. It is said that a polynomial satisfies the root condition if all its roots lie within or on the boundary of a unit circle, with those on the boundary being simple. That is, all roots must satisfy $|r| < 1$, and those whose modulus is unity must be simple.

Remark 3. *If λ is a simple root of $\rho(r)$ this is equivalent to say that $(\lambda - r)$ is a factor of $\rho(r)$ with multiplicity one.*

Definition 7. A LMM is called be zero-stable if its first characteristic polynomial $\rho(r)$ verifies the root condition.

Definition 8. The LMM Equation (2) is said to be convergent if, for all IVPs in Equation (1) having a unique solution $y(x)$,

$$\lim_{h \longrightarrow 0} y_n = y_{x^*}, \ nh = x^* - a, \tag{7}$$

holds for all $x^* \in [a, b]$.

For details, refer to the study by Lambert [13].

3. Derivation of the Method

Consider the uniform mesh

$$a = x_0 < x_1 < \cdots < x_N = b, \tag{8}$$

with $h = x_{j+1} - x_j$. Assume that the solution $y(x)$ of Equation (1) is approximated by the following polynomial $q(x)$, that is,

$$y(x) \simeq q(x) = \sum_{j=0}^{15} \rho_j x^j, \tag{9}$$

whose second and third derivatives are approximated by

$$\begin{aligned} y''(x) &\simeq q''(x) = \sum_{j=2}^{15} \rho_j j(j-1) x^{j-2}, \\ g(x) = y'''(x) &\simeq q'''(x) = \sum_{j=3}^{15} \rho_j j(j-1)(j-2) x^{j-3}, \end{aligned} \tag{10}$$

where $\rho_j \in \mathbb{R}$ are to be determined. We then consider the points $x_{n+\frac{i}{3}} = x_n + ih/3$, $i = 0(1)6$, for approximating the solution in the two-step interval $[x_n, x_{n+2}]$. Using Equation (9) and its first derivative at the point x_n, and the second and third derivatives in Equation (10), respectively, applied to the points $x_{n+\frac{i}{3}}$, $i = 0(1)6$, the following system of 16 equations is obtained $q(x_n) = y_n$; $q'(x_n) = y'_n$; $q''(x_{n+i/3}) = f_{n+i/3}$; $q'''(x_{n+i/3}) = g_{n+i/3}$; $i = 0(1)6$ with $y_n \simeq y(x_n)$, $y'_n \simeq y'(x_n)$, $f_n = f(x_n, y_n, y'_n)$, and $\{g_n \simeq df(x, y, y')/dx\}_{\{x=x_n, y=y_n, y'=y'_n\}}$.

Using the computer algebra system *Mathematica*, the above system is easily solved and the values of the coefficients ρ_j, $j = 0(1)15$ are obtained. These expressions are cumbersome and are not included here. After some algebraic simplification, we obtained

$$q(x) = \alpha_0 y_n + \widehat{\alpha}_0 h y'_n + h^2 \sum_{i=0}^{6} \left(\beta_{i/3} f_{n+i/3} + h\widehat{\beta}_{i/3} g_{n+i/3}\right), \tag{11}$$

where $\alpha_0 = 1, \widehat{\alpha}_0, \beta_{i/3}, \widehat{\beta}_{i/3}$ are coefficients that depend on x. By evaluating Equation (11) at the points $x = x_{n+i/3}$, $i = 1(1)6$, we obtain the following main formulas:

$$\begin{aligned} y_{n+\frac{1}{3}} = y_n + \frac{hy'_n}{3} + \frac{h^2}{50438868480000}(&1409950789503 f_n \\ &- 1206244188000 f_{n+\frac{1}{3}} - 2129246035875 f_{n+\frac{2}{3}} \\ &+ 1602588528000 f_{n+1} + 2494739750625 f_{n+\frac{4}{3}} \\ &+ 605115451872 f_{n+\frac{5}{3}} + 25255063875 f_{n+2} \\ &+ 38298466790 g_n h - 548081049600 g_{n+\frac{1}{3}} h \\ &- 1280080329750 g_{n+\frac{2}{3}} h - 1413689440000 g_{n+1} h \\ &- 577286813250 g_{n+\frac{4}{3}} h - 72516908160 g_{n+\frac{5}{3}} h \\ &- 1658209150 g_{n+2} h), \end{aligned} \tag{12}$$

$$\begin{aligned} y_{n+\frac{2}{3}} = y_n + \frac{2hy'_n}{3} + \frac{h^2}{98513415000}(&6271361196 f_n \\ &- 284106384 f_{n+\frac{1}{3}} - 9678481875 f_{n+\frac{2}{3}} \\ &+ 8910576000 f_{n+1} + 13328689500 f_{n+\frac{4}{3}} + 3210284304 f_{n+\frac{5}{3}} \\ &+ 133547259 f_{n+2} + 180107630 g_n h - 2902929120 g_{n+\frac{1}{3}} \\ &- 7110621000 g_{n+\frac{2}{3}} h - 7589200000 g_{n+1} h \\ &- 3072156750 g_{n+\frac{4}{3}} h - 384282720 g_{n+\frac{5}{3}} h \\ &- 8765020 g_{n+2} h), \end{aligned} \tag{13}$$

$$\begin{aligned} y_{n+1} = y_n + hy'_n + \frac{h^2}{7687680000}(&764760385 f_n + 231660864 f_{n+\frac{1}{3}} \\ &- 633794625 f_{n+\frac{2}{3}} + 1301872000 f_{n+1} + 1744797375 f_{n+\frac{4}{3}} \\ &+ 417234240 f_{n+\frac{5}{3}} + 17309761 f_{n+2} + 22324850 g_n h \\ &- 363718080 g_{n+\frac{1}{3}} h - 935246250 g_{n+\frac{2}{3}} h - 1003520000 g_{n+1} h \\ &- 400497750 g_{n+\frac{4}{3}} h - 49896000 g_{n+\frac{5}{3}} h - 1135730 g_{n+2} h), \end{aligned} \tag{14}$$

$$\begin{aligned} y_{n+\frac{4}{3}} = y_n + \frac{4hy'_n}{3} + \frac{h^2}{12314176875}(&1666430340 f_n \\ &+ 789473664 f_{n+\frac{1}{3}} - 618372000 f_{n+\frac{2}{3}} \\ &+ 3996288000 f_{n+1} + 4125820500 f_{n+\frac{4}{3}} + 947180160 f_{n+\frac{5}{3}} \\ &+ 39114336 f_{n+2} + 49029200 g_n h - 801089280 g_{n+\frac{1}{3}} h \\ &- 2086812000 g_{n+\frac{2}{3}} h - 2266240000 g_{n+1} h \\ &- 919548000 g_{n+\frac{4}{3}} h - 113068800 g_{n+\frac{5}{3}} h - 2565280 g_{n+2} h), \end{aligned} \tag{15}$$

$$\begin{aligned} y_{n+\frac{5}{3}} = y_n + \frac{5hy'_n}{3} + \frac{h^2}{80702189568}(&13817402355 f_n \\ &+ 8004319200 f_{n+\frac{1}{3}} - 981759375 f_{n+\frac{2}{3}} \\ &+ 40345200000 f_{n+1} + 41558878125 f_{n+\frac{4}{3}} \\ &+ 9002293920 f_{n+\frac{5}{3}} + 340040175 f_{n+2} \\ &+ 408493150 g_n h - 6671664000 g_{n+\frac{1}{3}} h \\ &- 17439918750 g_{n+\frac{2}{3}} h - 18807200000 g_{n+1} h \\ &- 7632506250 g_{n+\frac{4}{3}} h - 1019289600 g_{n+\frac{5}{3}} h \\ &- 22247750 g_{n+2} h), \end{aligned} \tag{16}$$

$$\begin{aligned} y_{n+2} = y_n + 2hy'_n + \frac{h^2}{15015000}(&3114974 f_n \\ &+ 2144880 f_{n+\frac{1}{3}} + 934875 f_{n+\frac{2}{3}} \\ &+ 10544000 f_{n+1} + 10226250 f_{n+\frac{4}{3}} \\ &+ 2869776 f_{n+\frac{5}{3}} + 195245 f_{n+2} \\ &+ 92570 g_n h - 1490400 g_{n+\frac{1}{3}} h \\ &- 3820500 g_{n+\frac{2}{3}} h \\ &- 3920000 g_{n+1} h - 1397250 g_{n+\frac{4}{3}} h \\ &-125280 g_{n+\frac{5}{3}} h - 9800 g_{n+2} h). \end{aligned} \tag{17}$$

Differentiating Equation (11) we obtain the approximating polynomial as follows:

$$q'(x) = \widehat{\alpha}'_0 h y'_n + h^2 \sum_{i=0}^{6} \left(\beta'_{\frac{i}{3}} f_{n+\frac{i}{3}} + h\widehat{\beta}'_{\frac{i}{3}} g_{n+\frac{i}{3}}\right). \tag{18}$$

Evaluating Equation (18) at the points $x = x_{n+i/3}$, $i = 1(1)6$, the following additional formulas are derived

$$\begin{aligned} y'_{n+\frac{1}{3}} = y'_n + \frac{h}{5604318720000}(&598106457531 f_n \\ &- 206855488656 f_{n+\frac{1}{3}} - 1141249411125 f_{n+\frac{2}{3}} \\ &+ 901926912000 f_{n+1} + 1371416995125 f_{n+\frac{4}{3}} \\ &+ 330981811344 f_{n+\frac{5}{3}} + 13778963781 f_{n+2} \\ &+ 17840980130 g_n h - 334886654880 g_{n+\frac{1}{3}} h \\ &- 715142076750 g_{n+\frac{2}{3}} h - 779350000000 g_{n+1} h \\ &- 316468860750 g_{n+\frac{4}{3}} h - 39630345120 g_{n+\frac{5}{3}} h \\ &- 904417630 g_{n+2} h), \end{aligned} \tag{19}$$

$$y'_{n+\frac{2}{3}} = y'_n + \frac{h}{21891870000}(2351162451f_n + 2120326272f_{n+\frac{1}{3}} - 1371175875f_{n+\frac{2}{3}} + 4163424000f_{n+1} + 5872060125f_{n+\frac{4}{3}} + 1400759424f_{n+\frac{5}{3}} + 58023603f_{n+2} + 70570130g_n h - 1176819840g_{n+\frac{1}{3}}h - 3379709250g_{n+\frac{2}{3}}h - 3367936000g_{n+1}h - 1346154750g_{n+\frac{4}{3}}h - 167425920g_{n+\frac{5}{3}}h - 3806290g_{n+2}h), \quad (20)$$

$$y'_{n+1} = y'_n + \frac{h}{7687680000}(826473395f_n + 775497456f_{n+\frac{1}{3}} + 688759875f_{n+\frac{2}{3}} + 2699264000f_{n+1} + 2168488125f_{n+\frac{4}{3}} + 508254480f_{n+\frac{5}{3}} + 20942669f_{n+2} + 24833650g_n h - 410101920g_{n+\frac{1}{3}}h - 1112298750g_{n+\frac{2}{3}}h - 1301872000g_{n+1}h - 491946750g_{n+\frac{4}{3}}h - 60631200g_{n+\frac{5}{3}}h - 1373070g_{n+2}h), \quad (21)$$

$$y'_{n+\frac{4}{3}} = y'_n + \frac{h}{1368241875}(147195378f_n + 140932800f_{n+\frac{1}{3}} + 141525000f_{n+\frac{2}{3}} + 700608000f_{n+1} + 594227250f_{n+\frac{4}{3}} + 95959872f_{n+\frac{5}{3}} + 3874200f_{n+2} + 4426340g_n h - 72662400g_{n+\frac{1}{3}}h - 194544000g_{n+\frac{2}{3}}h - 210496000g_{n+1}h - 100822500g_{n+\frac{4}{3}}h - 11352960g_{n+\frac{5}{3}}h - 253600g_{n+2}h), \quad (22)$$

$$y'_{n+\frac{5}{3}} = y'_n + \frac{h}{8966909952}(966379755f_n + 967797360f_{n+\frac{1}{3}} + 31455765f_{n+\frac{2}{3}} + 4853760000f_{n+1} + 5158693125f_{n+\frac{4}{3}} + 1828337040f_{n+\frac{5}{3}} + 1138426875f_{n+2} + 29120450g_n h - 471031200g_{n+\frac{1}{3}}h - 1229928750g_{n+\frac{2}{3}}h - 1246960000g_{n+1}h - 420648750g_{n+\frac{4}{3}}h - 128196000g_{n+\frac{5}{3}}h - 2021950g_{n+2}h), \quad (23)$$

$$y'_{n+2} = y'_n + \frac{h}{30030000}(3310219f_n + 5014656f_{n+\frac{1}{3}} + 11161125f_{n+\frac{2}{3}} + 21088000f_{n+1} + 11161125f_{n+\frac{4}{3}} + 5014656f_{n+\frac{5}{3}} + 3310219f_{n+2} + 102370g_n h - 1365120g_{n+\frac{1}{3}}h - 2423250g_{n+\frac{2}{3}}h + 2423250g_{n+\frac{4}{3}}h + 1365120g_{n+\frac{5}{3}}h - 102370g_{n+2}h). \quad (24)$$

4. Analysis of the Method

For any numerical scheme, it is very important to know the order of accuracy, the behavior of the local error, and the stability characteristics. To this end, the analysis of the proposed method is presented in this section.

For the order and truncation error of Equation (12), we consider the corresponding differential operators

$$\mathscr{L}_i[y(x);h] = y\left(x + i\frac{h}{3}\right) - y(x) - i\frac{h}{3}y'(x) - h^2\sum_{j=0}^{6}\left(\beta^i_{\frac{j}{3}}y''\left(x + j\frac{h}{3}\right) + h\widehat{\beta}^i_{\frac{j}{3}}y'''\left(x + j\frac{h}{3}\right)\right), \quad (25)$$

where $y(x)$ is sufficiently differentiable, and the coefficients $\beta^i_{\frac{j}{3}}, \widehat{\beta}^i_{\frac{j}{3}}$ are the corresponding constant coefficients in the formulas in Equation (12). Expanding Equation (25) in Taylor series about x_n and after collecting all the terms in h the local truncation errors take the form

$$\mathscr{L}_i[y(x);h] = C_{\frac{i}{3}}h^{p+2}y^{(p+2)}(x) + O(h^{p+3}), \quad (26)$$

where $C_{i/3}$ is the principal error constant and p is the order of the corresponding formula.

For instance, considering the first formula in Equation (12), the formula in Equation (26) takes the form

$$\mathscr{L}_{i/3}[y(x_n);h] = y\left(x_n + \frac{h}{3}\right) - y(x_n) - \frac{h}{3}y'(x_n) - h^2\left(\frac{1409950789503}{50438868480000}y''(x_n) + \frac{1602588528000}{50438868480000}y''(x_n + h) - \frac{1206244188000}{50438868480000}y''\left(x_n + \frac{h}{3}\right) + \frac{25255063875}{50438868480000}y''(x_n + 2h) - \frac{2129246035875}{50438868480000}y''\left(x_n + \frac{2h}{3}\right) + \frac{2494739750625}{50438868480000}y''\left(x_n + \frac{4h}{3}\right) + \frac{605115451872}{50438868480000}y''\left(x_n + \frac{5h}{3}\right) + \frac{38298466790}{50438868480000}hy'''(x_n) - \frac{1413689440000}{50438868480000}hy'''(x_n + h) - \frac{548081049600}{50438868480000}hy'''\left(x_n + \frac{h}{3}\right) - \frac{1658209150}{50438868480000}hy'''(x_n + 2h) - \frac{1280080329750}{50438868480000}hy'''\left(x_n + \frac{2h}{3}\right) - \frac{577286813250}{50438868480000}hy'''\left(x_n + \frac{4h}{3}\right) - \frac{72516908160}{50438868480000}hy'''\left(x_n + \frac{5h}{3}\right)\right). \quad (27)$$

So that, expanding Equation (27) in Taylor series, we arrive at

$$\mathscr{L}_{\frac{1}{3}}[y(x_n);h] = 6.94185 \times 10^{-16}h^{16}y^{(16)}(x_n) + O(h^{17}), \quad (28)$$

where the local truncation error is as expressed in Equation (26).

Following similar pattern for other formulas in Equation (12), the local truncation errors are given as

$$\mathscr{L}_{\frac{2}{3}}[y(x);h] = 1.85962\times10^{-15}h^{16}y^{(16)}(x_n) + O(h^{17}), \tag{29}$$

$$\mathscr{L}_{1}[y(x);h] = 3.06685\times10^{-15}h^{16}y^{(16)}(x_n) + O(h^{17}), \tag{30}$$

$$\mathscr{L}_{\frac{4}{3}}[y(x);h] = 4.29396\times10^{-15}h^{16}y^{(16)}(x_n) + O(h^{17}), \tag{31}$$

$$\mathscr{L}_{\frac{5}{3}}[y(x);h] = 5.56286\times10^{-15}h^{16}y^{(16)}(x_n) + O(h^{17}), \tag{32}$$

$$\mathscr{L}_{2}[y(x);h]7.30302\times10^{-15}h^{16}y^{(16)}(x_n) + O(h^{17}). \tag{33}$$

For the formulas in Equations (19)–(24), the local truncation errors may be obtained similarly. From the above results, the order of each of the formulas in Equations (12)–(17) is $p = 14$. This is the same order of the formulas in Equations (19)–(24).

4.1. Zero Stability. For a block method, zero stability is such that, the roots γ_i of the characteristic equation given by $\rho(\gamma) = 0$ must satisfy $|\gamma_i| \le 1$ and for those with $|\gamma_i| = 1$ the multiplicity does not exceed 2, [10].

We note that the method in Equations (12)–(17) and Equations (19)–(24) may be written in matrix form as follows:

$$P_1\bar{y}_{n+\tau} = P_0\bar{y}_n + h^2Q_1\bar{f} + h^3Q_2\bar{g}, \tag{34}$$

where

$$\bar{y}_{n+\tau} = \left(y_{n+\frac{1}{3}}, y_{n+\frac{2}{3}}, y_{n+1}, y_{n+\frac{4}{3}}, y_{n+\frac{5}{3}}, y_{n+2}\right)^T, \tag{35}$$

$$\bar{y}_{n} = \left(y_{n}, y_{n-\frac{1}{3}}, y_{n-\frac{2}{3}}, y_{n-1}, y_{n-\frac{4}{3}}, y'_{n-\frac{5}{3}}\right)^T, \tag{36}$$

$$\bar{f} = \left(f_n, f_{n+\frac{1}{3}}, f_{n+\frac{2}{3}}, f_{n+1}, f_{n+\frac{4}{3}}, f_{n+\frac{5}{3}}, f_{n+2}\right)^T, \tag{37}$$

$$\bar{g} = \left(g_n, g_{n+\frac{1}{3}}, g_{n+\frac{2}{3}}, g_{n+1}, g_{n+\frac{4}{3}}, g_{n+\frac{5}{3}}, g_{n+2}\right)^T, \tag{38}$$

and P_0, P_1, Q_1, Q_2, are corresponding matrices of coefficients. Zero-stability implies the stability of the system in Equation (34) as $h \longrightarrow 0$. Letting $h \longrightarrow 0$, the system in Equation (34) becomes

$$P_1\bar{y}_{n+\tau} - P_0\bar{y}_n = 0, \tag{39}$$

$$P_1 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \tag{40}$$

$$P_0 = \begin{pmatrix} 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}. \tag{41}$$

The roots of the characteristic equation $\rho(\gamma) = \det(\gamma P_1 - P_0) = 0$ are $\gamma_i = 0$ for $i = 1(1)5$ and $\gamma_6 = 1$. Consequently, the proposed method is zero stable.

4.2. Convergence. The convergence of a LMM is guaranteed if it is consistent (with order $p > 1$) and zero stable [13, 14]. Considering the analysis shown above, the method has order $p = 14$, and is zero stable. Hence the proposed method is convergent.

4.3. Linear Stability Analysis. For any numerical method, the linear stability analysis is an important aspect of the theoretical analysis of the method. The Dahlquist's test equation given by

$$y''(x) = \mu^2 y(x) \text{ with } \mu > 0, \tag{42}$$

is usually used in the linear stability analysis for numerical methods for second order differential equations. Nevertheless, since the general second order differential equations are the focus and Equation (42) does not contain the first derivative, then the following linear test equation is considered [15]

$$y''(x) = -2\mu y'(x) - \mu^2 y(x) \text{ with } \mu \in \mathbb{C}. \tag{43}$$

For $\mu > 0$, the solutions of Equation (43) are bounded and go to zero when $x \longrightarrow \infty$.

We employ the strategy by Singh and Ramos [16] to illustrate the procedure for obtaining the absolute stability region (the region in the $h\mu$-complex plane where the numerical method mimics the qualitative behavior of the exact solutions). The block method generated by the formulas in Equations (11) and (18) has 12 equations in which there are six different terms of derivatives: y'_n, y'_{n+1}, y'_{n+2}, $y'_{n+1/3}$, $y'_{n+2/3}$, $y'_{n+4/3}$, $y'_{n+5/3}$, and four intermediate values $y_{n+1/3}$, $y_{n+2/3}$, $y_{n+4/3}$, $y_{n+5/3}$. All these terms are eliminated from the system of equations using the *Mathematica* system so that the following recurrence equation with y_n, y_{n+1}, y_{n+2} is obtained

$$A(\eta)y_{n+2} + B(\eta)y_{n+1} + C(\eta)y_n = 0, \tag{44}$$

where $\eta = \mu h$, and

$$\begin{aligned} A(\eta) = {} & 532482786093750741849600000 + 532482786093750741849600000\eta + 246823217371661668657920000\eta^2 \\ & + 69328955340411421374720000\eta^3 + 12837801551690466711360000\eta^4 + 1561100573636676072000000\eta^5 \\ & + 106003774607933109600000\eta^6 - 6673266983257237440\eta^7 - 974845639058675076\eta^8 - 944578144342241640\eta^9 \\ & - 174298766158780200\eta^{10} + 48163646213512200\eta^{11} + 4903970942886750\eta^{12} + 95785561086390\eta^{13} \\ & - 16615431665595\eta^{14} - 1463480955720\eta^{15} + 19763818515\eta^16 + 15027183486\eta^{17} + 1325390871\eta^{18} + 58324896\eta^{19} \\ & + 5093158\eta^{20} + 735084\eta^{21} + 58450\eta^{22} + 2352\eta^{23} + 40\eta^{24}, \end{aligned} \tag{45}$$

$$\begin{aligned} B(\eta) = {} & 1331206965234376854624000 - 485454391880342556672000\eta^2 + 90026658847759233600\eta^4 \\ & - 11295787709726150400\eta^6 + 107847233733905820\eta^8 - 834531382409760\eta^{10} + 5477092678449\eta^{12} \\ & - 28291214574\eta^{14} + 8970070095\eta^{16} + 807362850\eta^{18} + 18968929\eta^{20} + 154504\eta^{22} + 400\eta^{24}, \end{aligned} \tag{46}$$

$$\begin{aligned} C(\eta) = {} & 532482786093750741849600000 - 532482786093750741849600000\eta + 246823217371661668657920000\eta^2 \\ & - 69328955340411421374720000\eta^3 + 12837801551690466711360000\eta^4 - 1561100573636676072000000\eta^5 \\ & + 106003774607933109600000\eta^6 + 6673266983257237440\eta^7 - 974845639058675076\eta^8 + 944578144342241640\eta^9 \\ & - 174298766158780200\eta^{10} - 48163646213512200\eta^{11} + 4903970942886750\eta^{12} - 95785561086390\eta^{13} \\ & - 16615431665595\eta^{14} + 1463480955720\eta^{15} + 19763818515\eta^{16} - 15027183486\eta^{17} + 1325390871\eta^{18} - 58324896\eta^{19} \\ & + 5093158\eta^{20} - 735084\eta^{21} + 58450\eta^{22} - 2352\eta^{23} + 40\eta^{24}. \end{aligned} \tag{47}$$

The characteristic Equation (44) is

$$A(\eta)\tau^2 + B(\eta)\tau + C(\eta) = 0. \tag{48}$$

To determining the region of stability, the roots $\tau_{1,2}$ of this equation must have absolute values less than unity. Solving Equation (48), the roots $\tau_{1,2}$ of the characteristic equation are

$$\tau_{1,2} = \sqrt{\frac{D(\eta)}{A(\eta)}}, \tag{49}$$

where

$$\begin{aligned} D(\eta) = {} & 532482786093750741849600000 + 532482786093750741849600000\eta - 285659568722089073191680000\eta^2 \\ & + 69328955340411421374720000\eta^3 - 12117588280908392842560000\eta^4 + 1561100573636676072000000\eta^5 \\ & - 115040404775714029920000\eta^6 - 6673266983257237440\eta^7 + 1061123426045799732000\eta^8 - 944578144342241640\eta^9 \\ & + 1075362555659994000\eta^{10} + 48163646213512200\eta^{11} - 4465803528610830\eta^{12} + 95785561086390\eta^{13} \\ & + 14352134499675\eta^{14} - 1463480955720\eta^{15} + 697841789085\eta^{16} + 15027183486\eta^{17} + 63263637129\eta^{18} + 58324896\eta^{19} \\ & + 1512421162\eta^{20} + 735084\eta^{21} + 12301870\eta^{22} + 2352\eta^{23} + 31960\eta^{24}. \end{aligned} \tag{50}$$

The plot of the region in which $|\tau_{1,2}|<1$, is shown in Figure 1. This shows the region of stability for the method derived, whose stability interval is (0, 8.69809). This region is in the complex μh-plane where the roots of the characteristic Equation (48) are bounded in modulus by unity.

4.4. Formulation in Variable Step-Size Mode and Error Estimation. To gain efficiency when using the method given by Equations (12)–(17) and (19)–(24), it is convenient to have it formulated in variable step-size mode (VHM). In this sense, a lower order formula (LOF) must be considered to have an estimation of the local error (LE) at the end point of the two-step interval $[x_n, x_{n+2}]$. The procedure is found to be less time consuming when the LOF uses values that have been previously obtained. To this end, for the block method constituted by Equations (12)–(17) and (19)–(24), we consider the LOF given by

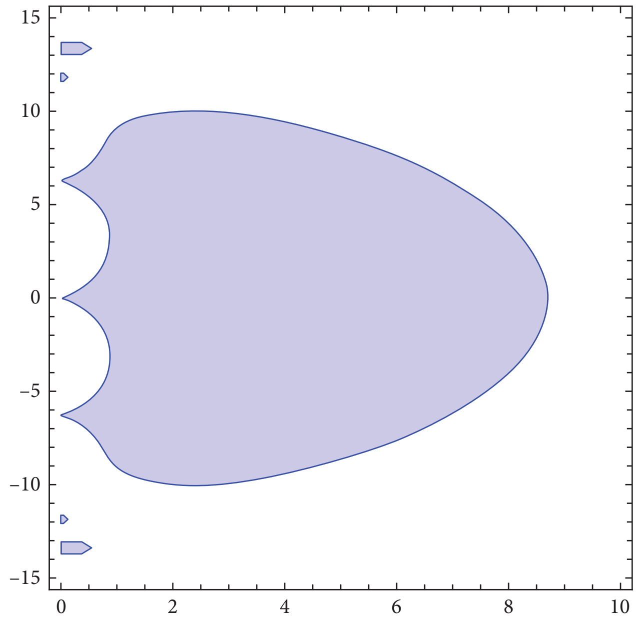

FIGURE 1: Absolute stability region.

$$\begin{aligned} y^*_{n+2} = {} & y_n + 2hy'_n + \frac{h^2}{7507500}(2356344f_n \\ & + 13333125f_{n+\frac{1}{3}} + 25440000f_{n+\frac{2}{3}} \\ & - 4965000f_{n+1} - 17940000f_{n+\frac{4}{3}} \\ & - 3209469f_{n+\frac{5}{3}} + 102370hg_n \\ & + 966750hg_{n+\frac{1}{3}} + 6870000hg_{n+\frac{2}{3}} \\ & + 10237000hg_{n+1} + 4242750hg_{n+\frac{4}{3}} \\ & + 420870hg_{n+\frac{5}{3}}), \end{aligned} \tag{51}$$

where y^*_{n+2} denotes another approximation of $y(x_{n+2})$. The LOF in (51) has the local truncation error

$$LTE = 6.94209 \times 10^{-12} y^{(14)}(x_n)h^{14} + O(h^{15}). \tag{52}$$

This was used to estimate the LE at the end-point x_{n+2}. This error value gives the basis on how the step-size for the subsequent steps are determined in the course of implementation. For a given defined tolerance *tol*, the algorithm will change the step-size according to the following strategy. The Algorithm 1 below is applied for implementation of the proposed block method in VHM:

4.5. Computational Procedure. The proposed block method is implemented in VHM in such a way that, on each block interval of the form $[x_n, x_{n+2}]$ and $n = 0, 2\dots, N-2$, where N is a multiple of 2 so as give an integer number of iterations to reach the end of the integration interval x_N. The system in (12)–(17) and (19)–(24) is solved by using the Newton's method and taking the approximations provided by the Taylor formulas as starting values. These values are given as follows:

$$\begin{aligned} y_{n+\frac{j}{3}} &= y_n + j\frac{h}{3}y'_n + \frac{1}{2}\left(j\frac{h}{3}\right)^2 f_n + \frac{1}{6}\left(j\frac{h}{3}\right)^3 g_n, \\ y'_{n+\frac{j}{3}} &= y'_n + j\frac{h}{3}f_n + \frac{1}{2}\left(j\frac{h}{3}\right)^2 g_n, \end{aligned} \tag{53}$$

for $j = 1(1)6$.

To apply the proposed method to a system of m second order ODEs, we have the following procedure

$$\bar{y}' = \bar{f}(x, \bar{y}^T, \bar{y}'^T), \ \bar{y}(a) = \bar{y}_0, \tag{54}$$

$$\bar{y}'(a) = \bar{y}'_0, \ x_0 \le x \le x_N, \tag{55}$$

where

$$\bar{y} = (y_1, y_2, \dots, y_m)^T, \ \bar{y}' = (y'_1, y'_2, \dots, y'_m)^T, \tag{56}$$

$$\bar{f}(x, \bar{y}^T, \bar{y}'^T) = (f_1(x, \bar{y}^T, \bar{y}'^T), f_2(x, \bar{y}^T, \bar{y}'^T), \dots, f_m(x, \bar{y}^T, \bar{y}'^T))^T \bar{y}_0 = (y_{1,0}, y_{2,0}, \dots, y_{m,0})^T, \ \bar{y}'_0 = (y'_{1,0}, y'_{2,0}, \dots, y'_{m,0})^T. \tag{57}$$

To solve this system we use again the Newton's method. To obtain the estimations of the third derivatives of each component at x_n and $n = 0(2)N-2$, we use the following formula

$$\begin{aligned} g_{i,n} \simeq y'''_i(x) &= \frac{df_i(x, y_1, y_2, \dots, y_m, y'_1, y'_2, \dots, y'_m)}{dx} \\ &= \frac{\partial f_i}{\partial x} + \sum_{j=1}^{m} \frac{\partial f_i}{\partial y_j} y'_j + \sum_{j=1}^{m} \frac{\partial f_i}{\partial y'_j} f_j. \end{aligned} \tag{58}$$

5. Numerical Examples

In this section we show the efficiency of the derived method implemented in variable and fixed step-size modes, respectively, where applicable. As mentioned earlier, our method has a higher order compared to other methods mentioned in this paper. The comparison of numerical results shall be based on the number of integration subintervals, the number of functions evaluations, accuracy in terms of global errors and CPU times, where applicable. There is no method of its order or of a higher order that has been found in the literature for comparisons. The examples considered show the robustness of the derived method in solving general second order systems of ODEs.

The following notations are used in the course of this section:

(.) *tol*: predefined tolerance
(.) $h_{initial}$: initial step-size
(.) N: number of steps

Require: a, b **(integration interval),** y_a, y'_a **(initial values),** $f, \frac{df}{dx}$
1. Set tol**, Let** $n = 0, h = h_{initial}$
2. Let $x_n = a, y_n = y_a, y'_n = y'_a$**, Let** $sol1 = \{(x_n, y_n)\}$.
3. Solve Equations (12)–(17), (19)–(24) to get $y_{n+\frac{i}{3}}, y'_{n+\frac{i}{3}}, \frac{i}{3} = 1(1)6$
4. Let $sol1 = sol1 \cup \left\{\left(x_{n+\frac{i}{3}}, y_{n+\frac{i}{3}}\right)\right\}_{i=1(1)6}$.
5. Let $x_n = x_n + 2h, y_n = y_{n+2}, y'_n = y'_{n+2}$
6. Let $n = n+2$
7. If $n<N$
8. then go to 3
9. else
10. go to 11 then
11. Obtain $sol1 = y_{n+2}$
12. end if else
13. Solve Equation (51) to get $sol2 = y^*_{n+2}$
14. Obtain LE using $est = |y_{n+2} - y^*_{n+2}|$
15. if $est \leq tol$ **then**
16. the result is accepted and the next step-size is taken as $h_{new} = 2 \times h_{initial}$
17. else
18. the result is rejected.
19. go to 15 with the new step-size
20. $h_{new} = \kappa h_{initial} \left(\frac{tol}{|est|}\right)^{\frac{1}{p+2}}$
21. where p **is the order of the LOF and** $0<\kappa<1$**, which serves as a safety factor to avoid failure steps**
22. end if else
23. End

Algorithm 1: Variable step-size mode implementation.

- (.) EMax: maximum absolute error along the integration interval
- (.) $EMax = \max_{n=0,1,\ldots,N} \{||y(x_n) - y_n||\}$
- (.) Time: CPU time in seconds
- (.) VHM: hybrid method derived in this paper using variable step size
- (.) FHM: hybrid method derived in this paper using fixed step size
- (.) NF: number of function evaluations
- (.) OPTBM: method derived by Singh and Ramos [16].

Problem 1. *Orbital problem considered by Singh and Ramos [16].*

$$y''(x) - y(x) = 0.001e^{ix};\ 0 \leq x \leq 40\,\pi, \tag{59}$$

$$y(0) = 1,\ y'(0) = \frac{9995}{10000}\, i. \tag{60}$$

The exact solution of the Stiefel and Bettis orbital problem is $y(x) = x\sin x/2000 + \cos x + i(\sin x - x\cos x/2000)$.

Table 1: Comparison of maximum absolute errors obtained for Problem 1.

Method	NF	EMax	CPU
VHM	1,645	2.05×10^{-14}	0.551
OPTBM	2,100	1.13×10^{-12}	0.825
SCOWE (6)	9,038	4.29×10^{-9}	–
I3P1B	16,755	4.10×10^{-9}	–

Table 2: Maximum absolute errors obtained for Problem 2.

HMM	NF	cdd	CPU
VHM	77	13.3	0.324
FHM	70	12.1	0.212
HLMM	3,601	11.9	0.602

We thus compare the proposed method named VHM with $h_{initial} = 0.1$, $tol = 10^{-10}$ and the methods reported by Singh and Ramos [16].

The comparison in Table 1 was done with the proposed method in VHM. The OPTBM in [16] was formulated in VHM. SCOWE(6) by Ramos and Vigo-Aguiar [3] and I3P1B by Ismail et al. [17] are methods reported by Singh and Ramos [16]. It shows the CPU times for VHM and OPTBM, with VHM performing the best. The number of function evaluations reveals that VHM performed well, being the one with the lowest number. Overall, this shows that the proposed method is the most accurate in comparison to the methods considered.

Problem 2. *Consider the nonlinear Duffing equation discussed by Jator [1].*

$$y''(x) - y(x) + (y(x))^3 = B\cos(\Omega x), \tag{61}$$

$$y(0) = C_0,\ y'(0) = 0, \tag{62}$$

with the solution $y(x) = C_1\cos(\Omega x) + C_2\cos(3\Omega x) + C_3\cos(5\Omega x) + C_4\cos(7\Omega x)$ where $\Omega = 1.01$, $B = 0.002$, $C_0 = 0.200426728069$, $C_1 = 0.200179477536$, $C_2 = 0.246946143 \times 10^{-3}$, $C_3 = 0.304016 \times 10^{-6}$, and $C_4 = 0.374 \times 10^{-9}$. For the sake of comparison as reported by Jator [1], the maximum global error is given in the form 10^{-cdd}, where $cdd = \log_{10}(AE)$, AE is the absolute error at the endpoint of the integration interval. Table 2 shows the maximum errors obtained from our method solved in the VHM with $h_{initial} = 0.1$ and $tol = 10^{-10}$ and fixed step-size mode, respectively, and compared to the hybrid linear multistep method (HLMM) of order seven derived by Jator [1]. It can be seen that the number of function evaluations is lower using our method.

Problem 3. *Consider the Van Der Pol oscillator discussed by Allogmany and Ismail [18].*

Table 3: Comparison of numerical solutions.

x	VHM	NDSolve
0	0	0
0.5	0.24030707	0.24030707
1.0	0.42277363	0.42277361
1.5	0.50228041	0.50228038
2.0	0.45888178	0.45888176
2.5	0.30272754	0.30272754
3.5	−0.17829328	−0.17829325
4.5	−0.49920271	−0.49920270
5.5	−0.36193141	−0.36193145
6.5	0.11085609	0.11085605
7.5	0.48578439	0.48578440
8.5	0.41539055	0.41539062
9.5	−0.03923230	−0.03923225
10.0	−0.28502433	−0.28502430

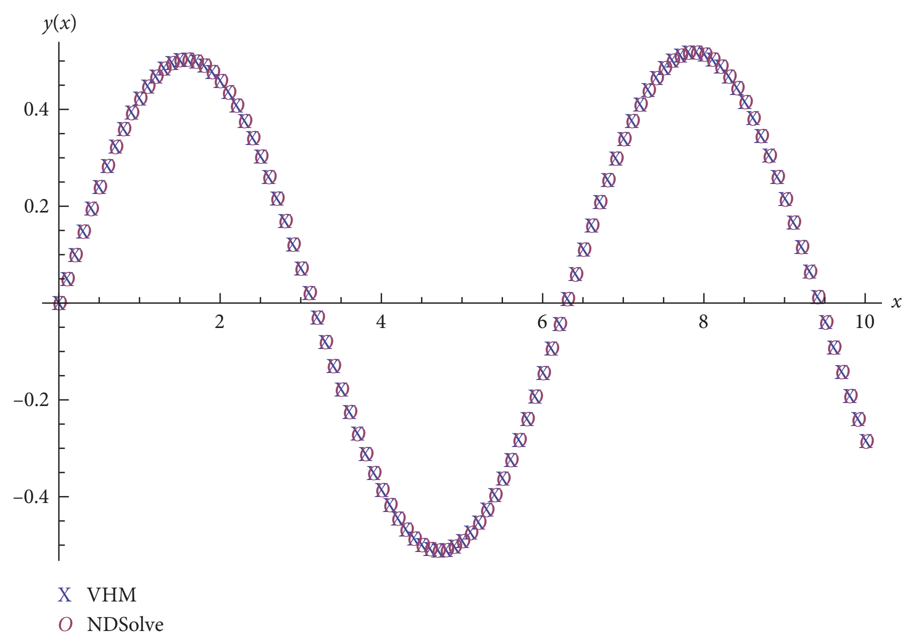

Figure 2: Solution of Van Der Pol oscillator using VHM and NDSolve.

$$y'' - 2\xi(1-y^2)y' + y = 0,\ 0 \le x \le 10, \tag{63}$$

$$y(0) = 0,\ y'(0) = 0.5, \tag{64}$$

where the parameter ξ shows the nonlinearity and the strength of damping with the value 0.005. It is a known fact that this problem does not have an exact solution. We compare the numerical solution obtained using our method implemented in the VHM and the in-built numerical scheme called up with NDSolve in *Mathematica* 12.0. Table 3 shows the solution obtained over the domain [0, 10] at varying points.

Figure 2 shows that the solution obtained using the VHM agrees with that obtained with NDSolve.

Problem 4. *The Kepler's problem is considered as discussed by Jator and King [19] and given by*

$$\begin{aligned} v_1'' &= -\frac{v_1}{(v_1^2+v_2^2)^{3/2}} - \frac{(2\epsilon+\epsilon^2)v_1}{(v_1^2+v_2^2)^{5/2}}, \\ v_2'' &= -\frac{v_2}{(v_1^2+v_2^2)^{3/2}} - \frac{(2\epsilon+\epsilon^2)v_2}{(v_1^2+v_2^2)^{5/2}}, \\ v_1(0) &= 1,\ v_1'(0) = 0, \\ v_2(0) &= 0,\ v_2'(0) = 1+\epsilon, \end{aligned} \tag{65}$$

whose exact solution is $v_1(t) = \cos(t+\epsilon t); v_2(t) = \sin(t+\epsilon t)$. The numerical results for this problem are obtained with $tol = 10^{-6}$ as shown in Table 4.

TABLE 4: Maximum absolute errors obtained for Problem 4.

Method	N	Max Err$\{v_1,v_2\}$	NF
	10	7.85×10^{-18}	70
VHM	8	0.00	56
	14	0.00	98
	8	3.49×10^{-12}	44
BHM [19]	16	6.94×10^{-12}	88
	32	1.92×10^{-12}	176

TABLE 5: Maximum absolute errors obtained for Problem 5.

Method	y_1–EMax	y_2–EMax	N	NF
VHM	$3.43E-14$	$3.88E-13$	16	112
VOPTBM	$2.66E-10$	$2.62E-10$	114	399
VJATOR	$4.13E-8$	$4.12E-8$	207	483
ode45	$4.44E-8$	$4.45E-8$	646	3877
ode113	$1.95E-6$	$1.95E-6$	207	419

For the proposed method, the initial step $h_{initial}$ taken determines the number of steps N. In Table 4, $h_{initial}$ used and N achieved are $h_{initial}=10^{-2}$, $N=10$, $h_{initial}=10^{-3}$, $N=8$, and $h_{initial}=10^{-4}$, $N=14$. It can be seen that the number of steps achieved in the proposed method and the maximum error are better than those by Jator and King [19].

Problem 5. *Consider the linear system of second order by Majid et al. and Singh and Ramos [2, 16].*

$$\begin{aligned} &y_1''(x)=-y_2(x)+\sin(\pi x),\\ &y_1(0)=0,\ y_1'(0)=-1,\\ &y_2''(x)=-y_1(x)+1-\pi^2\sin(\pi x),\\ &y_2(0)=1,\ y_2'(0)=1+\pi,\ x\in[0,10]. \end{aligned} \tag{66}$$

The exact solution is given as $y_1(x)=1-e^x$ and $y_2=e^x+\sin(\pi x)$. The numerical results of the problem are obtained for $h_{initial}=0.01$ with $tol=10^{-9}$ as shown in Table 5.

In Table 5, the VHM with a small number of integration subintervals produced a better result with small global error than other methods. This reveals clearly the efficiency of our method as compared to VOPTBM, VJATOR, and the MATLAB ODE solvers —ode45 and ode113.

6. Conclusion

A third derivative hybrid block method for the numerical solution of general second-order initial value problems has been derived in this paper and implemented in both variable and fixed step-size modes, respectively. First, the method is derived using the collocation approach at uniform off grid points and considering up to the third derivatives. A variable step-size formulation was obtained, seeking to improve the efficiency of the method. The examples presented displayed good performance of the proposed method which provides smaller errors and less number of steps as shown when compared with other methods in the cited literature.

Data Availability

The data used to support the findings in this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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