FEM MATRICES FOR PROBLEMS OF FREE VIBRATIONS AND BUCKLING OF A TRUNCATED CONE BEAM

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Abstract. *Proper use of materials is one of the most important criteria of a rational design and shaping of engineering constructions. It requires such dimensioning of each element of the construction which will ensure that the element is matched to its load – and this condition is fulfilled only for beams with variable cross section. Hence, it is essential to develop possibilities of calculations of beams with cross section varying along the beam longitudinal axis. This study provides relevant matrices (i.e. stiffness, mass and initial stress matrix) applied in the Finite Element Method for calculations of natural frequencies and buckling critical forces. The matrices have been derived for beams shaped as a truncated cone with a linear generatrix, supported in various ways. The results have been compared to those obtained for the stair-shaped beams approximating the conical ones; a good concordance of results has been stated.*

Keywords

Critical buckling force, Finite Element Method, first natural frequency, initial stress matrix, mass matrix, stiffness matrix, variable cross section beam.

1. Introduction

Problems of bending, stability and vibrations of variable cross section beams are being solved with many methods. The vibration mode shape and frequency as well as the deflection line after buckling and the buckling critical force can be determined from the Euler-Bernoulli differential equation of beam deflection. A classical solution by means of the Bessel functions was presented by Conway and Dubil [1]. Other works worth mentioning have been made by Ece et al. [2] (for an exponential variability of beam width), Laura et al. [3] (for beams of bilinearly varying thickness), Naguleswaran [4, 5] as well as Duan and Wang [6] – both for multi-segment, stair-shaped beams. A numerical solution for the natural vibration shape of beams with multiple step changes was provided by Vaz and Lima Junior [7]. If referred to approximated methods, it is worth to mention works by Jaworski et al. [8], Jaworski and Szlachetka [9] and Szlachetka et al. [10]. All of them apply the Rayleigh's method to find first natural frequencies for solid and hollow truncated cones with generatrices having the shape of straight line, concave parabola and convex parabola.

Naguleswaran [11] provided an exact solution for truncated cone and truncated wedge beams using the Frobenius method and submitted results for various types of beams. Due to an extensivity and accuracy of this study, it can be acknowledged as a benchmark in terms of the natural frequencies: the analysis is very extensive – concerns 16 combinations of supports (even unstable) and two types of stiffness variability (depending on the fourth and third power of the longitudinal coordinate).

Regarding the buckling, Qiusheng et al. [12] applied the Euler-Bernoulli differential equation to obtain exact solutions for stability analysis of beams with varying cross sections subjected to various axial loads (concentrated and variably distributed). Coşkun and Atay [13] used the variational iteration method (VIM) for seeking critical buckling loads of Euler columns with constant and variable cross-sections (three values of truncation factor) and support conditions (five combinations of supports). Soltani and Asgarian [14] used a combination of power series expansions and the Rayleigh-Ritz method for the analysis of stability and free vibration of axially functionally graded beams resting on an elastic foundation. The obtained results were compared to the results of Finite Element Method (FEM). The authors provided values of critical buckling forces and natural frequencies for three types of stiffness variability as well as three combinations of supports. Results obtained in [14] can be acknowledged as benchmark in terms of the critical buckling forces.

Solving problems of stability and vibrations of variable cross section beams with use of FEM is quite simple if the beam is divided into finite elements with a constant cross section (stair-shaped). However, in aim to enable the best possible approximation of the beam shape, it requires an assumption of a very large number of such elements what significantly increases the dimensions of matrices and the whole task. This problem can be overcome if shape functions, matrices of stiffness, mass and initial stress for a beam finite element with variable cross section are known; they allow to divide the beam into several finite elements. There is a lot of such works in the literature (Šapalas et al. [15], Katsikadelis and Tsiatas [16], Smoljanović et al. [17], etc.) but the basic problem is that these papers do not provide appropriate FEM matrices.

The present study is aimed to fill this gap. Due to a restricted space, however, this paper is limited only to one specific type of beams – where the second area moment depends on the fourth power of an axial coordinate; nevertheless, the presented derivation applies to all types of cross section variability. For the abovementioned variability type, all matrices (of stiffness, mass and initial stress) and shape functions, required in the Finite Element Method for calculations of statics, dynamics and stability of constructions consisting of variable cross section beams have been submitted. The formulas have been applied for calculations of first natural frequencies and critical buckling forces of beams supported in different ways and obtained results have been compared to results obtained for piecewise constant (stair-shaped) cross section beams.

2. Methods

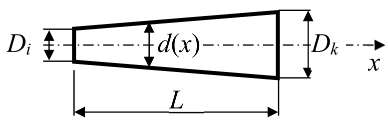

Fig. 1: General dimensions of a circular cross section beam with linearly changing diameter (truncated cone)

It is assumed in the paper that the beam cross section is circular with the diameter changing linearly (Fig. 1), i.e. the beam is a truncated cone. Hence, the second area moment depends on an axial coordinate in the fourth power. The related formulas for the diameter $d(x)$, cross section area $A(x)$ and second area moment $J(x)$ are following:

$$d(x) = D_i\left(1-\frac{1-\eta}{L}x\right),\quad A(x) = A_i\left(1-\frac{1-\eta}{L}x\right)^2, \quad J(x) = J_i\left(1-\frac{1-\eta}{L}x\right)^4, \tag{1}$$

where: D_i – a diameter of the beginning of the beam, $A_i = \frac{\pi D_i^2}{4}$, $J_i = \frac{\pi D_i^4}{64}$ – cross section area and second area moment at the beginning of the beam, L – length of the beam, $\eta = \frac{D_k}{D_i}$ – a truncation factor representing a ratio between the diameters at the end (D_k) and at the beginning of the beam. It is worth mentioning that a rectangular beam with both height and width changing linearly but maintaining the proportions between each other also has the second area moment depending on the fourth power of the axial coordinate, thus the relations obtained in this paper are also valid for it.

The beams calculated in this chapter are supported in 10 ways. The supports are classified according to the possibility of transversal movement ($w = 0$ or $w \neq 0$) or the possibility of rotation ($\varphi = 0$ or $\varphi \neq 0$). Along with their denotations, they are following:

- clamped support (C): $w = 0,\ \varphi = 0$,
- pinned support (P): $w = 0,\ \varphi \neq 0$,
- sliding support (S): $w \neq 0,\ \varphi = 0$,
- free end (F): $w \neq 0,\ \varphi \neq 0$.

Taking the aforementioned denotations into account, one can distinguish 10 combinations of supports: CC, CP, PC, PP, CS, SC, CF, FC, SP, PS (Fig. 2). As only simple beams (not connected with other beams) are being considered in the paper and the finite elements are not connected by hinges, it is enough for these types of support to provide a finite element clamped at both ends; a real support of a specific beam is modelled by removing appropriate rows and columns in the global matrices (according to the general rules of FEM).

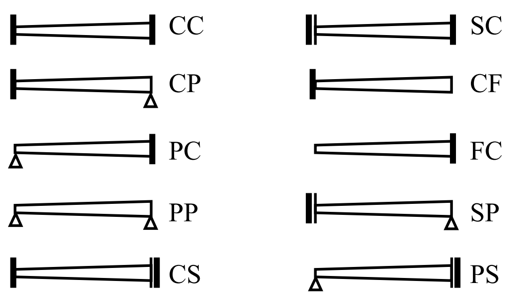

Fig. 2: Supporting of beams

The calculations of the FEM matrices start from solving two Euler-Bernoulli equations of displacements of a beam without load:

- for longitudinal displacements $u(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EA(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right) = 0, \tag{2}$$

- for transversal displacements $w(x)$:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EJ(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\right) = 0, \tag{3}$$

where E – Young modulus of the beam. As boundary conditions, it must be assumed:

$$u|_{x=0} = u_i,\ u|_{x=L} = u_k,\ w|_{x=0} = w_i,$$
$$\left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=0} = \varphi_i,\ w|_{x=L} = w_k,\ \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=L} = \varphi_k. \quad (4)$$

Following displacement formulas have been obtained:

$$u(x) = F_1 u_i + F_2 u_k, \quad (5a)$$

$$w(x) = F_3 w_i + F_4 \varphi_i + F_5 w_k + F_6 \varphi_k, \quad (5b)$$

where

$$F_1 = \frac{(L-x)\eta}{x+L\eta-x\eta},\quad F_2 = \frac{x}{x+L\eta-x\eta},$$
$$F_3 = \frac{(L-x)^2\eta(2x+L\eta)}{L(x+L\eta-x\eta)^2},\quad F_4 = \frac{(L-x)^2 x\eta^2}{(x+L\eta-x\eta)^2},$$
$$F_5 = \frac{x^2(L+2L\eta-2x\eta)}{L(x+L\eta-x\eta)^2},\quad F_6 = -\frac{x^2(L-x)}{(x+L\eta-x\eta)^2}$$

are shape functions constituting a shape function matrix **N**:

$$\mathbf{N} = \begin{bmatrix} F_1 & 0 & 0 & F_2 & 0 & 0 \\ 0 & F_3 & F_4 & 0 & F_5 & F_6 \end{bmatrix}. \quad (6)$$

Having the shape function matrix, one can calculate a stiffness matrix **K**, mass matrix **M** and initial stress matrix $\mathbf{K}_\sigma$ which are expressed as:

$$\mathbf{K} = \int_L \mathbf{B}^T \mathbf{E}\mathbf{B}\,\mathrm{d}x,\quad \mathbf{M} = \rho\int_L \mathbf{N}^T \mathbf{N} A(x)\,\mathrm{d}x,$$
$$\mathbf{K}_\sigma = \int_L S\left(\frac{\mathrm{d}\mathbf{N}_y}{\mathrm{d}x}\right)^T\left(\frac{\mathrm{d}\mathbf{N}_y}{\mathrm{d}x}\right)\mathrm{d}x, \quad (7)$$

where: $\mathbf{E} = \begin{bmatrix} EA(x) & 0 \\ 0 & EJ(x) \end{bmatrix}$ – an elasticity matrix, $\mathbf{B} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & \frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{bmatrix}\mathbf{N}$ – a strain matrix, ρ – mass density, $\mathbf{N}_y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_3 & F_4 & 0 & F_5 & F_6 \end{bmatrix}$, S – a longitudinal force acting on a beam. Execution of the calculations according to Form. (7) yields:

$$\mathbf{K} = \begin{bmatrix} \frac{EA_i}{L\eta} & 0 & 0 & -\frac{EA_i}{L\eta} & 0 & 0 \\ 0 & 4EJ_i\frac{1+\eta+\eta^2}{L^3\eta^3} & 2EJ_i\frac{1+2\eta}{L^2\eta^2} & 0 & -4EJ_i\frac{1+\eta+\eta^2}{L^3\eta^3} & 2EJ_i\frac{2+\eta}{L^2\eta^3} \\ 0 & 2EJ_i\frac{1+2\eta}{L^2\eta^2} & 4\frac{EJ_i}{L\eta} & 0 & -2EJ_i\frac{1+2\eta}{L^2\eta^2} & 2\frac{EJ_i}{L\eta^2} \\ -\frac{EA_i}{L\eta} & 0 & 0 & \frac{EA_i}{L\eta} & 0 & 0 \\ 0 & -4EJ_i\frac{1+\eta+\eta^2}{L^3\eta^3} & -2EJ_i\frac{1+2\eta}{L^2\eta^2} & 0 & 4EJ_i\frac{1+\eta+\eta^2}{L^3\eta^3} & -2EJ_i\frac{2+\eta}{L^2\eta^3} \\ 0 & 2EJ_i\frac{2+\eta}{L^2\eta^3} & 2\frac{EJ_i}{L\eta^2} & 0 & -2EJ_i\frac{2+\eta}{L^2\eta^3} & 4\frac{EJ_i}{L\eta^3} \end{bmatrix}, \quad (8)$$

$$\mathbf{M} = \rho A_i L\begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{6\eta} & 0 & 0 \\ 0 & \frac{a_{22}}{15(\eta-1)^7} & \frac{L\eta a_{23}}{60(\eta-1)^7} & 0 & \frac{a_{25}}{30\eta(\eta-1)^7} & \frac{La_{26}}{60\eta(\eta-1)^7} \\ 0 & \frac{L\eta a_{23}}{60(\eta-1)^7} & \frac{L^2\eta^2 a_{33}}{30(\eta-1)^7} & 0 & \frac{La_{35}}{60(\eta-1)^7} & \frac{L^2 a_{36}}{20(\eta-1)^7} \\ \frac{1}{6\eta} & 0 & 0 & \frac{1}{3\eta^2} & 0 & 0 \\ 0 & \frac{a_{25}}{30\eta(\eta-1)^7} & \frac{La_{35}}{60(\eta-1)^7} & 0 & \frac{a_{55}}{15\eta^2(\eta-1)^7} & \frac{La_{56}}{60\eta^2(\eta-1)^7} \\ 0 & \frac{La_{26}}{60\eta(\eta-1)^7} & \frac{L^2 a_{36}}{20(\eta-1)^7} & 0 & \frac{La_{56}}{60\eta^2(\eta-1)^7} & \frac{L^2 a_{66}}{30\eta^2(\eta-1)^7} \end{bmatrix}, \quad (9)$$

where: $a_{22} = -12 - 80\eta - 85\eta^2 + 155\eta^3 - 60\eta^4 + 112\eta^5 - 35\eta^6 + 5\eta^7 - 60\eta\left(1+2\eta+2\eta^2+\eta^3\right)\ln(\eta)$,

$a_{23} = -24 - 315\eta + 95\eta^2 + 100\eta^3 + 180\eta^4 - 41\eta^5 + 5\eta^6 - 60\eta\left(3 + 5\eta + 4\eta^2\right)\ln(\eta)$,

$a_{33} = -6 - 125\eta + 80\eta^2 + 60\eta^3 - 10\eta^4 + \eta^5 - 60\eta\left(1 + 2\eta\right)\ln(\eta)$,

$a_{25} = -5 + 49\eta + 125\eta^2 + 135\eta^3 - 135\eta^4 - 125\eta^5 - 49\eta^6 + 5\eta^7 + 120\eta\left(1 + 2\eta + 2\eta^2 + \eta^3\right)\ln(\eta)$,

$a_{26} = 6 - 75\eta - 275\eta^2 + 80\eta^3 + 210\eta^4 + 59\eta^5 - 5\eta^6 - 60\eta^2\left(4 + 5\eta + 3\eta^2\right)\ln(\eta)$,

$a_{35} = -5 + 59\eta + 210\eta^2 + 80\eta^3 - 275\eta^4 - 75\eta^5 + 6\eta^6 + 60\eta^2\left(3 + 5\eta + 4\eta^2\right)\ln(\eta)$,

$a_{36} = 1 - 15\eta - 80\eta^2 + 80\eta^3 + 15\eta^4 - \eta^5 - 60\eta^2\left(1 + \eta\right)\ln(\eta)$,

$a_{55} = -5 + 35\eta - 112\eta^2 + 60\eta^3 - 155\eta^4 + 85\eta^5 + 80\eta^6 + 12\eta^7 - 60\eta^3\left(1 + 2\eta + 2\eta^2 + \eta^3\right)\ln(\eta)$,

$a_{56} = 5 - 41\eta + 180\eta^2 + 100\eta^3 + 95\eta^4 - 315\eta^5 - 24\eta^6 + 60\eta^3\left(4 + 5\eta + 3\eta^2\right)\ln(\eta)$,

$a_{66} = -1 + 10\eta - 60\eta^2 - 80\eta^3 + 125\eta^4 + 6\eta^5 - 60\eta^3\left(2 + \eta\right)\ln(\eta)$,

$$\mathbf{K}_\sigma = \frac{S}{30}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4\left(1+7\eta+\eta^2\right)}{L\eta} & 4\eta-1 & 0 & -\frac{4\left(1+7\eta+\eta^2\right)}{L\eta} & \frac{4-\eta}{\eta} \\ 0 & 4\eta-1 & 4L\eta & 0 & 1-4\eta & -L \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{4\left(1+7\eta+\eta^2\right)}{L\eta} & 1-4\eta & 0 & \frac{4\left(1+7\eta+\eta^2\right)}{L\eta} & \frac{\eta-4}{\eta} \\ 0 & \frac{4-\eta}{\eta} & -L & 0 & \frac{\eta-4}{\eta} & \frac{4L}{\eta} \end{bmatrix}. \tag{10}$$

Then, having the matrices **K**, **M** and $\mathbf{K}_\sigma$, FEM equations for free vibrations and stability problem and their characteristic equations can be obtained:

$$\mathbf{M}\ddot{\mathbf{\Delta}} + \mathbf{K}\mathbf{\Delta} = 0 \;\Rightarrow\; \det(\mathbf{M}\omega^2 - \mathbf{K}) = 0, \tag{11a}$$

$$\left(\mathbf{K} + \mathbf{K}_\sigma\right)\mathbf{\Delta} = 0 \;\Rightarrow\; \det(\mathbf{K} + \mathbf{K}_\sigma) = 0, \tag{11b}$$

where $\mathbf{\Delta}$ – a displacement vector, ω – a vibration angular frequency. The characteristic equations enable to find a natural frequency ω_0 being a root of Eq. (11a) and a critical force P_{cr} being a root of Eq. (11b). The frequency ω_0 and the force P_{cr} are presented as a dimensionless (normalized) frequency ω_{nor} and a dimensionless (normalized) critical force P_{nor}, wherein

$$\omega_0 = \omega_{nor}^2\sqrt{\frac{EJ_i}{\rho A_i L^4}}, \quad P_{cr} = P_{nor}^2\frac{EJ_i}{L^2}. \tag{12}$$

3. Results

Results of application of the method presented in Section 2 are presented in Tab. 1. The dimensionless frequency ω_{nor} has been calculated for a beam consisting of 2 finite elements (i.e. the lowest possible number) characterized by the matrices (8), (9), (10); the dimensionless critical force P_{nor}, however, has been calculated for a beam consisting of 6 finite elements. These results are provided in Tab. 1 in the rows denoted as "v". As a comparison, analogical results have been provided for a beam consisting of 40 finite elements having the constant cross section area. These results are provided in Tab. 1 in the rows denoted as "c". Moreover, the benchmark solutions have been provided as well: for the frequencies – obtained by Naguleswaran [11] and denoted by "N"; for the critical buckling force – obtained by Soltani and Asgarian [14] and denoted by "S". The values printed with italic have been obtained by an extrapolation of the data provided in the relevant paper. Soltani and Asgarian provided data only for three types of supporting.

All results were calculated for the truncation factor η equal to 1, 2, …, 10. A small exception is the dimensionless frequency which was impossible to count for the variable cross section beam having $\eta = 1$, i.e. the constant cross section beam, probably due to certain limit passages existing for $\eta = 1$ in the matrix (10) (it is seen that $\eta = 1$ implies 0 in some denominators). In this case, $\eta = 1.1$ was assumed; assumption of values from the range 1.01–1.09 did not help either, although the reason is not known and requires a separate numerical analysis of the matrix (10).

Tab. 1: First natural frequencies and critical buckling forces of truncated cone beams for various values of truncation factor

η			Supporting									
			CC	CP	PC	PP	CF	FC	CS	SC	PS	SP
1 (1.1*)	ω_{nor}	v*	4.885	4.000	4.082	3.224	1.837	2.007	2.376	2.476	1.624	1.593
		c*	4.843	3.985	4.062	3.224	1.838	1.995	2.375	2.458	1.623	1.593
		N	*4.842*	*3.983*	*4.063*	*3.225*	*1.838*	*1.994*	*2.374*	*2.458*	*1.624*	*1.592*
	P_{nor}	v	6.288	4.488	4.494	3.142	1.571	1.571	3.142	3.142	1.571	1.571
		c	6.283	4.493	4.493	3.140	1.571	1.571	3.142	3.142	1.571	1.571
		S	–	–	*4.496*	*3.143*	–	*1.578*	–	–	–	–
2	ω_{nor}	v	5.795	4.406	5.108	3.737	1.611	3.045	2.503	3.374	1.966	1.695
		c	5.741	4.394	5.069	3.729	1.612	3.041	2.503	3.368	1.965	1.695
		N	5.741	4.393	5.070	3.730	1.611	3.041	2.502	3.368	1.966	1.695
	P_{nor}	v	12.59	8.995	8.993	6.284	2.331	4.058	6.573	6.573	4.058	2.331
		c	12.57	8.990	8.990	6.281	2.332	4.057	6.573	6.573	4.057	2.332
		S	–	–	8.996	6.286	–	4.060	–	–	–	–
3	ω_{nor}	v	6.623	4.744	6.007	4.126	1.468	4.007	2.657	4.248	2.212	1.720
		c	6.550	4.735	5.946	4.118	1.469	3.996	2.657	4.234	2.212	1.721
		N	*6.548*	*4.732*	*5.948*	*4.119*	*1.468*	*3.998*	*2.655*	*4.236*	*2.212*	*1.720*
	P_{nor}	v	18.91	13.56	13.51	9.428	2.902	6.867	10.43	10.43	6.867	2.902
		c	18.87	13.49	13.47	9.420	2.905	6.864	10.43	10.43	6.864	2.905
		S	–	–	*13.53*	*9.450*	–	*6.890*	–	–	–	–
4	ω_{nor}	v	7.340	5.017	6.771	4.425	1.372	4.846	2.796	5.031	2.396	1.712
		c	7.247	5.010	6.688	4.417	1.374	4.824	2.798	5.006	2.395	1.714
		N	7.244	5.006	6.690	4.419	1.372	4.826	2.796	5.007	2.395	1.712
	P_{nor}	v	25.21	18.12	18.08	12.58	3.379	9.825	14.53	14.53	9.825	3.379
		c	25.17	18.00	17.95	12.55	3.385	9.815	14.53	14.53	9.815	3.385
		S	–	–	*18.07*	*12.62*	–	*9.873*	–	–	–	–
5	ω_{nor}	v	7.984	5.249	7.449	4.672	1.301	5.599	2.920	5.745	2.546	1.692
		c	7.873	5.245	7.344	4.663	1.303	5.563	2.923	5.706	2.545	1.694
		N	7.868	5.239	7.348	4.665	1.301	5.566	2.920	5.708	2.546	1.692
	P_{nor}	v	31.54	22.62	22.72	15.74	3.800	12.86	18.77	18.77	12.86	3.800
		c	31.49	22.52	22.43	15.68	3.808	12.84	18.78	18.78	12.84	3.808
		S	–	–	*22.61*	*15.79*	–	*12.93*	–	–	–	–
6	ω_{nor}	v	8.573	5.452	8.066	4.883	1.245	6.286	3.031	6.405	2.673	1.667
		c	8.441	5.451	7.940	4.873	1.249	6.235	3.036	6.351	2.671	1.670
		N	*8.433*	*5.443*	*7.947*	*4.884*	*1.245*	*6.226*	*3.033*	*6.335*	*2.672*	*1.669*
	P_{nor}	v	38.28	27.08	27.46	18.91	4.173	15.94	23.09	23.09	15.94	4.173
		c	37.84	27.06	26.89	18.81	4.191	15.89	23.14	23.14	15.89	4.192
		S	–	–	*27.15*	*18.96*	–	*16.03*	–	–	–	–
7	ω_{nor}	v	9.120	5.634	8.636	5.070	1.200	6.922	3.132	7.020	2.784	1.641
		c	8.976	5.636	8.490	5.058	1.204	6.855	3.139	6.950	2.782	1.645
		N	*8.963*	*5.625*	*8.497*	*5.069*	*1.200*	*6.850*	*3.134*	*6.940*	*2.783*	*1.642*
	P_{nor}	v	44.50	31.52	32.23	22.10	4.518	19.05	27.47	27.47	19.05	4.518
		c	44.22	31.62	31.34	21.92	4.547	18.96	27.56	27.56	18.96	4.547
		S	–	–	*31.69*	*22.13*	–	*19.15*	–	–	–	–

Tab.1 (cont.)

η			Supporting									
			CC	CP	PC	PP	CF	FC	CS	SC	PS	SP
8	ω_{nor}	v	9.633	5.798	9.169	5.237	1.162	7.516	3.224	7.597	2.883	1.615
		c	9.467	5.805	9.003	5.224	1.167	7.431	3.233	7.510	2.880	1.621
		N	*9.460*	*5.790*	*9.011*	*5.232*	*1.162*	*7.433*	*3.226*	*7.509*	*2.882*	*1.616*
	P_{nor}	v	51.23	36.01	37.21	25.30	4.838	22.19	31.88	31.88	22.19	4.838
		c	50.64	36.20	35.76	25.02	4.880	22.04	32.04	32.04	22.04	4.880
		S	–	–	*36.22*	*25.30*	–	*22.26*	–	–	–	–
9	ω_{nor}	v	10.12	5.950	9.671	5.389	1.129	8.074	3.309	8.143	2.973	1.589
		c	9.945	5.961	9.486	5.375	1.136	7.972	3.321	8.038	2.969	1.597
		N	*9.930*	*5.941*	*9.495*	*5.380*	*1.129*	*7.983*	*3.310*	*8.049*	*2.973*	*1.590*
	P_{nor}	v	58.15	40.57	42.21	28.52	5.139	25.35	36.31	36.31	25.35	5.139
		c	57.10	40.81	40.17	28.11	5.195	25.11	36.56	36.56	25.11	5.195
		S	–	–	*40.75*	*28.46*	–	*25.37*	–	–	–	–
10	ω_{nor}	v	10.58	6.090	10.15	5.529	1.100	8.602	3.389	8.661	3.055	1.565
		c	10.39	6.106	9.943	5.513	1.108	8.482	3.402	8.539	3.050	1.575
		N	*10.38*	*6.082*	*9.954*	*5.515*	*1.100*	*8.503*	*3.387*	*8.562*	*3.055*	*1.565*
	P_{nor}	v	65.27	45.13	47.30	31.75	5.423	28.53	40.78	40.78	28.53	5.423
		c	63.61	45.45	44.56	31.19	5.496	28.18	41.12	41.12	28.19	5.496
		S	–	–	*45.28*	*31.62*	–	*28.46*	–	–	–	–

4. Analysis and conclusions

Table 1 shows a good conformity between the results obtained with use of the matrices (8)–(10) for beams consisting of the finite elements with variable cross section, results obtained with use of the known matrices for the finite elements with a constant cross section and the results obtained in the literature ([11], [14]) with use of other methods than FEM. It proves a correctness of the derived matrices. It also proves that the assumed number of the finite elements (2 for calculations of ω_{nor} and 6 for calculations of P_{nor}) is enough what – in turn – shows a high efficiency of the presented method.

Although there is a little literature data for the calculations of the buckling critical force, even the existing data for the three supporting ways allow to believe that the results for the remaining supporting are correct.

Similar calculations can be relatively easily performed for other types of variability of the cross section. For a construction practice, two types suggest themselves: 1) a truncated wedge with variable height (i.e. a cross section dimension changing in the bending plane) – then the second area moment depends on the 3rd power of the axial coordinate, 2) a truncated wedge with variable width (i.e. a cross section dimension changing transversely to the bending plane) – then the second area moment depends on the 1st power of the axial coordinate. However, a limited volume of this paper does not allow to present such calculations.

As a resumée, it can be concluded that the presented matrices can be applied in the Finite Element Method for calculations of the first natural frequency of transverse vibrations as well as critical buckling force of circular cross section beams with accuracy sufficient for engineering calculations. A certain difficulty arises from a quite complicated form of terms of the matrices for a finite element with variable stiffness – due to that, an introduction of these matrices into a computer program seems to be the most time-consuming part of the calculations. Nevertheless, two finite elements (as for calculations of ω_{nor}) described by these matrices generate a global matrix with dimensions 9 x 9 (i.e. 81 terms before reduction), six finite elements (as for calculations of P_{nor}) generate a global matrix with dimensions 21 x 21 (i.e. 441 terms before reduction). From the other hand, using 40 finite elements with a constant cross section (as for the comparative calculation results in the lines "c" of Table 1) generates a global matrix with dimensions 123 x 123 (i.e. 15129 terms before reduction). The difference and advantage is obvious.

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