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Finite time stability of tempered fractional systems with time delays

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ABSTRACT

We investigate the notion of finite time stability for tempered fractional systems (TFSs) with time delays and variable coefficients. Then, we examine some sufficient conditions that allow concluding the TFSs stability in a finite time interval, which include the nonhomogeneous and the homogeneous delayed cases. We present two different approaches. The first one is based on Hölder's and Jensen's inequalities, while the second one concerns the Bellman–Grönwall method using the tempered Grönwall inequality. Finally, we provide two numerical examples to show the practicability of the developed procedures.

1. Introduction

In the last decade, fractional calculus has gained increasing interests due to its crucial and efficient role in modeling various real world phenomena in different fields of science and engineering [1–3]. Fractional calculus involves the operation of convolution with a power law function. If one multiplies the fractional derivative and integral by an exponential term, the result will be a tempered fractional derivative (TFD) and integral [4], which provide undeviating generalization to the existing Caputo and Riemann–Liouville fractional operators and have many merits, both mathematically and practically. A truncated Lévy flight was investigated to capture the natural cutoff in real physical systems [5]. Without a sharp cutoff, the tempered Lévy flight was studied as a smoother alternative [6]. Cartea and del-Castillo-Negrete [7] explored the tempered fractional diffusion equation by the tempered Lévy flight. Furthermore, stochastic applications, such as tempered Lévy flights, present a complete set of statistical physics and numerical analysis tools including solving multi-dimensional partial differential equations [8]. TFD can be also found in geophysics [9], Brownian motion [10], and so on. For further information about tempered fractional calculus and its applications, we refer the reader to [4,7,11] and references therein.

As in classical calculus, stability is still one of the most extensively studied subjects in control theory and fractional systems analysis [12]. The stability analysis of fractional differential equations with the TFD is at its initial stage. Recently, some stability results of tempered fractional systems (TFSs) have been established in [13–15]. In [13], sufficient conditions that ensure the Mittag-Leffler stability of TFSs are investigated by means of a tempered fractional comparison principle and the extended Lyapunov direct method. Further, in [14], the Lyapunov approach is applied to analyze the generalized practical Mittag-Leffler stability of a class of fractional nonlinear systems involving TFDs. Moreover, in [15], the asymptotic and Mittag-Leffler stability of tempered fractional neural networks, with and without delay, are studied by using the Banach fixed point theorem.

Here, we consider the stability from a non-Lyapunov point of view, precisely, finite time stability. This approach concerns the system stability and, simultaneously, the bounds of its trajectories. In fact, a dynamical system could be stable but still entirely useless because of undesirable transient performances. Then, it may be important to consider the stability of such systems with regard to certain subsets of the state–space that are defined, a priori, in a given problem. From the engineering point of view, the boundedness properties of the system responses are very interesting. For example, constraining the state of a system in a transient regime to not exceed certain limits, to avoid saturations and excessive

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excitations of vital parts and nonlinear dynamics. In addition, it is of particular significance to analyze the behavior of dynamical systems over a finite time interval, especially when the systems lifetime is finite [16–18].

To the best of our knowledge, the finite time stability problem of TFSs, including time delay ones, has not yet been analyzed. This motivated us to write the current paper. Therefore, in this work we propose to study the finite time stability for nonlinear TFSs with variable coefficients and time delays. Compared to [15], here we study the finite time stability concept of a more general class of nonlinear tempered delayed systems and also by means of two different methods that were not used yet for the stability of any class of tempered systems [13–15]. Our first method concerns the Bellman–Grönwall approach using the tempered Grönwall inequality. The second one is based on Hölder's and Jensen's inequalities. It is important to mention that the problem of finite time stability of fractional-order time systems, including the delayed cases, is widely investigated in the literature using different approaches [19–22]. For example, based on Mittag-Leffler functions and the generalized Grönwall inequality, sufficient conditions that ensure the finite time stability of Caputo fractional order nonlinear systems with damping behavior are derived in [19]. Moreover, a finite time stability test procedure is presented for linear nonhomogeneous fractional time-delay systems though the Bellman–Grönwall approach [20]. Also, the stability in the finite range of time for Caputo linear fractional delayed systems is studied by means of a delayed Mittag-Leffler type matrix [21], while in [22], criteria of finite time stability of Hadamard fractional differential linear and nonlinear equations in weighted Banach spaces are established using the method of successive approximations and Beesack's inequality with a weakly singular kernel.

The paper is organized as follows. In Section 2, we recall necessary notions and results from the literature that will be useful in the sequel. Our original results are then given in Section 3: we first establish the existence result for a class of nonlinear delayed TFSs, then we prove a delay-dependent- (Theorem 1) and a delay-independent-criterion (Theorem 2) for the finite time stability of time delay nonhomogeneous TFSs with variable coefficients. The usefulness of the proved criteria is illustrated in Section 4 with two examples. We end with Section 5 of conclusions, pointing out some possible future directions of research.

2. Preliminaries

Tempered fractional operators, as we know them today, appear to have been introduced in [7]. However, other notions of "tempered" derivatives can be referred back to the seventies of the 20th century [23]. In this section, we first state some definitions and fundamental lemmas related to the tempered fractional order operators that are employed throughout this paper. For a more general tempered fractional calculus, that includes and generalizes what are usually called substantial, tempered, and shifted fractional operators, we refer to [24], where the reader can also find a discussion if the tempered fractional derivative can be considered as a class of fractional derivatives or not (see Section 3.4 of [24]).

Definition 1 (*See [4,25]*)**.** Let $\alpha > 0$, $\rho > 0$ and υ be an absolutely integrable function defined on $[a,b]$, $a,b \in \mathbb{R}$, and $a < b$ (if $b = \infty$, then the interval is half-open). The tempered fractional integral of the function υ is defined as follows:

$$^{T}I_{a}^{\alpha,\rho}\upsilon(t) = \frac{1}{\Gamma(\alpha)}\int_{a}^{t} e^{-\rho(t-s)}(t-s)^{\alpha-1}\upsilon(s)\,\mathrm{d}s,$$

where $\Gamma(\cdot)$ is the Euler Gamma function [1] defined by

$$\Gamma(r) = \int_{0}^{\infty} e^{-s}s^{r-1}\,\mathrm{d}s, \quad r \in \mathbb{C}.$$

Definition 2 (*See [4,25]*)**.** Let $\alpha \in (0,1)$ and $\rho > 0$. The Caputo tempered fractional order derivative of a function $\upsilon \in C^{1}([a,b],\mathbb{R})$ is given by

$$^{T}D_{a}^{\alpha,\rho}\upsilon(t) = \frac{1}{\Gamma(1-\alpha)}\int_{a}^{t} e^{-\rho(t-s)}(t-s)^{-\alpha}D^{1,\rho}\upsilon(s)\,\mathrm{d}s$$

with $D^{1,\rho}\upsilon(t) = \rho\upsilon(t) + d\upsilon'(t)$, where $d = 1$ and its dimension equals the dimension of the independent variable t.

Remark 1. If $\upsilon(t)$ stands for some chemical, geometrical or physical quantity as a function of time t, for example, concentration, coordinate or momentum, then its combination with its derivative $\upsilon'(t)$ does not preserve the correct dimensions. In order to ensure the dimensional homogeneity within the combination $\rho\upsilon(t) + d\upsilon'(t)$, we added the constant $d = 1$ with a dimension equal to the dimension of the independent variable t.

Remark 2. In the case $\rho = 0$, the TFD coincides with the left Caputo fractional derivative [1].

We have the following Grönwall inequality in the framework of tempered fractional integral.

Lemma 1 (*Tempered Grönwall Inequality [26]*)**.** *Suppose $\alpha > 0$, $\rho > 0$, g and f are nonnegative and locally integrable functions on $[0,t_f)$ ($t_f \leq \infty$) and h is a nonnegative, nondecreasing, and continuous function on $[0,t_f)$ satisfying $h(t) \leq L$, where L is a constant. Moreover, if*

$$g(t) \leq f(t) + h(t)\int_{0}^{t} e^{-\rho(t-s)}(t-s)^{\alpha-1}g(s)\,\mathrm{d}s,$$

then

$$g(t) \leq f(t) + \int_{0}^{t}\left[\sum_{n=1}^{+\infty}\frac{(h(t)\Gamma(\alpha))^{n}}{\Gamma(n\alpha)}e^{-\rho(t-s)}(t-s)^{n\alpha-1}f(s)\right]\mathrm{d}s, \quad t \in [0,t_f].$$

If, in addition, function f is nondecreasing on $[0,t_f)$, then

$$g(t) \leq f(t)E_{\alpha}(h(t)\Gamma(\alpha)t^{\alpha}), \quad t \in [0,t_f], \tag{1}$$

where $E_{\alpha}(\cdot)$ is the Mittag-Leffler function of one parameter [1] defined by

$$E_{\alpha}(s) = \sum_{n=0}^{+\infty}\frac{s^{n}}{\Gamma(\alpha n+1)}, \quad s \in \mathbb{C}.$$

We end this section by stating two inequalities that will be used to prove the finite time stability of nonlinear TFSs.

Lemma 2 (*Hölder's Inequality [27]*). *Let $g, q > 1$ such that $\frac{1}{g} + \frac{1}{q} = 1$. If $|h(\cdot)|^g, |k(\cdot)|^q \in L^1(M)$, then $h(\cdot)k(\cdot) \in L^1(M)$ and*

$$\int_M |h(r)k(r)|\,\mathrm{d}r \le \left(\int_M |h(r)|^g \mathrm{d}r\right)^{\frac{1}{g}} \left(\int_M |k(r)|^q \mathrm{d}r\right)^{\frac{1}{q}},$$

where $L^1(M)$ designs the Banach space of all Lebesgue measurable functions $h : M \longrightarrow \mathbb{R}$ with

$$\int_M |h(r)|^p \mathrm{d}r < \infty.$$

Lemma 3 (*Jensen's Inequality [28]*). *Let $n \in \mathbb{N}$ and $m_1, m_2, \dots, m_n$ be nonnegative real numbers. Then,*

$$\left(\sum_{k=1}^{n} m_k\right)^p \le n^{p-1} \sum_{k=1}^{n} m_k^p, \qquad \text{for } \ p \ge 0.$$

3. Main results

In this section, we shall analyze the finite time stability of the following class of time delay nonhomogeneous TFS with variable coefficients:

$$\begin{cases} {}^T D_0^{\alpha,\rho} y(t) = e^{-\rho t}(Ay(t) + By(t-\tau) + f(t, y(t), y(t-\tau))), & t \in [0, T], \\ y(t) = \omega(t), & t \in [-\tau, 0], \end{cases} \tag{2}$$

and the associated homogeneous TFS

$$\begin{cases} {}^T D_0^{\alpha,\rho} y(t) = e^{-\rho t}(Ay(t) + By(t-\tau)), & t \in [0, T], \\ y(t) = \omega(t), & t \in [-\tau, 0], \end{cases} \tag{3}$$

where $\alpha \in (0,1)$, $\rho \in (0,1]$, A and B are constant $n \times n$ matrices, $T > 0$ is a real number, $\tau > 0$ is a time delay, $\omega(\cdot)$ is a continuous function on $[-\tau, 0]$ with the norm $\|\omega\|_C = \sup_{t\in[-\tau,0]} \|\omega(t)\|$ such that $\|\cdot\|$ is the maximum norm, and $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a given nonlinear continuous function with $f(t, 0, 0) = 0$.

We present the following definition of the finite time stability of system (2).

Definition 3. The system (2) is finite time stable with regard to $\{\xi, \varepsilon, J\}$, $\xi \le \varepsilon$, if

$$\|\omega\|_C < \xi \tag{4}$$

implies

$$\|y(t)\| < \varepsilon, \quad \forall t \in J, \tag{5}$$

where ξ and ε are positive real numbers; and J is the time interval $J = [0, T] \subset \mathbb{R}$.

Remark 3. When the system (2) is of order $\rho = 0$, we retrieve the finite time stability definition of Caputo fractional delayed systems [20].

Remark 4. Intuitively, by defining two bounded sets in the state space, called "the initial set" and "the set of trajectories", the system is finite time stable if the trajectories of the system emanating from the initial set remain in the set of trajectories over a finite time interval.

Existence and uniqueness of solution for the nonlinear tempered fractional delayed system (2) is stated by the following lemma.

Lemma 4. *Let $\alpha \in (0,1)$, $\rho \in (0,1]$, $f \in C([0,T] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $\omega \in C([-\tau, 0], \mathbb{R}^n)$. The function $y : [-\tau, T] \longrightarrow \mathbb{R}^n$ is a solution of system (2) if, and only if, it satisfies*

$$\begin{cases} y(t) = \omega(0)e^{-\rho t} + {}^T I_0^{\alpha,\rho} e^{-\rho s}\left[Ay(t) + By(t-\tau) + f(t, y(t), y(t-\tau))\right], & t \in [0, T], \\ y(t) = \omega(t), & t \in [-\tau, 0]. \end{cases} \tag{6}$$

Moreover, if for any functions $y, z : [-\tau, T] \longrightarrow \mathbb{R}^n$, there exists a constant $L_f > 0$ such that

$$\|f(t, y(t), y(t-\tau)) - f(t, z(t), z(t-\tau))\| \le L_f\left(\|y(t) - z(t)\| + \|y(t-\tau) - z(t-\tau)\|\right), \quad t \in [0, T], \tag{7}$$

then system (2) has a unique mild solution.

Proof. The proof follows by using similar techniques of Lemma 1 and Theorem 2 in [29] and by taking into account Definition 2 and the tempered Grönwall inequality (Lemma 1). □

3.1. Time delay dependent criterion

Here, we establish a delay-dependent criterion that enables us to check the finite time stability of the nonhomogeneous time delay TFSs (2). Our proof uses Hölder's and Jensen's inequalities.

Theorem 1. *Let $\xi, \varepsilon > 0$ be given real numbers and consider $g = 1 + \alpha$ and $q = 1 + \frac{1}{\alpha}$. The system (2) is finite time stable with respect to $\{\xi, \varepsilon, J\}$, $\xi \leq \varepsilon$, if f satisfies condition (7) and*

$$\sqrt[q]{\frac{3^{\frac{1}{\alpha}} q + (3^{\frac{1}{\alpha}}\Psi + q\Phi + \Psi\Phi)e^{(\Psi+q)t}}{q+\Psi}} \leq \frac{\varepsilon}{\xi}, \quad \forall t \in J, \tag{8}$$

where

$$\Psi = \frac{3^{\frac{1}{\alpha}}((\lambda_{\max}(A) + L_f)^q + (\lambda_{\max}(B) + L_f)^q e^{-q\tau})V^q}{\Gamma^q(\alpha)}, \quad V = \left(\frac{\Gamma(\alpha^2)}{g^{\alpha^2}}\right)^{1/g} \tag{9}$$

and

$$\Phi = \frac{3^{\frac{1}{\alpha}}(\lambda_{\max}(B) + L_f)^q(1 - e^{-\tau q})}{q\Gamma^q(\alpha)}V^q \tag{10}$$

with $\lambda_{\max}(A)$ and $\lambda_{\max}(B)$ denoting the largest singular values of the matrices A and B, respectively.

Proof. According to Lemma 4, the solution of system (2) can be written as

$$y(t) = \omega(0)e^{-\rho t} + \frac{1}{\Gamma(\alpha)}\int_0^t e^{-\rho(t-s)}e^{-\rho s}(t-s)^{\alpha-1}\,[A(s)y(s) + B(s)y(s-\tau) + f(s, y(s), y(s-\tau))]\,\mathrm{d}s.$$

It follows that

$$\|y(t)\| \leq \|\omega(0)\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\lambda_{\max}(A)\|y(s)\| + \lambda_{\max}(B)\|y(s-\tau)\| + \|f(s, y(s), y(s-\tau))\|\right]\mathrm{d}s. \tag{11}$$

Using condition (7) with $f(s, 0, 0) = 0$, it implies

$$\|y(t)\| \leq \|\omega(0)\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[(\lambda_{\max}(A) + L_f)\|y(s)\| + (\lambda_{\max}(B) + L_f)\|y(s-\tau)\|\right]\mathrm{d}s,$$

which yields

$$\|y(t)\| \leq \|\omega(0)\| + \frac{\lambda_{\max}(A) + L_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^s e^{-s}\|y(s)\|\,\mathrm{d}s + \frac{\lambda_{\max}(B) + L_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^s e^{-s}\|y(s-\tau)\|\,\mathrm{d}s.$$

Applying Hölder's inequality (Lemma 2) with $g = 1 + \alpha$ and $q = 1 + \frac{1}{\alpha}$, one obtains that

$$\begin{aligned}\|y(t)\| \leq \|\omega(0)\| &+ \frac{\lambda_{\max}(A) + L_f}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{g(\alpha-1)}e^{gs}\,\mathrm{d}s\right)^{\frac{1}{g}} \times \left(\int_0^t e^{-qs}\|y(s)\|^q\,\mathrm{d}s\right)^{\frac{1}{q}}\\ &+ \frac{\lambda_{\max}(B) + L_f}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{g(\alpha-1)}e^{gs}\,\mathrm{d}s\right)^{\frac{1}{g}} \times \left(\int_0^t e^{-qs}\|y(s-\tau)\|^q\,\mathrm{d}s\right)^{\frac{1}{q}}.\end{aligned} \tag{12}$$

It is easy to show that

$$\int_0^t (t-s)^{g(\alpha-1)}e^{gs}\,\mathrm{d}s \leq \frac{e^{gt}\Gamma(g(\alpha-1)+1)}{g^{g(\alpha-1)+1}} = \frac{\Gamma(\alpha^2)e^{gt}}{g^{\alpha^2}}. \tag{13}$$

Combining inequality (12) and relation (13), one has

$$\begin{aligned}\|y(t)\| \leq \|\omega(0)\| &+ \frac{(\lambda_{\max}(A) + L_f)Ve^t}{\Gamma(\alpha)}\left(\int_0^t e^{-qs}\|y(s)\|^q\,\mathrm{d}s\right)^{\frac{1}{q}}\\ &+ \frac{(\lambda_{\max}(B) + L_f)Ve^t}{\Gamma(\alpha)}\left(\int_0^t e^{-qs}\|y(s-\tau)\|^q\,\mathrm{d}s\right)^{\frac{1}{q}}\end{aligned}$$

with V given in (9), which implies

$$\begin{aligned}\|y(t)\| \leq \|\omega(0)\| &+ \frac{(\lambda_{\max}(A) + L_f)Ve^t}{\Gamma(\alpha)}\left(\int_0^t e^{-qs}\|y(s)\|^q\,\mathrm{d}s\right)^{\frac{1}{q}}\\ &+ \frac{(\lambda_{\max}(B) + L_f)Ve^t}{\Gamma(\alpha)}\left(\int_{-\tau}^t e^{-q(s+\tau)}\|y(s)\|^q\,\mathrm{d}s\right)^{\frac{1}{q}}.\end{aligned} \tag{14}$$

Now, by applying Jensen's inequality (Lemma 3) to (14), one gets

$$\begin{aligned}\|y(t)\|^q \leq 3^{\frac{1}{\alpha}}\Bigg[\|\omega(0)\|^q &+ \frac{(\lambda_{\max}(A) + L_f)^q V^q e^{qt}}{\Gamma^q(\alpha)}\left(\int_0^t e^{-qs}\|y(s)\|^q\,\mathrm{d}s\right)\\ &+ \frac{(\lambda_{\max}(B) + L_f)^q V^q e^{qt}}{\Gamma^q(\alpha)}\left(\int_{-\tau}^t e^{-q(s+\tau)}\|y(s)\|^q\,\mathrm{d}s\right)\Bigg].\end{aligned}$$

Hence, it follows that

$$\|y(t)\|^q \leq 3^{\frac{1}{\alpha}}\|\omega(0)\|^q + \Psi e^{qt}\int_0^t e^{-qs}\|y(s)\|^q\,\mathrm{d}s + \frac{3^{\frac{1}{\alpha}}(\lambda_{\max}(B) + L_f)^q V^q e^{q(t-\tau)}}{\Gamma^q(\alpha)}\int_{-\tau}^0 e^{-qs}\|y(s)\|^q\,\mathrm{d}s,$$

where Ψ is defined in (9). This yields

$$\|y(t)\|^q \leq 3^{\frac{1}{\alpha}}\|\omega\|_C^q + \Psi e^{qt}\int_0^t e^{-qs}\|y(s)\|^q \, ds + e^{qt}\|\omega\|_C^q \Phi$$

with Φ given by (10). Hence,

$$e^{-qt}\|y(t)\|^q \leq \left(3^{\frac{1}{\alpha}} e^{-qs} + \Phi\right)\|\omega\|_C^q + \Psi\int_0^t e^{-qs}\|y(s)\|^q \, ds.$$

Furthermore, by virtue of Grönwall's inequality, one obtains that

$$e^{-qt}\|y(t)\|^q \leq \left(3^{\frac{1}{\alpha}} e^{-qs} + \Phi\right)\|\omega\|_C^q + \int_0^t \Psi\left(3^{\frac{1}{\alpha}} e^{-qs} + \Phi\right)\|\omega\|_C^q e^{\Psi(t-s)} \, ds,$$

which yields

$$\|y(t)\|^q \leq \frac{3^{\frac{1}{\alpha}} q + (3^{\frac{1}{\alpha}}\Psi + q\Phi + \Psi\Phi)e^{(\Psi+q)t}}{q+\Psi}\|\omega\|_C^q. \tag{15}$$

Consequently, from condition (8) and inequality (15), we obtain the finite time stability of system (2). □

Remark 5. One notes that Theorem 1 gives only a sufficient condition that ensures the finite time stability of the time delay TFS (2). If this condition does not hold, we cannot conclude that (2) is unstable.

For the homogeneous case, we obtain from Theorem 1 the following result.

Corollary 1. *Let $\xi, \varepsilon > 0$ be given real numbers and consider $g = 1+\alpha$ and $q = 1 + \frac{1}{\alpha}$. The homogeneous system (3) is finite time stable with respect to $\{\xi, \varepsilon, J\}$, $\xi \leq \varepsilon$, if*

$$\sqrt[q]{\frac{3^{\frac{1}{\alpha}} q + (3^{\frac{1}{\alpha}}\Psi + q\Phi + \Psi\Phi)e^{(\Psi+q)t}}{q+\Psi}} \leq \frac{\varepsilon}{\xi}, \quad \forall t \in J, \tag{16}$$

where

$$\Psi = \frac{3^{\frac{1}{\alpha}}(\lambda_{\max}(A)^q + \lambda_{\max}(B)^q e^{-q\tau})V^q}{\Gamma^q(\alpha)}, \quad V = \left(\frac{\Gamma(\alpha^2)}{g^{\alpha^2}}\right)^{1/J}$$

and

$$\Phi = \frac{3^{\frac{1}{\alpha}}\lambda_{\max}(B)^q(1-e^{-\tau q})}{q\Gamma^q(\alpha)}V^q.$$

3.2. Time delay independent criterion

Now, based on the Bellman–Grönwall approach, we shall formulate a sufficient condition that enables the TFS (2) trajectories to stay within a priori given sets.

Theorem 2. *Given real numbers $\xi > 0$ and $\varepsilon > 0$, the system (2) is finite time stable with respect to $\{\xi, \varepsilon, J\}$, $\xi \leq \varepsilon$, if f satisfies condition (7) and*

$$\left(1 + \frac{(\lambda_S + 2L_f)t^\alpha}{\Gamma(\alpha+1)}\right)E_\alpha\left((\lambda_S + 2L_f)t^\alpha\right) \leq \frac{\varepsilon}{\xi}, \quad \forall t \in J, \tag{17}$$

where $\lambda_S = \lambda_{\max}(A) + \lambda_{\max}(B)$.

Proof. For all $t \in [0, T]$, the system (2) admits a unique solution given by

$$y(t) = \omega(0)e^{-\rho t} + \frac{1}{\Gamma(\alpha)}\int_0^t e^{-\rho(t-s)}e^{-\rho s}(t-s)^{\alpha-1}\left[A(s)y(s) + B(s)y(s-\tau) + f(s, y(s), y(s-\tau))\right] ds.$$

Using the fact that $e^{-\rho s} \leq 1$ for all $s \in [0, t]$ and condition (7) holds, then

$$\|y(t)\| \leq \|\omega(0)\| + \frac{1}{\Gamma(\alpha)}\int_0^t e^{-\rho(t-s)}(t-s)^{\alpha-1}\left[(\lambda_{\max}(A) + L_f)\|y(s)\| + (\lambda_{\max}(B) + L_f)\|y(s-\tau)\|\right] ds. \tag{18}$$

Setting $Y(t) = \sup_{0\leq\zeta\leq t}\|y(\zeta)\|$ for all $t \in [0, T]$, one has

$$\|y(s-\tau)\| \leq Y(s) + \|\omega\|_C, \quad \forall s \in [0, t]. \tag{19}$$

Replacing (19) into inequality (18), it follows that

$$\|y(t)\| \leq \|\omega(0)\| + \frac{1}{\Gamma(\alpha)}\int_0^t e^{-\rho(t-s)}(t-s)^{\alpha-1}(\lambda_S + 2L_f)\left(Y(s) + \|\omega\|_C\right) ds, \tag{20}$$

which implies

$$\|y(t)\| \leq \|\omega\|_C + \frac{(\lambda_S + 2L_f)t^\alpha}{\Gamma(\alpha+1)}\|\omega\|_C + \frac{(\lambda_S + 2L_f)}{\Gamma(\alpha)}\int_0^t e^{-\rho(t-s)}(t-s)^{\alpha-1}Y(s) \, ds. \tag{21}$$

Using the change of variable $x = t - s$, one obtains

$$\|y(t)\| \leq \|\omega\|_C + \frac{(\lambda_S + 2L_f)t^\alpha}{\Gamma(\alpha+1)}\|\omega\|_C + \frac{(\lambda_S + 2L_f)}{\Gamma(\alpha)}\int_0^t e^{-\rho x}x^{\alpha-1}\Upsilon(t-x)\,\mathrm{d}x. \tag{22}$$

Also, by taking $t = \zeta$ in (22) with $\zeta \in [0,t]$ and using $\zeta^\alpha \leq t^\alpha$, we get

$$\|y(\zeta)\| \leq \left[1 + \frac{(\lambda_S + 2L_f)t^\alpha}{\Gamma(\alpha+1)}\right]\|\omega\|_C + \frac{(\lambda_S + 2L_f)}{\Gamma(\alpha)}\int_0^\zeta e^{-\rho x}x^{\alpha-1}\Upsilon(\zeta-x)\,\mathrm{d}x. \tag{23}$$

Since the function Υ is nonnegative, it follows that $\int_0^t e^{-\rho x}x^{\alpha-1}\Upsilon(t-x)\,\mathrm{d}x$ is an increasing function with respect to $t \geq 0$, which implies that $\int_0^\zeta e^{-\rho x}x^{\alpha-1}\Upsilon(\zeta-x)\,\mathrm{d}x \leq \int_0^t e^{-\rho x}x^{\alpha-1}\Upsilon(t-x)\,\mathrm{d}x$ and

$$\|y(\zeta)\| \leq \left[1 + \frac{(\lambda_S + 2L_f)t^\alpha}{\Gamma(\alpha+1)}\right]\|\omega\|_C + \frac{(\lambda_S + 2L_f)}{\Gamma(\alpha)}\int_0^t e^{-\rho x}x^{\alpha-1}\Upsilon(t-x)\,\mathrm{d}x.$$

It follows that

$$\Upsilon(t) = \sup_{0\leq\zeta\leq t}\|y(\zeta)\| \leq \left[1 + \frac{(\lambda_S + 2L_f)t^\alpha}{\Gamma(\alpha+1)}\right]\|\omega\|_C + \frac{(\lambda_S + 2L_f)}{\Gamma(\alpha)}\int_0^t e^{-\rho(t-s)}(t-s)^{\alpha-1}\Upsilon(s)\,\mathrm{d}s.$$

Now, let $f(t) = \left[1 + \frac{(\lambda_S + 2L_f)t^\alpha}{\Gamma(\alpha+1)}\right]\|\omega\|_C$, which is a nondecreasing function. By applying Lemma 1 with $h(t) = \frac{(\lambda_S + 2L_f)}{\Gamma(\alpha)}$, we get

$$\|y(t)\| \leq \Upsilon(t) \leq \|\omega\|_C\left[1 + \frac{(\lambda_S + 2L_f)t^\alpha}{\Gamma(\alpha+1)}\right]E_\alpha\left((\lambda_S + 2L_f)t^\alpha\right).$$

Then, by virtue of (4) and (17), one deduces that

$$\|y(t)\| < \varepsilon, \quad \forall t \in J = [0,T],$$

which proves the finite time stability of the nonhomogeneous TFS (2). □

Remark 6. The condition (17) can be written, in equivalent way, as follows:

$$\left(1 + \frac{(\|A\| + \|B\| + 2L_f)t^\alpha}{\Gamma(\alpha+1)}\right)E_\alpha\left((\|A\| + \|B\| + 2L_f)t^\alpha\right) \leq \frac{\varepsilon}{\xi}, \quad \forall t \in J.$$

In the homogeneous case, we obtain from Theorem 2 the following result.

Corollary 2. *Given real numbers $\xi > 0$ and $\varepsilon > 0$, the homogeneous system (3) is finite time stable with respect to $\{\xi, \varepsilon, J\}$, $\xi \leq \varepsilon$, if*

$$\left(1 + \frac{\lambda_S t^\alpha}{\Gamma(\alpha+1)}\right)E_\alpha\left(\lambda_S t^\alpha\right) \leq \frac{\varepsilon}{\xi}, \quad \forall t \in J. \tag{24}$$

Remark 7. From Corollary 2, if we let $\rho = 0$ in system (3), then one retrieves the condition

$$\left(1 + \frac{\lambda_S t^\alpha}{\Gamma(\alpha+1)}\right)E_\alpha\left(\lambda_S t^\alpha\right) \leq \frac{\varepsilon}{\xi}, \quad \forall t \in J,$$

for the finite time stability of the Caputo fractional order time-delay system

$$\begin{cases} {}^C D_0^\alpha y(t) = Ay(t) + By(t-\tau), & t \in [0,T], \\ y(t) = \omega(t), & t \in [-\tau, 0], \end{cases}$$

where ${}^C D_0^\alpha$ is the Caputo fractional derivative of order α. This result is proved in [20].

4. Illustrative examples

In this section, we present two expository examples in order to illustrate our previous results.

Our first example gives a situation where Theorem 1 allows us to conclude that the given TFS is finite time stable while Theorem 2 fails.

Example 1. Consider the nonhomogeneous tempered fractional system with time delay

$$\begin{cases} {}^T D_0^{0.3,0.8} y(t) = e^{-0.8t}(Ay(t) + By(t-\tau) + 2\sin(y(t)) - 3\sin(y(t-\tau))), & t \in [0,3], \\ y(t) = [0 \quad 0]^T, & t \in [-0.2, 0], \end{cases} \tag{25}$$

where

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2.$$

According to system (2), one has $\alpha = 0.3$, $\rho = 0.8$, $\tau = 0.2$, $T = 3$, $\omega(t) = [0 \quad 0]^T$ and

$$f(t, y(t), y(t-\tau)) = 2\sin(y(t)) - 3\sin(y(t-\tau)).$$

For all $y, z : [-0.2, 3] \longrightarrow \mathbb{R}^2$, one has

$$\begin{aligned} \|f(t,y(t),y(t-\tau)) - f(t,z(t),z(t-\tau))\| &= \|2(\sin(y(t)) - \sin(z(t))) - 3(\sin(y(t-\tau)) - \sin(z(t-\tau)))\| \\ &\leq 2\left[\|\sin(y(t)) - \sin(z(t))\| + 3\|\sin(y(t-\tau)) - \sin(z(t-\tau))\|\right] \\ &\leq 3\left[\|y(t) - z(t)\| + \|y(t-\tau) - z(t-\tau)\|\right], \quad t \in [0,3]. \end{aligned}$$

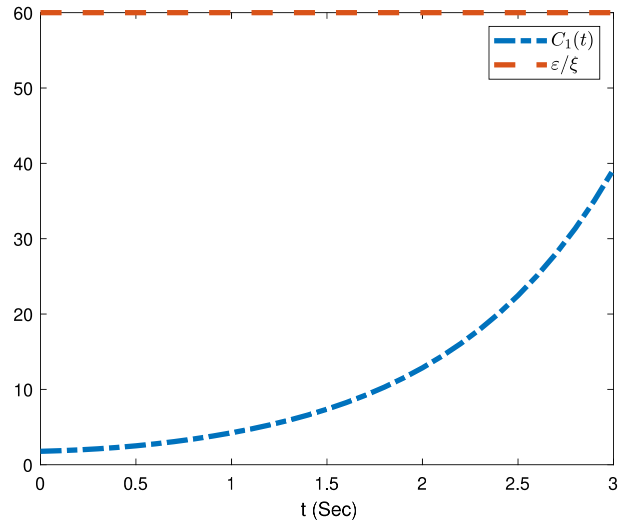

Fig. 1. Condition (8), over $[0,3]$, for $\alpha = 0.3$, $\rho = 0.8$, $\xi = 0.01$ and $\varepsilon = 0.6$.

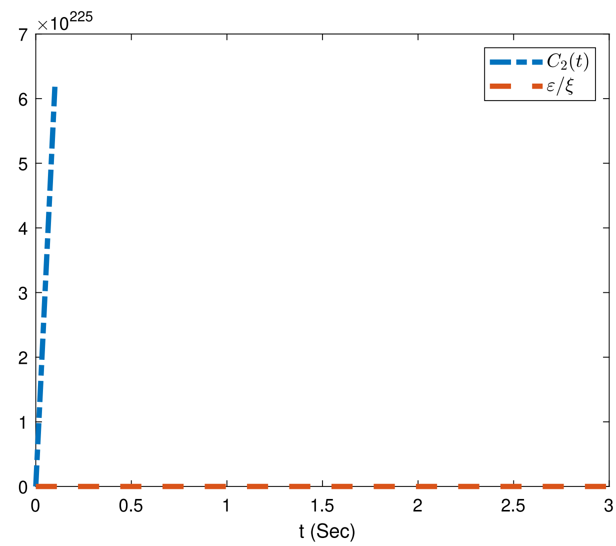

Fig. 2. Condition (17), over $[0,3]$, for $\alpha = 0.3$, $\rho = 0.8$, $\xi = 0.01$ and $\varepsilon = 0.6$.

It follows that the nonlinear function f satisfies the condition (7) with $L_f = 3$ and $f(t,0,0) = 0$. Also, one needs to check the finite time stability of system (25) with regard to $J = [0,3]$, $\xi = 0.01$, and $\varepsilon = 0.6$. From system (25), one gets

$$\|\omega\|_C < 0.01,$$

$$\lambda_{\max}(A) = 2, \lambda_{\max}(B) = 5 \quad \text{and} \quad \lambda_S = 7.$$

Hence, numerically, we have $q = 4.3333$, $\Psi = 0.4945$, $\Phi = 0.1201$ and

$$C_1(t) = \sqrt[q]{8.0658 + 4.1088 e^{(4.8279)t}} \leq 60 = \frac{\varepsilon}{\xi}, \quad \forall t \in [0,3].$$

Therefore, condition (8) holds, as it is illustrated in Fig. 1. Then, from Theorem 1, we deduce that system (25) is finite time stable with respect to $\{\xi = 0.01, \varepsilon = 0.6, J = [0,3]\}$. However, we cannot arrive at the same conclusion using Theorem 2, since

$$C_2(t) = \left(1 + 14.4847\, t^{0.3}\right) E_{0.3}\left(13\, t^{0.3}\right) \leq 60,$$

is not satisfied for all $t \in [0,3]$, as shown in Fig. 2, which means that condition (17) does not hold.

In contrast with Example 1, now we consider a problem where Theorem 2 allows us to deduce the finite time stability of the system while Theorem 1 does not.

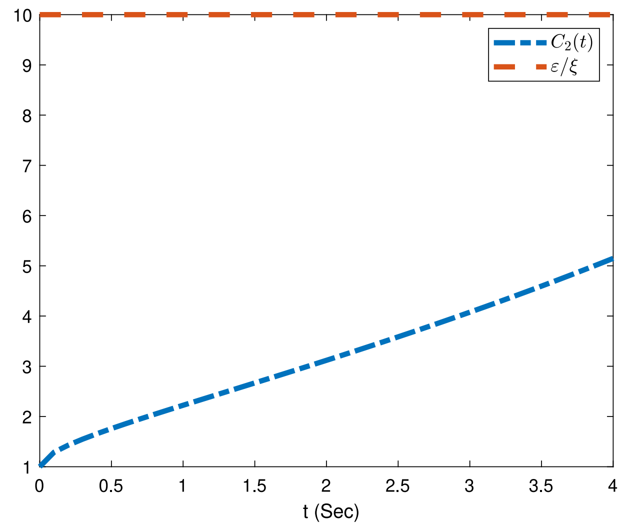

Fig. 3. Condition (17), over $[0,4]$, for $\alpha = \rho = 0.5$, $\xi = 0.02$ and $\varepsilon = 0.2$.

Example 2. Let us consider the following nonhomogeneous TFS time delay system:

$$\begin{cases} {}^{T}D_0^{0.5,0.5} y(t) = e^{-0.5t}(Ay(t) + By(t-\tau) + 0.03(y(t) + \sin(y(t-\tau)))), & t \in [0,4], \\ y(t) = [0.01\cos(\pi t) \quad 0.01]^T, & t \in [-0.2, 0], \end{cases} \tag{26}$$

with

$$A = \begin{pmatrix} 0 & 0.2 \\ -0.15 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.09 \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2.$$

According to system (2), one has $\alpha = \rho = 0.5$, $\tau = 0.2$, $T = 4$, $\omega(t) = [0.01\cos(\pi t) \quad 0.01]^T$ and

$$f(t, y(t), y(t-\tau)) = 0.03\,(y(t) + \sin(y(t-\tau))).$$

One has

$$\begin{aligned} \|f(t, y(t), y(t-\tau)) - f(t, z(t), z(t-\tau))\| &\le 0.03\|y(t) - z(t)\| + \|\sin(y(t-\tau)) - \sin(z(t-\tau))\| \\ &\le 0.03\,[\|y(t) - z(t)\| + \|y(t-\tau) - z(t-\tau)\|], \quad t \in [0,4], \end{aligned}$$

for all $y, z : [-0.2, 4] \longrightarrow \mathbb{R}^2$. It follows that condition (7) holds with $L_f = 0.03$ and $f(t,0,0) = 0$.

We need to check the finite time stability of system (26) with regard to $J = [0,4]$, $\xi = 0.02$, $\varepsilon = 0.2$. From system (26), it follows that

$$\|\omega\|_C < 0.02$$

and

$$\lambda_{\max}(A) = 0.2, \lambda_{\max}(B) = 0.1, \lambda_S = 0.3.$$

On the other hand, for $m \in \mathbb{N}$, one has

$$\Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{(\pi)}\Gamma(2m+1)}{2^{2m}\Gamma(m+1)} = \frac{\sqrt{(\pi)}(2n)!}{2^{2n}m!}.$$

Therefore, for $m = 1$, we have

$$\Gamma\left(1 + \frac{1}{2}\right) = \frac{\sqrt{(\pi)2!}}{2^2 1!} = 0.886.$$

Moreover, by using the numeric computing environment MATLAB, we obtain that

$$C_2(t) = \left(1 + 0.4063\,t^{0.5}\right) E_{0.5}\left(0.36\,t^{0.5}\right) \le 10 = \frac{\varepsilon}{\xi}, \quad \forall t \in [0,4],$$

which means that condition (17) holds, as it is shown in Fig. 3. Then, from Theorem 2, we deduce that system (26) is finite time stable with respect to $\{\xi = 0.02, \varepsilon = 0.2, J = [0,4]\}$.

However, by using Theorem 1, we cannot conclude that the system (26) is finite time stable. Indeed, for $[3.5, 4] \subset [0,4]$ one has $\Psi = 0.0075$, $\Phi = 1.8486 \times 10^{-4}$ and

$$C_1(t) = \sqrt[3]{8.9776 + 0.0226e^{(3.0075)t}} > 10,$$

which means that condition (8) does not hold as it is illustrated in Fig. 4.

Our examples show that Theorems 1 and 2 are different and both useful, depending on the systems under study.

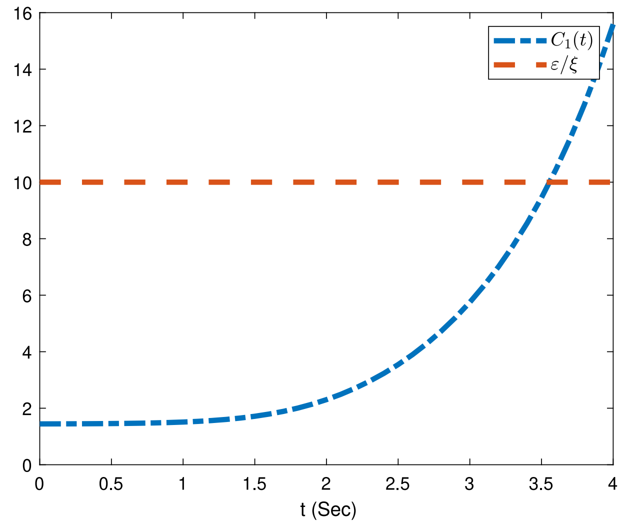

Fig. 4. Condition (8), over $[0, 4]$, for $\alpha = \rho = 0.5$, $\xi = 0.02$ and $\varepsilon = 0.2$.

5. Conclusion

For some engineering systems whose operation is time limited and should be done within prescribed bounds on system variables, the only meaningful stability concept is finite time stability. Since the concept of a change given in terms of the tempered fractional derivative (TFD) is more appropriate for some specific applications, in this work we provided two finite time stability test procedures for fractional differential equations with time delays involving the TFD. One stability criterion depends on the time delay while the second one is delay independent. We used mainly two different approaches for nonhomogeneous time delay TFSs over a finite time interval: (i) one is based on Hölder's and Jensen's inequalities; (ii) the second one on Bellman–Grönwall method using the tempered fractional Grönwall inequality. The effectiveness of the proposed procedures was illustrated through two numerical examples, showing that the obtained criteria are different and relevant. Our developed stability results may be applied to investigate the stability, over a finite time, of different mathematical delayed models, e.g., neural networks with a bounded activation function or tuberculosis epidemic models.

The generalized kernel idea started with Boltzmann in 1874 [30] and can be seen, e.g., in [31]. As future work, we plan to investigate necessary conditions for the finite time stability of TFSs (2) and also to analyze their finite time stabilization, developing numerical methods to approximate the solution of the considered problems and to study the stability of fractional delayed systems with more general types of kernels.

CRediT authorship contribution statement

Hanaa Zitane: Conceptualization, Formal analysis, Investigation, Methodology, Validation, Visualization, Writing – original draft, Writing – review & editing. **Delfim F.M. Torres:** Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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