Neural Computing and Applications
https://doi.org/10.1007/s00521-024-09552-x

REVIEW

A systematic literature review of soft set theory

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Received: 29 September 2023 / Accepted: 22 January 2024

Abstract

Soft set theory, initially introduced through the seminal article "Soft set theory—First results" in 1999, has gained considerable attention in the field of mathematical modeling and decision-making. Despite its growing prominence, a comprehensive survey of soft set theory, encompassing its foundational concepts, developments, and applications, is notably absent in the existing literature. We aim to bridge this gap. This survey delves into the basic elements of the theory, including the notion of a soft set, the operations on soft sets, and their semantic interpretations. It describes various generalizations and modifications of soft set theory, such as N-soft sets, fuzzy soft sets, and bipolar soft sets, highlighting their specific characteristics. Furthermore, this work outlines the fundamentals of various extensions of mathematical structures from the perspective of soft set theory. Particularly, we present basic results of soft topology and other algebraic structures such as soft algebras and σ-algebras. This article examines a selection of notable applications of soft set theory in different fields, including medicine and economics, underscoring its versatile nature. The survey concludes with a discussion on the challenges and future directions in soft set theory, emphasizing the need for further research to enhance its theoretical foundations and broaden its practical applications. Overall, this survey of soft set theory serves as a valuable resource for practitioners, researchers, and students interested in understanding and utilizing this flexible mathematical framework for tackling uncertainty in decision-making processes.

Keywords Soft set · Fuzzy soft set · Soft topology · Data filling · Parameter reduction · Decision-making

1 Introduction

Despite the growing prominence of soft set theory, the existing literature is conspicuously lacking in a comprehensive overview of its development. In this paper, we fill this gap with an up-to-date and organized review of the bibliography on soft set theory. There is no doubt that its genesis can be traced back to an article by Molodtsov [179]. It is only appropriate to begin with some biographical notes. Then, in this section, we will show how this theory became popular, charting its evolution and uptake within the mathematical community and beyond. Afterward we will present a catalog of the key areas of development that we will overview and their main proponents. This Introduction concludes with an outline of our literature review.

1.1 Biographical notes

The founder of soft set theory, Professor Dmitri Anatol'evich Molodtsov passed away on December 4, 2020.

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Published online: 25 February 2024

We confirmed the death in an email exchange with Prof. Vladimir Tsurkov, the head of the department where Prof. Molodtsov worked.

Professor Molodtsov was a mathematician, born on June 2, 1949, in Moscow. His entry at the world biographical encyclopedia Prabook [1] explains that Professor Molodtsov was the son of Anatoliy and Julia (Sedova) Molodtsova. He earned an Honors Degree from Moscow University in 1971, and went on to obtain a Candidate of Science degree in 1974, and a Doctor of Science degree in 1990, both from Moscow University. He worked as a researcher at the Computer Center of the Russian Academy of Sciences in Moscow, from 1974 until his passing.

According to the second 2022 update of the zbMATH Open (formerly known as Zentralblatt MATH) interface, Professor Molodtsov produced 46 publications since 1972, including 2 books, with 6 coauthors: D. V. Kovkov (4 publications), V. V. Fedorov (2 publications), Santanu Acharjee (1 publication), V. M. Kolbanov (1 publication), V. Yu. Leonov (1 publication), and A. A. Sokolov (1 publication). Specifically, the All-Russian portal Math-Net.Ru lists 30 articles written by him.

We shall not describe his specific contributions to the field of mathematics here. Suffice to say that emerging from [179], soft set theory has become a thriving field of research. According to Google Scholar (consulted December 4, 2023), his article, titled "Soft set theory—First results," has been cited by more than 6650 scholarly works. The Science Direct site counts more than 3500 citations to [179] on December 4, 2023. From this vast literature, we can only hope to make an adequate selection of works in this survey.

Section 4.5 acknowledges Professor Molodtsov's attempt to launch soft probability too.

1.2 The development of a new theory

It took some time for the new theory to gain traction. Its first steps were slow, and we can safely conclude that it received little attention in its origins. To this end, let us examine the chronological development of its early years. We shall conclude that the use of this subject has steadily increased over the past two decades.

Both fuzzy soft sets and intuitionistic fuzzy soft sets were defined in 2001 by [169] and [170], respectively. The next year, [171] showed that soft sets may be applied to solve decision-making problems, and also, these authors proposed the idea of reducing soft sets (we should be aware of the fact that it was later reformulated by [81]). These new contributions went largely unnoticed at the time. However, they are now highly cited and gave rise to many different branches of the literature.

The year 2007 witnessed the introduction of a basic formulation of soft group theory [29, 30] and decision-making with fuzzy soft sets [199]. This year appears to be a watershed in the development of soft set theory. Until then, the number of Scopus citations to the seminal [179] did not exceed 3 per year. In 2008, the number of Scopus citations rose to 12, and it was 22 in 2009, the year when [131] defined soft p-ideals of soft BCI-algebras. Since 2012, the annual number of Scopus citations has not fallen below 100. At that time, some important guidelines had already been established. For evidence, note that many interesting problems such as data filling of incomplete soft sets [191, 256], utilization of soft set theory in association rules mining [121], hybridization with rough sets [63, 108], algebraic structures associated with soft sets [65], and soft topology [79, 210] were studied before 2012. It is safe to say that all these works contributed greatly to the success of soft set theory. In support of this assertion, it is worth noting that Google Scholar counts more than 1500 citations to [210] on December 4, 2023. It is for this reason that we dedicate a full section to an overview of soft topological spaces. Also [256] has over 600 citations in Google Scholar, a remarkable achievement that speaks to the interest in the topic of incomplete soft sets.

1.3 Basic elements of soft set theory: a brief description

What is the core content of soft set theory? There exists no updated, organized study establishing the standards of this theory. In fact, there is no systematic literature review for soft set theory, not even an outdated one. For newcomers to the subject, there are no organized presentations of its fundamental facts and achievements, the current state of the art, and promising lines for future research. These gaps are addressed in this paper.

The takeaway message of soft set theory is that concepts that hinge on the idea of "belongingness" can be extended by making them dependent on a set of parameters. In its inception, soft sets over a set were described not by one indicator function (as in the case of standard subsets), but by a multiplicity of indicators (one for each "attribute" pertaining to another reference set). The interpretation was that each attribute produced an "aproximate description" of the subset of elements that the soft set jointly describes [179], although explicit discussions of their semantic interpretation came much later [52, 235]. The basic operations with soft sets and their properties were described in [172] and later clarified in [66].

We should include [169] and [170] among the proponents of this field of research. Their merit was to prove that

[1] See https://prabook.com/web/dmitri.molodtsov/447824 (consulted February 22, 2024).

Molodtsov's idea could be combined with other successful concepts (fuzzy sets, intuitionistic fuzzy sets). Relationships among these models continued to come to light [46, 160]. Many other extensions and hybrid models have been developed ever since, inclusive of generalized intuitionistic fuzzy soft sets [106], bipolar soft sets [211] and fuzzy bipolar soft sets [183], probabilistic and dual probabilistic soft sets [102, 253], N-soft sets [100], hesitant fuzzy soft sets [73, 224] and intertemporal hesitant fuzzy soft sets [159], Dempster-Shafer fuzzy soft sets [94], m-polar fuzzy soft sets [244], ranked soft sets [208], soft rough sets and rough soft sets [108, 110], *et cetera*.

An important branch of the literature has to do with decision-making. The prioneer approach must be credited to [171] and [81], and also [199] launched decision-making in the hybrid fuzzy soft set framework, which was improved with [107] and [237]. [102] pioneered decision-making in (dual) probabilistic soft sets, and [106] did the same in generalized intuitionistic fuzzy soft sets. [138] and [92] overview these topics from a recent perspective. A distinctly different approach related to Weierstrass extreme value theorem is [50]. Soft topological knowledge is a prerequisite for this technique.

As a matter of fact, topology has certainly become the most successful extension of a mathematical structure with the quality of soft set theory. It was launched by [79] and [210], and further developed in a long series of papers including [39–41, 47–49, 60, 96, 125, 174, 175, 177, 184, 185, 202, 223], and [254]. Fuzzy soft topologies [86, 201, 214], soft metric spaces [1, 2, 90], and further generalizations have been defined and studied too. Other structures have been enriched with the idea of parametric belongingness, which gave raise to: (1) algebraic structures such as soft groups [29, 30], soft semirings [101], soft rings [5], etcetera; (2) set-theoretic structures such as soft ideals and filters in [206], or soft algebras and soft σ-algebras, respectively, defined in [198] and [136]; (3) extensions of convexity in finite environments [51], or (4) graph-theoretic models [11, 12, 139, 212].

Two other topics that are worth mentioning are data filling and parameter reduction. The first problem arose from the occurrence of missing data in the framework of soft set information. The model known as incomplete soft sets emerged from this situation. It is in this context that [256] and [191] started the problem of filling the missing data in order to put incomplete soft sets into practice. Different techniques have continued to appear both for this basic model [153–155, 205] and many of its extensions inclusive of fuzzy soft sets [91, 93], interval-valued fuzzy soft sets [109, 163] or interval-valued intuitionistic fuzzy soft sets [165, 190]. Although data filling has become a crucial topic in soft set theory, it has been argued that for the purpose of decision-making under incomplete soft information, this problem can be bypassed [54]. Concerning parameter reduction, it is appropriate to explain that the problem has been imported from the theory of rough sets. In both fields, the purpose is the simplification of the space of parameters, in such way that the ultimate goal (typically, decision-making) remains unaffected. The problem was stated very soon after the emergence of soft set theory, as explained above. [171] and then [81] established the main concept. We underline the role of [149] with a normal parameter reduction algorithm, and [151], with a concern for computational efficiency in the presence of large datasets. Reviews of this literature include [88] and [246]. In addition to this case, the problem has been approached from the perspective of fuzzy soft sets [75, 116, 137, 148], interval-valued fuzzy soft sets [163, 193], bipolar fuzzy soft sets [64], and N-soft sets [21].

This shortlist of general topics is far from being exhaustive. Rule mining from the perspective of soft set theory has been approached by [105, 157] among others. [161] have shown that soft set theory can be related to machine learning methods. These and many other articles have produced original approaches to other topics, that have gained generality with the inclusion of parametric membership.

1.4 Outline of the paper

By shedding light on the underexplored aspects of soft set theory, this first-ever survey consists of six sections. Section 2 summarizes three elements: the definition and fundamental operations of soft sets, a (non-exhaustive) list of extensions and variations, and the semantic interpretations of soft sets. Section 3 is dedicated to the fundamentals of soft topology, its foundational literature, and its relationship with topology. Other topics in soft set theory are overviewed in Sect. 4. These subjects include: other mathematical structures with a soft-set-based approach, data filling of incomplete soft sets, parameter reduction of soft sets and fuzzy soft sets, aggregation, plus an outline of soft probability. Section 5 is dedicated to decision-making. Section 6 concludes our survey and identifies potential areas for future research.

2 Preliminary concepts

Henceforward, X will denote a nonempty set (that is usually called the "universe of discourse"). A set E will be usually interpreted as a list of characteristics, properties, or attributes (although Sect. 2.3 explains that other interpretations are possible). When U is a set, $\mathcal{P}(U)$ will denote the set of parts of U.

This section has three parts. In the next Sect. 2.1, we review basic elements of soft set theory. Then, Sect. 2.2 summarizes a number of extensions of soft sets. We neither intend to give an exhaustive list nor formally describe all of them, because the current number of extensions is too high. The semantic interpretation of soft sets is the subject of Sect. 2.3.

2.1 Elements of soft set theory

We begin by explaining that two standard modelings of a soft set exist in the literature. Both are trivially equivalent.

1. Under the first presentation, soft sets on X are defined by a pair (F, E), the set E being formed by all the properties that characterize the members of the universe of discourse, and F is a function $F : E \longrightarrow \mathcal{P}(X)$. As presented by [179], a soft set on X can be described as a parameterized collection of subsets of X, the set of parameters being E.
2. A second presentation uses the fact that $\mathcal{P}(X)$ can be identified with $\{0,1\}^X$, which is the set of all functions from X to $\{0,1\}$. Indeed, every subset A of X is uniquely determined by $\chi_A : X \longrightarrow \{0,1\}$, which is its indicator or characteristic function. With this identification the soft set (F, E) can be interpreted as a mapping $F : E \longrightarrow \{0,1\}^X$. Under this presentation, if we observe $F(a)(x) = 1$ then we interpret that x satisfies property $a \in E$, and $F(a)(x) = 0$ means the opposite.

Whatever the presentation that we choose, a soft set on the universe of discourse X is simply a multi-function F from E —a set of properties that identify the alternatives– to X [177]. This apparently casual comment is in fact very important: for example, it is the key to transform concepts from the language of soft topology to the language of topology. This correspondence will be discussed in Sect. 3.3. Multi-functions or multifunctions are referred to as correspondences, point-to-set mappings, or multi-valued mappings in the specialized literature.

For this reason, a soft set (F, E) can also be regarded as a subset of the Cartesian product $E \times X$, namely, the graph of F. Recall that when $F : E \longrightarrow \mathcal{P}(X)$ is a multi-function, its graph is $Gr(F) = \{(e,y) | y \in F(e)\}$. So, there is an exact correspondence between (F, E) and $Gr(F)$.

In the literature, it is common to use the representation $\{(e, F(e)) : e \in E\}$ for the aforementioned (F, E). Note that this presentation is reminiscent of the graph representation explained above. When $e \in E$, $F(e)$ is a subset of X that is sometimes expressed with the more accurate notation (F, E)(e). It is referred to as the set of e-approximate elements of the universe of discourse, or alternatively, as the subset of the universe of discourse approximated by e.

Henceforth $SS_E(X)$ will represent the set of all soft sets on X. If E, the set of relevant characteristics of the elements of X, is common knowledge, we can drop the subindex and just use the notation $SS(X)$.

Two basic examples of soft sets are the full or absolute, and the null soft sets on X. The full soft set $\tilde{X}$ is such that $\tilde{X}(e) = X$ for each $e \in E$. In the interpretation by graphs, one has $\tilde{X} = E \times X$. The null soft set Φ is such that $\Phi(e) = \varnothing$ for all $e \in E$. So, in the interpretation by graphs, $\Phi = \varnothing \subseteq E \times X$. Soft points are special types of soft sets that we discuss in Sect. 3.

Operations from set theory were soon transferred to the theory of soft sets. Predictably, union, intersection, and inclusion within $SS_E(X)$ were defined in the following way [66, 172]: for any $(F_1, E), (F_2, E) \in SS_E(X)$,

(1) $(F_1, E) \sqcup (F_2, E)$ is $(F_3, E) \in SS_E(X)$, the soft set such that $F_3(e) = F_1(e) \cup F_2(e)$ for each $e \in E$. We can write this concept as $((F_1, E) \sqcup (F_2, E))(e) = F_1(e) \cup F_2(e)$ when $e \in E$.
(2) $(F_1, E) \sqcap (F_2, E)$ is the soft set $(F_4, E) \in SS_E(X)$ such that $F_4(e) = F_1(e) \cap F_2(e)$ for each $e \in E$. We can write this concept as $((F_1, E) \sqcap (F_2, E))(e) = F_1(e) \cap F_2(e)$ when $e \in E$.
(3) $(F_1, E) \sqsubseteq (F_2, E)$ means $F_1(e) \subseteq F_2(e)$ whenever $e \in E$.

$(F, E)^c$ holds for complement of $(F, E) \in SS_E(X)$, and it is $(F^c, E) \in SS_E(X)$ for which every $e \in E$ defines $F^c(e) = X \setminus F(e)$.

Remark 1 It is trivial to extend the union and intersection operations on soft sets to either finite or infinite lists of soft sets.

Relatedly, soft inclusion produces a natural idea of soft equality whereby $(F_1, E) = (F_2, E)$ is equivalent to both $(F_1, E) \sqsubseteq (F_2, E)$ and $(F_2, E) \sqsubseteq (F_1, E)$. Put shortly, soft equality of (F_1, E) and (F_2, E) boils down to $F_1(e) = F_2(e)$ for all $e \in E$.

The soft sets $(F, E), (F', E) \in SS_E(X)$ are disjoint if their intersection is the null soft set, i.e., if $(F, E) \sqcap (F', E) = \Phi$. And in this case, the two soft sets must be forcefully different, in the sense that they cannot be soft equal.

It is important to bear in mind that [66] corrected some wrong assertions given in [172]. This analysis produced new concepts (such as restricted intersection, union, and difference, plus extended intersection, of a pair of soft sets). Also, the reader should be aware that in addition to the definitions given above, other concepts of intersection, union, and inclusions have been defined. For more details, see [33] and [34].

2.2 Extensions of soft sets

Let us now outline the ideas that have produced extensions of the model presented in Sect. 2.1.

The addition of other feasible items to $\{0,1\}$ in the second modelization presented in Sect. 2.1 produces extended soft set models in three different directions, namely, incomplete soft sets, N-soft sets, and fuzzy soft sets.

1. Incomplete soft sets were introduced by Zou and Xiao [256] and later studied by authors such as Qin et al. [191] and Alcantud and Santos-García [54]. They are defined by replacing $\{0,1\}$ with $\{0,1,*\}$ as follows:

Definition 1 ([191]) The pair (F, E) is an *incomplete soft set* over X when $F : E \longrightarrow \{0,1,*\}^X$, Recall that $\{0,1,*\}^X$ represents the mappings from X to the set $\{0,1,*\}$.

So, if we observe $F(a)(x) =^*$, then we interpret that we do not know whether x satisfies property $a \in E$ or not.

2. Fatimah *et al.* [100] defined N-soft sets and provided the first examples with real data. This model uses $\{0,1,\ldots,N-1\}^X$ for the codomain, although the numbers are a convenient default and can be replaced with N distinctive items, and the triple (F, E, N) becomes an N-soft set. The formal concept has the following structure:

Definition 2 ([100]) Let $G = \{0,1,2,\cdots,N-1\}$ denote a set of grades, for some $N \in \{2,3,\ldots\}$. The triple (F, E, N) is an *N-soft set* on X when $F : E \rightarrow 2^{X\times G}$ meets the property that for every $a \in E$ and $x \in X$, there is a unique $(x, g_a) \in X \times G$ such that $(x, g_a) \in F(a)$, $g_a \in G$.

What N-soft sets add to the original model is the ability to differentiate among options that satisfy the properties in a variety of manners, which are captured by the grades in G. In addition to the real examples provided by the founding [100], other articles have shown the adequacy of N-soft sets to capture real situations, e.g., [52, 59, 61, 62].

From direct inspection, soft sets can be identified with 2-soft sets, and incomplete soft sets can be identified with 3-soft sets. Table 1 presents the tabular representation of these models.

Although [100] defined incomplete N-soft sets, there is virtually no research about this model.

Alcantud *et al.* [62], Sect. 1.1, presented four arguments that prove the superiority of N-soft sets over the original soft set model. Put briefly, these arguments are:

(a) Hesitation in N-soft sets is natural, but in soft sets, it is equivalent to allowing for incompleteness. Therefore new problems can be set up in the field of hesitant N-soft set.
(b) Aggregation of N-soft sets is natural (and we shall dedice Sect. 4.4 to this problem), but in soft sets, this topic has been disregarded due to its simplicity.
(c) Soft sets are confined to Aristotelian binary logic but N-soft sets have been linked to many-valued logic (v. Sect. 2.3 for more details on this issue).
(d) An ordinal improvement of soft sets is the ranked soft set structure defined in [208]. Then a cardinal improvement of this model produces N-soft sets. It can be said that ranked soft sets give us for each property in E, an ordered list of the alternatives, in such way that we can know for each pair of options which one performs better in terms of that property. The improvement brought by N-soft sets is that we can make comparisons that disclose how much better the alternatives are.

Table 1 Tabular representation of a general N-soft set

(F,E,N)	t_1	t_2	$\ldots$	t_q
o_1	r_{11}	r_{12}	$\ldots$	r_{1q}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
o_p	r_{p1}	r_{p2}	$\ldots$	r_{pq}

Each r_{jk} is in $\{0,1,\ldots,N-1\}$. When $N=2$, every element in the table is either 0 or 1, hence producing a soft set. When $N=3$, we can (by convention) interpret that 0 holds for 'false', 1 holds for 'true', and 2 holds for 'indeterminate'. This is a simple reformulation of an incomplete soft set that replaces the * symbol with 2

3. Fuzzy soft sets were presented by [169] in order to extend soft sets with the ability to capture partial membership. So with respect to the second modelization presented in Sect. 2.1, this model replaces $\{0,1\}$ with [0, 1]. And we can extend the tabular representation in Table 1 correspondingly. In formal terms:

Definition 3 ([169]) The pair (F, E) is a *fuzzy soft set* over X when $F : E \longrightarrow [0,1]^X$. Recall that the notation $[0,1]^X$ represents the set of all mappings from X to the interval $[0,1]$.

Under this extended presentation, the number $F(a)(x) \in [0,1]$ is interpreted as the degree of membership of x to the set of alternatives that satisfy property $a \in E$.

In this category we can also insert the probabilistic soft sets defined by [253]. Fatimah *et al.* [102] investigated them, introduced dual probabilistic soft sets, and—especially important—gave the first decision-making methodologies in both frameworks. In formal terms:

Definition 4 ([253]) Denote by $D(X)$ the set of all probability distributions over X. The pair (F, E) is a *probabilistic soft set* over X when $F : E \to D(X)$.

Fatimah *et al.* [102] defined one-probabilistic soft sets: these are the particular cases of Definition 4 such that for every $a \in E$, a unique $x \in X$ exists with $F(a)(x) = 1$ (therefore $F(a)(x') = 0$ when $x \in X \backslash \{x\}$). With this new tool, their Proposition 1 drew a bridge among probability distributions, probabilistic soft sets, and soft sets. As said above, their work proposed the next concept:

Definition 5 ([102]) In the conditions of Definition 4, the pair (F, E) is a *dual probabilistic soft set* over E when $F : X \to D(E)$, i.e., for each $x \in X$, $F(x) \in \mathscr{D}(E)$.

The authors explained that this dual model is motivated by examples such as behavioral phenotypes or in general, by cases where the characteristics classify the alternatives

In addition, m-polar fuzzy soft sets [139, 244] were investigated for the first time to deal with multi-polar data.

Other authors used alternative strategies for extending the reach of soft sets. For example, hesitant fuzzy soft sets [73, 224] added the ability to show hesitation in the case of fuzzy soft sets. Ashraf *et al.* [70] have recently defined complex probabilistic hesitant fuzzy soft sets which extend the probabilistic hesitant fuzzy soft set model introduced by [158]. A different type of extensions came from bipolar soft sets. They were defined by [211] and then [36] improved the knowledge about their role in the expression of dual thinking. Bipolar soft sets are built with the help of two soft sets that provide positive and negative information. For each attribute, this information cannot overlap. In formal terms:

Definition 6 ([211]) The triplet (F, F', E) is a *bipolar soft set* over X when $F : E \to \mathcal{P}(X)$, $F' : \neg E \to \mathcal{P}(X)$, and the condition $F(a) \cap F'(\neg a) = \varnothing$ is met for each $a \in E$. Here $\neg E$ denotes the NOT set of E, defined by the negation of the properties in E: $\neg E = \{\neg a \,|\, a \in E\}$.

Their fuzzy version (fuzzy bipolar soft sets) was given in [183]. This model has been generalized until [168] proposed bipolar complex fuzzy soft sets. In a related line of inspection, intuitionistic fuzzy soft sets [170] and generalized intuitionistic fuzzy soft sets [106, 170] split membership and non-membership in the evaluation of the satisfaction of the characteristics. By extending the bipolar scale into a multi-polar scale (i.e., to the space of m different categories or states) a new concept of m-polar fuzzy soft set, investigated by [244], was introduced. [188] defined Pythagorean fuzzy soft sets with the aim of expanding the set of admissible evaluations in intuitionistic fuzzy soft sets. This idea has been further developed until [42] defined (a, b)-fuzzy soft sets. An alternative expansion came from [239], who designed picture fuzzy soft sets with the help of a third "neutral" evaluation. Also, this model witnessed a generalized version by [145], and a further improvement came from [127] who designed multi-valued picture fuzzy soft sets. Vague sets [114] are the germ of vague soft sets [82, 84, 232]. Alkhazaleh and Salleh [67] introduced soft expert sets and fuzzy soft expert sets [68]. Their novelty is arguably limited: they incorporate the opinions by a group of experts in one unique structure. Nevertheless the later model was eventually extended by e.g., m-polar fuzzy soft expert sets [22], picture fuzzy soft expert sets [222], Fermatean fuzzy soft expert sets [24], or hesitant fuzzy soft expert sets [17].

Extensions of N-soft sets abound nowadays too. The hybridization of fuzzy soft sets and N-soft sets led [18] to define fuzzy N-soft sets. [99] contributed with the multi-fuzzy N-soft set model that extends the multi-fuzzy soft set notion defined by [237]. In addition, N-soft sets were amplified with the help of hesitancy too. This was a remarkable improvement, because hesitancy in the strict framework of soft sets reduced to producing incomplete soft sets. The extension that emerged was called hesitant N-soft set in [19]. Informally, if we start with Table 1 as a joint representation of soft sets, incomplete soft sets, and N-soft sets, then the hesitant extension uses a multiplicity of values from $\{0, 1, \ldots, N-1\}$ at each cell. Relatedly, the combination of both fuzzy and hesitant generalization produces hesitant fuzzy N-soft sets [20]. Zhang *et al.* [249] defined Pythagorean fuzzy N-soft sets and designed multi-attribute group decision-making methods in this context (see also [28]), whereas [194] defined picture fuzzy N-soft sets. Wang *et al.* [231] defined probabilistic hesitant N-soft sets which have the ability to express the occurrence probability of hesitant grades. They consider group decision-making methodologies (such as TOPSIS and VIKOR) in this framework. Akram *et al.* [26] have designed the complex Fermatean fuzzy N-soft set model, [13] have established the complex m-polar fuzzy N-soft model, and [71] have defined complex probabilistic hesitant fuzzy N-soft sets. [134] defined bipolar N-soft set theory, whereas [23] extended their idea with the introduction of bipolar fuzzy N-soft sets. The amalgamation with soft expert knowledge has been formulated too, and it has given rise to spherical and Pythagorean fuzzy N-soft expert sets [25, 27].

2.3 Semantic interpretations

Yang and Yao [235] have been the first authors to discuss the semantics of soft sets explicitly. Their contribution was supplemented with a pioneering analysis of three-way decision in the framework of soft sets. Both ideas were extended to the N-soft set arena in the recent Alcantud [52]. Let us summarize the contributions to this debate given in both articles:

- The 'multi-context' semantics is surely the original semantic interpretation of both soft sets and N-soft sets. Soft sets offer taxonomies since they describe the options in terms of their attributes. Other authors [105] had explored this idea in more depth than the original [179]. There is no real difference with the more modern idea of N-soft set.
- The 'possible worlds' semantics (of both soft sets and N-soft sets) is applied when the set of characteristics is made of possible worlds for explication of a partially-known notion. This meaning is reminiscent of Savage's 'states of nature' [209], as reported in [52]. So, in the first version provided by Yang and Yao, precisely one of the possible worlds is the world that defines the set of occurrences of the notion. In Savage's adapted version, states of nature capture future events whose probabilities are unknown, and the decision-maker cannot affect them.

Still there is a third semantic interpretation in the generalized model of N-soft sets. This is the 'values of truth' semantics described in Sect. 3.3 of [52]. In cases where we cannot decide whether an option has a property or not (such as movies that cannot be called 'funny', but we cannot say that they 'are not funny' either), binary logic should be replaced with multi-valued logic. And the rates in an N-soft set can be interpreted as the possible values of truth in an N-valued logic. Under this description, N-soft sets extend incomplete soft sets, which only admit one level of indeterminacy. Or as mentioned above, incomplete soft sets can be identified with 3-soft sets.

3 Soft topology: first results

Topology has become a renowned mathematical discipline whose origins are in the early 1900 s, when it was simply a part of set theory. In fact, Felix Hausdorff introduced the name "topological space" in 1914. As an evidence that decision-making can benefit from topological ideas, the Weierstrass Extreme Value Theorem comes to mind easily. We shall return to this line of inspection in Sect. 5.

We have argued above that sets are "extended" to soft sets via parameterized belongingness. It is therefore only natural to expect that the insights provided by topology would be extended to produce a recognizable part of soft set theory. Indeed, with the help of the extended set-theoretic operations described in Sect. 2.1, one can define soft topology on X as follows:

Definition 7 ([79, 210]) A *soft topology* τ on X is a collection $\tau \subseteq SS_E(X)$ of soft sets on X (with a set of attributes formed by E), called *soft open sets*, such that:

(1) Both Φ and $\tilde{X}$ are soft open sets.
(2) Arbitrary unions of soft open sets are soft open sets too.
(3) Finite intersections of soft open sets are soft open sets too.

The triple (X, τ, E) is called soft topological space, and $ST_E(X)$ defines the set of all soft topological spaces on X (whose set of attributes is formed by E).

As explained in the Introduction, the expansion of soft topology has been considerable. Hence we can only attempt to summarize a small proportion of the many articles that it has produced. Instead, our main purpose in this part of our survey is to emphasize the correspondence between critopological and soft topological concepts.

One apparent difference between crisp (or ordinary) and soft topologies that one can perceive easily is the basic notion of "point". In set theory and topology, there is no argument about what a point can mean. However, things are different in soft set theory. References [90, 185], and [254] defined a concept of soft point that was very useful to analyze soft neighborhood systems and soft interior points. However both soft point [72] and element [184] were defined differently, and the names change with the authors. Alcantud [48] discusses this issue at length. Here is a short summary:

1. The special soft set $(\{y\}_e, E)$ such that $\{y\}_e(e) = \{y\}$ and $\{y\}_e(e') = \varnothing$ when $e' \in E \backslash \{e\}$ was used by [90] under the term 'soft point', and [184], Definition 3.1, referred to it as a 'soft element'.
2. In [254], a soft point is a soft set (F, E) with the property that $e \in E$ exits for which $F(e) \neq \varnothing$, and if $a \in E$, $a \neq e$, then $F(a) = \varnothing$. The previous notion is a particular case of this concept.
3. In [223] and in [210], a soft point is a soft set (F, E) with the property that there exists $y \in X$ such that $F(e) = \{y\}$ for each $e \in E$. This soft set is expressed as (y, E).
4. The previous definition is generalized by [72], Definition 2.11, who nevertheless use the same name (soft point). In [72], soft points are (G, E) for which $y \in X$ and $E' \subseteq E$ exist, such that $G(e) = \{y\}$ when $e \in E'$. The soft sets $(\{y\}_e, E)$ given above satisfy this definition too.

3.1 Construction of soft topologies

In this overview, it is worth noting that there are at least two general procedures for the generation of soft topological spaces from topologies on the universe of discourse, in the standard sense.

Procedure 1 is quite direct:

Definition 8 ([47, 223]) Fix $\Sigma = \{\Sigma_e\}_{e \in E}$, a set of topologies on X whose set of indices is E. We say that

$$\tau(\Sigma) = \Big\{ \{(e, G(e)) : e \in E\} \in SS_E(X) \quad \text{with the property } G(e) \in \Sigma_e \text{ for all } e \in E \Big\} \tag{1}$$

is the soft topology on X defined from Σ.

In the case where Σ_e is independent from e, i.e., when there is Σ with $\Sigma = \Sigma_e$ for all $e \in E$, the above notation is simplified to $\tau(\Sigma) = \tau(\Sigma)$.

Example 1 When Σ is the cofinite crisp topology on X, Definition 8 produces the cofinite soft topology [47].

Interestingly, [39] studied $\tau(\Sigma)$ but called it extended soft topology. In addition, a soft topology τ is enriched when τ includes every (F, E) with the property $F(e) \in \{X, \varnothing\}$ whenever $e \in E$. Theorem 2 of [39] proved an equivalence between extended and enriched soft topologies.

Procedure 2 constructs soft topological spaces using crisp topologies with the help of soft open bases. These are sets whose elements are soft sets, that produce soft topologies which are formed with all their soft unions. More explicitly, the soft topology τ has an soft base $\mathcal{B} \subseteq \tau$ when every $(F, E) \in \tau$ is a union of soft sets in $\mathcal{B}$ [79]. From a soft open base, we can gemerate a soft topology, for which the soft open base is in fact a soft base: v., [202], Theorem 16.

With each base for a topology on the universe of discourse, it is now possible to generate a soft topology in a two-step process. The base gives rise to a soft open base (for the definition, see Proposition 1 in [47]). And Theorem 13 in [202] explains how to generate a soft topology with this soft open base.

Additionally to the production of soft topologies from standard topologies, soft topologies can be combined to generate new soft topological spaces. This goal can be achieved for example, through the sum of soft topological spaces [41].

3.2 First results and more concepts in soft topological spaces

The two constructions of soft topologies that we have recalled above can be related to each other. If we select a base for a topology, then we can define both a soft open base and a crisp topology from it. Definition 8 can be applied to the new topology. We can define a soft topology from the soft open base too. The respective soft topologies obtained from these procedures coincide [47, Theorem 3].

We can draw theoretical consequences from these two designs. For example, let us define that a soft topological space is soft second-countable when the soft topology has a base that contains a countable number of elements [90, Definition 4.32]. Then Definition 8 generates a soft second-countable soft topological space when E is either finite or countable, and all Σ_e's are second-countable [47, Sect. 4]. And the reverse of this result is also true [47, Corollary 2].

Other concepts in soft topology are motivated by separation axioms, which are defining properties that attempt to "separate" either sets or points (or sets and points), in a "topological" sense. Not surprisingly, separation axioms have been reformulated in the soft topological framework in different ways by many authors (e.g., [38, 96, 125, 210, 223]). For example, the soft points defined by [210] and [223] are the key to define the next axioms:

Definition 9 ([210, 223]) The soft topology τ on the universe of discourse X is:

1. T_0 when for each distinct $x, y \in X$, either there is $(G, E) \in \tau$ such that $(x, E) \sqsubseteq (G, E)$ but it is not true that $(y, E) \sqsubseteq (G, E)$, or there is $(F, E) \in \tau$ such that $(y, E) \sqsubseteq (F, E)$ but it is not true that $(x, E) \sqsubseteq (F, E)$.
2. T_1 when for each distinct $x, y \in X$, there are $(F, E), (F', E) \in \tau$ such that $x \in (F', E)$, $y \notin (F', E)$, and also, $y \in (F, E)$, $x \notin (F, E)$.
3. T_2 when for all distinct $x, y \in X$, disjoint soft open sets $(F, E), (F', E)$ exist with the properties $x \in (F', E)$, $y \in (F, E)$.

Many other related axioms were defined and studied, for example, [175] discussed soft regularity. We shall not produce a comprehensive presentation here.

Soft compactness was first studied in [72] and [254]. Other remarkable references are Al-shami [37] and Al-shami *et al.* [40]. It is worth noting that 7 generalized types of soft semi-compact spaces were defined by these authors.

Another classical topic in topology is the analysis of separability axioms. They have motivated Alcantud [47] to construct soft topological spaces that are well-behaved in relation to these ideas. In addition, two classes of axioms generalizing suitable soft separability ideas were introduced by [60]. Their inspiration was the countable chain condition and the topological concept of caliber. In their article, the role of cardinality in both the countable chain condition and the finite chain condition is deeply investigated. Also, Alcantud *et al.* [60] prove that both calibers (their cardinality being fixed), the countable chain condition, and the finite chain condition, are topological properties (i.e., they are preserved by bijective soft continuous functions).

We also mention that [147] have defined soft Menger spaces in their pioneering analysis of selection principles in the soft framework. Soft compact spaces are soft Menger spaces, which in turn are soft Lindelöf spaces.

Finally, [74] have recently associated a fundamental group with each soft topological space.

3.3 The fundamental theorem of soft topology

Matejdes [174] soon realized that a bijective correspondence exists between the set of soft topological spaces (with characteristics E) defined on a set X, and the set of topological spaces either on $E \times X$ or on $X \times E$ (see also [176]). Put shortly, if we start with a soft topology, then by identifying a multifunction with its graph, we find ourselves in a classical topology on a Cartesian product.[2] In this section we recall the explicit bijection that has been used in [48] to prove this remarkable result, that can be rightfully called the fundamental theorem of soft topology.

To make this section self-contained, the next elements are needed.

Definition 10 With any $S \subseteq X \times E$ we associate $(\Upsilon_S, E) \in SS_E(X)$ defined by: for all $e \in E$, $\Upsilon_S(e) = \{y \in X \mid (y, e) \in S\}$. And with any $(F, E) \in SS_E(X)$ we associate $A_{(F,E)} = \{(y, e) \in X \times E \mid y \in F(e)\}$, a subset of $X \times E$.

These simple notions allow us to associate a soft topology to any crisp topology, and a crisp topology to any soft topology too. The next results formalize both constructions:

Proposition 1 *When Σ is a standard topology on $X \times E$, $\tau^\Sigma = \left\{(\Upsilon_A, E) \mid A \in \Sigma\right\}$ is a soft topology on X.*

If τ' is a soft topology on X, then $\Sigma(\tau') = \left\{A_{(G,E)} \mid (G, E) \in \tau'\right\}$ is a standard topology on $X \times E$.

We are ready to state the fundamental theorem of soft topology:

Theorem 1 ([48, *174*, *176*])

$$T : \{\text{ Topological spaces on } X \times E\} \longrightarrow ST_E(X)$$
$$(X \times E, \Sigma) \qquad \tau^\Sigma$$

defines a bijective function with inverse function

$$T^{-1} : ST_E(X) \longrightarrow \{\text{ Topological spaces on } X \times E\}$$
$$\tau \qquad (X \times E, \Sigma(\tau))$$

Since [174] it is known that with this theorem one can transform results from general topology to soft topology (see also [48, 60, 175] for more arguments supporting this claim). Nevertheless, one can also argue that this process does not exhaust all possible developments in soft set theory because the soft topological framework is semantically richer (as shown by our discussion about soft points given above). We do not intend to settle this controversy here. However, we feel obliged to draw the attention of researchers to this fundamental transformation so that forthcoming studies can give more explicit arguments about what sets them apart from concepts known from classical topology.

3.4 Extensions of soft topologies

Many combinations of topological models with generalized soft set theory have produced further generality. For example, the incorporation of soft sets to fuzzy topological spaces [80] led [218] to design fuzzy soft topologies. Khameneh *et al.* [140] studied the concepts of fuzzy soft interior, closure, and boundary in fuzzy soft topological spaces. Later, fuzzy soft product topologies and fuzzy soft Hausdorff spaces were investigated by [141]. These authors discussed the notion of fuzzy soft point and redefined it as an extension of the fuzzy point. Based on [141], Definition 6.1, a fuzzy soft set $\tilde{x}_E$, given by the map $\tilde{x} : E \longrightarrow I^X$ where $I = [0, 1]$, is a fuzzy soft point if for all $z \in X$, any $e \in E$,

$$\tilde{x}_e(z) = \begin{cases} \lambda_e & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$$

that $\lambda_e \in (0, 1]$ for each $e \in E$. In other words, for each parameter e, y^{λ_e} is a typical fuzzy point whose support is y and whose degree of membership is λ_e.

Alcantud [49] has produced the first investigation of relationships between soft topologies and fuzzy soft topologies.

More generally, [196] benefitted from the generalized approach by N-soft sets in order to design N-soft topologies. In addition, [98] defined bipolar soft topological spaces and [195] defined bipolar fuzzy soft topological spaces. Besides, [197] introduced hesitant fuzzy soft topological spaces.

A different type of generalizations has been motivated by the success of topological ordered spaces. With this spirit, [35] have designed their soft topological counterpart.

Sect. 4.1.3 discusses other extensions of soft topologies that are born from the structure of sets called "primal".

[2] We owe this crucial remark to a personal communication with the author.

It is worth underlining that the aforementioned [143] and [196, 197] state applications to decision-making.

4 Other topics in soft set theory

This section presents some other ideas that have amplified the soft set narrative from different perspectives. First, we shall underline that many other mathematical structures have been extended with a soft set approach. We have made a short introduction to a sample of recent trends, which may be especially beneficial for the reader due to their novelty. Afterward, we present brief summaries of several branches of the literature with varying level of importance. Data filling in the case of incomplete soft sets, as well as parameter reduction, have produced a number of impactful scholarly works. Meanwhile, topics such as aggregation or the theory of soft probability are less developed at present. Soft graphs [11] and their extensions to fuzzy soft graphs [12] and intuitionistic fuzzy soft graphs [212] are beyond the scope of this summary of literature.

4.1 Other soft algebraic structures

In addition to topology, algebraic structures such as groups or (semi)rings have been exported to the soft setting. These extensions produced soft groups [29, 30], soft semirings [101], and soft rings [5]. Feng et al. [104] demonstrated the applicability of soft binary relations to the theory of semigroups and [248] defined soft rough hemirings, which they utilized for multi-attribute multi-person decision-making. Afterward, the algebraic structures called soft topological soft groups/rings [216] and soft topological rings [217] were proposed. Still the reader can explore other topics like fuzzy soft Lie algebras [16], (intuitionistic) fuzzy soft K-algebras [14, 15], the soft ideals and filters that extend the corresponding crisp notions [206], characterization of hemirings in terms of fuzzy soft h-ideals [240], soft BL-algebras defined from fuzzy sets [247], or the introduction to N-soft algebraic structures (e.g., N-soft groups or N-soft rings and ideals) given by [133].

Recent contributions to extended mathematical structures include soft generalizations of algebras and σ-algebras, and convex geometries. Soft grills, soft primals, and many others have been considered too. We proceed to recall the respective rudiments of these new concepts.

4.1.1 Soft algebras and soft σ-algebras

Measures assign numerical values to sets. In Lebesgue integration, measures are fundamental for defining integrals, enabling the computation of quantities such as areas or volumes. In probability theory, they help define probability spaces, providing a foundation for analyzing random events and calculating probabilities for different outcomes. The right structured framework for defining measurable sets within a given space is a σ-algebra. Therefore this idea is pivotal in both mathematical analysis and probability theory. Let us recall this notion and the weaker structure of a set algebra:

Definition 11 An algebra $\mathbf{A}$ on X, a non-empty set, is a collection $\mathbf{A} \subseteq \mathcal{P}(X)$ that satisfies:

(1) $\varnothing \in \mathbf{A}$.
(2) $\mathbf{A}$ is closed under complement: $X \backslash A \in \mathbf{A}$ whenever $A \in \mathbf{A}$.
(3) $\mathbf{A}$ is closed under finite unions: for each $k \in \mathbb{N}$ and $A_1, \ldots, A_k \in \mathbf{A}$, $\bigcup_{i=1}^{k} A_i \in \mathbf{A}$.

A σ-algebra on X is an algebra on X that is closed under countable unions, i.e., (3) above is replaced with

(3') when $A_i \in \mathbf{A}$ $(i = 1, 2, 3 \ldots)$, it must be the case that $\bigcup_{i=1}^{+\infty} A_i \in \mathbf{A}$.

A routine application of De Morgan's laws proves that algebras must be closed under finite intersections. And σ-algebras must be closed under countable intersections too.

These two concepts have been extended in a natural way to the soft framework, hence producing soft algebras and soft σ-algebras.

Definition 12 ([198]) A soft algebra $\mathcal{A}$ on X is a collection $\mathcal{A} \subseteq SS_E(X)$ with the properties:

(A.1) $\Phi \in \mathcal{A}$.
(A.2) When $(G, E) \in \mathcal{A}$, $(G, E)^c \in \mathcal{A}$.
(A.3) When $k \in \mathbb{N}$ and $(F_1, E), \ldots, (F_k, E) \in \mathcal{A}$, it must be the case that $\sqcup_{i=1}^{k} (F_i, E) \in \mathcal{A}$.

Soft σ-algebras are defined in [136] by replacing (A.3) above with the stronger requirement

(A.3') when $(F_i, E) \in \mathcal{A}$ for al $i = 1, 2, \ldots$,then $\sqcup_{i=1}^{+\infty} (F_i, E) \in \mathcal{A}$.

Relationships between soft and ordinary algebras have been recently studied by [69]. Also recently, relationships between soft and ordinary σ-algebras have come to light in [44].

Extensions include picture fuzzy soft σ-algebras [182] and q-rung orthopair fuzzy soft σ-algebras [118].

4.1.2 Convex soft geometries

Convex soft geometries were defined in Alcantud [51] by inspiration of convex geometries. This interesting structure bridges the gap between convexity and finiteness, hence for this model, X is required to be finite. Then a convex geometry is a combinatorial abstraction of the idea of convexity, for which not only the standard techniques from convexity apply, but also one can resort to the theory of ordered sets and graph theory. Although there are several equivalent definitions, we only present the next one here:

Definition 13 ([95]) $\mathbf{C} \subseteq \mathcal{P}(X)$ is a convex geometry on X, a non-empty finite set, if:

(1) $\varnothing \in \mathbf{C}$.
(2) $G_1 \cap G_2 \in \mathbf{C}$ whenever $G_1, G_2 \in \mathbf{C}$.
(3) If $G_1 \in \mathbf{C}$, $G_1 \neq X$, then there exists $y \in X \backslash G_1$ with $G_1 \cup \{y\} \in \mathbf{C}$.

Then the convex sets of X are the subsets of X that are in $\mathbf{C}$.

Section 2 of [95] proves various characterizations of this model.

A mandatory construction is the convex hull of a set in this framework. It is defined as follows: for any convex geometry $\mathbf{C}$ on X, the convex hull of $X' \subseteq X$ in $\mathbf{C}$ is $\operatorname{conv}(X') = \bigcap\{G \in \mathbf{C} \text{ such that } X' \subseteq G\}$.

An important result in the basic theory of convex geometries is that there are extreme elements in all nonempty subsets of X. They are defined as follows:

Definition 14 ([95]) In the conditions of Definition 13, if $\varnothing \neq Y \subseteq X$, then $x \in Y$ is extreme element of Y when $x \notin \operatorname{conv}(Y \setminus \{x\})$.

In our framework one has the following extension of Definition 13:

Definition 15 (Alcantud [51]) $\mathcal{C} \subseteq SS_E(X)$ is a convex soft geometry on X if:

(C.1) $\Phi \in \mathcal{C}$.
(C.2) When $(G_1, E), (G_2, E) \in \mathcal{C}$ then $(G_1, E) \sqcap (G_2, E) \in \mathcal{C}$.
(C.3) When $(G, E) \in \mathcal{C} \setminus \tilde{X}$, there is $y \in X$ for which $(y, E) \sqsubseteq (G, E)$ is not true, and $(y, E) \sqcup (G, E) \in \mathcal{C}$.

The soft $\mathcal{C}$-convex sets, or simply soft convex sets for simplicity, are the members of $\mathcal{C}$.

Sections 3.2 and 3.3 in [51] discuss the construction of convex geometries from convex soft geometries, and the reverse process. To prove that the theory of convex soft geometries is meaningful and promising, we recall that with respect to any convex soft geometry, extreme elements exist for all soft sets (except for Φ) by Theorem 2 of [51]. In this short summary we omit the soft variations of the concepts involved in this statement. Suffice to say that this result is a non-trivial extension of the corresponding theorem in the theory of convex geometries mentioned above. Its proof relies on a non-trivial generalization to the soft setting of the anti-exchange property of convex geometries [51, Theorem 1].

Section 5 in [51] gives a long research program for this topic.

Finally in this section, we note that [55] have used convex geometries to design the first valid combination of convexity and rough set theory in a finite setting.

4.1.3 Soft extensions of filters, ideals, primals and grills

Filters and ideals are useful mathematical structures that found applications across the field of soft set theory too. Filters originate with topology, providing a unified concept of limit accross topological spaces. But they are important in order theory, set theory, model theory, mathematical analysis, or lattice theory too. Intuitively, they are commonly used to describe the subsets that are "large enough" to contain points that might be difficult to write down. The dual notion of a filter is an ideal. Therefore ideals describe "negligible" or "sufficiently small" subsets in set theory. The formal definitions are as follows:

Definition 16 A collection $\varnothing \neq \mathbf{F} \subseteq \mathcal{P}(X)$ is a filter on X if:

(1) $\varnothing \notin \mathbf{F}$.
(2) $F_1 \cap F_2 \in \mathbf{F}$ whenever $F_1, F_2 \in \mathbf{F}$.
(3) If $F \in \mathbf{F}$, $G \subseteq X$, and $F \subseteq G$ then $G \in \mathbf{F}$.

Thus a filter is a collection of non-empty sets, that is closed under finite intersections (2) and the superset operation (3). If in addition, it is closed under countable intersections, then it is a σ-filter, or countably complete filter.

As an example, in a topological space, the neighborhood system of any point (i.e., the collection of all subsets such that the point is in their topological interior) is a filter.

Definition 17 A collection $\varnothing \neq \mathbf{I} \subseteq \mathcal{P}(X)$ is an ideal on X if:

(1) $X \notin \mathbf{I}$.
(2) $I_1 \cup I_2 \in \mathbf{I}$ whenever $I_1, I_2 \in \mathbf{I}$.
(3) If $I \in \mathbf{I}$, $G \subseteq X$, and $G \subseteq I$ then $G \in \mathbf{I}$.

Thus an ideal is a collection of subsets of X that is closed under finite unions (2) and the subset operation (3). If in addition, it is closed under countable unions, then it is a σ-ideal.

As an example, the set of μ-negligible sets (or sets with null μ-measure) is a σ-ideal when μ is a measure on (X, Σ) and Σ is a σ-algebra on X.

Soft ideals were defined by [242], whereas soft filters appeared in [206]. Their formal definitions follow:

Definition 18 A collection of soft sets $\mathcal{F} \subseteq SS_E(X)$ is a soft filter on X if:

(1) $\Phi \notin \mathcal{F}$.

(2) The collection $\mathcal{F}$ is closed under finite intersections: $(F_1,E)\sqcap(F_2,E)\in\mathcal{F}$ whenever $(F_1,E),(F_2,E)\in\mathcal{F}$.
(3) The collection $\mathcal{F}$ is closed under the superset operation: when $(F_1,E)\in\mathcal{F}$ and $(F_1,E)\sqsubseteq(F_2,E)$ then $(F_2,E)\in\mathcal{F}$.

Definition 19 A collection of soft sets $\mathcal{I}\subseteq SS_E(X)$ is a soft ideal on X if:

(1) $\tilde{X}\notin\mathcal{I}$.
(2) The collection $\mathcal{I}$ is closed under finite unions: $(F_1,E)\sqcup(F_2,E)\in\mathcal{I}$ when $(F_1,E),(F_2,E)\in\mathcal{I}$.
(3) The collection $\mathcal{I}$ is closed under the subset operation: when $(F_1,E)\in\mathcal{I}$ and $(F_2,E)\sqsubseteq(F_1,E)$ then $(F_2,E)\in\mathcal{I}$.

Al-shami *et al.* [43] have inaugurated the study of the next concept in the vein of soft ideals:

Definition 20 ([43]) $\mathcal{F}\subseteq SS_E(X)$ is a soft primal on X when:

(Pr.1) $\tilde{X}\notin\mathcal{F}$.
(Pr.2) When $(F_1,E)\in\mathcal{F}$ and $(F_1,E)\sqsubseteq(F_2,E)$ then $(F_2,E)\in\mathcal{F}$.
(Pr.3) If $(F_1,E)\sqcap(F_2,E)\in\mathcal{F}$, then either $(F_1,E)\in\mathcal{F}$ or $(F_2,E)\in\mathcal{F}$.

This concept is the counterpart of set-theoretic primals, defined as follows:

Definition 21 A collection $\varnothing\neq\mathbf{F}\subseteq\mathcal{P}(X)$ is a primal on X if:

(1) $X\notin\mathbf{F}$.
(2) If $F\in\mathbf{F}$, $G\subseteq X$, and $G\subseteq F$ then $G\in\mathbf{F}$.
(3) If $F_1\cap F_2\in\mathbf{F}$ with $F_1,F_2\subseteq X$, then either $F_1\in\mathbf{F}$ or $F_2\in\mathbf{F}$.

In relation with Sect. 3.4, we emphasize that [6] and [31] defined and studied primal topological spaces. In a similar manner, the notion in Definition 20 allowed [43] to study primal soft topologies.

Primals have appeared as the dual concept of grills [6, Theorem 3.1] whose definition follows:

Definition 22 ([85]) A collection $\varnothing\neq\mathbf{G}\subseteq\mathcal{P}(X)$ is a grill on X if:

(1) $\varnothing\notin\mathbf{G}$.
(2) If $F\in\mathbf{G}$, $G\subseteq X$, and $F\subseteq G$ then $G\in\mathbf{G}$.
(3) If $G_1\cup G_2\in\mathbf{G}$ with $G_1,G_2\subseteq X$, then either $G_1\in\mathbf{G}$ or $G_2\in\mathbf{G}$.

Therefore it is unsurprising that Definition 20 bears comparison with the structure defined by soft grills in the following way:

Definition 23 $\mathcal{G}\subseteq SS_E(X)$ is a soft grill on X when:

(Gr.1) $\Phi\notin\mathcal{G}$.
(Gr.2) If $(G_1,E)\in\mathcal{G}$ and $(G_1,E)\sqsubseteq(G_2,E)$ then $(G_2,E)\in\mathcal{G}$.
(Gr.3) If $(G_1,E)\sqcup(G_2,E)\in\mathcal{G}$, then either $(G_1,E)\in\mathcal{G}$ or $(G_2,E)\in\mathcal{G}$.

One relationship is given by the next property [43, Theorem 2, Corollary 1], that establishes that the family formed by all complements of the soft sets belonging to a fixed soft grill on X is itself a soft primal on X, and conversely:

Proposition 2 *When $\mathcal{G}\subseteq SS_E(X)$ is a soft grill, then $\mathcal{F}=\{(F,E)|(F,E)^c\in\mathcal{G}\}$ defines a soft primal on X.*

Conversely: when $\mathcal{F}\subseteq SS_E(X)$ is a soft primal, then $\mathcal{G}=\{(F,E)|(F,E)^c\in\mathcal{F}\}$ defines a soft grill on X.

Al-Saadi and Al-Malki [32] have defined generalized primal topologies. This recent extension benefits from the generalized topological spaces developed since [87]. Their idea paves the way for the introduction of generalized primal soft topologies in the future.

4.2 Data filling: incomplete soft sets

Motivated by the occurrence of missing information in the soft set scenario, [256] and [191] initiated this area of research for operational utilization of incomplete soft sets. These authors and others [153–155, 205] developed various techniques to predict the missing data from the information available in the incomplete soft set. This problem has later been expanded to take into account the occurrence of missing information in extensions such as fuzzy soft sets [91, 93], interval-valued fuzzy soft sets [109, 163] or interval-valued intuitionistic fuzzy soft sets [165, 190].

As said above, a good proportion of authors are concerned with data filling as a stage that justifies the subsequent decision-making. However [54] argued that this step may not be necessary if one only needs to make decisions with incomplete soft information. These authors focused directly on decision, and they bypassed the data filling problem.

Table 2 summarizes techniques available to the practitioner.

Although [100] introduced incomplete N-soft sets, there seems to be no literature about data filling or decision-making in this case.

4.3 Parameter reduction in soft sets

Parameter reduction is a well-established branch within rough set theory. As in that case, in the framework of soft

Table 2 A brief summary of contributions to incomplete soft sets and extensions. FSS holds for fuzzy soft set, IVFSS holds for interval-valued fuzzy soft set and IVIFSS stands for interval-valued intuitionistic fuzzy soft set (v., Table 4)

Methodology	Context	Articles	Comments
Weighted average of choice values	Soft sets	[256]	Seminal article. High computational complexity
Data filled by association between attributes	Soft sets	[54]	Adequate in the presence of related alternatives
Choice made by choice values of completion		[191]	
		[256]	
Incomplete soft sets produced from restricted intersections: elicitation criteria	Soft sets	[119]	Different approach to related problem
	Soft sets	[56]	*Sampling:* Adequate under independence of objects
Dominated alternatives are removed			and attributes. User controls sample dimension
Assumption: equiprobable completed tables			
Choice values determine choice	Soft sets	[54]	*Brute force:* Computational costs grow quickly
			Application in computational biology
			[207]
Probability analysis	Soft sets	[155]	Data filling problem
Total values of association degrees	Soft sets	[153]	Simplified approach to data filling problem in [155]
Object-parameter approach	FSS	[93]	Use of relative dominance degree between parameters
Algorithmic approach	FSS/IVFSS	[91]	Improvement of entropy (lower degree of fuzziness) with respect to [93]
Data analysis approach	IVIFSS	[190]	Missing data can be either ignored or filled
K complete nearest neighbors by			
attribute-based combining rule	IVFSS	[165]	KNN data filling algorithm

set theory it has a practical importance: its goal is the elimination of redundant parameters without affecting the optimal decisions. Since [171] defined the idea of reducing soft sets, a number of scholars have investigated this problem. Chen *et al.* [81] argued that another concept was better suited for this purpose. Kong *et al.* [149] contributed to the problem with an algorithm of normal parameter reduction. Xu *et al.* [233] designed a parameter reduction technique aiming at the selection of financial ratios for the prediction of business failure. Kong *et al.* [151] argued that for large datasets, the previous algorithms were not computationally tractable. Consequently, they proposed that the particle swarm optimization algorithm could be more useful for parameter reduction of soft sets in this case. Other works that continue this trend include [204], who resort to the hybrid binary particle swarm and biogeography optimizer, and [189], whose metodology leans on the chi square distribution. For further reading about this topic, the reader can consult [88] and [246], who produced respective surveys of parameter reduction in the case of soft set theory. Other papers on this topic continued to appear, e.g., [167] and [144]. Relatedly, [21] have launched the problem of parameter reduction in N-soft sets.

Distinctive positions have been taken for the study of this problem with fuzzy soft sets. Basu *et al.* [75] suggested a parameter reduction algorithm which is inspired by relational algebra. Also in this framework, [137] produced an adjustable approach to parameter reduction of fuzzy soft sets which is inspired by three-way decision. Kong *et al.* [148] studied normal parameter reduction from a new perspective. And [116] have defined a difference-based parameter reduction algorithm in fuzzy soft set theory too. In the field of interval-valued fuzzy soft sets, [163] gave four heuristic algorithms for respective definitions of reduction (inclusive of normal and approximate normal parameter reduction). And [193] considered an approach based on the Euclidean distance. Relatedly, in the framework of bipolar fuzzy soft sets, [64] have investigated 4 types of parameter reductions.

4.4 Aggregation

Aggregation of soft sets is probably overly simplistic, because it restricts the result to one of two values (either 0 or 1). We are not aware of any contribution dedicated to this topic. Nevertheless, if we extend the scope to include more general models such as N-soft sets or m-polar fuzzy soft sets, then the literature about aggregation has already produced flexible approachs, namely, Alcantud *et al.* [61] and Zahedi Khameneh and Kiliçman [244]. The former article pioneers the theory of aggregation of N-soft sets. Aggregation operators are designed with the assistance of OWA operators. And with this tool, the first mechanisms for multi-agent decisions based on N-soft sets are provided. In [244], the authors developed weighted aggregation functions for the m-polar fuzzy soft case and discussed two new operators, called M-pFSIOWG and M-pFSIOWA, as generalizations of IOWG and IOWA operators. Then, the authors designed with these new tools an algorithm to solve decision-making problems.

We underline that this seminal approach relies on [162]. These authors generalized the construction of OWA operators [234] to apply on complete lattices with t-norm/conorm.

It is to be expected that further contributions will be made in this area of inspection in the near future.

4.5 Soft probability

The literature about this topic is rather reduced.

It was launched when [181] introduced a notion of soft probability. In this framework, his paper presented an analogue of Chebyshev's inequality, and the computation of soft large deviation probabilities for nonnegative random variables under a mean hypothesis. Relatedly, [180] proved an analogue of the central limit theorem in the context of soft probability.

Recently, [255] has produced a hybrid approach that combines Bayesian decision theory with soft probability. Then the new methodology is applied in a numerical exercise motivated by medical diagnosis.

We are not aware of any other notable contributions to this branch of the literature.

5 Applications and utilization in decision-making

This section focuses on reviewing the techniques and models that connect soft set theory with decision-making.

There exist a few reviews of decision-making based on soft set theory. Ma *et al.* [164] have surveyed decision-making methods with soft sets, fuzzy soft sets, rough soft sets and soft rough sets. They used them to generate new algorithms that merge these hybrid models. Khameneh and Kiliçman [138] produced a systematic review of multi-attribute decision-making based on the soft set model and its fuzzy extensions. Their review considers individual and group decision-making approaches, each of them from two angles: single approaches (use a unique method to yield decision-making solutions) and hybrid approaches (use a blend of methods). [92] also published a short survey of decision-making with fuzzy soft sets.

In general, multi-attribute decision-making (abbreviated MADM) is a process in which a list of alternatives is evaluated in terms of several criteria to select the most appropriate alternative. These basic evaluations can be expressed in linguistic terms, fuzzy model, rough set template, and soft set format; however, we focus here on soft set-based decision-making approaches. Regardless of the model utilized to express the input information, decision problems are divided into three general categories. A decision problem may be handled by an individual (IDM) or involves a few decision-makers, known as group decision-making (GDM). However, if the number of invited decision-makers is no less than 20, the GDM problem is renamed as large-scale group decision-making (abbreviated LSGDM).

All decision-making situations comprise the alternative set, attribute set and decision-maker set. Note that the DMs and attributes may have different weights, which are presented by the weighting vectors, or all have the same importance. In addition, attributes can be heterogeneous. This means that we can have both cost (less is better) and benefit (more is better) criteria in one problem. Formally, in an IDM process one agent needs to choose from a set of options $X = \{x_1, \ldots, x_n\}$ (with $n > 1$). To do so, the agent considers a finite number of characteristics $P = \{p_1, \ldots, p_m\}$ (with $m > 1$) of the alternatives. The degrees of importance of these features may vary. GDM is different because there is a finite list of agents $D = \{d_1, \ldots, d_k\}$ (with $k > 1$). The weights of their opinions may be different. They also assess X in terms of P, whose interpretations are the same as in an IDM problem. Note that the weights or importance degree of different attributes in a DM problem are usually determined by the experts. However, there are several systematic methods, known as Standard Deviation method, Entropy method [213], and AHP (for Analytic Hierarchy Process) [203], to compute the weights of the attributes.

Any IDM problem follows the below steps to find the final solution:

1. Providing the initial evaluations/preferences: DM provides his/her evaluation/opinion of the alternatives.
2. Selection phase:
   (a) Comparing and ranking the options: DM compares the alternatives' preference levels to rank them from best to worst.
   (b) Solution recommendation: DM selects one or several alternatives as the final solution.

Table 3 reports on the most well-known methods that solve decision-making problems.

When the decision is made by a group of experts rather than an individual, another stage called consensus, which

Table 3 Summary of DM approaches

Selection methods	Abbreviation	Definition
Simple additive weighting [112]	SAW	The alternatives are evaluated by a weighted score (weighted sum-based model: the score sums up the evaluation by each attribute multiplied by their respective weights)
Weighted product method [178]	WPM	The alternatives are evaluated by a product weighted score (geometric mean-based model: the score multiplies the evaluation by each attribute powered by their respective weights)
ELECTRE method [200]	ELECTRE family	A family of weighted-sum-techniques-based selection methods where the final solution is selected regarding comparison tables which are formulated based on the concordance (number of criteria that an object is preferred to another) set and its complement, i.e., discordance sets for each pair of alternatives
Analytic hierarchy process [203]	AHP	Method based on pairwise comparisons: it utilizes a system of hierarchies of targets, attributes and alternatives
Technique for order of preference by similarity to ideal solution [126]	TOPSIS	The selected alternative has the shortest distance from the ideal solution and the farthest distance from the negative ideal solution
Gray relational analysis [130]	GRA	Compromise ranking method that owes to gray system theory. It may be utilized to resolve complex inter-relationships among a multiplicity of performances
Compromise ranking method [241]	VIKOR	The compromise output is a feasible result whose distance to the ideal solution is the shortest

refers to the judgment arrived at by “most of” those concerned, is added to the resolution process. Typically the consensus process (CP) is monitored by a moderator. This person provides the DMs with feedback conerning the state of the negotiation. To solve a GDM/LSGDM problem, the following steps should be addressed.

1. Gathering the initial evaluations or preferences of the DMs. They provide their respective evaluations of the alternatives which are then collected by the moderator.
2. Consensus phase:
   (a) *Aggregation* An aggregation function combines the individual evaluations into one joint evaluation.
   (b) *Consensus measuring* a consensus index measures the current level of agreement within the group.
   (c) *Consensus control* The mentor checks whether the present level of agreement is greater than or equal to a predefined minimum consensus level. If yes, the group meets the acceptable agreement level, the CP is finished, and the selection process is started; otherwise, the following stage must be compiled.
   (d) *Feedback process* The moderator recommends which DMs should vary their opinions. Then the consensus process goes back to the consensus control stage.
3. Selection phase.

Now we proceed to consider 43 papers in the field of decision-making with soft sets and its fuzzy extensions. Afterward, we dwell on applicability in Sect. 5.3. It is worth noting that the SAW-based selection methods in decision-making with soft set theory usually proceed with the help of a notion known as choice value of an alternative. Let us present this idea, both when the alternatives are evaluated by a soft set and a fuzzy soft set.

Definition 24 ([171, 199]) Consider the set X (of objects or alternatives). Then:

1. In soft set-based decision-making, the choice value of an alternative equals the number of benefit attributes that it possesses.
2. In fuzzy soft set-based decision-making, the fuzzy choice value of an alternative is the sum of the membership values associated to all attributes for that alternative.

Table 4 List of acronyms

Abbreviation	Description
SS	Soft set
FSS	Fuzzy soft set
IFSS	Intuitionistic fuzzy soft set
IVFSS	Interval-valued fuzzy soft set
IVIFSS	Interval-valued intuitionistic fuzzy soft set
m-PFSS	m-Polar fuzzy soft set

3. In these two situations, the weighted (fuzzy) choice value of an alternative is the weighted sum of its (fuzzy) choice values.

We classify the articles that we overview into either IDM and GDM approaches. In this way, we can realize how these techniques have evolved over the past decades, and which ones have received more attention. To simplify our surveys, we use the abbreviations defined in Table 4.

5.1 Individual decision-making (IDM) approaches

The first touch on using the SS format for expressing the IDM problems backs to the founding [179] through an example. However, [171] was the first paper that dwelled on applying soft sets for decision-making. This article proved that the soft set format was potentially applicable for expressing the information about the inputs in IDM problems. In general, the early efforts in this line adapted widely known DM methodologies, such as SAW and TOPSIS, for the SS format and its fuzzy extensions, but usually without any contribution on new solving methods. Table 5 lists the most important papers that consider the (fuzzy) soft set-based individual decision-making.

5.2 Group decision-making (GDM) approaches

Let us now consider studies that discussed group (multi-observer) decision-making problems. To this purpose, Table 6 contains short descriptions of such methods. It is observed from this table that in years, most researchers have paid attention to group decision-making problems rather than to the individual case. Meanwhile, most authors have tried to introduce new methods instead of considering a simple adoption of existing approaches for (fuzzy) soft sets.

5.3 Applicability

To conclude this section, we present a sample of areas where soft set theory found applications.

Economic applications were published as soon as in 2009, when [227] considered a problem in international trade (forecasting imports and exports) that was solved with the help of fuzzy soft information. Using this model, a methodology for the valuation of assets was proposed by [57].

Financial applications have been covered in works such as [132], where the authors collected data from one hundred female employees working in Coimbatore, India. Their technique was improved in [186]. Xu *et al.* [233] were concerned with predicting business failure via the selection of financial ratios. They use an explicitly designed parameter reduction technique for this purpose. Their methodology is applied to real datasets from listed firms in China. Tas *et al.* [221] applied both soft set and fuzzy soft set theory to the problem of stock management.

Applications in the medical sciences include [215] (who classify microcalcifications on mammograms with the help of soft set theory), [173] (that resort to bipolar soft information), [75] (diagnosis using the mean potentiality approach in fuzzy soft set theory), [117] (diagnosis of mouth cancer with the help of fuzzy soft similarity indices), [7] (use of fuzzy N-soft sets to detect tumor cells and estimate their severity), [146] (application of fuzzy soft set decision-making to Cleveland heart disease dataset), [89, 124] (use of intuitionistic fuzzy soft information for diagnosis), and many others [56, 58, 226].

Other fields of application include logistics [128], sport competitions [61], environmental policy [118], and awards [62].

Table 7 gives a non-exhaustive summary of applications of soft-set-based decision-making approaches to other fields.

Table 5 Summary of existing (fuzzy) soft-set-based IDM approaches

Data format	Selection techniques	Type of attributes	Description
SS: [171] and [81]	SAW (choice value)	Benefit	Before finding the optimal choice, a reduction of the parameter set is obtained
FSS: [107, 237], IFSS: [129, 252], IVIFSS: [251], Bipolar Multi-Fuzzy Soft Set (BMFSS): [238]	SAW	Benefit	Converting to a SS by using the α-level (cut) technique; Compared with [199]
Trapezoidal FSS (TFSS): [229], Interval-valued hesitant FSS: (IVHFSS) [187, 250], Interval-valued intuitionistic FSS (IVIFSS): [113]	TOPSIS	Benefit	
Softarisons: [50]	Min-max weighted inverse score	Benefit	The optimum choice has the minimum overall importance of attributes in which is dominated by other alternatives

Table 6 Summary of existing (fuzzy) soft-set-based GDM approaches

Data format	Consensus techniques	Selection techniques	Type of discussed attributes	Description
SS [76, 77], FSS [78]	AND-Product operator	Union-intersection or min-max decision matrix	Benefit	A set of feasible solutions
FSS [150, 199]	Minimum operator	Scores from a comparison matrix (ELECTRE-I)	Benefit	Comparison table is computed based on number of parameters in which an object dominates or is dominated by other objects
FSS [152]	Weighted sum	GRA	Benefit	Computing the grey relational degree of each object as a weighted sum of its grey relative coefficient, generating based on choice value [171] or score value [199]; Compared with [77]
FSS [45, 53]	Product operator	Scores from a comparison matrix	Benefit and cost	Comparison table is computed based on relative differences of the membership values of objects; Compared with [199]
FSS [137, 142], IFSS [143]	Fuzzy soft topology (Min and Max operators)	Scores from upper and lower comparison matrices	Benefit and cost	Comparison tables are established based on upper and lower preorder relations, induced by the fuzzy soft topology. The overall score value of each alternative is computed based on the number of objects which dominate the alternative and are dominated by it; In [137] the problem of parameter reduction is solved by deleting the attributeswithout significant effect on alternatives ranking and partitioning induced by the upper and lower preorder matrices; Compared with [45, 107, 199]
IVFSS [236]	Minimum operator	Scores from a comparison matrix	Benefit	Comparison table is computed based on SAW method (choice value)
IVFSS [230]	Aggregation operators	SAW	Benefit	Using an optimization model to determine the unknown attribute weights; Converting the IVFSS data into the FSS
2-Tuple linguistic SS [220]	Aggregation operators	Scores from comparison matrix		Using the linguistic quantifiers to computing the experts weights and solving an optional problem to find the weight of attributes
Probabilistic Soft Set (PSS) [102]	Aggregation operators	Maximizing the eigenvectors of the comparison matrix	Benefit	By combining soft sets and probability theory, the concept of probabilistic soft set was defined as a parameterized family of probability distributions subsets of the universe of discourse

6 Concluding remarks

Much has been written about what defines the field of soft computing. Zadeh [243] assured that its guiding tenet is to take advantage of uncertainty, imprecision, or partial truth to attain robustness with small solution burden [83]. Magdalena [166], Sect. 2.1, argued that the most recognizable feature of soft computing is its ability to generate hybrid systems that integrate existing technologies. In relation with both positions, we believe that our survey has given arguments proving that soft set theory can become a recognizable part of this toolbox.

We have attempted to give an integrated view of the vast literature inspired by the soft set approach, with a unified notational and semantical description of both fundamental concepts and problems. Practitioners, researchers, and students will find here a selection of topics and works that are likely to be of particular value to newcomers to the field. This allows the non-specialist to bypass the difficulties of a vast literature with remarkable differences in notation and terminology.

A further benefit for the reader is that our survey has identified some promising lines of future research. New models, such as ranked soft sets [208], are still in their early stages of development. Aggregation in the case of *N*-soft sets and its generalizations seems to be particularly attractive, since works on the topic are in short supply. Data filling for *N*-soft sets appears to be an unexplored field. For those wanting to explore soft algebraic concepts, Sect. 4.1 has listed new areas including convex soft structures [51], soft grills, and the soft primals that are at the root of the novel primal soft topological spaces [43]. Others will probably be designed in the near future (e.g., generalized primal soft topologies, borrowing from the idea in [87]).

Table 7 Summary of applications of (fuzzy) soft-set-based DM approaches

Data format	Consensus techniques	Selection techniques	Application	Dataset
SS [56, 58]	AND-Product operator	Deriving soft rules from the inputs, then using the probability function to estimate the risk of disease	Medical diagnosis: Glaucoma [56], Lung Cancer [58]	In [56], 53 eyes were randomly selected among the total of 106 eyes in order to configure the soft expert system and remaining eyes were used for validation purposes; in [58], 170 patients with known survival status and a surgical procedure other than pneumonectomy are used from an initial database of 403 patients to estimate and the rest for validation process
SS and FSS [156, 219, 225, 226]	Dempster's rule of combination	A ranking method obtained through a mass function of different independent alternatives with different criteria	Medical diagnosis	Result in [226] was compared with [156, 225]
FSS [120, 135]	Minimum operator	Deriving fuzzy soft rules from the inputs then calculating the scores obtained from the comparison matrix (using the discussed method in [150])	Medical diagnosis: Coronary Artery [120], Lung Cancer [135]	In [120] 200 patients from the Cardiac Unit, Department of Cardiology, Faculty of Medicine, Assiut University, Egypt; In [135] 190 patients from Nanjing Chest Hospital in China
FSS [228]	Particle Swarm Optimization (PSO) over fuzzy cognitive map (FCM)	Scores from the collective comparison matrix (using the discussed method in [150])	Supplier selection	–
Generalized IFSS (GIFSS) [8]	Intuitionistic fuzzy weighted averaging (IFWA)	A score value based on the accuracy degrees of GIFSS	Supplier selection; Medical diagnosis	National Crime Records Bureau (NCRB) project under Ministry of Home Affairs of India
m-PFSSs [244]	New aggregation operators *m*-PFSIOWA and *m*-PFSIOWG (the proposed m-polar fuzzy soft weighted aggregation operators act over the binomial coefficient to cover the partial agreement with required consensus degree)	Scores from *m*-PFS preference matrix	Hotel booking problem	dataset collected from www.booking.com
N-SS [7]	–	TOPSIS and ELECTRE-I	Medical diagnosis (detection and severity of tumor cells)	Synthetic data
Hesitant *N*-SS [19]	Union operator	SAW	Medical diagnosis	Synthetic data
Multi-agent *N*-SS [61, 62]	*Merge-then-decide* strategy in [61]. The aggregation process is done by top/bottom and OWA operators, among others. *Decide-then-merge* aggregation strategy in [62]: aggregation of ordinal information uses voting theory	A new SAW-based method called WAOWA score according to OWA operator [61]; ranking with the help of *N*-soft set prioritizations in [62]	Sports [61], Valuation of assets [57]	In [61]: data collected from 2019 USA Diving Senior National Diving Championships. In [62]: prize awarded in the 20th European Conference on Operational Research

Another applicable topic is the investigation of parameter reduction with large datasets initiated by [151] and [204]. Their methodologies resort to particle swarm optimization and its hybrid approach with the biogeography optimizer, respectively. It is therefore reasonable to foresee that new approaches to this problem can take advantage of optimization algorithms that have emerged in recent literature, such as those presented by [4, 10, 9, 97, 115, 122, 123], and [245]. Statistical arguments can be added too, aligning with the recent [189].

Bibliometric studies are conspicuosly absent, and they may shed light on the development of this area of research.

In addition to the topics that we have explored, other techniques that have been integrated with soft set theory include clustering [192] and classification [3], association rules mining [103, 105, 121, 157], three-way decision [3, 52, 111, 235], or neural networks [161]. It is important that these issues continue to be discussed and debated in the future too.

Author contributions JCRA, AZK, GSG and MA conceptualized and designed the study, analyzed the data, and wrote the manuscript.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. Alcantud thanks the European Regional Development Fund and the Junta de Castilla y León (Grant CLU-2019-03) for financial support to the Research Unit of Excellence GECOS ("Economic Management for Sustainability"). Santos-García is grateful to the Spanish Ministerio de Ciencia e Innovación project ProCode-UCM (PID2019-108528RB-C22).

Data availability No data were used to support this study.

Declarations

Conflict of interest The authors declare no conflict of interest.

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