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Classification of thin regular map representations of hypermaps*

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Abstract

There are two well known maps representations of hypermaps, namely the Walsh and the Vince map representations, being dual of each other. They correspond to normal subgroups of index two of a free product $\Gamma = (C_2 \times C_2) * C_2$ which decompose as "elementary" free product $C_2 * C_2 * C_2$. However Γ has three normal subgroups that decompose as "elementary" free product $C_2 * C_2 * C_2$, the third of these sbgroups giving the less known petrie-path map representation. By relaxing the "elementary" free product condition to free product of rank 3, and under the extra condition "words of smaller length" on the generators, we prove that the number of map representations of hypermaps increases to 15 (up to a restrictedly dual), all of which described in this paper.

Keywords: Map representation, hypermaps, maps, regularity, restricted regularity, orientably regular.

Math. Subj. Class.: 05C10, 05C25, 05C65, 05E18, 20F65

1 Introduction

Using maps to describe hypermaps is not new. The well known Walsh [8] bipartite map representation uses a bipartite map $\mathcal{M}$ to describe a hypermap $\mathcal{H}$ by interpreting the two monochromatic vertices of the map as hypervertices and hyperedges (respectively), and the faces of $\mathcal{M}$ as the hyperfaces of $\mathcal{H}$. The Vince 2-face bipartite map [7], a dual of a bipartite map, also describes a hypermap by assigning the two monochromatic faces to hyperedges and hyperfaces respectively, and vertices to hypervertices. These are two, out of three, Θ-marked map representations realised by an index 2 normal subgroups Θ of the free product

$$\Gamma = \Delta(\infty, 2, \infty) = \langle R_0, R_1, R_2 \mid R_0^2, R_1^2, R_2^2, (R_0R_2)^2 \rangle = C_\infty * (C_2 \times C_2),$$

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which are isomorphic to $\Delta = \Delta(\infty,\infty,\infty) = \langle S_0, S_1, S_2 \mid S_0^2, S_1^2, S_2^2\rangle$ (see [1] and section 3). They are namely, $\Gamma_{2.4} = \langle R_1^{R_0}, R_1, R_2\rangle$ and $\Gamma_{2.1} = \langle R_0, R_1, R_1^{R_2}\rangle$. The third subgroup of Γ of index 2 isomorphic to Δ is $\Gamma_{2.5} = \langle R_1^{R_0}, R_1, R_0R_2\rangle$ (see Subsection 4.2). This induces the third less known representation, succinctly described in [2], given by $\Gamma_{2.5}$-marked maps. In this representation Petrie-path-bipartite maps represent hypermaps by assigning the two monochromatic Petrie polygons (closed zig-zag paths turning alternately left and right) to hypervertices and hyperedges, and faces to hyperfaces.

More generally, a *regular* representation of hypermaps by maps is given by an epimorphism ρ from a finite index *normal* subgroup Θ of Γ to Δ.

This paper is inspired by the work of Lynne James on map representation of topological categories (see [5]) and is organised as follow: In Section 2 we give an introduction to the theory of hypermaps and maps focusing on restrictedly marked hypermaps and maps, a theory developed in [1]. In particular, we focus on Θ-marked maps for normal subgroups Θ of finite index in Γ. Section 3 is devoted to define the notion of clean and thin Θ-marked representation of a hypermap by a map. As we will focus on Θ-marked representations for rank 3 normal subgroups Θ of Γ, in Section 4 we derive a rank formula and classify the rank 3 normal subgroups of Γ. The rank formula is derived using presentations for NEC groups (see [3]). Last section is devoted to thin representations (given in Table 2) and its geometric description by means of an example.

In what follows by "representation" we always mean "regular representation". Note that we use right notation, that is, we denote by xf the image of x by the function f.

2 Preliminaries

Hypermaps are 4-tuples $\mathcal{H} = (F; r_0, r_1, r_2)$ where F is a finite set and r_0, r_1, r_2 are involutory permutations of F ($r_i^2 = 1$) generating a transitive group on F. The elements of F are called *flags* and the transitive group $Mon(\mathcal{H}) = \langle r_0, r_1, r_2\rangle$ is the *monodromy group* of $\mathcal{H}$. The orbits of the action of the subgroups of $Mon(\mathcal{H})$ generated by $\{r_0, r_1, r_2\} \setminus \{r_i\}$ for $i = 0, 1, 2$ are respectively the *hypervertices*, *hyperedges* and *hyperfaces* of the hypermap $\mathcal{H}$, called respectively 0-*cells*, 1-*cells* and 2-*cells* of $\mathcal{H}$. The *valency* of a $i-$cell is the length of the orbit of one of its flag by $r_j r_k$ where $\{i, j, k\} = \{0, 1, 2\}$. If, for some positive integers k, ℓ, m all hypervertices have valency k, all hyperedges have valency ℓ and all hyperfaces have valency m, then we say that $\mathcal{H}$ is a *uniform* hypermap (of type (k, ℓ, m)). In this case, $(k, \ell, m) = (|r_1r_2|, |r_2r_0|, |r_0r_1|)$, where $|g|$ denotes the order of g. If r_0, r_1 and r_2 have no fixed point then we say that $\mathcal{H}$ has *no boundary*. Thus, a uniform hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ without boundary has $V = \frac{|F|}{2|r_1r_2|}$ hypervertices, $E = \frac{|F|}{2|r_2r_0|}$ hyperedges and $F = \frac{|F|}{2|r_0r_1|}$ hyperfaces.

A *morphism* or *covering* from the hypermap $\mathcal{H}_1 = (E; r_0, r_1, r_2)$ to the hypermap $\mathcal{H}_2 = (F; s_0, s_1, s_2)$ is a function $\phi\colon E \to F$ satisfying

$$xr_i\phi = x\phi s_i$$

for any $x \in E$ and any $i \in \{0, 1, 2\}$. We say that the hypermap $\mathcal{H}_1$ *covers* the hypermap $\mathcal{H}_2$ if there is a covering from $\mathcal{H}_1$ to $\mathcal{H}_2$. It is straightforward to see that any covering is onto and uniquely determined by the image of a flag. Injective coverings are therefore called *isomorphisms*. An *automorphism* of $\mathcal{H}$ is an isomorphism from $\mathcal{H}$ to itself. We will denote by $Aut(\mathcal{H})$ the set of automorphisms of $\mathcal{H}$, which is obviously a group under composition.

Topologically, a hypermap $\mathcal{H}$ can be seen as a triangulation of a compact surface $\mathcal{S}$ with vertices labelled 0,1 and 2 such that each triangle (a flag of $\mathcal{H}$) has labels $0, 1$ and 2 assigned to its vertices; the vertices labelled $0, 1$ and 2 are respectively the hypervertices, hyperedges and hyperfaces. For each $x \in F$ the two triangles x and xr_i share the common edge e opposite to the vertices labelled i if $x \neq xr_i$; if $x = xr_i$, then the edge e is on the boundary of $\mathcal{S}$ and so $\mathcal{S}$ is a bordered surface. This triangulation is a topological map representation of hypermaps whose dual is the James topological map representation of hypermaps [4]; here the faces are labelled 0 (grey faces), 1 (dotted faces) and 2 (white faces) (see Figure 3). The hypermap $\mathcal{H}$ has (no) boundary if and only if $\mathcal{S}$ has (no) boundary. The *characteristic* of $\mathcal{H}$ is the Euler characteristic of $\mathcal{S}$. In particular, if $\mathcal{H} = (F; r_0, r_1, r_2)$ is a uniform hypermap without boundary, then the Euler characteristic of $\mathcal{H}$ is

$$\chi(\mathcal{H}) = \frac{|F|}{2}\left(\frac{1}{|r_1r_2|} + \frac{1}{|r_2r_0|} + \frac{1}{|r_0r_1|} - 1\right).$$

Alternatively, a hypermap is a cellular embedding of a hypergraph in a compact connected surface.

The monodromy group $Mon(\mathcal{H})$ of a hypermap $\mathcal{H}$ is a quotient of the triangle group Δ. Hence we have an epimorphism $\pi : \Delta \to Mon(\mathcal{H})$ and an action

$$F \times \Delta \to F,\ (x, d) \mapsto x(d\pi)$$

of Δ on the set F of flags of $\mathcal{H}$. The stabiliser H of a flag under this action is a subgroup of Δ called a *hypermap subgroup* of $\mathcal{H}$. As the action of Δ is transitive, hypermap subgroups of $\mathcal{H}$ are conjugate. The hypermap $\mathcal{H}$ is then isomorphic to the hypermap $(\Delta/H; H_\Delta R_0, H_\Delta R_1, H_\Delta R_2)$, where Δ/H denotes the set of right cosets of a hypermap subgroup H of $\mathcal{H}$ in Δ, H_Δ is the normal core of H in Δ and $(Hd)H_\Delta R_i = HdR_i$ for any $d \in \Delta$ and any $i \in \{0, 1, 2\}$ (see, for instance [1]).

Let Θ be a normal subgroup of finite index n in Δ and let $\mathcal{H}$ be a hypermap with hypermap subgroup H. Then Θ acts (as a subgroup of Δ) on the set $F = \Delta/H$ of flags of $\mathcal{H}$ partitioning it into at most n orbits, called Θ-*orbits*; in fact, suppose that H is not a subgroup of Θ and let $b \in H \setminus \Theta$. Then $Hb = H$ and $b\Theta \neq \Theta$. Therefore the Θ-orbit $\{Hbt : t \in \Theta\}$ is equal to the Θ-orbit $\{Ht : t \in \Theta\}$, forcing the number of Θ-orbits being at most n. The number of Θ-orbits is n if and only if $H < \Theta$; in this case we say that $\mathcal{H}$ is Θ-*conservative*. A Θ-conservative hypermap $\mathcal{H}$ is Θ-*regular* if the group $Aut^\Theta(\mathcal{H})$ of automorphisms preserving Θ-orbits acts transitively on each Θ-orbit, or equivalently, if H is normal in Θ. However if H is normal in Δ, then $\mathcal{H}$ is a *regular* hypermap, that is, Δ-regular. We shall say that a hypermap $\mathcal{H}$ is *restrictedly-marked* if it is Θ-conservative for some normal subgroup Θ of finite index in Δ. Ought to emphasise that not every hypermap is restrictedly-marked (see [1] for examples).

A hypermap $(F; r_0, r_1, r_2)$ satisfying $(r_0r_2)^2 = 1$ is called a *map*. The hypervertices, hyperedges and hyperfaces of a map are called *vertices*, *edges* and *faces*, since topologically a map is a cellular embedding of a graph on a compact surface. The monodromy group of a map $\mathcal{M}$ is then a quotient of the "right" triangle group Γ. This group acts on the set of flags of $\mathcal{M}$ via the canonical projection $\pi\colon \Gamma \to Mon(\mathcal{M})$ sending R_i to r_i. The stabiliser of a flag under this action will be called a *map subgroup* of $\mathcal{M}$. Keeping the same notation as already used for hypermaps, we have that a map $\mathcal{M}$ is then isomorphic to the map $(\Gamma/M; M_\Gamma R_0, M_\Gamma R_1, M_\Gamma R_2)$, where M is a map subgroup of $\mathcal{M}$. The theory of restrictedly-marked maps unfolds in the same way as the theory of restrictedly-marked hypermaps by taking finite index normal subgroups Θ of Γ instead of Δ. The group Γ is a free product of $C_2 = \langle R_1 \rangle$ with $D_2 = \langle R_0, R_2 \rangle$ and by the Kurosh's Subgroup Theorem, any normal subgroup Θ of Γ freely decomposes uniquely (up to a permutation of factors) in a (indecomposable) free product (see [6] pg 243 and 245)

$$C_2 * \cdots * C_2 * D_2 * \cdots * D_2 * C_\infty * \cdots * C_\infty =$$
$$\langle A_1 \rangle * \cdots * \langle A_s \rangle * \langle B_1, C_1 \rangle * \langle B_t, C_t \rangle * \langle Z_1 \rangle * \cdots * \langle Z_u \rangle$$

for a certain numbers s, t and u of factors $\langle A_i \rangle = C_2$, $\langle B_j, C_j \rangle = D_2$ and $\langle Z_u \rangle = C_\infty$ respectively, whereas s, t or u may be zero. Let $m = s + 2t + u = rank(\Theta)$ and let

$$\{A_1, \ldots, A_s, B_1, \ldots, B_t, C_1, \ldots, C_t, Z_1, \ldots, Z_u\} = \{X_1, \ldots, X_m\}.$$

Then a Θ-conservative map $\mathcal{M}$ with map subgroup M can be represented by the Θ-*marked map*

$$\mathcal{Q} = (\Omega; x_1, \ldots, x_m)$$

where $\Omega = \Theta/M$ is the set of right cosets of M in Θ and $x_1, \ldots, x_m$ are permutations of Ω generating a group G acting transitively on Ω such that the function

$$X_1 \mapsto x_1\,, \ldots\,, X_m \mapsto x_m$$

extends to an epimorphism from Θ to G.

Any Θ-regular map $\mathcal{M}$ covers the regular map

$$\mathcal{T}_\Theta = (\Gamma/\Theta; \Theta R_0, \Theta R_1, \Theta R_2)$$

called the Θ-*trivial map*. As $\mathcal{T}_\Theta$ is a regular map, we have that

- any two vertices of $\mathcal{T}_\Theta$ have same valency, say k,
- any two edges of $\mathcal{T}_\Theta$ have same valency, say $l \in \{1, 2\}$,
- any two faces of $\mathcal{T}_\Theta$ have same valency, say m.

The triple (k, l, m) is called *the type* of the regular map $\mathcal{T}_\Theta$. As $\mathcal{M}$ is Θ-regular and covers $\mathcal{T}_\Theta$, we also have that:

- the vertices of $\mathcal{M}$ covering a vertex v of $\mathcal{T}_\Theta$ also have same valency, say k_v (which is a multiple of k),
- the faces of $\mathcal{M}$ covering a face f of $\mathcal{T}_\Theta$ also have same valency, say m_f (which is a multiple of m).

Denoting by V, E and F the sets of vertices, edges and faces of $\mathcal{T}_\Theta$ and assuming that $\mathcal{M}$ has no boundary, then this together with Euler formula gives that the characteristic of $\mathcal{M}$ is

$$\chi(\mathcal{M}) = \frac{|\Theta : M|}{2} \left(\sum_{v \in V} \mu_v \frac{k}{k_v} + \sum_{e \in E} \mu_e \frac{l}{2} + \sum_{f \in F} \mu_f \frac{m}{m_f} - |\Gamma : \Theta| \right)\,, \tag{2.1}$$

where $\mu_v = 1$ or 2 according as the vertex v is on the boundary or not and similarly for μ_e and μ_f. For details we refer the reader to [1].

3 Thin map representations of hypermaps

Let Θ be a finite index normal subgroup of Γ of rank 3 and let $\{X_1, X_2, X_3\}$ be a set of generators of Θ. The pair $R = (\Theta, \{X_1, X_2, X_3\})$ will be called a Θ-*marked representation (of hypermaps by maps)* if the function

$$X_1 \mapsto S_0\,,\ X_2 \mapsto S_1\,,\ X_3 \mapsto S_2$$

extends to an epimorphism ρ from Θ onto Δ. We call ρ the *canonical epimorphism* of the representation R. Two representations $(\Theta_1, \{X_1, X_2, X_3\})$ and $(\Theta_2, \{Y_1, Y_2, Y_3\})$ are to be considered equal if $\Theta_1 = \Theta_2 = \Theta$ and their canonical epimorphisms $\rho_1, \rho_2 : \Theta \to \Delta$ are such that $\rho_1 = \iota\rho_2$ for some inner automorphism ι of Θ. For example, since S_0, S_1, S_2 are involutions, inverting one or more generators of R give the same representation.

Given a hypermap $\mathcal{H}$ with hypermap subgroup H, setting $\Omega = \{(H\rho^{-1})t : t \in \Theta\}$ and

$$x_i : \Omega \to \Omega,\ (H\rho^{-1})t \mapsto (H\rho^{-1})tX_i, \qquad i = 1, 2, 3$$

we get a Θ-marked map $(\Omega; x_1, x_2, x_3)$ called a Θ-*marked map representation of* $\mathcal{H}$.

Remark 3.1. In fact, denoting by N the normal core of $H\rho^{-1}$ in Θ, the group $G = \langle x_1, x_2, x_3\rangle$ is isomorphic to Θ/N by an isomorphism φ mapping x_i to NX_i for any $i \in \{1, 2, 3\}$. Hence $\pi\varphi^{-1}$, where $\pi : \Theta \to \Theta/N$ is the canonical epimorphism, is an epimorphism from Θ to G extending the function $X_1 \mapsto x_1, X_2 \mapsto x_2, X_3 \mapsto x_2$. Remark also that ρ induces a bijection $\widetilde{\rho}$ from Ω to $\{Hd : d \in \Delta\}$ which sends $(H\rho^{-1})t$ to $H(t\rho)$ and satisfies $x_i\,\widetilde{\rho} = \widetilde{\rho}\,r_{i-1}$, where r_{i-1} maps Hd to HdR_{i-1} for any $i \in \{1, 2, 3\}$. Thus, we say that $\widetilde{\rho}$ is an *isomorphism* from the Θ-marked map representation $(\Omega; x_1, x_2, x_3)$ of $\mathcal{H}$ to $\mathcal{H}$.

A (Θ-marked) representation $R = (\Theta, (X_1, X_2, X_3))$ will be called *clean* if Θ is the free product of the cyclic groups $\langle X_1\rangle, \langle X_2\rangle, \langle X_3\rangle$, in which case we write

$$\Theta = \langle X_1\rangle * \langle X_2\rangle * \langle X_3\rangle\,.$$

A clean representation is called *thin* if the sum of the lengths of its generators (as words in the free group over $\{R_0, R_1, R_2\}$) is minimal.

The number of rank 3 normal subgroups Θ of Γ is finite, but there are infinitely many clean representations given by all possible sets $\{X_1, X_2, X_3\}$ such that $\Theta = \langle X_1\rangle * \langle X_2\rangle * \langle X_3\rangle$. On the other hand the number of thin representations is finite (see Sections 4 and 5).

4 The rank 3 normal subgroups of Γ

4.1 Rank computation

(see also [3])

In order to compute the rank of a normal subgroup Θ of finite index in Γ, we remark that Γ acts as a group of isometries on the hyperbolic plane $\mathbf{H}$, regarding its generators R_0, R_1, R_2 as the reflections on the geodesics given in Figure 1 in the Poincaré disk model. The action of Θ on $\mathbf{H}$ gives rise to a quotient orbifold $\mathbf{H}/\Theta$ which is a punctured surface (with or without boundary) punctured at the vertices and at the face centers of the

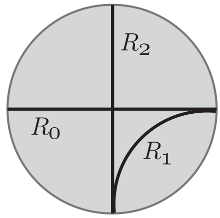


Figure 1: The generators of Γ as hyperbolic reflections.

regular map $\mathcal{M} = (\Gamma/\Theta; \Theta R_0, \Theta R_1, \Theta R_2)$ with underlying surface $\mathbf{S}$. If $\Theta R_0 = \Theta R_2$, then the covering $\mathbf{H} \to \mathbf{H}/\Theta$ is also branched at the edge centers of $\mathcal{M}$. The group Θ, being the fundamental group of $\mathbf{H}/\Theta$, has a presentation P with $p + 2 - \chi$ generators $X_1, \ldots, X_p, Y_1, \ldots, Y_{2-\chi}$, where p is the total number of punctures and branching points of $\mathbf{H}/\Theta$ and χ is the characteristic of $\mathcal{S}$. The presentation P has a relator $S = X_1 \cdots X_p \cdot \prod_{i=1}^{k} Y_i \cdot W(Y_{k+1}, \ldots, Y_{2-\chi})$, where

$$\langle Y_1, \ldots, Y_{2-\chi} \mid W(Y_{k+1}, \ldots, Y_{2-\chi}) \rangle$$

is a presentation of the fundamental group of the surface $\mathbf{S}$ with k boundary components (setting $\prod_{i=1}^{k} Y_i = 1$ if $k = 0$), and eventually e relators $X_1^2, \ldots, X_e^2$ if $\Theta R_0 = \Theta R_2$, where e is the number of edges of $\mathcal{M}$.
Hence $rank(\Theta) = p + 2 - \chi - 1 = p + 1 - \chi$. More precisely:

- If $\mathbf{S}$ has no boundary ($k = 0$) and is non-orientable, then $2 - \chi$ is the genus g of $\mathbf{S}$, $W(Y_1, \ldots, Y_{2-\chi}) = W(Y_1, \ldots, Y_g) = \prod_{i=1}^{g} Y_i^2$ and therefore

$$S = X_1 \cdots X_p \cdot \prod_{i=1}^{g} Y_i^2 \,.$$

- If $\mathbf{S}$ has no boundary and is orientable, then $2 - \chi$ is even and the genus g of $\mathbf{S}$ is $\frac{2-\chi}{2}$. Replacing $(Y_1, \ldots, Y_{2-\chi})$ by $(A_1, B_1, \ldots, A_g, B_g)$ we have

$$S = X_1 \cdots X_p \cdot \prod_{i=1}^{g} [A_i, B_i] \,.$$

  In the particular case of $\chi = 2$ (sphere) the word $W(Y_1, \ldots, Y_{2-\chi})$ is empty and therefore

$$S = X_1 \cdots X_p.$$

- If $\mathbf{S}$ has boundary, then $\{R_0, R_1, R_2\} \cap \Theta \neq \emptyset$. Thus, any triangle of the triangulation of $\mathbf{S}$ given by the flags of $\mathcal{M}$ has at least an edge on the boundary, since Θ is normal in Γ. This shows that $\mathbf{S}$ is a closed disk, that is, a bordered surface on a sphere with only one boundary component ($k = 1$). Hence $\chi = 2 - k = 1$ and therefore, setting $Y = Y_1$ we have that $\prod_{i=1}^{k} Y_1 = Y$, $W(Y_{k+1}, \ldots, Y_{2-\chi})$ is the empty word and $\langle Y \rangle \cong C_\infty$ is the fundamental group of $\mathbf{S}$. Hence

$$S = X_1 \cdots X_p \cdot Y .$$

In particular, $rank(\Theta) = p$ in this case.

The next proposition relates the rank of Θ with its index n in Γ for $n > 4$. Relating rank with all indices will give a clumsy formula which does not give more information about the index bound for fixed rank.

Proposition 4.1. If Θ is a normal subgroup of finite index $n > 4$ in Γ, then n is even and

$$rank(\Theta) = \begin{cases} 1+n & \text{if } \mathbf{H}/\Theta \text{ has boundary and branching points;} \\ 1+\frac{n}{2} & \text{if } \mathbf{H}/\Theta \text{ has boundary and no branching point;} \\ & \text{or } \mathbf{H}/\Theta \text{ has no boundary but has branching points;} \\ 1+\frac{n}{4} & \text{if } \mathbf{H}/\Theta \text{ has no boundary and no branching point.} \\ & \text{(in this case } n \text{ is a multiple of } 4). \end{cases}$$

Proof. Using the above notations and remarks we have the following:
If $\mathbf{S}$ has boundary, then $\Theta R_1 \neq \Theta$ since $|\Gamma/\Theta| = n > 4$. Hence $\Gamma/\Theta = \langle \Theta R_1, \Theta R_j \rangle$ for some $j \in \{0, 2\}$, that is, Γ/Θ is dihedral of even order n. The total number of vertices and faces of the map $\mathcal{M} = (\Gamma/\Theta; \Theta R_0, \Theta R_1, \Theta R_2)$ is then $1 + \frac{n}{2}$. This gives

$$rank(\Theta) = p = \begin{cases} 1+\frac{n}{2} & \text{if } \mathbf{H}/\Theta \text{ has no branching points,} \\ 1+n & \text{if } \mathbf{H}/\Theta \text{ has branching points,} \end{cases}$$

since in the case when $\mathbf{H}/\Theta$ has branching points, $\mathcal{M}$ has $\frac{n}{2}$ edges.

If $\mathbf{S}$ has no boundary, then n is a multiple of 4 and from Euler formula we have that

$$\chi = \begin{cases} p - \frac{n}{4} & \text{if } \mathbf{H}/\Theta \text{ has no branching points,} \\ p - \frac{n}{2} & \text{if } \mathbf{H}/\Theta \text{ has branching points.} \end{cases}$$

Therefore

$$rank(\Theta) = p + 1 - \chi = \begin{cases} 1+\frac{n}{4} & \text{if } \mathbf{H}/\Theta \text{ has no branching points,} \\ 1+\frac{n}{2} & \text{if } \mathbf{H}/\Theta \text{ has branching points.} \end{cases}$$ □

Corollary 4.2. *If $rank(\Theta) = 3$, then the index n is 2, 4 or 8 and Γ/Θ is isomorphic to C_2, $C_2 \times C_2$, $C_2 \times C_2 \times C_2$ or D_4.*

Proof. Proposition 4.1 guaranties that $n \in \{2, 4, 8\}$ if $rank(\Theta) = 3$. The groups of order $2, 4$ and 8 not listed in the statement are not generated by involutions. □

4.2 The rank 3 normal subgroups of Γ

(1) $n = 2$: As mentioned in the introduction, there are seven epimorphisms from Γ to C_2 having kernels $\Gamma_{2.1}, \ldots, \Gamma_{2.7}$. Only three of them have rank 3, as it is easily checked by

applying the Reidemeister-Schreier rewriting process. In this way, one gets that the rank 3 kernels $\Theta = \langle X\rangle * \langle Y\rangle * \langle Z\rangle$ are

$$\Gamma_{2.1} = \langle R_0\rangle * \langle R_1\rangle * \langle R_1^{R_2}\rangle\,,\quad \Gamma_{2.4} = \langle R_1\rangle * \langle R_2\rangle * \langle R_1^{R_0}\rangle \quad \text{and}$$
$$\Gamma_{2.5} = \langle R_1\rangle * \langle R_0R_2\rangle * \langle R_1^{R_0}\rangle\,.$$

These three groups are isomorphic to the free product $C_2 * C_2 * C_2$ and therefore isomorphic to Δ. The remaining four epimorphisms have kernels

$$\Gamma_{2.2} = \langle R_0, R_2\rangle * \langle R_0^{R_1}, R_2^{R_1}\rangle\,,\quad \Gamma_{2.3} = \langle R_0\rangle * \langle R_1R_2\rangle\,,\quad \Gamma_{2.6} = \langle R_2\rangle * \langle R_0R_1\rangle \quad \text{and}$$
$$\Gamma_{2.7} = \langle R_0R_2\rangle * \langle R_1R_2\rangle\,.$$

The group $\Gamma_{2.2}$ has rank 4 and is isomorphic to the free product $D_2 * D_2$, while the other three groups $\Gamma_{2.3}$, $\Gamma_{2.6}$ and $\Gamma_{2.7}$ have rank 2 and are all isomorphic to $C_2 * C_\infty$.

(2) $n = 4$: Up to an automorphism of $G = C_2 \times C_2$ there are seven epimorphisms from Γ to G with kernels $\Gamma_{4.1}, \ldots, \Gamma_{4.7}$. One can check that three of them have rank 3, namely

$$\Gamma_{4.1} = \langle R_0\rangle * \langle R_0^{R_1}\rangle * \langle (R_1R_2)^2\rangle\,,\quad \Gamma_{4.4} = \langle R_2\rangle * \langle R_2^{R_1}\rangle * \langle (R_0R_1)^2\rangle \quad \text{and}$$
$$\Gamma_{4.5} = \langle R_0R_2\rangle * \langle (R_0R_1)^2\rangle * \langle (R_0R_2)^{R_1}\rangle\,.$$

These groups are all isomorphic to the free product $C_2 * C_2 * C_\infty$ so that Δ is an epimorphic image of each of them.

Remark 4.3.
$$\begin{aligned}\Gamma_{4.1} &= \Gamma_{2.3} \cap \Gamma_{2.2} = \Gamma_{2.3} \cap \Gamma_{2.1} = \Gamma_{2.2} \cap \Gamma_{2.1} = \Gamma_{2.3} \cap \Gamma_{2.2} \cap \Gamma_{2.1}\,,\\ \Gamma_{4.4} &= \Gamma_{2.4} \cap \Gamma_{2.2} = \Gamma_{2.4} \cap \Gamma_{2.6} = \Gamma_{2.2} \cap \Gamma_{2.6} = \Gamma_{2.4} \cap \Gamma_{2.2} \cap \Gamma_{2.6}\,,\\ \Gamma_{4.5} &= \Gamma_{2.7} \cap \Gamma_{2.5} = \Gamma_{2.7} \cap \Gamma_{2.2} = \Gamma_{2.5} \cap \Gamma_{2.2} = \Gamma_{2.7} \cap \Gamma_{2.5} \cap \Gamma_{2.2}\,.\end{aligned}$$

(3) $n = 8$, $G = D_4$: Up to an automorphism of G there are six epimorphism from Γ to G with kernels $\Gamma_{8.1}, \ldots, \Gamma_{8.6}$. Three of them have rank 3 and are all free groups, namely

$$\begin{aligned}\Gamma_{8.4} &= \langle R_0R_1R_2R_1\rangle * \langle R_1R_0R_1R_2\rangle * \langle (R_0R_1)^2R_0R_2\rangle\,,\\ \Gamma_{8.5} &= \langle (R_1R_2)^2\rangle * \langle R_2(R_1R_0)^2\rangle * \langle R_2(R_0R_1)^2\rangle \quad \text{and}\\ \Gamma_{8.6} &= \langle (R_0R_1)^2\rangle * \langle R_0(R_1R_2)^2\rangle * \langle R_0(R_2R_1)^2\rangle\,.\end{aligned}$$

Remark 4.4. $\Gamma_{4.5}$ is the unique normal subgroup of index 4 containing $\Gamma_{8.4}$, while $\Gamma_{4.4}$ is the unique normal subgroup of index 4 containing $\Gamma_{8.5}$ and $\Gamma_{4.1}$ is the unique normal subgroup of index 4 containing $\Gamma_{8.6}$.

(4) $n = 8$, $G = C_2 \times C_2 \times C_2$: Up to an automorphism of G there is only one epimorphism from Γ to G with kernel isomorphic to the rank 3 free group $C_\infty * C_\infty * C_\infty$, namely

$$\Gamma_{8.7} = \langle (R_0R_1)^2\rangle * \langle (R_1R_2)^2\rangle * \langle R_0(R_1R_2)^2R_0\rangle\,.$$

Remark 4.5. $\Gamma_{8.7} = \Gamma_{4.i} \cap \Gamma_{4.j}$ for any distinct $i, j \in \{1, \ldots, 7\}$.

The following table gives a overall description of Θ and the Θ-trivial map for each normal subgroup Θ of Γ of index $2, 4, 6$ and 8.

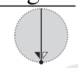
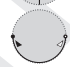
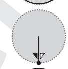
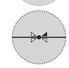
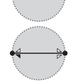
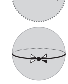
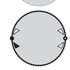
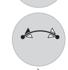
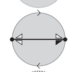
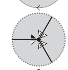
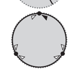
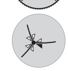
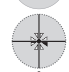
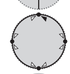
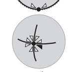
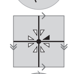
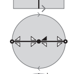
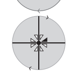
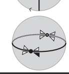
Θ	index	rank	Free-Product dec.	Type of $\mathcal{T}_\Theta$	surface	χ	fig
$\Gamma_{2.1}$	2	3	$C_2 * C_2 * C_2$	(2,2,1)	border	1	
$\Gamma_{2.2}$	2	4	$D_2 * D_2$	(2,1,2)	border	1	
$\Gamma_{2.3}$	2	2	$C_2 * C_\infty$	(1,2,2)	border	1	
$\Gamma_{2.4}$	2	3	$C_2 * C_2 * C_2$	(1,2,2)	border	1	
$\Gamma_{2.5}$	2	3	$C_2 * C_2 * C_2$	(2,1,2)	border	1	
$\Gamma_{2.6}$	2	2	$C_2 * C_\infty$	(2,2,1)	border	1	
$\Gamma_{2.7}$	2	2	$C_2 * C_\infty$	(1,1,1)	orient.	2	
$\Gamma_{4.1}$	4	3	$C_2 * C_2 * C_\infty$	(2,2,2)	border	1	
$\Gamma_{4.2}$	4	4	$C_2 * C_2 * C_2 * C_2$	(2,2,2)	border	1	
$\Gamma_{4.3}$	4	2	$C_\infty * C_\infty$	(2,2,1)	orient.	2	
$\Gamma_{4.4}$	4	3	$C_2 * C_2 * C_\infty$	(2,2,2)	border	1	
$\Gamma_{4.5}$	4	3	$C_2 * C_2 * C_\infty$	(2,1,2)	orient.	2	
$\Gamma_{4.6}$	4	2	$C_\infty * C_\infty$	(1,2,2)	orient.	2	
$\Gamma_{4.7}$	4	2	$C_\infty * C_\infty$	(2,2,2)	nonori.	1	
$\Gamma_{6.1}$	6	4	$C_2 * C_2 * C_2 * C_\infty$	(3,2,2)	border	1	
$\Gamma_{6.2}$	6	4	$C_2 * C_2 * C_2 * C_\infty$	(2,2,3)	border	1	
$\Gamma_{6.3}$	6	4	$C_2 * C_2 * C_2 * C_\infty$	(3,1,3)	orient.	2	
$\Gamma_{8.1}$	8	5	$C_2 * C_2 * C_2 * C_2 * C_\infty$	(4,2,2)	border	1	
$\Gamma_{8.2}$	8	5	$C_2 * C_2 * C_2 * C_2 * C_\infty$	(2,2,4)	border	1	
$\Gamma_{8.3}$	8	5	$C_2 * C_2 * C_2 * C_\infty * C_\infty$	(4,1,4)	orient.	2	
$\Gamma_{8.4}$	8	3	$C_\infty * C_\infty * C_\infty$	(4,2,4)	orient.	0	
$\Gamma_{8.5}$	8	3	$C_\infty * C_\infty * C_\infty$	(2,2,4)	nonori.	1	
$\Gamma_{8.6}$	8	3	$C_\infty * C_\infty * C_\infty$	(4,2,2)	nonori.	1	
$\Gamma_{8.7}$	8	3	$C_\infty * C_\infty * C_\infty$	(2,2,2)	orient.	2	

Table 1: Normal subgroups of indices 2, 4, 6 and 8 in $\Gamma = C_2 * D_2$.

5 Description of the thin map representations

In the previous section we computed all rank 3 normal subgroups $\Theta = \Gamma_{i.j}$ together with a set of generators $\{X_1, X_2, X_3\}$ such that $\Gamma_{i.j} = \langle X_1 \rangle * \langle X_2 \rangle * \langle X_3 \rangle$ is a thin representation $Ri.j$. Since some $\Gamma_{i.j}$ gives rise to more than one thin representation, we label the corresponding representations by $Ri.ja$, $Ri.jb$, etc. Note that the generators of a thin representation can be read out as fundamental group generators (written as words on $\{R_0, R_1, R_2\}$) from the respective trivial map (Section 4). The classification is done up to a restrictedly dual, that is, the generators of a Θ-marked representation are computed up to the usual map dual if its restriction to Θ is an automorphism of Θ (see also Remark below). The following table gives all the thin Θ-marked representations. Generators of $\Gamma_{i.j}$ which are involutions will be labelled by A, B, C and those which are not will be denoted by X, Y, Z.

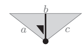
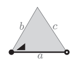
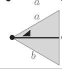
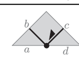
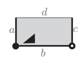
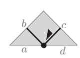
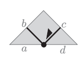
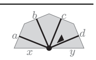
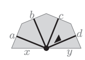
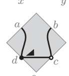
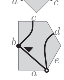
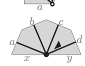
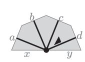
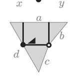
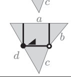
#	Rep.	Generators			Epim.	Θ-slice
1	$R2.1$	$A{=}R_0$	$B{=}R_1$	$C{=}R_2R_1R_2$	$A\to r_0$ $B\to r_1$ $C\to r_2$	
2	$R2.4$	$A{=}R_0R_1R_0$	$B{=}R_1$	$C{=}R_2$	$A\to r_0$ $B\to r_1$ $C\to r_2$	
3	$R2.5$	$A{=}R_0R_1R_0$	$B{=}R_1$	$C{=}R_0R_2$	$A\to r_0$ $B\to r_1$ $C\to r_2$	
4	$R4.1$	$A{=}R_0$	$B{=}R_1R_0R_1$	$X{=}R_1R_2R_1R_2$	$A\to r_0$ $B\to r_1$ $X\to r_2$	
5	$R4.4$	$A{=}R_2$	$B{=}R_1R_2R_1$	$Z{=}R_0R_1R_0R_1$	$A\to r_2$ $B\to r_1$ $Z\to r_0$	
6	$R4.5a$	$A{=}R_0R_2$	$B{=}R_1R_0R_2R_1$	$Z{=}R_0R_1R_0R_1$	$A\to r_0$ $B\to r_2$ $Z\to r_1$	
7	$R4.5b$	$A{=}R_0R_2$	$B{=}R_1R_0R_2R_1$	$X{=}R_0R_1R_2R_1$	$A\to r_0$ $B\to r_2$ $X\to r_1$	
8	$R8.4a$	$X{=}R_0R_1R_2R_1$	$Y{=}R_1R_0R_1R_2$	$Z{=}R_0R_1R_0R_1R_0R_2$	$X\to r_0$ $Y\to r_1$ $Z\to r_2$	
9	$R8.4b$	$X{=}R_0R_1R_2R_1$	$Y{=}R_1R_0R_1R_2$	$Z{=}R_0R_2R_1R_0R_2R_1$	$X\to r_0$ $Y\to r_1$ $Z\to r_2$	
10	$R8.5a$	$X{=}R_1R_2R_1R_2$	$Y{=}R_0R_1R_0R_1R_2$	$Z{=}R_0R_2R_1R_0R_1$	$X\to r_0$ $Y\to r_1$ $Z\to r_2$	
11	$R8.5b$	$X{=}R_1R_2R_1R_2$	$Y{=}R_0R_1R_0R_2R_1$	$Z{=}R_0R_2R_1R_0R_1$	$X\to r_0$ $Y\to r_1$ $Z\to r_2$	
12	$R8.6a$	$X{=}R_0R_1R_2R_1R_2$	$Y{=}R_0R_1R_0R_1$	$Z{=}R_0R_2R_1R_2R_1$	$X\to r_1$ $Y\to r_0$ $Z\to r_2$	
13	$R8.6b$	$X{=}R_0R_2R_1R_2R_1$	$Y{=}R_0R_1R_0R_1$	$Z{=}R_1R_0R_2R_1R_2$	$X\to r_1$ $Y\to r_0$ $Z\to r_2$	
14	$R8.7a$	$X{=}R_1R_2R_1R_2$	$Y{=}R_0R_1R_2R_1R_2R_0$	$Z{=}R_0R_1R_0R_1$	$X\to r_1$ $Y\to r_2$ $Z\to r_0$	
15	$R8.7b$	$X{=}R_1R_2R_1R_2$	$Y{=}R_0R_2R_1R_2R_0R_1$	$Z{=}R_0R_1R_0R_1$	$X\to r_1$ $Y\to r_2$ $Z\to r_0$	

Table 2: The 15 thin representations.

Remark 5.1. The assignments

$$R_0 \mapsto R_2\,, \qquad R_1 \mapsto R_1\,, \quad R_2 \mapsto R_0 \quad \text{and}$$
$$R_0 \mapsto R_0R_2\,, \quad R_1 \mapsto R_1\,, \quad R_2 \mapsto R_2$$

extend to automorphisms of Γ and give rise to the map dualities D (the usual map duality) and P (the Petrie duality). Together they generate the outer automorphism group $Out(\Gamma) = \langle D, P\rangle \cong S_3$. The following diagram graphically pictures the action of $Out(\Gamma)$ on the set of rank 3 normal subgroups of Γ, where lines and dash lines represent the action of D and P, respectively. Note that D, or P, fixes some Θ and therefore for those Θ's it is a Θ-

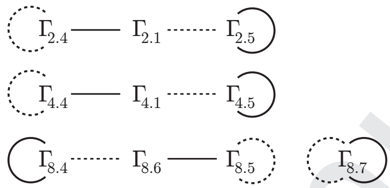


Figure 2: The actions of D and P on the Θ's.

restrictedly duality. The Petrie duality is not a thin-preserving duality except in the case of $\Gamma_{8.7}$; here $R8.7a$ and $R8.7b$ are Petrie duals of each other. The duality D fixes $\Gamma_{2.5}$, $\Gamma_{4.5}$, $\Gamma_{8.4}$ and $\Gamma_{8.7}$. These give rise to the restrictedly-dual representations given in the following Table, but not listed in Table 2.

Θ-dual of Rep.	Generators		
$R2.5$	$A = R_2R_1R_2$	$B = R_1$	$C = R_0R_2$
$R4.5a$	$A = R_0R_2$	$B = R_1R_0R_2R_1$	$Z = R_2R_1R_2R_1$
$R4.5b$	$A = R_0R_2$	$B = R_1R_0R_2R_1$	$Z = R_2R_1R_0R_1$
$R8.4a$	$X = R_2R_1R_0R_1$	$Y = R_1R_2R_1R_0$	$Z = R_2R_1R_2R_1R_0R_2$
$R8.7a$	$X = R_1R_0R_1R_0$	$Y = R_0R_2R_1R_2R_0R_1$	$Z = R_2R_1R_2R_1$

Table 3: The dual representations.

To illustrate each thin representation, we exhibit the Θ-marked map representation of the toroidal regular hypermap $\mathcal{H}$ pictured in Figure 3 using the James hypermap representation [4], where hypervertices, hyperedges and hyperfaces of $\mathcal{H}$ are represented by simply connected regions colored grey, dotted and white, respectively, and flags are the numbered points. We note that Lynne James hypermap representation is actually the $\Gamma_{6.1}$-marked map representation sending $(R_2R_1)^3$ to 1. Here $\Gamma_{6.1}$ is the normal subgroup of index 6 of Γ isomorphic to $C_2 * C_2 * C_2 * C_\infty$ generated by R_0, $R_0^{R_1}$, $R_0^{R_1R_2}$ and $(R_2R_1)^3$ (given in Table 1). This representation is not listed in Table 2 because this restrictedly marked representation is not thin (it is not even clean).

Thus $\mathcal{H} = (F; r_0, r_1, r_2)$ with $F = \{1, \ldots, 6\}$ and up to permutation (coloring)

$$r_0 = (1,4)(2,5)(3,6)\,, \quad r_1 = (1,2)(3,4)(5,6)\,, \quad r_2 = (1,6)(2,3)(4,5)\,.$$

The hypermap $\mathcal{H}$ has one hypervertex, one hyperedge and one hyperface all of valency 3. The monodromy group of $\mathcal{H}$ is $G = \langle r_0, r_1, r_2\rangle \cong S_3$. The Euler characteristic of a map

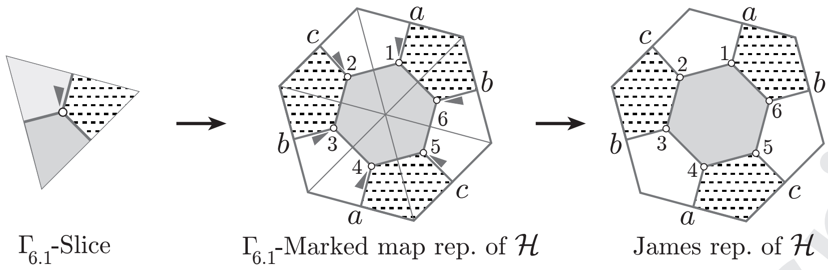


Figure 3: The toroidal regular hypermap $\mathcal{H}$.

representation of $\mathcal{H}$ is given by (2.1) taking into account the Θ-trivial map given in Table 2 and using the isomorphism $\widetilde{\rho}$ given in 3.1.

As an example, we give a detailed construction of the thin representation $R4.1$ of $\mathcal{H}$ following the generic description given in [1]:
The words R_0, $R_0^{R_1}$ and $(R_1R_2)^2$, in this order, generate the subgroup $\Theta = \Gamma_{4.1}$ as a free product $C_2 * C_2 * C_\infty$ (Table 2). A rooted Θ-slice can be obtained from a Schreier transversal of Θ in Γ, or alternatively by a cut-opening of the trivial Θ-map (see Table 1). The rooted Θ-slice we are taking here is the one given by the Schreier transversal $\{1, R_1, R_2, R_1R_2\}$. Another Schreier transversal may lead to a different rooted Θ-

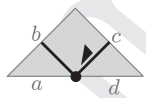
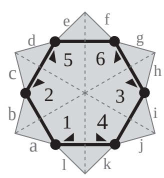


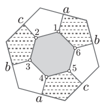
Figure 4: The rooted $\Gamma_{4.1}$-slice.

slice, and a choice of another flag as root corresponds to take another Schreier transversal, and both will lead to "similar" Θ-marked maps, in the sense that the underlying map is the same. The Θ-marked map representation of $\mathcal{H}$ is obtained by the isomorphism $\rho\colon \Theta/H\rho^{-1} \to \Delta/H$ given by $R_0 \mapsto r_0$, $R_0^{R_1} \mapsto r_1$ and $(R_1R_2)^2 \mapsto r_2$. So we have $R_0 = (1,4)(2,5)(3,6)$, $R_0^{R_1} = (1,2)(3,4)(5,6)$ and $(R_1R_2)^2 = (1,6)(2,3)(4,5)$. Now we take 6 rooted Θ-slices labelled 1, 2, 3, 4, 5 and 6 and join them through their sides a, b, c and d accordingly to the action of the words R_0, $R_0^{R_1}$ and $(R_1R_2)^2$ on the root flag of the slices. In this way, the word R_0 joins the slices 1 and 4, 2 and 5, and 3 and 6, by their sides labelled c, while $R_0^{R_1}$ joins the slices 1 and 2, 3 and 4, and 5 and 6, by their sides b. This leaves to an incomplete picture:



Now $(R_1R_2)^2$, which is an involution, says that the slices 1 and 6, 2 and 3, and 4 and 5, are joined together through their sides a and d, that is, in the picture above we have the following equality between labels: $g = a$ and $f = l$, $h = b$ and $i = c$, and $d = j$ and $k = e$. This lead to the final picture of $R4.1$ in Table 5.

In the following tables we illustrate the fifteen map representations Rep of the toroidal regular hypermap $\mathcal{H}$, we display the general Euler's characteristic formula for the map representation Rep of any hypermap, the actual Euler's characteristic of $\mathrm{Rep}(\mathcal{H})$ and the orientability (up to restricted dual) of $\mathrm{Rep}(\mathcal{H})$ - and when possible we record their overall orientability behaviour in parenthesis.

$\mathcal{H} =$  Euler characteristic of $\mathcal{H} = 0$ oriented

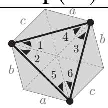
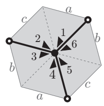
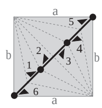
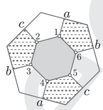
Rep	Rep($\mathcal{H}$)	Euler characteristic of Rep($\mathcal{H}$)	orient.
$R2.1$		$\lvert G\rvert \left(\frac{1}{2\lvert AB\rvert} + \frac{1}{2\lvert BC\rvert} + \frac{1}{2\lvert CA\rvert} - \frac{1}{2}\right) = 0$	yes
$R2.4$		$\lvert G\rvert \left(\frac{1}{2\lvert AB\rvert} + \frac{1}{2\lvert BC\rvert} + \frac{1}{2\lvert CA\rvert} - \frac{1}{2}\right) = 0$	yes
$R2.5$		$\lvert G\rvert \left(\frac{1}{2\lvert BCAC\rvert} + \frac{1}{2\lvert BA\rvert} - \frac{1}{2}\right) = 1$	no

Table 4: The Θ-marked map representations of $\mathcal{H}$ for $\lvert\Gamma : \Theta\rvert = 2$.

$\mathcal{H} =$  Euler characteristic of $\mathcal{H} = 0$ oriented

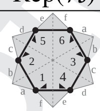
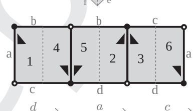
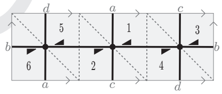
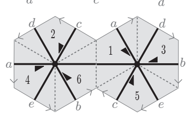
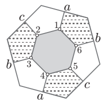
Rep	Rep($\mathcal{H}$)	Euler characteristic of Rep($\mathcal{H}$)	orient.
$R4.1$		$\lvert G\rvert \left(\frac{1}{2\lvert AB\rvert} + \frac{1}{2\lvert AXBX\rvert} - \frac{1}{2}\right) = 1$	no
$R4.4$		$\lvert G\rvert \left(\frac{1}{2\lvert BA\rvert} + \frac{1}{2\lvert AZBZ\rvert} - \frac{1}{2}\right) = 1$	no
$R4.5a$		$\lvert G\rvert \left(\frac{1}{\lvert AZB\rvert} - \frac{1}{2}\right) = 0$	yes (always)
$R4.5b$		$\lvert G\rvert \left(\frac{1}{\lvert AX\rvert} + \frac{1}{\lvert XB\rvert} - 1\right) = -2$	yes (always)

Table 5: The Θ-marked map representations of $\mathcal{H}$ for $\lvert\Gamma : \Theta\rvert = 4$.

$\mathcal{H} =$  Euler characteristic of $\mathcal{H} = 0$ oriented

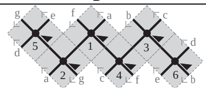
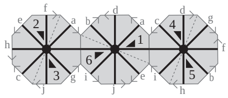
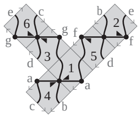
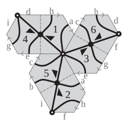
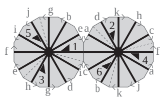
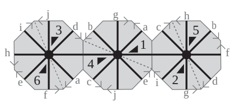
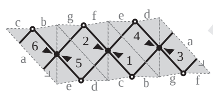
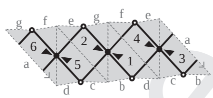
Rep	Rep($\mathcal{H}$)	Euler's charac. form. on Rep($\mathcal{H}$)	orient.
$R8.4a$		$\lvert G\rvert\left(\frac{1}{\lvert XZXY\rvert}+\frac{1}{\lvert YZ\rvert}-2\right)=-4$	yes (always)
$R8.4b$		$\lvert G\rvert\left(\frac{1}{\lvert XZY\rvert}+\frac{1}{\lvert ZXY\rvert}-2\right)=-6$	yes (always)
$R8.5a$		$\lvert G\rvert\left(\frac{1}{\lvert YXZ\rvert}+\frac{1}{\lvert ZY\rvert}-\frac{3}{2}\right)=-4$	no
$R8.5b$		$\lvert G\rvert\left(\frac{1}{\lvert YZ\rvert}+\frac{1}{\lvert YXZ\rvert}-\frac{3}{2}\right)=-4$	no
$R8.6a$		$\lvert G\rvert\left(\frac{1}{\lvert XZ\rvert}+\frac{1}{\lvert ZYX\rvert}-\frac{3}{2}\right)=-4$	no
$R8.6b$		$\lvert G\rvert\left(\frac{1}{\lvert XYZ\rvert}+\frac{1}{\lvert ZX\rvert}-\frac{3}{2}\right)=-4$	no
$R8.7a$		$\lvert G\rvert\left(\frac{1}{\lvert YZX\rvert}-\frac{1}{2}\right)=0$	yes (always)
$R8.7b$		$\lvert G\rvert\left(\frac{1}{\lvert XY\rvert}+\frac{1}{\lvert YZ\rvert}-1\right)=-2$	yes (always)

Table 6: The Θ-marked map representations of $\mathcal{H}$ for $\lvert\Gamma:\Theta\rvert = 8$.

We discuss now orientability in more details. The first two thin representations $R2.1$ and $R2.4$ (Vince and Walsh representations) are the unique orientation-preserving representations, that is, if they are orientable they represent orientable hypermaps and if they are nonorientable they represent nonorientable hypermaps. However, the maps coming out from the representations $R4.5a$, $R4.5b$, $R8.4a$, $R8.4b$, $R8.7a$ and $R8.7b$ are always orientable, since the Θ-trivial maps for $\Theta \in \{\Gamma_{4.5}, \Gamma_{8.4}, \Gamma_{8.7}\}$ are orientable. This poses the question: when they represent non-orientable hypermaps? The same question hang over the other representations with an additional hitch, both orientable and non-orientable maps can represent orientable and nonorientable hypermaps. This means that for these representations we no longer have the clue given by $R2.1$ and $R2.4$, and for this reason we need to make a local teste. In general, a Θ-marked map representation $\mathcal{M} = (\Omega; x_1, x_2, x_3)$ is a representation of an orientable hypermap if and only if x_1x_2 and x_2x_3 act on the set of Θ-slices with two orbits (Θ-orbits). As a hypermap $\mathcal{H}$ is orientable if and only if $\mathcal{H}$

covers the orientably-trivial hypermap $\mathcal{T}_{\mathcal{H}}^{+}$ (Figure 5), a thin map representation $\text{Rep}(\mathcal{H})$ of $\mathcal{H}$ represents an orientable hypermap if and only if $\text{Rep}(\mathcal{H})$ covers the corresponding representation $\text{Rep}(\mathcal{T}_{\mathcal{H}}^{+})$ of the orientably-trivial hypermap, call this representation *RoriT-map*. In the cases of $R2.1$ and $R2.4$, the RoriT-map is spherical and so for any hypermap

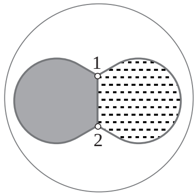


Figure 5: The orientably-trivial hypermap $\mathcal{T}_{\mathcal{H}}^{+}$.

$\mathcal{H}$ the representations $R2.1(\mathcal{H})$ and $R2.4(\mathcal{H})$ are orientable if and only if $\mathcal{H}$ is orientable. For the other cases, and specially the cases in which the representation R is always orientable ($R4.5a$, $R4.5b$, $R8.4a$, $R8.4b$, $R8.7a$, $R8.7b$), the representation $\mathcal{M} = R(\mathcal{H})$ is an orientable hypermap $\mathcal{H}$ if $\mathcal{M}$ covers the respective RoriT-map. Any RoriT-map has two flags, so a Θ-marked map representation $\mathcal{M} = (\Omega; x_1, x_2, x_3)$ is a representation of an orientable hypermap if and only if the triple (x_1, x_2, x_3) induce a two blocks system on the set of Θ-slices Ω (the two Θ-orbits) such that each x_i permutes the two blocks exactly as the permutation $\overline{x}_i$ of the two flags of the RoriT-map does. That is, x_i exchanges the two blocks if and only if $\overline{x}_i$ exchanges the two flags.

Take for example the two map representations (always orientable) given by $R4.5a$ and $R4.5b$ on the non-orientable hypermap $\mathcal{H}$ pictured in Figure 6 left, a non-regular, but uniform of type $(3, 3, 3)$, 6 flag hypermap with monodromy group generated by

$$r_0 = (1,5)(2,4)(3,6)\,, \quad r_1 = (1,2)(3,4)(5,6)\,, \quad r_2 = (1,6)(2,3)(4,5)\,.$$

It is simple to see that $R4.5a(\mathcal{H})$ represents a non-orientable hypermap because by having

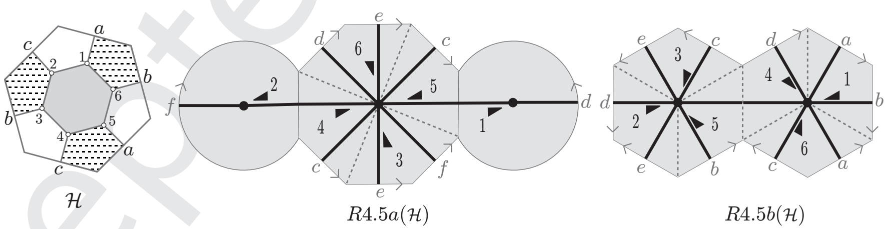


Figure 6:

two vertices of valency 2 the map $R4.5a(\mathcal{H})$ does not cover the uniform (regular) toroidal map $R4.5a(\mathcal{T}_{\mathcal{H}}^{+})$ of type $\{4, 4\}$.

For the case $R4.5b(\mathcal{T}_{\mathcal{H}}^{+})$, the argument is not so simple as before because this map is uniform of type $\{6, 6\}$ and the trivial oriented map $R4.5b(\mathcal{T}_{\mathcal{H}}^{+})$ is also uniform of type $\{2, 2\}$. However, the word $R_0R_2R_0R_1R_0R_1$ fix the root flag 1 in the map $R4.5b(\mathcal{T}_{\mathcal{H}}^{+})$, but does not fix any flag on the RoriT-map $R4.5b(\mathcal{T}_{\mathcal{H}}^{+})$.

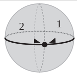
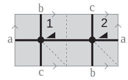
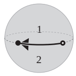
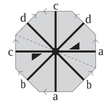
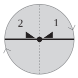
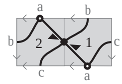
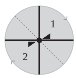
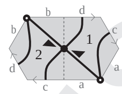
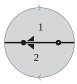
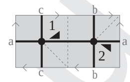
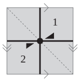
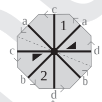
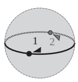
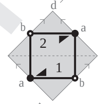
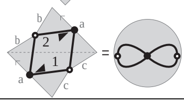
Rep	Rep($\mathcal{T}_{\mathcal{H}}^{+}$)	Rep	Rep($\mathcal{T}_{\mathcal{H}}^{+}$)
$R2.1$		$R8.4a$	
$R2.4$		$R8.4b$	
$R2.5$		$R8.5a$	
$R4.1$		$R8.5b$	
$R4.4$		$R8.6a$	
$R4.5a$		$R8.6b$	
$R4.5b$		$R8.7a$	
		$R8.7b$	

Table 7: The 15 RoriT-maps (thin representations of the orientably-trivial hypermap).

Alternatively, following the block system argument described above, painting by red and blue the possible two blocks, we have for $R4.5a(\mathcal{T}_{\mathcal{H}}^{+})$

$$1(red) \xrightarrow{A} 5(blue) \xrightarrow{Z} 6(red) \xrightarrow{B} 1(blue)$$

and for $R4.5a(\mathcal{T}_{\mathcal{H}}^{+})$

$$1(red) \xrightarrow{A} 5(blue) \xrightarrow{X} 6(red) \xrightarrow{B} 1(blue)$$

In both cases such two block system does not exist.

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