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Scores of hesitant fuzzy elements revisited: *"Was sind und was sollen"*

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ABSTRACT

This paper revolves around the notion of score for hesitant fuzzy elements, the constituent parts of hesitant fuzzy sets. Scores allow us to reduce the level of uncertainty of hesitant fuzzy sets to classical fuzzy sets, or to rank alternatives characterized by hesitant fuzzy information. We propose a rigorous and normative definition capable of encapsulating the characteristics of the most important scores introduced in the literature. We systematically analyse different types of scores, with a focus on coherence properties based on cardinality and monotonicity. The hesitant fuzzy elements considered in this analysis are unrestricted. The inspection of the infinite case is especially novel. In particular, special attention will be paid to the analysis of hesitant fuzzy elements that are intervals.

1. Introduction

In this manuscript we attempt to perform a systematic and normative study of the concept of score defined on a family of hesitant fuzzy elements (i.e., subsets of the unit interval). This position is a jarring departure from earlier approaches, which we believe overlooked some important facts that distorted research on the topic. As an example, here is just a missing argument in the best studied case, namely, scores of hesitant fuzzy elements (HFEs) that are finite (typical HFEs):

For each $\{x, y\} \subseteq [0, 1]$ with $x \neq y$, the score of the HFE $\{x, y\}$ is a number (say, z) which is universally accepted to be in $[0, 1]$. It is also universally accepted that the score of the HFE $\{z\}$ must be z, for each $z \in [0, 1]$. Therefore the score of $\{x, y\}$ must be equal to the score of a certain $\{z\}$ even though $\{x, y\} \neq \{z\}$. In conclusion: every score must produce ties (and in fact, ties must appear on a regular basis).

This novel observation is in sheer contrast with the fact that the validity of many scores on the set of finite HFEs has been rejected because they produce ties (see Rodríguez et al. [27, section 2]). Ties cannot be avoided in the application of scores, and researchers must be aware of this. Nevertheless, we will show that we can still build scores that avoid any tie on a special family of subsets of the unit interval $[0, 1]$, namely, that of closed non-degenerate intervals $\{[a, b] : 0 \leq a < b \leq 1\}$.

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Hence we are motivated to launch a systematic inspection of the notion of score, paying particular attention to its properties –a reflection of the expected behavior of a score– and scope of applicability.

We will work here in the spirit of Richard Dedekind in his 1888 main work entitled "Was sind und was sollen die Zahlen?"[1] [12]. At that time any mathematician had an idea of what a number was. However, a rigorous and clear definition was compulsory to give an ultimate notion of what a number actually is (*"was sind"*). In addition, also the main properties of numbers were known, but it was necessary to establish why they are good, how we can use them, and what we can expect from them (*"was sollen"*).

Scores of hesitant fuzzy sets appear to be in a similar stage of development. We have witnessed how authors tend to investigate particular scores without explicitly defining what we should understand by "score". Agreement on an appropriate definition that is both flexible and meaningful needs to be reconciled with mathematical rigor ("was sind", again). Furthermore, a number of properties of scores have been introduced, scattered across many papers. It is now time to produce a systematic account and discussion of axioms in order to organize the debate about the performance of particular scores in terms of their properties. This would be the "was sollen", that is, an examination of what scores are good for, what they should be, and their scope of application.

Motivated by these concerns, this paper makes two specific contributions.

First, although the literature has mainly focused on typical scores, we will not be bound by such restriction. For instance, we shall consider scores acting on (finite unions of) intervals contained in $[0,1]$. This wider analysis connects the investigation of scores on HFEs with e.g., the literature on scores defined for intuitionistic fuzzy numbers [39] –because Atanassov and Gargov [4] and Deschrijver and Kerre [13] proved that they are in bijection with the set of all interval-valued fuzzy numbers– or interval-valued intuitionistic fuzzy numbers [24,34,37].

Secondly, arguments will be couched in terms of features of scores, rather than in terms of ad-hoc inconveniences of particular formulas. This position parallels research on other important analytical tools such as aggregation operators, or t-norms and t-conorms. The goals of our axiomatic analysis are similar. On one hand, it establishes bounds to what can be done (recall the argument concerning the unavoidable existence of ties explained above). On the other hand, it helps the researcher to select the most appropriate tool with a precise description of the main characteristics (and setbacks) of explicit expressions. This objective has a crucial importance in practice. In the presence of a multiplicity of alternative scores for application in a practical situation, its characteristics should govern the features of the selected score. In summary, an organized theoretical analysis of the properties of relevant families of scores should help the researchers to single out a particular score in practice.

The structure of the paper is as follows. In Section 2 (Preliminaries) we recall some classical key definitions. In Section 3 we formalize the key concept of a score. In the next Section 4 we analyze different kinds of scores, with special attention to their coherence features and scope of application. Also in this main section, we introduce some compatibility and incompatibility results concerning properties and features of scores. Particularly, Example 3 below in that section takes advantage of measure theory to define suitable scores on infinite HFEs. A final Section 5 closes the paper with conclusions and lines for future inspection.

2. Preliminaries

In this section we collect some classical basic terminology (see, e.g. Alcantud and Giarlotta [2]) that will be used in the sequel.

Henceforward X will denote a nonempty set, also known as the universe.

Definition 1. (Zadeh [40]). A (type-1) fuzzy subset H of X is defined as a function $\mu_H : X \to [0,1]$. The function μ_H is called the membership function of H. In the particular case when μ_H is dichotomous and takes values in $\{0,1\}$, the corresponding subset defined by means of μ_H is a subset of X in the classical crisp sense.[2]

Along the paper, we will use the following notation:
- $\mathcal{P}([0,1])$ is the family of all subsets of the unit interval $[0,1]$,
- $\mathcal{P}^*([0,1])$ is the family of all nonempty subsets of $[0,1]$,
- $\mathcal{I}([0,1])$ is the subset formed by all intervals in $[0,1]$,
- $\mathcal{I}_C([0,1])$ is the subset formed by all the closed intervals in $[0,1]$,
- $\mathcal{I}^{\cup}([0,1])$ is the subset formed by all finite unions of intervals in $[0,1]$,
- $\mathcal{F}^*([0,1])$ is the family of all nonempty finite subsets of $[0,1]$, and
- $\mathcal{F}^*_n([0,1])$ is the family of all nonempty subsets of $[0,1]$ with n or fewer elements.

When $A, C \subseteq \mathbb{R}$, we write $C > A$ to mean $c > a$ for all $c \in C$, $a \in A$. Obviously, $C > A$ implies $A \cap C = \emptyset$.

The main objects of our analysis are HFEs:

Definition 2 *(Xia and Xu [35], Torra [30]).* A hesitant fuzzy element (HFE) is a subset E of $[0,1]$.

A hesitant fuzzy set (HFS) over X is a function $h: X \to \mathcal{P}([0,1])$.

Remark 1. Classical fuzzy sets extend crisp subsets with the assistance of a first level of uncertainty: the membership function μ_H maps any element x of the universe X with its "uncertainty degree". Mapping x into a number from $[0,1]$ graduates the acceptability

[1] "What are numbers and what should they be?".

[2] The term crisp is usually understood in these contexts to mean non-fuzzy.

of the claim that this element belongs to the fuzzy subset H, which therefore generalizes the idea of classical subsets. HFSs are a particular case of "type-2 fuzzy sets" [8,23,31]. Either because we hesitate, or because we collect opinions from various sources, we finally assess the membership of x not by a number, but by a subset of values in $[0,1]$, which introduces a second level of uncertainty. In the terminology of type-2 fuzzy sets, $h(x)$ is the primary membership of x, whereby its secondary membership is crisp or binary-valued.

In many practical applications, the employed HFSs map each element of X to a finite subset of $[0,1]$.
This case is well known in the literature:

Definition 3 *(Bedregal et al. [6]).* A typical hesitant fuzzy set (THFS) over X is a function $h: X \to \mathcal{F}^*([0,1])$. A typical hesitant fuzzy element (THFE) is a nonempty finite subset of $[0,1]$.

Remark 2. Notice that the family of THFEs is equal to $\mathcal{F}^*([0,1])$.

It is customary to list the elements of a THFE in ascending order, i.e., E is a set of the type $\{e_1,\dots,e_k\} \subseteq [0,1]$ for some $k \geqslant 1$, where $e_1 < \dots < e_k$. Noteworthy examples of THFEs are $E = \{1\}$ (full HFE) and $E = \{0\}$ (empty HFE).

As usually done, we often represent an HFS h over X as a set of ordered pairs, i.e., $h = \{(x, h(x)) : x \in X\}$. For example, the ideal or full HFS over X is $\{(x,\{1\}) : x \in X\}$, and the anti-ideal or empty HFS over X is $\{(x,\{0\}) : x \in X\}$.

The following definition provides a strengthening of the notion of a typical HFS:

Definition 4 *(Alcantud and Torra [3]).* A THFS h over X is said to be uniformly typical if there is $n \in \mathbb{N}$ such that $|h(x)| \leqslant n$ for each $x \in X$. Equivalently, a uniformly THFS is a function $h: X \to \mathcal{F}_n^*([0,1])$ for some $n \geqslant 1$. We shall abbreviate "uniformly typical HFS" by UTHFS.

3. Scores of hesitant fuzzy elements: definition, a brief reminder, and a few novel extensions

In this section we introduce a formal definition of the concept of a score. Then we establish some new facts concerning scores on families of HFEs.

Formally, scores can be defined as follows[3]:

Definition 5. Given a family $\mathcal{G} \subseteq \mathcal{P}([0,1])$, a score on $\mathcal{G}$ is a function $s: \mathcal{G} \to [0,1]$ with the following properties[4]:

1. $s(\emptyset) = 0$ whenever $\emptyset \in \mathcal{G}$;
2. Boundedness [B]: for all $E \in \mathcal{G}$, we have $\inf(E) \leqslant s(E) \leqslant \sup(E)$.

A score on $\mathcal{G} = \mathcal{P}([0,1])$ is called total, and a score on $\mathcal{G} = \mathcal{F}^*([0,1])$ is called typical. We can also consider interval scores which are defined on $\mathcal{G} = \mathcal{I}([0,1])$.

Remark 3. Notice that boundedness implies the next property:

Compatibility [C]: $s(\{a\}) = a$ for each $a \in [0,1]$ such that $\{a\} \in \mathcal{G}$.

Remark 4. An HFS is a particular case of "type-2 fuzzy set" ([8,23]) hence it conveys two degrees of uncertainty. A first, but important, application of scores defined on HFSs is that they allow us to reduce their uncertainty in one level. To see why, note that if $h: X \to \mathcal{P}([0,1])$ defines an HFS, and s is a score defined on the set $\mathcal{P}([0,1])$ of HFEs, the composition $s \circ h$ immediately defines a (type-1) fuzzy set over the universe X.

The most used scores (see e.g. [10,14,15,25–27,29,33,35,36,38]) are typical (in the sense of Definition 5), since they are defined on the family of all finite subsets of the unit interval. The next example collects some simple instances of this kind.

Example 1. The following expressions define functions $s: \mathcal{F}^*([0,1]) \to [0,1]$ that are typical scores: for each $E = \{e_1,\dots,e_n\} \in \mathcal{F}^*([0,1])$, with $e_1 < \dots < e_n$,

- (min) $s(E) = e_1$.
- (max) $s(E) = e_n$.

[3] The generic term "the score function for HFEs" has often been used to designate specific examples, as explained in [27]. Alcantud et al. [1] formally define score operators for typical HFEs: here we employ a slight weakening of their definition.

[4] Farhadinia [15, Section 2] defines scores for the special case of THFEs with the same number of (possibly repeated) elements. Besides this restriction, which we eschew, his definition imposes some unnecessary boundary conditions.

- (second worst) $s(E) = e_1$ if $|E| = 1$, and $s(E) = e_2$ otherwise.
- (second best) $s(E) = e_n$ if $|E| = 1$, and $s(E) = e_{n-1}$ otherwise.
- (simple average) $s(E) = \frac{e_1+e_n}{2}$.
- (average)[5] $s(E) = (e_1 + \ldots + e_n)/n$.
- (cut average) $s_a(E)$ is the average score of E if $|E| \leqslant 2$, otherwise we set $s(E) = (e_2 + \ldots + e_{n-1})/(n-2)$.

The literature on typical scores is abundant: see Xia and Xu [35], and Farhadinia [14,15], as well as [27, section 2] for further examples. Three acclaimed scores are defined as follows:

Definition 6. For each $E = \{e_1, \ldots, e_n\}$, with $e_1 < \ldots < e_n$,

(i) the (typical) Xia–Xu score of E is $s_{\text{XX}}(E) = \sum_{i=1}^{n} e_i/n$. In other words, this is the average score introduced in the previous Example 1,
(ii) for a nondecreasing sequence $(\delta_n)_{n\geqslant 1}$ of positive numbers, the (typical) Farhadinia score of E is $s_{\text{F}}(E) = \sum_{i=1}^{n}(\delta_i e_i)/\sum_{i=1}^{n}\delta_i$, and
(iii) the (typical) geometric-mean score of E is $s_{\text{gm}}(E) = \left(\prod_{i=1}^{n} e_i\right)^{1/n}$.

Clearly, (ii) includes (i) as a special case, taking the constant sequence with value 1.

The next Example 2 gives novel scores on a framework that consists of possibly infinite HFEs with a specific structure, namely, on $\mathcal{I}^{\cup}([0,1])$. Note that when $E \subseteq \mathcal{I}^{\cup}([0,1])$ there are $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in [0,1]$, $b_1, \ldots, b_n \in [0,1]$, such that E is the union of n intervals with respective extremes a_i, b_i. These intervals might have the form $[a_i, b_i]$, $[a_i, b_i)$, $(a_i, b_i]$, or (a_i, b_i). The first case allows for $a_i = b_i$, otherwise $a_i < b_i$. Thus in particular typical HFEs are in $\mathcal{I}^{\cup}([0,1])$.

Example 2. The following expressions produce functions from $\mathcal{I}^{\cup}$ onto $[0,1]$ that are scores on $\mathcal{I}^{\cup}$. Suppose $E \in \mathcal{I}^{\cup}$ is the union of n intervals (for any arbitrary $n \in \mathbb{N}$) with respective extremes a_i, b_i and with $b_i < a_{i+1}$ for $i = 1, \ldots, n-1$, then we define:

- (min) $s(E) = a_1$.
- (max) $s(E) = b_n$.
- (second worst) $s(E) = a_1$ if $n = 1$, and $s(E) = a_2$ otherwise.
- (second best) $s(E) = b_n$ if $n = 1$, and $s(E) = b_{n-1}$ otherwise.
- (simple average) $s(E) = \frac{a_1+b_n}{2}$.
- (extended Xia–Xu) $s_{XX}(E) = \frac{1}{2n}(a_1 + \ldots + a_n + b_1 + \ldots + b_n) = \frac{1}{n}\sum_{i=1}^{n}\frac{a_i+b_i}{2}$.
- (extended Farhadinia score) $s_{\text{F}}(E) = \frac{\sum_{i=1}^{n}\delta_i \frac{a_i+b_i}{2}}{\sum_{i=1}^{n}\delta_i}$, for a non-decreasing sequence $(\delta_n)_{n\geqslant 1}$ of positive numbers.
- (extended geometric-mean score) $s_{\text{gm}}(E) = \left(\prod_{i=1}^{n}\frac{a_i+b_i}{2}\right)^{1/n}$.

Note that Example 2 extends typical scores, using the embedding of $\mathcal{F}^*([0,1])$ into $\mathcal{I}^{\cup}([0,1])$ described above. To be precise, when $E = \{e_1, \ldots, e_n\}$ with $e_1 < \ldots < e_n$ is a typical HFE, then we can write $E = \cup_{i=1}^{n}[e_i, e_i] \in \mathcal{I}^{\cup}([0,1])$. And the min (respectively, max, second worst, second best, simple average, and extended versions of Xia–Xu, Farhadinia, geometric-mean) scores of E defined in Example 2 with $a_i = b_i = e_i$ for each $i = 1 \ldots, n$, and in Example 1–Definition 6, coincide.

Let us introduce other scores for the infinite case. These new scores are based on some aspects of measure theory. Note that already in Section IV of [23], scores based upon integrals have been designed.

Example 3.

- Suppose that we deal with measurable subsets of $[0,1]$ with respect to the Lebesgue measure (see e.g., Halmos [18] for a further account). This family contains $\mathcal{I}^{\cup}([0,1])$. We may now define the following score s_M:

$$\text{for each measurable } A \subseteq [0,1],\ s_M(A) = \begin{cases} \sup A, & \text{if } \int_A 1\,dx = 0, \text{ and} \\ \frac{\int_A x\,dx}{\int_A 1\,dx}, & \text{otherwise.} \end{cases}$$

Notice that if A is an interval $[a,b]$ then this score coincides with the simple average $\frac{a+b}{2}$. Moreover, for a finite union of non-degenerate intervals $E = [a_1, b_1] \cup \ldots \cup [a_n, b_n]$, its extended Xia-Xu score also appears to be the arithmetic mean of the numbers $s_M([a_i, b_i])$ $(i = 1, \ldots, n)$.

By the way, not all the subsets of $[0,1]$ are measurable: one example is the Vitali set discovered in 1905 (see [32]).

[5] This is the Xia–Xu score: see Definition 6(i).

- This construction can be generalized as follows. If $\delta : [0,1] \to [0,1]$ is a continuous function, we may define the following score s_δ:

$$\text{for each measurable } A \subseteq [0,1],\ s_\delta(A) = \begin{cases} \sup A, & \text{if } \int_A 1\ dx = 0, \text{ and} \\ \frac{\int_A x\cdot\delta(x)\ dx}{\int_A 1\ dx}, & \text{otherwise.} \end{cases}$$

- Again, let A be a measurable subset of $[0,1]$. We may now define a new score s_p that reflects the idea of a percentile p, namely: for each measurable $A \subseteq [0,1]$,

$$s_p(A) = \begin{cases} \sup A, & \text{if } \int_A 1\ dx = 0, \text{ and} \\ \sup\left\{ b \in A : \int_{A\cup[0,b]} 1\ dx \le \frac{p}{100}\int_A 1\ dx \right\}, & \text{otherwise.} \end{cases}$$

- Although the scores will give rise to ties, we may still suitably design some score that never provokes a draw on remarkable families of subsets of $[0,1]$. Consider the following example, where a score s_I is defined on $\mathcal{I}_{\subsetneq}([0,1])$, namely the set of closed intervals of $[0,1]$, as follows: Given an interval $[a,b]$ we define $s_I([a,b]) = a$ if $a = b$. Otherwise, if $a < b$ we consider the decimal expansions[6] of the numbers a and b as $0.a_1a_2\dots a_n\dots$ and $0.b_1b_2\dots b_n\dots$ and then we define $s_I([a,b]) = 0.a_1b_1a_2b_2\dots a_nb_n\dots$.

  At this point, the good news is that if we consider closed non-degenerate intervals, then it must be the case that if either $a \neq a'$ or $b \neq b'$ we have that $s_I([a,b]) \neq s_I([a',b'])$ by construction.

Scores were originally introduced because they permit to rank HFEs: the higher the score of a HFE, the higher its relevance. Note that however, this ranking must fail to be fully discriminative due to the existence of ties:

- Ties must always appear for total scores, because Cantor's theorem assures that there is no injection from $\mathcal{P}(X)$ to X for any nonempty X [20, Theorem 3.1].
- In the case of scores on families that contain all typical HFEs, the argument given in Introduction proves that there must be ties too. This proof uses the Compatibility property that any score must satisfy.

  Rodríguez et al. [27, section 2] has attested this issue both with Xia–Xu's and Farhadinia's score, and examples can be produced to show that the geometric-mean score displays the same behavior (existence of different typical HFEs with the same score).

Thus, even if the typical scores given in Definition 6 are not fully satisfactory, we cannot discard them on account of the fact that they produce ties, as aforementioned and explained in the Introduction.

Independently of this issue, scores can be introduced to lower the type of uncertainty of HFSs, as mentioned in Remark 4.

4. Different classes of scores: definitions, hierarchies and incompatibility results

One crucial idea that carries the mere definition of a score is that of a "mean value" (see e.g. Campión et al. [9] for a further account). This is captured by property [B] or boundedness, whereby a score s is such that $s(A) \in [\inf A, \sup A]$ when $A \subseteq [0,1]$. If a person is hesitating in assigning a membership degree to an element x of a universe X, and instead of assigning just one number in the unit interval $[0,1]$, she/he defines a HFS on X, through a map $h : X \to \mathcal{P}([0,1])$, a score s should retrieve a suitable and feasible membership degree for x, and this, obviously and merely by common sense, should lie between the infimum and the supremum of $h(A)$.

At this stage, it might be interesting to study what a score should mean (hopefully Definition 5 is acceptable), what other properties might be sensible, and especially, whether acceptable sets of properties are incompatible (i.e., they lead to an impossibility). If this is done for typical scores, the impact will be much higher. So far there is not too much real progress in that direction. The problem with scores for more general HFEs has been little studied.

Later on in this Section 4 we furnish a table that visualizes the properties of the most used scores.

Looking for a classification of scores on HFEs we may focus on properties of the following kinds:

i) Properties based on some coherence features of the score (e.g., the addition of worse elements should never increase the score).
ii) Properties based on the special kind of HFEs where the score will be considered (e.g., scores defined for finite HFEs, for interval HFEs, etc.).

Remark 5. Here, the classification not necessarily will give rise to separate and mutually exclusive sets. That is, first one score can be viewed paying attention to some coherence features, and then, on the other hand, focusing on the kind of HFSs to which it will be applied.

[6] Here we will avoid infinite sequences of "9". That is, instead of, say, 0.299999999999999… we will write 0.300000000000000….

4.1. On coherence features of scores

To start with, first we introduce some definitions and results relative to coherence properties of scores.

Definition 7. Suppose $\mathcal{G} \subseteq \mathcal{P}([0,1])$. A score s on $\mathcal{G}$ is said to be best-worst monotonic for elements [BWME] when for any $x, y \in [0,1]$ such that $x < y$, and in addition $\{x\}, \{x,y\}$ and $\{y\}$ belong to $\mathcal{G}$, it holds true that $s(\{x\}) < s(\{x,y\}) < s(\{y\})$.

Next we discuss a property that will be unacceptable for two reasons explained below:

Definition 8. Suppose now that $\mathcal{G} \subseteq \mathcal{P}^*([0,1])$. We say that a score s on $\mathcal{G}$ is monotonic by addition of sets [MAS] when for each $A, B, C \in \mathcal{G}$ such that $s(A) < s(B)$ and $C \cap (A \cup B) = \emptyset$ it holds true that $s(A \cup C) < s(B \cup C)$.

Now one example and a proof of incompatibility with an admissible property will serve us to claim that [MAS] should be rejected as a desirable property of scores:

Example 4. Being $\mathcal{G} = \mathcal{F}^*([0,1])$, the classical Xia-Xu score does not fulfill the property of monotonicity by addition of sets. To see this, consider $A = \{0.2\}$; $B = \{0, 0.5\}$ and $C = \{0.7\}$. Here $s(A) = 0.2$, $s(B) = 0.25$, $s(A \cup C) = 0.45$ and $s(B \cup C) = 0.4$.

Proposition 1. *Consider $s : \mathcal{G} \to [0,1]$ with $\mathcal{F}^*([0,1]) \subseteq \mathcal{G} \subseteq \mathcal{P}^*([0,1])$. Then s cannot satisfy both best-worst monotonicity for elements (Definition 7) and monotonicity by addition of sets. (Definition 8).*

Proof. We argue by contradiction, assuming that there is some score s that satisfies both those properties. Now notice that $s(\{0\}) = 0$; $s(\{0.5\}) = 0.5$; $s(\{1\}) = 1$. By best-worst monotonicity for elements we get that $0 = s(\{0\}) < s(\{0, 0.5\}) < s(\{0.5\}) = 0.5 < s(\{0.5, 1\}) < s(\{1\}) = 1$. Now, by monotonicity by addition of sets we have that $s(\{0\}) < s(\{0, 0.5\}) \Rightarrow s(\{0,1\}) < s(\{0, 0.5, 1\})$. But also $s(\{0.5, 1\}) < s(\{1\}) \Rightarrow s(\{0, 0.5, 1\}) < s(\{0, 1\})$. Therefore we get $s(\{0,1\}) < s(\{0, 0.5, 1\}) < s(\{0,1\})$, which is a contradiction. □

Definition 9. Suppose that $\mathcal{G} \subseteq \mathcal{P}^*([0,1])$. We say that a score s on $\mathcal{G}$ is strongly monotonic with respect to unions [SMU] when for each $A, B, A \cup B \in \mathcal{G}$ and such that $a < b$ for every $a \in A$, $b \in B$, it holds true that: $s(A) < s(A \cup B) < s(B)$.

[SMU] captures the following intuition: Adding better elements to a subset should increase its score, whereas deleting worse elements should also increase the score.

Remark 6. Notice that [SMU] implies [BWME]. However, the converse is not true in general. A counterexample is the score s_M defined in Example 3.

Given a non-empty subset A of the unit interval $[0,1]$ and real numbers $\alpha, \beta > 0$, we define the set αA as

$$\alpha A = \{\alpha \cdot t : t \in A\}.$$

Also, we define the set $\beta + A$ as follows:

$$\beta + A = \{\beta + t : t \in A\}.$$

Obviously, depending on A and α, β the resulting sets αA and/or $\beta + A$ may or may not be subsets of $[0,1]$.

Definition 10. Suppose now that $\mathcal{G} \subseteq \mathcal{P}^*([0,1])$. We say that a score s on $\mathcal{G}$ is algebraically coherent with respect to a dilatation [ACD] when for each $A \in \mathcal{G}$ and $\alpha > 0$ such that αA also belongs to $\mathcal{G}$ it holds true that $s(\alpha A) = \alpha s(A)$.

Similarly, we say that a score s on $\mathcal{G}$ is algebraically coherent with respect to a translation [ACT] when for each $A \in \mathcal{G}$ and $\beta > 0$ such that $A + \beta$ also belongs to $\mathcal{G}$ it holds true that $s(A + \beta) = s(A) + \beta$.

Remark 7. All the typical scores introduced in Example 1 as well as in Definition 6 satisfy both [ACD] and [ACT], with the exception of s_{gm}. To show that s_{gm} contradicts [ACT], consider $A = \{0.1, 0.3\}$ and $\beta = 0.5$, then $s_{gm}(A) \approx 0.1732$ and $s_{gm}(A + \beta) \approx 0.6928$, so $s_{gm}(A) + \beta = s_{gm}(A + \beta)$ does not hold.

Definition 11. Suppose $\mathcal{G} \subseteq \mathcal{P}([0,1])$. We say that a score s on $\mathcal{G}$ satisfies translation invariance [TI] when for each $A \in \mathcal{G}$ such that $A + \varepsilon \in \mathcal{G}$ (with $\varepsilon > 0$) it holds true that $s(A) < s(A + \varepsilon)$.

Remark 8. All the scores introduced in Example 2 satisfy [TI].

Table 1
Main scores and their properties.

	Main scores					
Property	Min/ Max	Second Worst/Best	Simple average	Xia-Xu	Farhadinia	Geometric-mean
[B]	YES	YES	YES	YES	YES	YES
[C]	YES	YES	YES	YES	YES	YES
[BWME]	NO	NO	YES	YES	YES	YES
[SMU]	NO	NO	YES	YES	YES	YES
[ACD]	YES	YES	YES	YES	YES	YES
[ACT]	YES	YES	YES	YES	YES	NO
[TI]	YES	YES	YES	YES	YES	YES
[G]	NO	NO	YES	YES	YES	YES
[WM]	NO	NO	NO	NO	NO	NO

Definition 12. Let $\mathcal{G} \subseteq \mathcal{P}([0,1])$. We say that a score s on $\mathcal{G}$ satisfies the (adapted[7]) Gärdenfors property [G] if for every $A \in \mathcal{P}^*([0,1])$ and any element $x \notin A$ such that $A, A \cup \{x\} \in \mathcal{G}$ the following two conditions hold true:

i) [G1] $x < \inf A \Rightarrow s(A \cup \{x\}) < s(A)$,
ii) [G2] $\sup A < x \Rightarrow s(A) < s(A \cup \{x\})$.

Remark 9. Among the classical scores defined in Example 1 for finite subsets of the unit interval (TFHE's), the minimum satisfies [G1] but not [G2]. Similarly, the maximum satisfies [G2] but not [G1]. The simple average defined in Example 2, however, is a clear example of a score accomplishing the Gärdenfors property [G].

Definition 13. Let $\mathcal{G} \subseteq \mathcal{P}([0,1])$. We say that a score s on $\mathcal{G}$ satisfies the weak monotonicity property [WM] if for every $A, B \in \mathcal{P}^*([0,1])$ and $x \notin A \cup B$, such that $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{G}$, it holds true that $s(A) < s(B) \Rightarrow s(A \cup \{x\}) < s(B \cup \{x\})$.

Remark 10. Notice that [WM] is weaker than the property [MAS] of monotonicity by addition of sets, introduced in Definition 8.

Example 5. It is not easy to find scores satisfying [WM] except maybe in situations in which we work with special classes of finite subsets of the unit interval (see, e.g., [7,22]).[8]

Suppose for instance that we want to define a score s acting only for $\mathcal{F}_2^*([0,1])$, namely the nonempty subsets of $[0,1]$ with at most two elements. Obviously, we should start by declaring $s(\{a\} = a$ for every $a \in [0,1]$. If we define now $s(\{x,y\}) = \frac{x+y}{2}$, for every $x, y \in [0,1]$ this score, since it has been defined just for $\mathcal{F}_2^*([0,1])$, actually satisfies [WM].

With other scores [WM] usually fails to be true. Consider, for instance, the subsets $A = \{0.1, 0.15, 0.2, 0.25, 0.3, 0.35\}$ and $B = \{0.1, 0.25, 0.3\}$. The Xia–Xu score does not satisfy [WM] because it gives rise to $s(A) = 0.225 > s(B) \approx 0.21667$ but $s(A \cup \{0.95\}) \approx 0.32857 < s(B \cup \{0.95\}) = 0.4$. The geometric-mean score does not satisfy [WM] either since now $s(A) \approx 0.207 > s(B) \approx 0.19574$ but $s(A \cup \{0.95\}) \approx 0.25737 < s(B \cup \{0.95\}) \approx 0.2905$. And the simple average does not satisfy [WM] because $s(A) = 0.225 > s(B) = 0.2$ but $s(A \cup \{0.95\}) = 0.525 = s(B \cup \{0.95\}) = 0.525$.

A classical theorem by Kannai and Peleg [21] states the following incompatibility result.

Proposition 2. *Let $\mathcal{G} \subseteq \mathcal{P}([0,1])$ such that there exists a subset $A \subseteq [0,1]$ whose cardinality is at least 6, such that A and all its subsets belong to $\mathcal{G}$. Then there is no score s on $\mathcal{G}$ that satisfies both the adapted Gärdenfors property [G] and the weak monotonicity property [WM].*

Proof. See Kannai and Peleg [21], main Theorem on p. 174. □

Table 1 summarizes the features of the main scores introduced before.

4.2. Scores defined on special classes of sets

Scores sometimes are not defined on the whole set $\mathcal{P}([0,1])$, but, instead, they may act only on THFEs (namely, $\mathcal{F}^*([0,1])$), or on intervals $\mathcal{I}([0,1])$. In spite of most part of the classical literature dealing on scores defined on finite subsets of $[0,1]$, we believe that is worthwhile to analyze some properties of scores defined on remarkable families of infinite subsets of the unit interval (but not

[7] We say here "adapted" Gärdenfors property since this property was introduced by Gärdenfors [16] in 1976 just to deal with finite subsets, so that the original definition was stated making reference to maxima and minima instead of suprema and infima.

[8] The property [WM] has been considered in the literature (see [5,7,22]) when trying to construct some algorithms that, stepwise, extend a score defined on a family $\mathcal{F}_k^*([0,1])$ to the next $\mathcal{F}_{k+1}^*([0,1])$.

necessarily the whole set $\mathcal{P}([0,1])$). In particular we may pay attention to scores defined on families $\mathcal{G}$ that include the set of intervals $\mathcal{I}([0,1])$.

This approach will give rise to new definitions and results of compatibility and/or incompatibility of the newly defined properties about these kinds of scores.

To start with, we consider scores defined on intervals (and possibly, other HFEs).

Definition 14. Suppose $\mathcal{I}([0,1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0,1])$. We say that a score $s : \mathcal{G} \to [0,1]$ satisfies the property of interval compatibility [IC] when the following two properties hold true:

i) [IC1]: When $\{a\} \in \mathcal{G}$ with $a \in [0,1]$, then $s(\{a\}) = a$.
ii) [IC2]: When $a < b$ and $a, b \in [0,1]$, the scores of the intervals $[a,b]$, $[a,b)$, $(a,b]$ and (a,b) coincide (provided that they are admissible in $\mathcal{G}$).[9]

Remark 11. Note that [IC] implies property [C] or compatibility, but the converse is not true in general.

Definition 15. Suppose $\mathcal{I}([0,1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0,1])$. We say that a score $s : \mathcal{G} \to [0,1]$ is extremes monotonic [EM] when it satisfies the following two conditions:

i) [EM1]: $0 \leqslant b < b' \leqslant 1$ implies $s([a,b]) < s([a,b'])$ for each $a \in [0,b]$,
ii) [EM2]: $0 \leqslant a < a' \leqslant 1$ implies $s([a,b]) < s([a',b])$ for each $b \in [a',1]$.

Proposition 3. *Consider an interval score $s : \mathcal{G} \to [0,1]$ (i.e., s is a score such that $\mathcal{G} = \mathcal{I}([0,1])$). If s satisfies extremes monotonicity and interval compatibility, then s satisfies translation invariance.*

Proof. Let us check the property with $A \in \mathcal{I}([0,1])$.

It suffices to proceed when $A = [a,b]$ for some $a, b \in [0,1]$ with $a < b$, due to interval compatibility. Notice that $a = b$ produces the claim $s([a,a]) < s([a+\varepsilon, a+\varepsilon])$ which boils down to $s(\{a\}) < s(\{a+\varepsilon\})$. This is equivalent to $a < a+\varepsilon$ by interval compatibility. Besides, the case where A is half-open or open with $a < b$ reduces to $A = [a,b]$ by interval compatibility.

Fix $\varepsilon > 0$ such that $b + \varepsilon \leqslant 1$. Then we need to show $s([a,b]) < s([a+\varepsilon, b+\varepsilon])$. We just need to invoke extremes monotonicity twice:

$$s([a+\varepsilon, b+\varepsilon]) > s([a, b+\varepsilon]) > s([a,b]). \quad \square$$

The next property will be shown to be excessively demanding. Afterwards we shall discuss its rejection.

Definition 16. Suppose $\mathcal{I}([0,1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0,1])$. We say that a score s on $\mathcal{G}$ is strongly extremes monotonic [SEM] when for each $a, b \in [0,1]$ with $a < b$ it holds true that $s([0,a)) < s([0,a]) < s([0,b))$.

Lemma 1. *Consider a score $s : \mathcal{G} \to [0,1]$ with $\mathcal{I}([0,1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0,1])$. Then s cannot satisfy the property [SEM] of strong extremes monotonicity.*

Proof. We proceed by contradiction. Suppose $s : \mathcal{G} \to [0,1]$ satisfies [SEM]. Then we can associate with each $a \in [0,1]$, a rational number $q_a \in (s([0,a)), s([0,a]))$. Now $a, b \in [0,1]$ with $a < b$ implies $q_a < s([0,a]) < s([0,b)) < q_b$. This is a contradiction with the fact that the set $\mathbb{Q}$ of rationals is countably infinite. $\square$

This impossibility can be traced back to the fact that under strong extremes monotonicity, adding/removing one single point to/from an uncountable subset (with a very specific form) modifies the score assigned to the subset. For this reason we believe that Lemma 1 supports the view that the property of interval compatibility [IC] should be imposed on scores defined on any $\mathcal{G}$ containing intervals.

A similar argument proves:

Lemma 2. *Consider a score $s : \mathcal{G} \to [0,1]$ with $\mathcal{I}([0,1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0,1])$. Then s cannot satisfy both [SMU] and [EM1].*

Additionally, in view of this setback we are motivated to consider a variation of Definition 16 in the garb of a more technical cardinality issue:

[9] We need to impose the condition $a < b$ in [IC2] to avoid a mathematical contradiction. Otherwise, [IC] might impose the condition $s((a,a)) = s([a,a]) = s(\{a\}) = a$, against the fact that when $a > 0$, the interval (a,a) is empty, therefore $s((a,a)) = s(\emptyset) = 0 \neq a$.

Definition 17. We say that $s : \mathcal{G} \to [0,1]$ is star monotonic [STM] when for each uncountable $A \in \mathcal{G}$ it holds true that $s(A) > s(B)$ when $B \subseteq A$, $B \neq A$, and $A \setminus B$ is infinite.

So in Definition 17 we consider a property where the addition/removal of an infinite set of evaluations guarantees the improvement/reduction of the score assigned to the subset of evaluations (the HFE).

We will discuss it, and give arguments proving that this is not a "reasonable" property for a score to fulfill. Notice that when passing from $A \setminus B$ to A, we might add worse-off evaluations, which (at least intuitively) should reduce the score! This setback will be the germ of Proposition 4 below. Besides, for "internal consistency", probably we should accompany this property with a requirement like $s(A) = s(C)$ when A is infinite (or uncountably infinite), $C \subseteq A$, and $A \setminus C$ is finite. These considerations will lead to the formulation of Definition 21 which hopefully, is far more acceptable than Definition 17 above.

We shall show that star monotonicity is in contradiction with the next requirement, which does appear to be far more reasonable than Definition 17:

Definition 18. Suppose $\mathcal{I}([0,1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0,1])$. We say that a score $s : \mathcal{G} \to [0,1]$ satisfies trade-off [TO] when for each $a, b, b' \in [0,1]$ with $a < b < b'$, and each countable $A \subseteq [0,1]$ the following two properties are fulfilled:

i) Trade-off-1 [TO1]: $s([a,b]) \leqslant s([a,b'] \setminus A)$,
ii) Trade-off-2 [TO2]: $s([a,b] \setminus A) \leqslant s([a,b'])$.

Trade-off-1 [TO1] considers the addition of an uncountable number of evaluations, namely, $(b,b']$, which are all higher than any evaluation in $[a,b]$. Then the score cannot strictly decrease when moving from $[a,b]$ to $[a,b] \cup (b,b']$, even if simultaneously we remove a countable number of evaluations. Hence the name "trade-off": we replace an (at most) countable number of worse-off evaluations (i.e., those in $[a,b] \cap A$) with an uncountable number of better-off evaluations (i.e., those in $(b,b'] \setminus A$). After this exchange, the score of the set of evaluations under consideration cannot be strictly worse.

A similar defense of trade-off-2 [TO2] can be presented: if we remove a countable number of (potentially "good") evaluations from $[a,b]$, and we compare the score that arises (i.e., the score of $[a,b] \setminus A$) with the score of a set where an uncountable number of better-off evaluations is added (i.e., those in $(b,b']$), we cannot be strictly worse-off after the addition.

Due to the discussion about how poor star monotonicity is, probably the next Proposition 4 and Proposition 5 should be regarded as being of little value. Nevertheless, from a theoretical point of view, they can be used to support once more the idea that from a formal (not only intuitive) point of view, Definition 17 should be rejected in our context.

Proposition 4. *Consider a score $s : \mathcal{G} \to [0,1]$ with $\mathcal{I}([0,1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0,1])$. Then s cannot satisfy both star monotonicity [STM] and [EM2].*

Proof. We proceed by contradiction. Suppose $s : \mathcal{G} \to [0,1]$ satisfies star monotonicity and [EM2] (part 2 of extremes monotonicity). Then $s([0,0.5]) < s([0.1,0.5])$ by [EM2]. However $s([0,0.5]) > s([0.1,0.5])$ by [STM] (take here $A = [0,0.5]$, $B = [0.1,0.5]$). □

Proposition 5. *Consider a score $s : \mathcal{G} \to [0,1]$ with $\mathcal{I}([0,1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0,1])$. Then s cannot satisfy both star monotonicity [STM] and trade-off-1 [TO1].*

Proof. Again we proceed by contradiction. Suppose $s : \mathcal{G} \to [0,1]$ satisfies star monotonicity and trade-off-1. Let $A = [0,a]$ be an interval with $a > 0$, then $B = [0,a] \setminus \mathbb{Q} = A \setminus \mathbb{Q}$ is the set of irrational numbers of A. Due to star monotonicity, $s(A \setminus \mathbb{Q}) < s(A)$. Thus, the interval $((s(A \setminus \mathbb{Q}), s(A))$ is non-degenerate and it must contain rational numbers (see e.g., Courant and Robbins [11], pp. 79 and ff.). Using the Axiom of Choice, we select a rational number $q_a \in (s(A \setminus \mathbb{Q}), s(A))$.

Given $a, b \in (0,1]$ with $a < b$, by trade-off-1 and taking into account that $\mathbb{Q}$ is denumerable, it follows that $s([0,a]) \leqslant s([0,b] \setminus \mathbb{Q})$. Hence $q_a < s([0,a]) \leqslant s([0,b] \setminus \mathbb{Q}) < q_b$. This proves that the map $q : (0,1] \to \mathbb{Q}$ defined by $q(a) = q_a$ for each $a \in (0,1]$ is injective. A contradiction arises from the fact that $(0,1]$ is not denumerable (see [11], Chapter II, Section 4). □

In view of the discussion above, we are now more attracted by another form of trade-off. The limitations imposed by Definition 18 could be an argument to take sides with the next formulation. Henceforth we will always refer to the next definition when we mention trade-off:

Definition 19. Suppose $\mathcal{I}([0,1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0,1])$. We say that a score $s : \mathcal{G} \to [0,1]$ satisfies reinforced trade-off [RTO] when for each $a, b \in [0,1]$ with $a < b$, and each $\varepsilon > 0$ such that $1 \geqslant b + \varepsilon$ and $b - a \geqslant \varepsilon$ it holds true that

$$s([a,b] \setminus A) < s([a,b+\varepsilon]) \text{ for each } A \subseteq [b-\varepsilon, b] \text{ such that } [b-\varepsilon, b] \setminus A \text{ is infinite.}$$

Definition 19 considers the addition of an uncountable number of evaluations, namely, $(b, b+\varepsilon]$, which are all higher than any evaluation in $[a,b]$. Then the score cannot strictly decrease when moving from $[a,b]$ to $[a,b] \cup (b, b+\varepsilon]$, even if simultaneously we remove evaluations which do no separate from b more than ε units (and in a way such that at least an infinite number of evaluations, namely, those in $[b-\varepsilon, b] \setminus A$, are preserved). Hence the name "(reinforced) trade-off": we replace worse-off evaluations (i.e., those in $[a,b] \cap A \subseteq [b-\varepsilon, b]$) with an uncountable number of better-off evaluations (i.e., those in $(b, b+\varepsilon]$). After this exchange, the score of the set of evaluations under consideration must be strictly better.

Definition 20. Suppose $\mathcal{I}([0,1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0,1])$. We say that a score s on $\mathcal{G}$ is interval monotonic [IM] when for each $a, b, a', b' \in [0,1]$ with $a \leqslant b$ and $a' \leqslant b'$, and either $b < a'$ or $b = a' < b'$, it holds true that $s([a,b]) < s([a',b'])$.

To conclude this section, we introduce some other potentially interesting properties for scores defined on certain collections that contain infinite HFEs, not necessarily intervals.

Definition 21. We say that a score $s: \mathcal{G} \to [0,1]$ is positively monotonic [PM] when for each $A \in \mathcal{G}$ the following two properties are satisfied:

i) [PM1]: $s(C \cup A) > s(C)$ when $A > C$ and either A is infinite or C is finite,
ii) [PM2]: $s(A \cup C) = s(A)$ when C is finite and A is uncountably infinite.

Positive monotonicity is a more justifiable attempt to improve upon start monotonicity. Consider its first part [PM1]. It is concerned with the addition of better-off evaluations to a set A. Then it claims that this extension always produces a set of evaluations with a strictly higher score, when either (a) the original set of evaluations is finite –irrespective of any other cardinality consideration– or (b) we add an infinite number of better-off evaluations. Consider now its second part [PM2]. It captures the fact that adding a finite number of evaluations to an uncountably infinite set of evaluations should not modify the score (because, in terms of cardinality, the change in the set of evaluations is remarkably negligible).

Observe that star monotonicity implies one of the claims contained in positive monotonicity, namely, under star monotonicity one has $s(C \cup A) > s(C)$ when $A > C$ and A is infinite. So in positive monotonicity, we retain a scarcely controversial consequence of star monotonicity, and also we add other related properties with a similar concern.

Finally in this section, we inquire whether positive monotonicity is compatible with reinforced trade-off. The next Proposition 6 shows that the answer to this question is negative:

Proposition 6. *Consider a score $s: \mathcal{G} \to [0,1]$ with $\mathcal{I}([0,1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0,1])$. Then s cannot satisfy both reinforced trade-off (Definition 19) and positive monotonicity (Definition 21).*

Proof. We proceed by contradiction (and quite similarly to our previous incompatibility results). Hence suppose $s: \mathcal{G} \to [0,1]$ satisfies both positive monotonicity and reinforced trade-off.

With each $b \in (\frac{1}{4}, \frac{3}{4})$, we fix a strictly decreasing sequence $\{x_n^b\}_{n\in\mathbb{N}}$ of numbers in the interval $(\frac{b}{2}, \frac{3}{4})$ convergent to $\frac{b}{2}$ and let $A_b = \{x_n^b | n \in \mathbb{N}\}$. Then [PM1] ensures that $s([0, \frac{b}{2}]) < s([0, \frac{b}{2}] \cup A_b)$. We select a rational number q_b with $s([0, \frac{b}{2}]) < q_b < s([0, \frac{b}{2}] \cup A_b)$.

A contradiction will arise if we prove that $\frac{1}{4} < b < b' < \frac{3}{4}$ implies $q_b < q_{b'}$, since this would yield an injection from $(\frac{1}{4}, \frac{3}{4})$ into $\mathbb{Q}$, contradicting the fact that $\mathbb{Q}$ is countably infinite.

Fix $b < b' < \frac{3}{4}$. Select $\varepsilon = \frac{b'-b}{4}$, so $\varepsilon < \frac{1}{8}$ and it satisfies $\frac{b}{2} + \varepsilon = \frac{b'}{2} - \varepsilon = \frac{b+b'}{4}$. We set $a = \frac{b}{2} + \varepsilon = \frac{b'}{2} - \varepsilon$, so a is the midpoint of $\frac{b}{2}$ and $\frac{b'}{2}$.

By the construction of A_b, $A_{b-b'} := (\frac{b}{2}, a] \cap A_b$ is infinite, while $(a, 1] \cap A_b$ is finite. Now, [PM2] ensures that $s([0, \frac{b}{2}] \cup A_b)) = s([0, \frac{b}{2}] \cup A_{b-b'})$.

Observe that $1 \geqslant a + \varepsilon$ (because $a + \varepsilon = \frac{b'}{2} < \frac{3}{8}$) and $a \geqslant \varepsilon$ (because $a > \frac{b}{2} > \frac{1}{8} > \varepsilon$). Therefore, [RTO] applies to the interval $[0, a]$, $\varepsilon > 0$, and the set $(\frac{b}{2}, a] \setminus A_{b-b'} \subseteq (\frac{b}{2}, a] \subseteq [a-\varepsilon, a]$. It guarantees $s([0, \frac{b}{2}] \cup A_{b-b'}) < s([0, a+\varepsilon] = s([0, \frac{b'}{2}])$.

We have thus shown $q_b < s([0, \frac{b}{2}] \cup A_b) = s([0, \frac{b}{2}] \cup A_{b-b'}) < s([0, \frac{b'}{2}]) < q_{b'}$. This finishes the proof. □

Remark 12. We have not used the full force of Definition 21, part 1. The consequence for C finite can be removed from the Definition without affecting the validity of the impossibility result. However, we have kept it in the statement because we think that it is natural enough and innocuous.

Observe also that positive monotonicity implies interval monotonicity in the presence of interval compatibility. Note that interval compatibility is needed here to prove $s([a,b]) < s([b,b']) = s((b,b'])$ only.

5. Concluding remarks and lines for future research

From inspection of the existing literature, one can observe that the study of scores defined on infinite HFEs is still in its childhood. We have tried to fill this gap, analyzing some possible features and properties of scores, at least in the case where the scores act on finite unions of intervals contained in $[0,1]$. This framework connects our investigation to the most fundamental reaction to a paradox involving the utilization of fuzzy sets. Zadeh [41] remarked that although fuzziness attempts to capture vague expressions, one precise membership degree (i.e., a number) must be ascribed for each element. As a reaction to this methodological inconsistency, in 1975 various articles introduced interval-valued fuzzy sets [17,19,28,41], which happen to be a specific type of HFS. Both are type-2 fuzzy sets. Future research can be done to exploit these connections, because many authors [24,34,37,39] have investigated scores in models that expand fuzzy sets with the utilization of closed subintervals of the unit interval.

Besides, our work has demonstrated that desirable properties are not compatible with other interesting characteristics for a score to fulfill. This is important because the researcher must be aware that properties that are individually attractive happen to be jointly incompatible. This debate helps understand why certain properties should be rejected, or mitigated, if we need to retain other characteristics of a score. In this way, we inaugurate the axiomatic approach to the study of scores on HFEs with both possibility and impossibility results. This novel approach can be exported to the models mentioned above in the future.

Finally, we have also shown that fundamental scores defined on typical HFEs, remarkably, the geometric-mean, Xia-Xu and Farhadinia's constructions, can be extended to a more general setting with an uncountable number of evaluations.

Here is a brief summary of the main achievements and contributions of our article:

1) Furnishing a formal and rigorous definition of the concept of a score. This corresponds to the *"Was sind?"* of the title.
2) Proving that the existence of ties when dealing with scores defined on finite sets of $[0, 1]$ is unavoidable.
3) Proving by example that a score on non-degenerate closed intervals of the unit interval exists that does not produce ties.
4) Extending the concept of score to infinite subsets of $[0, 1]$ and analyzing its main features.
5) Classifying the main features that the scores may have, paying attention to several criteria, and then studying their hierarchies and mutual compatibility. This point, jointly with point 4), corresponds to the *"Was sollen?"* of the title.
6) Proving new possibility/impossibility results when more than one feature are imposed a priori on a family of scores.

Concerning other lines for future research, we point out that this study bears comparison with the analysis of extensions of total orders from a set to its power set, following an list of criteria imposed ex-ante (see [5]). Thus, incompatibility results such as those proved in Section 4 could give rise to incompatibility results about extensions of total orders, and vice versa.

CRediT authorship contribution statement

José Carlos R. Alcantud: Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Resources, Validation, Writing – original draft, Writing – review & editing. **María J. Campión:** Conceptualization, Formal analysis, Investigation, Methodology, Resources, Validation, Writing – original draft. **Esteban Induráin:** Conceptualization, Formal analysis, Investigation, Methodology, Resources, Validation, Visualization, Writing – original draft, Writing – review & editing. **Ana Munárriz:** Conceptualization, Formal analysis, Investigation, Methodology, Resources, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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