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Existence and uniqueness of solution for fractional differential equations with integral boundary conditions and the Adomian decomposition method

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We propose an Adomian decomposition method to solve a class of nonlinear differential equations of fractional-order with modified Caputo derivatives and integral boundary conditions. Our approach uses the integral boundary conditions to derive an equivalent nonlinear Volterra integral equation before establishing existence and uniqueness of solution and a recursion scheme for the solution. The convergence of the method is proved and an error analysis given. Two numerical examples are solved by obtaining a rapidly converging sequence of analytical functions to the solution.

KEYWORDS

Adomian decomposition method, convergence analysis, error estimate, integral boundary conditions, nonlinear fractional differential equations, series solution

MSC CLASSIFICATION

26A33, 45J05, 65L99

1 | INTRODUCTION

During several decades, the topic of fractional calculus has attracted researchers from various fields of science and engineering.[1–3] In particular, fractional differential equations appear frequently in various fields of science and engineering, namely, in signal processing, control theory, diffusion, thermodynamics, biophysics, blood flow phenomena, rheology, electrodynamics, electrochemistry, electromagnetism, continuum and statistical mechanics, and dynamical systems. For more details on the applications of fractional differential equations, see, for example, previous work.[4–6] For recent applications of fractional order models to the study of the COVID-19, see Ndaïrou et al.[7,8]

In 2012, Cabada and Wang considered a class of nonlinear fractional differential equations with integral boundary value conditions given by

$$\begin{cases} {}^{C}D^{\alpha}u(t) + f(t, u(t)) = 0, \ t \in (0,1), \\ u(0) = u''(0) = 0, \ u(1) = \lambda \int_0^1 u(s)ds, \end{cases} \tag{1}$$

where $2 < \alpha < 3$, $0 < \lambda < 2$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function, establishing the existence of positive solutions with the help of the Guo–Krasnoselskii fixed point theorem.[9] Given the fact that problems with integral boundary conditions arise naturally in many applied fields of science, like thermal conduction problems, semiconductor problems, chemical engineering, blood flow problems, underground water problems, hydrodynamic problems and population dynamics, and include multipoint and nonlocal integral boundary value conditions as special cases, the work of Cabada and Wang[9] originated a strong research on integral boundary value problems of nonlinear multiterm fractional differential equations; see, for example, previous studies.[10–13] Among recent methods that are useful for such kind of fractional differential equations, we can mention the monotone iterative technique,[14] the topological degree theory,[15] and fixed point approaches.[16,17]

Generally, most fractional differential equations do not have exact/analytical solutions. For this reason, numerical and approximative methods become increasingly important for studying fractional differential equations.[18–20] With the purpose of solving fractional differential equations numerically, several algorithms have been investigated. These include the power series method, fractional Adams–Moulton methods, explicit Adams multistep algorithms, the Adomian decomposition method (ADM), variational iteration methods, fractional difference methods, decomposition methods, and least squares finite element solutions.[21–24]

In 2020, Jong et al. studied the following nonlinear problem involving nonlinear integral conditions:

$$\begin{cases} D^{\alpha}y(t) = f(t, y(t), D^{\beta_1}y(t), \dots, D^{\beta_n}y(t)), \ t \in (0,1), \\ y(0) = 0, \ y(1) = \int_0^1 g(s, u(s))ds, \end{cases} \tag{2}$$

where $1 < \alpha < 2$, $0 < \beta_1 < \cdots < \beta_n < 1$, $\alpha - \beta_n > 1$, $f : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}$, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ and D^{α} is the Riemann–Liouville fractional derivative.[25] They discuss the existence and uniqueness of solutions and propose a new method to obtain their approximate solutions. More precisely, their existence results are established by the Banach fixed point theorem and approximate solutions are determined by Daftardar–Gejji, Jafari, and Adomian iterative methods.[25] The results admit generalizations with Stieltjes integral boundary conditions.[26]

Here, we consider nonlinear problems with integral boundary conditions of form

$$\begin{cases} D_*^{\alpha}u(t) + \mu F(t, u(t)) = 0, \quad t \in (0, T), \\ u^{(m)}(0) = u_0^m, \quad u(T) = \lambda \int_0^T u(s)ds, \quad m = 0, \dots, n-2, \end{cases} \tag{3}$$

where D_*^{α} is the modified Caputo fractional derivative of order α (see Definition 4) with $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $n \geq 2$, $0 < \lambda < \alpha$, and $F(t, u(t)) = a(t)f(t, u(t))$ with function a continuous and non-negative on $[0, T]$ and $f \in C([0, T] \times [0, \infty), [0, \infty))$. We prove existence and uniqueness of solution and a recursion scheme to approximate it based on an ADM. The convergence of the method is proved and an error analysis given.

The paper is organized as follows. In Section 2, we recall necessary definitions and useful results about modified Caputo derivatives. Our original results begin with Section 3, where we rewrite our nonlinear fractional integral boundary value problem (3) as an equivalent Volterra integral equation (Theorem 7) and we prove existence and uniqueness of solution (Theorem 10). Then, in Section 4, we apply the ADM to approximate the solution of the considered problem. We prove convergence of the proposed recursive scheme (Theorem 11) as well as an upper bound for the error (Theorem 12). We finish with two numerical examples in order to illustrate the usefulness of the suggested method.

2 | PRELIMINARIES

For the concept of fractional derivative, we will adopt the definition of Caputo, which is a modification of the definition of Riemann–Liouville and has the advantage of correctly treating the problem of initial values in which the initial conditions are given in terms of field variables and their entire order, which is the case in most physical processes.[27,28] We briefly recall here the necessary definitions and results from fractional calculus theory; the interested reader can find all the details in the classical books.[27,28]

Definition 1. The Riemann–Liouville fractional integral of order α for a function f is defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0,$$

provided such integral exists.

Definition 2. For a function $f : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 3. The Riemann–Liouville fractional derivative of order α for a function f is defined by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \quad n = [\alpha] + 1,$$

provided the right-hand side of the previous equation is pointwise defined on $(0, \infty)$.

In order to have equivalence of solutions between a Caputo fractional equation and the fixed points of an integral equation, one needs to consider a modified version of the Caputo derivative (see Webb[29, Theorem 5.1]).

Definition 4 (See Webb[29,30] and Diethelm[31]). Let $\alpha \geq 0$ and $n = [\alpha]$. The modified Caputo derivative of order α of a function $f \in AC^n$, denoted by $D_*^{\alpha} f$, is defined by

$$D_*^{\alpha} f = D^{\alpha} \left[f - T_{n-1} f \right],$$

where $T_{n-1} f$ is the Taylor polynomial of degree $n-1$ of f, that is,

$$T_{n-1} f(t) := \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.$$

The following results are useful in the proof of our existence result (cf. proof of Theorem 7).

Theorem 5 (See Theorem 3.7 of Diethelm[31]). *If f is continuous and $\alpha \geq 0$, then*

$$D_*^{\alpha} I^{\alpha} f = f.$$

Theorem 6 (See Theorem 3.8 of Diethelm[31]). *Assume that $\alpha \geq 0$, $n = [\alpha]$, and $f \in AC^n[0, T]$. Then*

$$I^{\alpha} D_*^{\alpha} f(t) = f(t) + \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k.$$

3 | EXISTENCE AND UNIQUENESS OF SOLUTION

We begin by rewriting our nonlinear fractional integral boundary value problem (3) as an equivalent Volterra integral equation (Theorem 7). This gives an implicit formula (4) for $u(t)$, where the right-hand side also depends on the unknown function u. Such formula is useful to prove existence and uniqueness of solution (Theorem 10) but is not directly useful for approximation purposes: Many numerical approaches, like the predictor–corrector method, fail because of the dependence of our boundary condition on an integral from the initial time 0 to final time T that also depends on the unknown function u (therefore, to compute $u_{n+1}(t)$, we need to know the values of $u_n(t)$ for all $t \in [0, T]$). A good way to deal with this difficulty is to use the Adomian decomposition method, which will be the subject of Section 4.

Theorem 7. *Function $u \in C[0,1]$ is a solution of the boundary value problem (3) if, and only if, u satisfies*

$$\begin{aligned} u(t) = {} & \sum_{j=0}^{n-2} \frac{u_0^j}{j!} t^j + \frac{\lambda t^{n-1}}{T^{n-1}\left(1-\lambda \frac{T}{n}\right)} \sum_{j=0}^{n-2} \frac{u_0^j}{(j+1)!} T^{j+1} + \frac{t^{n-1}}{T^{n-1}} \left(\frac{-\lambda}{nT^{n-2}\left(1-\lambda\frac{T}{n}\right)} - 1 \right) \sum_{j=0}^{n-2} \frac{u_0^j}{j!} T^j \\ & - \mu I^\alpha F(t,u(t)) - \frac{\lambda \mu t^{n-1}}{T^{n-1}\left(1-\lambda\frac{T}{n}\right)} I^{\alpha+1} F(t,u(t))|_{t=T} + \left(\frac{\lambda}{T^{n-2}\left(1-\lambda\frac{T}{n}\right)} \frac{\mu}{n} + \frac{\mu}{T^{n-1}} \right) t^{n-1} \\ & \times I^\alpha F(t,u(t))|_{t=T} \end{aligned} \tag{4}$$

for all $t \in [0,T]$, where $F(t,u) = a(t)f(t,u)$.

Proof. We begin by proving the first implication.

($\Rightarrow$) Following the proof of Lemma 6.2 of Diethelm,[31] and applying Theorem 6, we reduce problem (3) to an equivalent integral equation. Precisely, by applying the Riemann–Liouville fractional integral I^α to both sides of (3), we get

$$u(t) = -I^\alpha F(t,u(t)) + \sum_{j=0}^{n-1} \frac{u^j(0)}{j!} t^j,$$

where $F(t,u(t)) = a(t)f(t,u(t))$. Because $u^{(m)}(0) = u_0^{(m)}$, $0 \le m \le n-2$, it follows that

$$u(t) = -\mu I^\alpha F(t,u(t)) + \sum_{j=0}^{n-2} \frac{u_0^j}{j!} t^j + \frac{u^{(n-1)}(0)}{(n-1)!} t^{n-1}.$$

The condition $u(T) = \lambda \int_0^T u(s)ds$ implies

$$u^{(n-1)}(0) = \frac{(n-1)!}{T^{n-1}} \left[\lambda \int_0^T u(s)ds + \mu \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} F(s,u(s))ds - \sum_{j=0}^{n-2} \frac{u_0^j}{j!} T^j \right].$$

Therefore,

$$u(t) = -\mu I^\alpha F(t,u(t)) + \sum_{j=0}^{n-2} \frac{u_0^j}{j!} t^j + \frac{t^{n-1}}{T^{n-1}} \left[\lambda \int_0^T u(s)ds + \mu \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} F(s,u(s))ds - \sum_{j=0}^{n-2} \frac{u_0^j}{j!} T^j \right].$$

Precisely, we have

$$\begin{aligned} u(t) = {} & \frac{-\mu}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s,u(s))ds + \frac{t^{n-1}}{T^{n-1}} \\ & \times \left[\lambda \int_0^T u(s)ds + \mu \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} F(s,u(s))ds - \sum_{j=0}^{n-2} \frac{u_0^j}{j!} T^j \right] + \sum_{j=0}^{n-2} \frac{u_0^j}{j!} t^j. \end{aligned} \tag{5}$$

Let $A = \int_0^T u(s)ds$. Then,

$$\begin{aligned} \int_0^T u(t)dt = {} & \frac{-\mu}{\Gamma(\alpha)} \int_0^T \int_0^t (t-s)^{\alpha-1} F(s,u(s))dsdt + \frac{\mu}{\Gamma(\alpha)T^{n-1}} \int_0^T \int_0^T (T-s)^{\alpha-1} t^{n-1} F(s,u(s))dsdt \\ & + \int_0^T \sum_{j=0}^{n-2} \frac{u_0^j}{j!} t^j dt + \frac{\lambda}{T^{n-1}} A \int_0^T t^{n-1} dt - \frac{1}{T^{n-1}} \int_0^T \left(\sum_{j=0}^{n-2} \frac{u_0^j}{j!} T^j \right) t^{n-1} dt. \end{aligned}$$

Hence,

$$A\left(1-\lambda\frac{T}{n}\right)=\frac{-\mu}{\alpha\Gamma(\alpha)}\int_0^T (t-s)^{\alpha}F(s,u(s))ds+\frac{\mu T}{n\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s,u(s))ds$$
$$+\sum_{j=0}^{n-2}\frac{u_0^j}{j!}\frac{T^{j+1}}{(j+1)}-\frac{T}{n}\sum_{j=0}^{n-2}\frac{u_0^j}{j!}T^j,$$

and we obtain that

$$A=\frac{1}{\left(1-\frac{\lambda T}{n}\right)}\left(\frac{-\mu}{\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}F(s,u(s))ds+\frac{\mu T}{n\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s,u(s))ds\right.$$
$$\left.+\sum_{j=0}^{n-2}\frac{u_0^j}{j!}\frac{T^{j+1}}{(j+1)}-\frac{T}{n}\sum_{j=0}^{n-2}\frac{u_0^j}{j!}T^j\right).$$

Replacing this relation into (5), we obtain the intended equality (4).

($\Leftarrow$) Applying the fractional differential operator D_*^{α} to (4), and recalling that the operator is linear and $D_*^{\alpha}t^n=0$ for all $n=0,1,\ldots,\lceil\alpha\rceil-1$, it follows from Theorem 5 that

$$D_*^{\alpha}u(t)=-\mu F\left(t,u(t)\right).$$

For the initial condition, we substitute $t=0$ in (4). It is clear that we only need to analyze the first term on the right-hand side of (4), because all the remaining terms will vanish at $t=0$:

$$u^{(m)}(0)=\frac{d^m}{dt^m}\left[\sum_{j=0}^{n-2}\frac{u_0^j}{j!}t^j\right]\Bigg|_{t=0},\quad m=0,\ldots,n-2.$$

The result follows by direct computations:

$$u(0)=\sum_{j=0}^{n-2}\frac{u_0^j}{j!}t^j\Bigg|_{t=0}=u_0^0,$$
$$u^{(1)}(0)=\frac{d}{dt}\left[\sum_{j=0}^{n-2}\frac{u_0^j}{j!}t^j\right]\Bigg|_{t=0}=u_0^1,$$
$$\vdots$$
$$u^{(n-2)}(0)=\frac{d^{n-2}}{dt^{n-2}}\left[\sum_{j=0}^{n-2}\frac{u_0^j}{j!}t^j\right]\Bigg|_{t=0}=u_0^{n-2}.$$

The boundary condition $u(T)=\lambda\int_0^T u(s)ds$ is an immediate consequence of (5). □

Remark 8. Our Theorem 7 is true for the $D_*^{\alpha}u$ derivative but to be true for the standard Caputo derivative ${}^CD^{\alpha}u(t)$ one should prove that $u^{(m)}\in AC$. It is not difficult to verify that the fixed points of the integral operator are in C^m but in the definition of ${}^CD^{\alpha}u(t)$ it appears $u^{(m+1)}$, which must be at least AC to be well defined. Despite this, the definition of $D_*^{\alpha}u$ is valid of $u\in C^m$ and one needs $u^{(m)}$ to be absolutely continuous to ensure that ${}^CD^{\alpha}u(t)=D_*^{\alpha}u$. This is not automatic if f is only continuous (see Webb[32, Remark 4.7]).

Now, let $E=C([0,T])$ be a Banach space endowed with the norm $\|u\|=\sup_{t\in[0,T]}|u(t)|$. In view of Theorem 7, we define the following operator $\mathcal{T}$:

$$\mathcal{T}:E\to E$$
$$u\mapsto\mathcal{T}u(t),\ t\in[0,T],$$

where

$$\mathcal{T}u(t) := \sum_{j=0}^{n-2} \frac{u_0^j}{j!} t^j + \frac{\lambda t^{n-1}}{T^{n-1}\left(1-\lambda\frac{T}{n}\right)} \sum_{j=0}^{n-2} \frac{u_0^j}{(j+1)!} T^{j+1} - \frac{t^{n-1}}{T^{n-1}} \left(\frac{\lambda}{nT^{n-2}(1-\lambda\frac{T}{n})} + 1 \right) \sum_{j=0}^{n-2} \frac{u_0^j}{j!} T^j$$
$$- \mu I^{\alpha} F(t,u(t)) - \frac{\lambda \mu t^{n-1}}{T^{n-1}\left(1-\lambda\frac{T}{n}\right)} I^{\alpha+1} F(t,u(t))_{|_{t=T}} + \left(\frac{\lambda}{T^{n-2}(1-\lambda\frac{T}{n})} \frac{\mu}{n} + \frac{\mu}{T^{n-1}} \right) t^{n-1} \tag{6}$$
$$\times I^{\alpha} F(t,u(t))_{|_{t=T}}$$

for all $t \in [0,T]$. It is clear that Equation (4) is equivalent to

$$u(t) = \mathcal{T}u(t), \ \ t \in [0,T]. \tag{7}$$

Moreover, the fixed points of operator $\mathcal{T}$ coincide with the solutions of problem (3) as assured by the results of Webb.[30]

Remark 9. The results of Webb[30] are for absolutely continuous functions, where singularities at time $t = 0$ are possible. Here, we are in the continuous and nonsingular case: Our solution u is continuous (see Theorem 7) and the integral operator $\mathcal{T}$ given by (6) has the necessary regularity to ensure the uniqueness of solution of the integral operator. In the present situation, where continuity holds and no singular term occurs, the equivalence between the two problems (fixed points of $\mathcal{T}$ and solutions of (3)) is known: see paragraph before Theorem 4.6 of Webb[30] and the reference therein.

Hereinafter, we assume that F is a continuous function that satisfies the following Lipschitz condition:

(H) There exists a constant $L_1 > 0$ such that

$$|F(t,u) - F(t,v)| \le L_1 |u - v| \ \text{ for each } \ t \in [0,T] \ \text{ and } \ u, v \in E.$$

Theorem 10 (Existence and uniqueness of solution to problem (3))**.** *Suppose that function F satisfies the Lipshitz condition (H) with constant L_1. If*

$$k = L_1 \left[\mu \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \right) + \left(\frac{\lambda}{T^{n-2}\left(1-\lambda\frac{T}{n}\right)} \frac{\mu}{n} + \frac{\mu}{T^{n-1}} \right) \left(\frac{T^{\alpha+1}}{\Gamma(\alpha+2)} \right) T^{n-1} \right.$$
$$\left. + \left(\frac{\lambda}{T^{n-2}\left(1-\lambda\frac{T}{n}\right)} \frac{\mu}{n} + \frac{\mu}{T^{n-1}} \right) \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \right) T^{n-1} \right] \tag{8}$$

is positive and less than one, that is, $0 < k < 1$, *then problem (3) has a unique solution.*

Proof. The result is a consequence of Theorem 7. Let $\mathcal{T}$ be the operator defined by (6). Then, for $u_1, u_2 \in E$, we have

$$\begin{aligned} ||\mathcal{T}u_1 - \mathcal{T}u_2|| &= \max |\mathcal{T}u_1(t) - \mathcal{T}u_2(t)| \\ &= \mu \left| I^{\alpha} F(t,u_1(t)) - I^{\alpha} F(t,u_2(t)) \right| \\ &\quad + \frac{\lambda \mu t^{n-1}}{T^{n-1}\left(1-\lambda\frac{T}{n}\right)} \left| I^{\alpha+1} F(t,u_1(t))_{|_{t=T}} - I^{\alpha+1} F(t,u_2(t))_{|_{t=T}} \right| \\ &\quad + \left(\frac{\lambda}{T^{n-2}\left(1-\lambda\frac{T}{n}\right)} \frac{\mu}{n} + \frac{\mu}{T^{n-1}} \right) t^{n-1} \left| I^{\alpha} F(t,u_1(t))_{|_{t=T}} - I^{\alpha} F(t,u_2(t))_{|_{t=T}} \right|. \end{aligned}$$

Using the fact that F satisfies hypothesis (H), then we get

$$||\mathcal{T}u_1 - \mathcal{T}u_2|| \leq k||u_1 - u_2||,$$

where k is given by (8). Under the condition $0 < k < 1$, the mapping $\mathcal{T}$ is a contraction and, therefore, by the Banach fixed-point theorem for contractions, there exist a unique solution to problem (4), which completes the proof. □

4 | APPROXIMATION OF THE SOLUTION

The ADM[33–36] is a powerful method developed for solving nonlinear differential equations. It requires the division of the unknown function $u(t)$ into components, which are infinite and expressed in the form $u_0, u_1, u_2, \dots$. For the nonlinear terms, the Adomian polynomials, noted by A_n, are calculated in terms of the nonlinearity. Precisely, assume one writes (4) as

$$u(t) = L(u(t)) + N(u(t)) + G(t), \tag{9}$$

where L is a linear operator, to be inverted, N is a nonlinear operator, assumed to be analytic, and G is a known given function. We will decompose the solution $u(t)$ into a rapidly convergent series of solution components, and then we decompose the analytic nonlinearity Nu into a series of Adomian polynomials:

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{10}$$

$$Nu(t) = \sum_{n=0}^{\infty} A_n, \tag{11}$$

where $A_n = A_n(u_0, u_1, \dots, u_n)$ are the well-known Adomian polynomials, defined by Adomian and Rach in 1983,[37] and which are given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{n} u_k \lambda^k\right)\Bigg|_{\lambda=0}, \quad n \geq 0. \tag{12}$$

For convenient reference, we list here the first five Adomian polynomials for the general analytic nonlinearity $N[u] = f(u)$:

$$\begin{aligned}
A_0 &= f(u_0),\\
A_1 &= f'(u_0)\cdot u_1,\\
A_2 &= f'(u_0)\cdot u_2 + f''(u_0)\cdot \frac{u_1^2}{2!},\\
A_3 &= f'(u_0)\cdot u_3 + f''(u_0)\cdot u_1u_2 + f^{(3)}(u_0)\cdot \frac{u_1^3}{3!},\\
A_4 &= f'(u_0)\cdot u_4 + f''(u_0)\cdot\left(u_1u_3 + \frac{u_2^2}{2!}\right) + f^{(3)}(u_0)\cdot \frac{u_1^2u_2}{2!} + f^{(4)}(u_0)\cdot \frac{u_1^4}{4!},\\
&\vdots
\end{aligned}$$

Therefore, by (10) and (11), Equation (9) becomes

$$\sum_{n=0}^{\infty} u_n(t) = L\left(\sum_{n=0}^{\infty} u_n(t)\right) + \sum_{n=0}^{\infty} A_n + G(t). \tag{13}$$

From (13), the u_n are determined by the following recursion scheme:

$$\begin{cases} u_0(t) = G(t),\\ u_{n+1}(t) = L(u_n(t)) + A_n, \ n = 0, 1, 2, \dots \end{cases} \tag{14}$$

If we define the N-term approximation to the solution as

$$\phi_N(t) = \sum_{n=0}^{N-1} u_n(t), \tag{15}$$

then the ADM asserts that the exact solution $u(t)$ is given by

$$u(t) = \lim_{N\to+\infty} \phi_N(t). \tag{16}$$

4.1 | Convergence and error estimate

In practical terms, we obtain approximations to the solution asserted by Theorem 10 by using our Theorem 7 together with the ADM and the convergence of the series solution given by (10).

Theorem 11 (Convergence of the method)**.** *Assume that the operators L and N, defined in Equation (9) for our problem (3) in form (4), satisfy, respectively, Lipshitz conditions with constants L_1 and L_2. The solution (10) of Equations (9) and (3) given by the ADM converges if* $0 < L_1 + L_2 < 1$ *and* $||u_i|| < \infty$, $i \geq 0$, *where u_i are given by (14).*

Proof. Let S_n be the partial sum of the series, that is, $S_n = \sum_{i=0}^{n} u_i(t)$. We prove that S_n is a Cauchy sequence in the Banach space E. One has

$$\begin{aligned}
\|S_{n+p} - S_n\| &= \max_{t\in[0,T]} \left| \sum_{i=n+1}^{n+p} u_i(t) \right|, \\
\|S_{n+p} - S_n\| &= \max_{t\in[0,T]} |N(S_{n+p-1}) - N(S_{n-1}) + L(S_{n+p-1}) - L(S_{n-1})| \\
&\leq L_1\|S_{n+p-1} - S_{n-1}\| + L_2\|S_{n+p-1} - S_{n-1}\| \\
&\leq (L_1 + L_2)\|S_{n+p-1} - S_{n-1}\|,
\end{aligned}$$

and

$$\begin{aligned}
\|S_{n+p-1} - S_{n-1}\| &= \max_{t\in[0,T]} \left| \sum_{i=n+1}^{n+p} u_i(t) \right|, \\
\|S_{n+p} - S_n\| &= \max_{t\in[0,T]} \left|N(S_{n+p-1}) - N(S_{n-1}) + L(S_{n+p-1}) - L(S_{n-1})\right| \\
&\leq L_1\|S_{n+p-1} - S_{n-1}\| + L_2\|S_{n+p-1} - S_{n-1}\| \\
&\leq (L_1 + L_2)\|S_{n+p-1} - S_{n-1}\|.
\end{aligned}$$

Thus,

$$\|S_{n+p} - S_n\| \leq C\|S_{n+p-1} - S_{n-1}\|,$$

where $C = L_1 + L_2$. Similarly,

$$\begin{aligned}
\|S_{n+p} - S_n\| &\leq C^2\|S_{n+p-2} - S_{n-2}\|, \\
\|S_{n+p} - S_n\| &\leq C^n\|S_p - S_0\|.
\end{aligned}$$

Now, for $n > m$, we have

$$\|S_n - S_m\| \leq \|S_{m+1} - S_m\| + \ldots + \|S_n - S_{n\text{-}1}\|.$$

Finally, we get

$$\|S_n - S_m\| \leq \frac{C^m}{1 - C}\|u_1\|. \tag{17}$$

Since u is bounded, as $n \to \infty$ one has $\|S_n - S_m\| \to 0$. Hence, S_n is a Cauchy sequence in E and, therefore, the series is convergent. □

Theorem 12 (Upper bound error). *Under the assumptions of Theorem 11, the maximum absolute truncation error of the series solution (10) to problem (3) is estimated to be*

$$\max_{t\in[0,T]}\left|u(t)-\sum_{i=0}^{m}u_i(t)\right|\leq\frac{(L_1+L_2)^m}{1-(L_1+L_2)}\max_{t\in[0,T]}|u_1(t)|. \tag{18}$$

Proof. Let $C := L_1 + L_2$. From (17) in Theorem 11, we have

$$\|S_n - S_m\| \leq \frac{C^m}{1-C}\max_{t\in[0,T]}|u_1(t)|.$$

As $n \to \infty$, then $S_n \to u(t)$. Thus, we get

$$\|u(t) - S_m\| \leq \frac{C^m}{1-C}\max_{t\in[0,T]}|u_1(t)|.$$

Therefore, the maximum absolute truncation error of the series solution (10) to problem (3) is estimated to be (18), which completes the proof. □

4.2 | Applications and numerical results

We illustrate our results with two examples. For Example 1, we are able to prove existence and uniqueness of solution with Theorem 10 and such solution is then approximated using the recursion scheme (14).

Example 1. Consider the following BVP with integral boundary condition

$$\begin{cases} D_*^{\alpha}u(t)+\frac{1}{5}\left(u(t)+1\right)=0,\ t\in(0,1),\ 1<\alpha\leq 2,\\ u(0)=0,\ u(1)=\int_0^1 u(s)ds. \end{cases} \tag{19}$$

This is problem (3) with $\mu = 1$, $\lambda = 1$, $T = 1$ and $n = 2$. Define the function

$$F(t,u(t)) = a(t)f(t,u(t)) = \frac{1}{5}\left(u(t)+1\right),$$

where $a(t) = 1/5$ and $f(t,u) = u + 1$. This means that our function F satisfies hypothesis (H) of Theorem 10 with $L_1 = 1/5$. Moreover, here k of Theorem 10 is given by

$$k = \frac{1}{5}\left(\frac{3}{\Gamma(\alpha+1)}+\frac{2}{\Gamma(\alpha+2)}\right), \tag{20}$$

which takes values between 0 and 1 for $\alpha \in (1,2]$. The Volterra integral equation to problem (19) is, according with Theorem 7, given by

$$u(t) = \frac{1}{5}\left(-I^{\alpha}[u(t)+1] - 2tI^{\alpha+1}[u(t)+1]_{|_{t=1}} + 2tI^{\alpha}[u(t)+1]_{|_{t=1}}\right). \tag{21}$$

To solve this equation by the proposed method, the right-hand side of (21) is decomposed as follows:

$$\begin{aligned} G(t) &= \frac{1}{5}\left(-I^{\alpha}1 - 2tI^{\alpha+1}[1]_{|_{t=1}} + 2tI^{\alpha}[1]_{|_{t=1}}\right)\\ &= \frac{2t-t^{\alpha}}{5\alpha\Gamma(\alpha)} - \frac{2t}{5(\alpha+1)\Gamma(\alpha+1)}, \end{aligned} \tag{22}$$

$$L(u(t)) = \frac{1}{5}\left(-I^{\alpha}u(t) - 2tI^{\alpha+1}[u(t)]_{|_{t=1}} + 2tI^{\alpha}[u(t)]_{|_{t=1}}\right), \tag{23}$$

and $N = 0$. So, Equation (21) can be rewritten as

$$u(t) = G(t) + L(u(t)). \tag{24}$$

From (10), Equation (24) becomes

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(t) &= G(t) + L\left(\sum_{n=0}^{\infty} u_n(t)\right)\\ &= G(t) + \frac{1}{5}\left(-I^{\alpha}\left[\left(\sum_{n=0}^{\infty} u_n(t)\right)\right] - 2tI^{\alpha+1}\left[\left(\sum_{n=0}^{\infty} u_n(t)\right)\right]\Bigg|_{t=1} + 2tI^{\alpha}\left[\left(\sum_{n=0}^{\infty} u_n(t)\right)\right]\Bigg|_{t=1}\right).\end{aligned} \tag{25}$$

Hence, the recursion scheme (14) is given by

$$\begin{cases} u_0(t) = \frac{2t-t^{\alpha}}{5\alpha\Gamma(\alpha)} - \frac{2t}{5(\alpha+1)\Gamma(\alpha+1)},\\ u_{n+1}(t) = -\frac{1}{5}I^{\alpha}u_n(t) - \frac{2}{5}tI^{\alpha+1}[u_n(t)]_{|_{t=1}} + \frac{2}{5}tI^{\alpha}[u_n(t)]_{|_{t=1}}. \end{cases} \tag{26}$$

Theorem 11 with $L_1 = 0.05$ and $L_2 = 0$ assert that the algorithm (26) converges. From (26), we obtain

$$\begin{aligned} u_0(t) =& \frac{2t - t^{\alpha}}{5\alpha\Gamma(\alpha)} - \frac{2t}{5(\alpha+1)\Gamma(\alpha+1)},\\ u_1(t) =& \frac{2\alpha t\,(2(\alpha+1) - t^{\alpha}(\alpha+2))}{5^2\,(\Gamma(\alpha+2) + \Gamma(\alpha+3))} + \frac{4\alpha t + t^{2\alpha}(1+2\alpha)}{5^2\Gamma(2\alpha+2)},\\ u_2(t) =& \frac{4\alpha t(\alpha+1)(\alpha+2)\,(t^{\alpha}(\alpha+2) - 2(\alpha+1))}{5^3\Gamma^3(\alpha+3)} + \frac{2t\left(5 + (t^{\alpha}(t^{\alpha}+2) - 6)\,(\alpha+1) - \frac{4}{\alpha+2}\right)}{5^3(\alpha+1)^2\Gamma(\alpha)\Gamma(2\alpha+2)}\\ &+ \frac{6\alpha t - t^{3\alpha}(3\alpha+1)}{5^3\Gamma(3\alpha+2)}. \end{aligned}$$

The components u_i with $i \geq 3$ are computed similarly but are cumbersome. Below, we approximate the solution $u(t)$ of our problem (19) by $u(t) \approx u_0(t) + u_1(t) + u_2(t) + u_3(t) + u_4(t) = \phi_5(t)$ for various values of $\alpha \in (1, 2]$: for $\alpha = 1.1$,

$$\begin{aligned} u(t) \approx\ & 0.202274t - 0.191116t^{1.1} - 0.0184085t^{2.1} + 0.0165019t^{2.2} + 0.00104309t^{3.2} - 0.000903409t^{3.3}\\ & - 0.000042484t^{4.3} + 0.0000358754t^{4.4} + 1.33016 \times 10^{-6}t^{5.4} - 1.11155 \times 10^{-6}t^{5.5} =: g_1; \end{aligned}$$

for $\alpha = 1.3$,

$$\begin{aligned} u(t) \approx\ & 0.198563t - 0.171422t^{1.3} - 0.0147992t^{2.3} + 0.0107613t^{2.6} + 0.000593533t^{3.6} - 0.000387083t^{3.9}\\ & - 0.0000156705t^{4.9} + 9.44476 \times 10^{-6}t^{5.2} + 2.95196 \times 10^{-7}t^{6.2} - 1.71008 \times 10^{-7}t^{6.5} =: g_2; \end{aligned}$$

for $\alpha = 1.5$,

$$\begin{aligned} u(t) \approx\ & 0.186317t - 0.150451t^{1.5} - 0.0112126t^{2.5} - 0.000152839t^{4.5} - 5.17045 \times 10^{-6}t^{5.5}\\ & - 2.28011 \times 10^{-8}t^{7.5} + t^3\left(0.00666667 + 0.000310509t + 2.22222 \times 10^{-6}t^3 + 5.73145 \times 10^{-8}t^4\right) =: g_3; \end{aligned}$$

for $\alpha = 1.7$,

$$\begin{aligned} u(t) \approx\ & 0.168594t - 0.129476t^{1.7} - 0.0080476t^{2.7} + 0.00394629t^{3.4} + 0.0001512t^{4.4} - 0.0000561593t^{5.1}\\ & - 1.54996 \times 10^{-6}t^{6.1} + 4.73804 \times 10^{-7}t^{6.8} + 9.90397 \times 10^{-9}t^{7.8} - 2.68248 \times 10^{-9}t^{8.5} =: g_4; \end{aligned}$$

for $\alpha = 2$,

$$u(t) \approx 0.137484\,t - 0.1\,t^2 - 0.00458279\,t^3 - 0.00458279\,t^4 + 0.00166667\,t^5 + 4.58261 \times 10^{-5}\,t^6 - 1.11111 \times 10^{-5}\,t^7 - 2.1798899999999999 \times 10^{-7}\,t^8 + 3.96825 \times 10^{-8}\,t^9 + 5.87889 \times 10^{-10}\,t^{10} - 8.818339999999999 \times 10^{-11}\,t^{11} =: g_5.$$

In Figure 1, we plot the numerical solutions $g_1, \dots, g_5$ given above. One can see that when $\alpha \to 2$, our approximated curves tend to the exact solution of the classical integer-order problem of order $\alpha = 2$. Figure 2 illustrates the fact that our numerical method converges very fast. In fact, in this case, one has $\max_{t\in[0,1]}|u_1(t)| < 0.0018$ for all $\alpha \in (1, 2]$ and from (18) of Theorem 12, we know that the error of our approximation is less than 1.19×10^{-8}.

Now, we consider a more challenging example, where the proof of existence and uniqueness of solution is still an open problem: for Example 2, the Lipschitz hypothesis (H), necessary to have a contraction and apply the Banach fixed-point theorem, is not satisfied. However, as we shall see, the sequence of functions $\{u_n\}$ given by our recursion scheme (14) does converge.

Example 2. Consider the following nonlinear fractional differential equation with integral boundary condition:

$$\begin{cases} D_*^{\alpha} u(t) + \mu F(t, u(t)) = 0, \quad t \in (0, 1), \\ u(0) = 1,\ u'(0) = -1,\ u(1) = \lambda \int_0^1 u(s)ds, \end{cases} \tag{27}$$

where $2 < \alpha \le 3$, $\mu = \frac{1}{2}$, $\lambda = \frac{3}{2}$ and $T = 1$. Define the function

$$F(t, u(t)) = a(t)f(t, u(t)) = t\left(u^3(t) + 1\right).$$

Theorem 7 tell us that the solution to Equation (27) is given by:

$$u(t) = (1 - t) + \frac{3}{4}t^2 - \frac{1}{2}I^{\alpha}[t(u^3(t) + 1)] - \frac{3}{2}t^2 I^{\alpha+1}[t(u^3(t) + 1)]_{|_{t=1}} + t^2 I^{\alpha}[t(u^3(t) + 1)]_{|_{t=1}}. \tag{28}$$

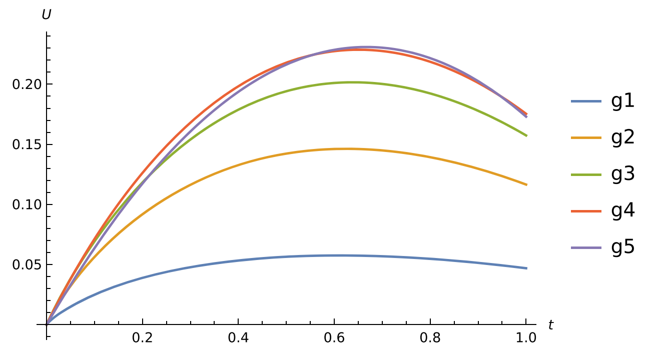

FIGURE 1 Numerical solutions g_i to problem (19) for various values of $\alpha \in (1, 2]$: g_i, $i = 1, \dots, 5$, is the approximated solution with $\alpha_1 = 1.1$, $\alpha_2 = 1.3$, $\alpha_3 = 1.5$, $\alpha_4 = 1.7$, and $\alpha_5 = 2$, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

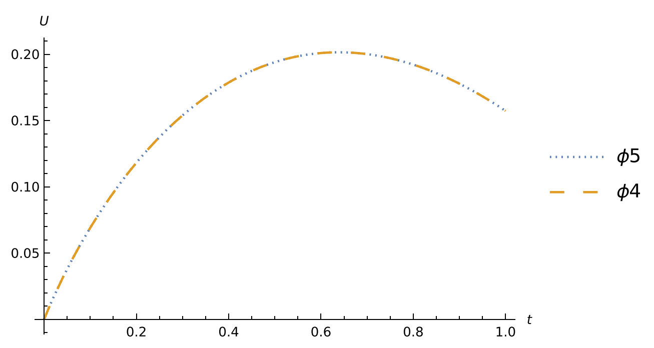

FIGURE 2 Convergence of the approximate solutions (15) to problem (19) with $\alpha = \frac{3}{2}$ [Colour figure can be viewed at wileyonlinelibrary.com]

To apply the ADM, we write Equation (28) as

$$u(t) = G(t) + N(u(t)), \tag{29}$$

with

$$G(t) = (1-t) - \frac{3}{4}t^2 - 2I^{\alpha}[t] - \frac{3}{2}t^2 I^{\alpha+1}[t]_{|_{t=1}} + t^2 I^{\alpha}[t]_{|_{t=1}}$$

and

$$N(u(t)) = -\frac{1}{2}I^{\alpha}[t(u^3(t))] - \frac{3}{2}t^2 I^{\alpha+1}[t(u^3(t))]_{|_{t=1}} + t^2 I^{\alpha}[t(u^3(t))]_{|_{t=1}}.$$

By Equations (10) and (11), (28) becomes

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(t) &= G(t) + \left(\sum_{n=0}^{\infty} A_n\right)\\ &= (1-t) - \frac{3}{4}t^2 - 2I^{\alpha}[t] - \frac{3}{2}t^2 I^{\alpha+1}[t]_{|_{t=1}} + t^2 I^{\alpha}[t]_{|_{t=1}}\\ &\quad - \frac{1}{2}I^{\alpha}\left[t\left(\sum_{n=0}^{\infty} A_n(t)\right)\right] - \frac{3}{2}t^2 I^{\alpha+1}\left[t\left(\sum_{n=0}^{\infty} A_n(t)\right)\right]\Bigg|_{t=1} + t^2 I^{\alpha}\left[t\left(\sum_{n=0}^{\infty} A_n(t)\right)\right]\Bigg|_{t=1}.\end{aligned} \tag{30}$$

Now, we set the following recursion scheme:

$$\begin{cases} u_0(t) = (1-t) - \frac{3}{4}t^2 - 2I^{\alpha}[t] - \frac{3}{2}t^2 I^{\alpha+1}[t]_{|_{t=1}} + t^2 I^{\alpha}[t]_{|_{t=1}},\\ u_{n+1}(t) = -\frac{1}{2}I^{\alpha}\left[t\left(A_n(t)\right)\right] - \frac{3}{2}t^2 I^{\alpha+1}[t\left(A_n(t)\right)]_{|_{t=1}} + t^2 I^{\alpha}[t\left(A_n(t)\right)]_{|_{t=1}}.\end{cases} \tag{31}$$

The first few Adomian polynomials for the nonlinear term $u^3(t)$ are given by

$$\begin{aligned} A_0(t) &= u_0^3(t),\\ A_1(t) &= 3u_0^2(t)\cdot u_1(t),\\ A_2(t) &= 3u_0^2(t)\cdot u_2(t) + 6u_0(t)\cdot\frac{u_1^2(t)}{2},\\ A_3(t) &= 3u_0^2(t)\cdot u_3(t) + 6u_0(t)\cdot u_1(t)u_2(t) + 6\cdot\frac{u_1^3(t)}{6},\\ A_4(t) &= 3u_0^2(t)\cdot u_4(t) + 6u_0(t)\cdot\left(u_1(t)u_3(t) + \frac{u_2^2(t)}{2!}\right) + 6\cdot\frac{u_1^2(t)u_2(t)}{2!} + 0\cdot\frac{u_1^4(t)}{24}.\end{aligned}$$

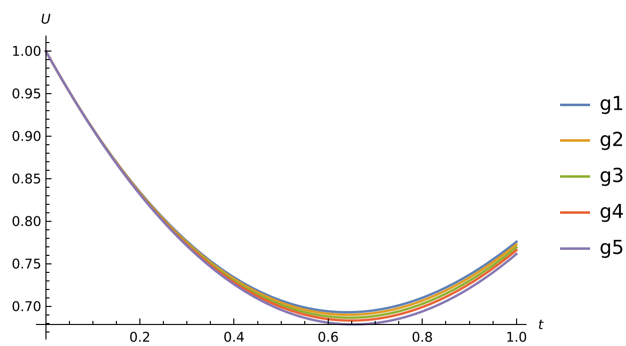

FIGURE 3 Numerical solutions g_i to problem (27) for various values of $\alpha \in (2, 3]$: g_i, $i = 1, \ldots, 5$, is the approximated solution with $\alpha_1 = 2.1$, $\alpha_2 = 2.3$, $\alpha_3 = 2.5$, $\alpha_4 = 2.7$, and $\alpha_5 = 3$, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

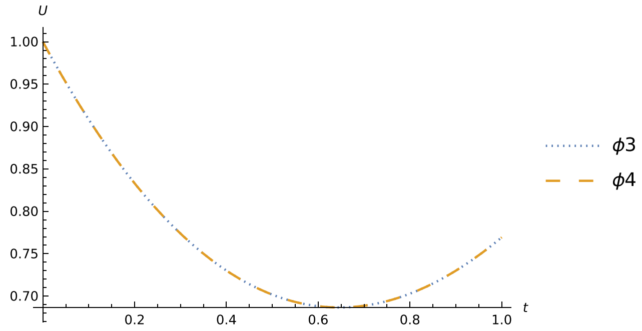

FIGURE 4 Convergence of the approximate solutions (15) to problem (27) with $\alpha = \frac{5}{2}$ [Colour figure can be viewed at wileyonlinelibrary.com]

In view of Adomian polynomials decomposition and (31), we can write the components of the solution for problem (27) as follows:

$$u_0(t) = (1-t) - \frac{3}{4}t^2 + \frac{t\,(t + 2\alpha t - t^{\alpha}(\alpha+2))}{2\Gamma(\alpha+3)},$$

$$\vdots$$

In Figure 3, we plot

$$g(t) := u_0(t) + u_1(t) + u_2(t) + u_3(t) \approx u(t)$$

for various values of α: $\alpha = 2.1$ (g_1), $\alpha = 2.3$ (g_2), $\alpha = 2.5$ (g_3), $\alpha = 2.7$ (g_4), and $\alpha = 3$ (g_5).

One can see that when $\alpha \to 3$ our approximated curves tend to the exact solution of the classical integer-order problem of order $\alpha = 3$. Figure 4 shows that our numerical method also converges very fast for this nonlinear problem. The proof that such limit function is the unique solution of (27) remains, however, a nontrivial open problem.

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CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

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