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Elliptic isometries of the manifold of positive definite real matrices with the trace metric

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Abstract

We study the differential-geometric properties of the loci of fixed points of the elliptic isometries of the manifold of positive definite real matrices with the trace metric. We also give an explicit description of such loci and in particular we find their De Rham decomposition.

Keywords Positive definite matrices · Trace metric · Hadamard manifolds · (Elliptic) isometries · (Irreducible) symmetric Riemannian spaces · De Rham decomposition

Mathematics Subject Classification 15B48 · 53C35

Introduction

The Riemannian manifold $(\mathcal{P}_n, g)$ of symmetric positive definite real matrices endowed with the trace metric has been object of interest in many frameworks, for instance in the theory of metric spaces of non-positive curvature, in the theory of diffusion tensor imaging, in geometry of manifold of probability distributions and more generally in matrix information geometry (see for instance [1, 7–10, 21, 23, 24, 26, 27]). We began the study of the trace metric on the manifold of non-singular real matrices in [12], next we considered the same metric on the manifolds of orthogonal real matrices and of non-singular symmetric real matrices respectively in [13] and in [14].

In the present paper we focus our attention on the elliptic isometries (i.e. the isometries having fixed points) of $(\mathcal{P}_n, g)$ and we study their loci of fixed points, providing explicit descriptions of them.

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A first general result can be obtained as consequence of ordinary (but not trivial) facts of Riemannian geometry and without regarding any explicit description of such fixed loci; precisely (Theorem 2.10):

if Φ is an elliptic isometry of $(\mathcal{P}_n, g)$, then $(Fix(\Phi), g)$ is a closed totally geodesic simply connected symmetric Riemannian submanifold of $(\mathcal{P}_n, g)$ and so $(Fix(\Phi), g)$ is a symmetric Hadamard manifold.

An explicit description of $(Fix(\Phi), g)$ needs more careful studies of the different types of elliptic isometries. In [14] we have already determined and described geometrically the full group of isometries of $(\mathcal{P}_n, g)$ (see also Proposition 2.3 and Remark 2.12):

there are four types of elliptic isometries of $(\mathcal{P}_n, g)$, consisting in

- $\Gamma_M : X \mapsto MXM^T$, the *congruence* by an arbitrary non-singular real matrix M (such isometries form a group acting transitively on $(\mathcal{P}_n, g)$);
- $\Gamma_M \circ \delta$ with $M \in GL_n$ and where $\delta : X \mapsto det(X)^{-2/n} X$ can be interpreted as the orthogonal symmetry with respect to the totally geodesic hypersurface $SL\mathcal{P}_n$ of matrices in $\mathcal{P}_n$ with determinant 1;
- $\Gamma_M \circ j$ with $M \in GL_n$ and where $j : X \mapsto X^{-1}$ can be interpreted as the central symmetry with respect to I_n;
- $\Gamma_M \circ j \circ \delta$ with $M \in GL_n$ and where $j \circ \delta$ can be interpreted as the orthogonal symmetry with respect to the geodesic $\mathcal{R} = \{tI_n : t \in \mathbb{R}, t > 0\}$ (i.e. the geodesic through I_n and orthogonal to $SL\mathcal{P}_n$).

In particular: $Fix(j) = \{I_n\}$, $Fix(\delta) = SL\mathcal{P}_n$ and $Fix(j \circ \delta) = \mathcal{R}$.

We describe the loci of fixed points of all elliptic isometries and, as consequence, we are able to list the De Rham decompositions and the De Rham factors of all fixed loci.

This paper is organized following the different types of elliptic isometries Φ: Sect. 3 is devoted to Γ_M, Sect. 4 to $\Gamma_M \circ \delta$, Sect. 5 to $\Gamma_M \circ j$ and Sect. 6 to $\Gamma_M \circ j \circ \delta$.

The explicit descriptions of $(Fix(\Phi), g)$ are obtained in Propositions 3.4, 4.4, 5.9 and 6.2 respectively, while the complete lists of the DeRham factors are in Propositions 3.7, 4.5, 5.10 and 6.3 respectively.

Our methods involve the theory of matrices and the actions of suitable classical Lie groups. In Sect. 1 we resume some facts on matrices. In particular we point out two particular canonical forms for matrices which are similar to a multiple of an orthogonal matrix: the real Jordan standard form and the real Jordan auxiliary form (see Remarks-Definitions 1.7 and 1.8); the reason is that the fixed loci are related to certain closed Lie subgroups of GL_n, consisting in matrices commuting with the real Jordan standard form or fixing by congruence the real Jordan auxiliary form of suitable matrices. Some relevant properties of $(\mathcal{P}_n, g)$ and of its totally geodesic submanifolds are resumed in Sect. 2; these, together with some ordinary facts of Riemannian geometry, allow to obtain the general result (Theorem 2.10), quoted above.

1 Notations and recalls on matrices

Notations 1.1 I_n: the identity matrix of order n;

A^T: the *transpose* of any matrix A;

M_n (and Sym_n): the vector space of the real square matrices of order n (which are symmetric);

GL_n (and SL_n): the multiplicative group of the non-singular real matrices of order n (and with determinant 1);

$\mathcal{P}_n$ (and $SL\mathcal{P}_n$): the manifold of *symmetric positive definite* matrices of order n (and with determinant 1);

$\mathcal{O}_n$ (and $S\mathcal{O}_n$): the multiplicative group of real *orthogonal* matrices of order n (with determinant 1);

$\mathcal{O}(p, n-p)$ (and $S\mathcal{O}_0(p, n-p)$): the *generalized orthogonal group* of *signature* $(p, n-p)$ (and its connected component of the identity);

Sp_{2n}: *the real symplectic group* given by matrices $W \in GL_{2n}$ such that
$W \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} W^T = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$;

$M_n(\mathbb{C})$ (and $Herm_n$): the vector space of the complex square matrices of order n (which are *hermitian*);

$GL_n(\mathbb{C})$ (and $SL_n(\mathbb{C})$): the multiplicative group of the non-singular complex matrices of order n (with determinant 1);

$\mathcal{H}_n$ (and $SL\mathcal{H}_n$): the real manifold of hermitian positive definite matrices of order n (and with determinant 1);

U_n (and SU_n): the multiplicative group of complex *unitary* matrices of order n (with determinant 1);

$U(\mu, \nu)$ (and $SU(\mu, \nu)$): the *generalized unitary* group of signature (μ, ν) (with determinant 1).

For every $A \in M_n(\mathbb{C})$, $tr(A)$ is its *trace*, $A^* := \overline{A}^T$ is its *transpose conjugate*, $det(A)$ is its *determinant* and, provided that $det(A) \neq 0$, A^{-1} is its *inverse* and we denote $A^{-T} = (A^T)^{-1} = (A^{-1})^T$.

When $A \in \mathcal{P}_n$, $\sqrt{A}$ is its unique *square root* contained in $\mathcal{P}_n$.

For every $\theta \in \mathbb{R}$, we denote
$E_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $E := E_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so $E_\theta = (\cos\theta)I_2 + (\sin\theta)E$.

If $B_1, \dots, B_m$ are square matrices (of possible distinct orders), $B_1 \oplus \cdots \oplus B_m$ is the block diagonal square matrix with $B_1, \dots, B_m$ on its diagonal and, for every square matrix B, $B^{\oplus m}$ denotes $B \oplus \cdots \oplus B$ (m times). The notations $(\pm I_0) \oplus B$ and $B \oplus (\pm I_0)$ simply indicate the matrix B.

If $\mathcal{S}_1, \dots, \mathcal{S}_m$ are sets of square matrices, then $\mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_m$ denotes the set of all matrices $B_1 \oplus \cdots \oplus B_m$ with $B_j \in \mathcal{S}_j$ for every j.

For every matrix $X \in GL_n$ we denote

$$\mathcal{C}_X := \{B \in GL_n : BX = XB\} \textit{ and}$$
$$\mathcal{K}_X := \{K \in GL_n : KXK^T = X\}.$$

It is easy to check that both $\mathcal{C}_X$ and $\mathcal{K}_X$ are closed Lie subgroups of GL_n.

For any other notation and for information on the matrices, not explicitly recalled here, we refer to [16].

Definition 1.2 For every matrix $C \in GL_n(\mathbb{C})$ we denote by Γ_C, by j and by δ the maps: $GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ given by

- $\Gamma_C(X) := CXC^T$ (the *congruence* by C),
- $j(X) := X^{-1}$ and
- $\delta(X) := |det(X)|^{-2/n}X$.

The restrictions of these maps to any subset of $GL_n(\mathbb{C})$ will be still denoted by the same letters.

Remarks-Definitions 1.3 a) Two matrices $A, B \in M_n(\mathbb{C})$ are *similar* if there exists a matrix $C \in GL_n(\mathbb{C})$ such that $A = CBC^{-1}$.

When A, B are real, it does not matter if C is real or complex. Indeed, even if C is complex, then we can find a real matrix C' satisfying $A = C'BC'^{-1}$.

b) Two matrices $A, B \in M_n(\mathbb{C})$ are $\mathbb{K}$-*congruent* with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, if there is a non-singular matrix C, with entries in $\mathbb{K}$, such that $A = CBC^T$.

Two real matrices, e.g. I_2 and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, can be $\mathbb{C}$-*congruent*, but not $\mathbb{R}$-*congruent*.

c) It is known that two matrices $A, B \in GL_n(\mathbb{C})$ are $\mathbb{C}$-congruent if and only if AA^{-T} and BB^{-T} are similar (see for instance in [16, Thm. 4.5.27 p. 295], via the fact that MM^{-T} and $M^{-T}M$ are similar for every $M \in GL_n(\mathbb{C})$).

Assume furthermore that A, B are real; if they are $\mathbb{R}$-congruent, then AA^{-T} and BB^{-T} are similar too, but the converse is not generally true.

d) Finally we recall that a real matrix $A \in M_n$ is said to be *normal*, if $AA^T = A^TA$.

Theorem 1.4 (Polar decomposition, see [16, Thm. 7.3.1 p. 449]) *Let $A \in GL_n$. Then there exist, and are uniquely determined, $Q, Q' \in \mathcal{P}_n$ and $U, U' \in \mathcal{O}_n$ such that $A = QU = U'Q'$.*

Moreover $U = U'$, $Q = \sqrt{AA^T}$ and *$Q' = \sqrt{A^TA}$.*

In particular $A \in GL_n$ is real normal if and only if $Q = Q'$, i.e. if and only if Q and *U commute.*

Remark 1.5 Let $A = QU = UQ$ be a real normal matrix in GL_n together with its polar decompositions. From $Q = UQU^T$, we get $\sqrt{Q} = U\sqrt{Q}U^T$; therefore also $\sqrt{Q}$ commutes with U and $A = \sqrt{Q}\,U\sqrt{Q}$. Hence every real normal non-singular matrix is $\mathbb{R}$-congruent to the orthogonal matrix of its polar decomposition.

Theorem 1.6 *For every $A \in GL_n$ the following facts are equivalent*:

i) *$A = \lambda P$, where P is an orthogonal matrix and $\lambda \neq 0$ is a real number*;
ii) *there is a matrix $Q \in \mathcal{O}_n$ such that $Q^TAQ = |\lambda|\left(I_p \oplus E_{\theta_1}^{\oplus m_1} \oplus \cdots \oplus E_{\theta_r}^{\oplus m_r} \oplus (-I_q)\right)$, where $\lambda \neq 0$ is a real number,*

with $p, q, r \geq 0$, $m_j > 0$ for $1 \leq j \leq r$ (if $r \geq 1$), $p + q + 2m_1 + \cdots + 2m_r = n$ and $0 < \theta_1 < \theta_2 < \cdots < \theta_r < \pi$.

Proof This is essentially the Real Spectral Theorem for matrices which are multiple of orthogonal matrices (see for instance [16, Cor. 2.5.11 pp. 136–137], except for an irrelevant change of sign), because the matrix on the right side of (ii) is multiple of an orthogonal matrix.

Remark–Definition 1.7 A matrix in GL_n is similar to a multiple of an orthogonal matrix if and only if it is semisimple and its eigenvalues have constant modulus. By Theorem 1.6, such a matrix, A, is similar to a matrix of the form

$$J_A := |\lambda|\left(I_p \oplus E_{\theta_1}^{\oplus m_1} \oplus \cdots \oplus E_{\theta_r}^{\oplus m_r} \oplus (-I_q)\right)$$

with $\lambda \neq 0$, $p, q, r \geq 0$, $m_j > 0$ for $1 \leq j \leq r$ (if $r \geq 1$), $p + q + 2m_1 + \cdots + 2m_r = n$ and $0 < \theta_1 < \theta_2 < \cdots < \theta_r < \pi$.

Hence, for every matrix $A \in GL_n$, similar to a multiple of an orthogonal matrix, we call such matrix J_A *the real Jordan standard form* (shortly: RJS form) of A.

We remark that the eigenvalues of A (and of J_A) are: $|\lambda|$ with multiplicity p, $-|\lambda|$ with multiplicity q and $|\lambda|e^{\pm i\theta_j}$ each with multiplicity m_j, for $j = 1, \ldots, r$ (if $r \geq 1$).

Finally, from the similarity between A and A^T (see for instance [16, Thm. 3.2.3.1, p. 177]), we get $J_A = J_{A^T}$.

Remark–Definition 1.8 By technical reasons, for every matrix $A \in GL_n$, similar to a multiple of an orthogonal matrix, we are interested in introducing another Jordan-type form, $\widetilde{J}_A$, having the property: $(\widetilde{J}_A)^2 = J_{A^2}$.

By means of congruences given by orthogonal matrices, we can arbitrarily permute the direct addends of its RJS form J_A.

Moreover, for $\Xi := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have $\Gamma_\Xi(E_\theta) = E_{-\theta} = -E_{\pi-\theta}$. Hence for every $\theta_j \in (\frac{\pi}{2}, \pi)$, up to an orthogonal congruence, we can replace each E_{θ_j} with $-E_{\pi-\theta_j}$. Now we reorder the values $\theta_i \in (0, \frac{\pi}{2})$ together with the new values $\pi - \theta_j \in (0, \frac{\pi}{2})$ following the increasing order of θ_i's and of $(\pi - \theta_j)$'s; hence, after renaming them ϕ_t, we obtain the following matrix:

$$\widetilde{J}_A := |\lambda|\Big(I_p \oplus (-I_q) \oplus E_{\phi_1}^{\oplus\mu_1} \oplus (-E_{\phi_1}^{\oplus\nu_1}) \oplus \cdots \oplus E_{\phi_h}^{\oplus\mu_h} \oplus (-E_{\phi_h}^{\oplus\nu_h}) \oplus E_{\pi/2}^{\oplus k}\Big)$$

where $p \geq 0$ is the multiplicity of the eigenvalue $|\lambda|$, $q \geq 0$ is the multiplicity of the eigenvalue $-|\lambda|$, $k \geq 0$ is the multiplicity of the eigenvalues $\pm i|\lambda|$ and (when $h > 0$), $\mu_j \geq 0$ is the multiplicity of the eigenvalues $|\lambda|e^{\pm i\phi_j}$, $\nu_j \geq 0$ is the multiplicity of the eigenvalues $|\lambda|e^{\pm i(\pi-\phi_j)}$ with $\mu_j + \nu_j \geq 1$ for every $j \leq h$ and $0 < \phi_1 < \cdots < \phi_h < \frac{\pi}{2}$.

Note that $\widetilde{J}_A = C^TAC$ for some $C \in \mathcal{O}_n$.

Finally since $(\widetilde{J}_A)^2 = \lambda^2\Big(I_{p+q} \oplus E_{2\phi_1}^{\oplus(\mu_1+\nu_1)} \oplus \cdots \oplus E_{2\phi_h}^{\oplus(\mu_h+\nu_h)} \oplus (-I_{2k})\Big)$, we get that:

$$(\widetilde{J}_A)^2 = J_{A^2}.$$

We call the matrix $\widetilde{J}_A$ *the real Jordan auxiliary form* (shortly: RJA form) of A.

As in the case of the *RJS* forms, we have $\widetilde{J}_A = \widetilde{J}_{A^T}$.

2 Recalls on the trace metric and a first general result

From now on, and for the remaining part of this paper, n is a fixed integer, $n \geq 2$.

Remark–Definition 2.1 The C^∞-tensor γ of type (0, 2) on $\mathcal{H}_n$, defined by

$$\gamma_A(V, W) = tr(A^{-1}VA^{-1}W)$$

for every $A \in \mathcal{H}_n$ and for every $V, W \in T_A\mathcal{H}_n = Herm_n$, is called *trace metric*.

For convenience, the restriction of γ to $\mathcal{P}_n$ will be denoted by g (always called trace metric), so that $(\mathcal{P}_n, g)$ is a Riemannian submanifold of $(\mathcal{H}_n, \gamma)$.

Instead we will denote again by g the restriction of g to any submanifold of $\mathcal{P}_n$.

For the differential-geometric properties of $(\mathcal{P}_n, g)$ and of $(\mathcal{H}_n, \gamma)$ we refer to [7–9, 14, 23, 24, 26, 27].

Definition 2.2 A *Hadamard manifold* is a simply connected, complete, smooth Riemannian manifold without boundary and with non-positive sectional curvature.

An isometry of a Hadamard manifold is said to be *elliptic*, if it has a fixed point.

For more information on Hadamard manifolds and on their isometries we refer for instance to [6, Lecture I § 2 and Lecture II § 6] and to [2, Ch. 1 § 5 and Ch. 2 § 6].

Proposition 2.3

a) *$(\mathcal{P}_n, g)$ is a symmetric Hadamard manifold.*

b) *A mapping $\Phi : (\mathcal{P}_n, g) \to (\mathcal{P}_n, g)$ is an isometry if and only if there exists a matrix $M \in GL_n$ such that one of the following cases occurs*: $\Phi(X) = \Gamma_M(X) = MXM^T$; $\Phi(X) = (\Gamma_M \circ \delta)(X) = \dfrac{MXM^T}{det(X)^{2/n}}$; $\Phi(X) = (\Gamma_M \circ j)(X) = MX^{-1}M^T$; $\Phi(X) = (\Gamma_M \circ j \circ \delta)(X) = det(X)^{2/n}\, MX^{-1}M^T$.

The part (a) is well-known (see for instance [20, Ch. XII]). For the part (b) see [14, Thm. 4.2].

Remark 2.4 It is well-known that also $(\mathcal{H}_n, \gamma)$ is a symmetric Hadamard manifold and that $(\mathcal{P}_n, g)$ is a totally geodesic Riemannian submanifold of $(\mathcal{H}_n, \gamma)$.

The description of the isometries of $\mathcal{H}_n$ endowed with a class of metrics which includes γ, is given in [22, Thm. 3]. As already remarked in [14], from the comparison with the previous result, it follows that every isometry of $(\mathcal{P}_n, g)$ is the restriction of an isometry of $(\mathcal{H}_n, \gamma)$.

Definition 2.5 Let G be a closed subgroup of GL_n.

G is said to be *reductive*, if $A^T \in G$ as soon as $A \in G$.

G is said to be *algebraic*, if there is a finite system of polynomials (in the entries of M_n) such that G is the intersection of GL_n with the set of common zeroes of this system.

Proposition 2.6 (see [7, Thm.10.58]) *Let G be a reductive subgroup of GL_n satisfying the following property*:

(*) if $X \in Sym_n$ and $e^X \in G$, then $e^{sX} \in G$ for every $s \in \mathbb{R}$.

Then

i) *$(G \cap \mathcal{P}_n, g)$ is a totally geodesic submanifold of $(\mathcal{P}_n, g)$*;

ii) *$G \cap \mathcal{P}_n$ is the orbit of I_n under the action of G by congruence, so that $G \cap \mathcal{P}_n$* is *diffeomorphic to $G/(G \cap O_n)$*;

iii) *$(G \cap \mathcal{P}_n, g)$ is a symmetric Riemannian manifold with non-positive sectional curvature.*

Remark 2.7 Let G be a closed sugbroup of GL_n, then G satisfies the condition (*) of Proposition 2.6 if and only if G_0 (the connected component of the identity of G) satisfies (*). Moreover any algebraic subgroup of GL_n satisfies the condition (*) (see [7, Lemma 10.59]).

Remark 2.8 The mapping $\rho : \mathbb{C} \to M_2$, given by $\rho(z) = Re(z)I_2 + Im(z)E$, is a monomorphism of $\mathbb{R}$-algebras between $\mathbb{C}$ and M_2. Note that $\rho(\overline{z}) = \rho(z)^T$ and that $\rho(z) \in GL_2$ as soon as $z \neq 0$.

More generally, for any $h \geq 1$, we denote again by ρ the mapping: $M_h(\mathbb{C}) \to M_{2h}$, which maps the $h \times h$ complex matrix $Z = (z_{ij})$ to the $(2h) \times (2h)$ block real matrix $(\rho(z_{ij}))$, having h^2 blocks of order 2×2.

In literature there are other ways, essentially equivalent to ρ, to embed $M_h(\mathbb{C})$ into M_{2h} (see for instance [11, Prop. 2.12]). It seem to us that the mapping ρ, used here, is more useful for the purposes of present paper.

Standard arguments show that $tr(\rho(Z)) = 2Re(tr(Z))$, $det(\rho(Z)) = |det(Z)|^2$ and that ρ is a monomorphism of $\mathbb{R}$-algebras, whose restriction to $GL_h(\mathbb{C})$ has image into GL_{2h} and it is a monomorphism of Lie groups.

We have: $\rho(Z^*) = \rho(Z)^T$ and, so, the restriction of ρ to U_h is again a monomorphism of Lie groups and $\rho(U_h) = \rho(GL_h(\mathbb{C})) \cap \mathcal{SO}_{2h}$; analogously the restriction of ρ to $Herm_h$ has image into Sym_{2h} and it is a monomorphism of $\mathbb{R}$-vector spaces.

Finally ρ maps injectively $\mathcal{H}_h$ into $\mathcal{P}_{2h}$. Indeed $\rho(ZZ^*) = \rho(Z)\rho(Z)^T$.

Moreover, for $A \in \mathcal{H}_h$ and $Z, W \in Herm_h$, we get:

$g_{\rho(A)}(d\rho(Z), d\rho(W)) = tr(\rho(A)^{-1}\rho(Z)\rho(A)^{-1}\rho(W)) = tr(\rho(A^{-1}ZA^{-1}W)) =$
$= 2Re\, tr(A^{-1}ZA^{-1}W) = 2Re\, \gamma_A(Z, W) = 2\gamma_A(Z, W)$.

Hence the restriction of ρ from $(\mathcal{H}_h, 2\gamma)$ into $(\mathcal{P}_{2h}, g)$ is an isometry onto its image $\rho(\mathcal{H}_h) = \rho(GL_h(\mathbb{C})) \cap \mathcal{P}_{2h}$.

Definition 2.9 For every isometry $\boldsymbol{\Phi}$ of $(\mathcal{P}_n, g)$ we denote by $Fix(\boldsymbol{\Phi})$ the set of points of $\mathcal{P}_n$ fixed by $\boldsymbol{\Phi}$.

Theorem 2.10 *If $\boldsymbol{\Phi}$ is an elliptic isometry of $(\mathcal{P}_n, g)$, then $(Fix(\boldsymbol{\Phi}), g)$ is a closed totally geodesic simply connected symmetric Riemannian submanifold of $(\mathcal{P}_n, g)$ and so $(Fix(\boldsymbol{\Phi}), g)$ is a symmetric Hadamard manifold.*

Proof Each connected component of $Fix(\boldsymbol{\Phi})$ is a closed totally geodesic submanifold of $\mathcal{P}_n$ by [19, Thm. 5.1 p. 59]. By completeness of $\mathcal{P}_n$, points in different components of $Fix(\boldsymbol{\Phi})$ should be mutually *cut points* (see [19, Cor. 5.2 p. 60] and, for more information on cut points, [18, Ch. VIII § 7]). Now, since $\mathcal{P}_n$ is a Hadamard manifold, by Cartan-Hadamard Theorem (see for instance [25, Thm. 22 p. 278] and [6, Lecture 1, Sect. 2]) any two points are joined by a unique minimizing geodesic. Hence $\mathcal{P}_n$ has no cut points and therefore $Fix(\boldsymbol{\Phi})$ is connected too. Moreover $(Fix(\boldsymbol{\Phi}), g)$ is complete, its curvature is non-positive and it has no non-trivial geodesic loop, because it is closed and totally geodesic in the Hadamard manifold $(\mathcal{P}_n, g)$. Hence, by [3, Cor. 9.2.8], $(Fix(\boldsymbol{\Phi}), g)$ is simply connected and so it is a Hadamard manifold. Finally $(Fix(\boldsymbol{\Phi}), g)$ is symmetric, because it is a totally geodesic Riemannian submanifold of the symmetric Riemannian manifold $(\mathcal{P}_n, g)$.

Remark–Definition 2.11 The previous Theorem implies that, if $\boldsymbol{\Phi}$ is an elliptic isometry of $(\mathcal{P}_n, g)$, then $(Fix(\boldsymbol{\Phi}), g)$ has a *De Rham decomposition* into a Riemannian product: one of its factors (called *flat* or *Euclidean factor*) may be isometric to some Euclidean space $\mathbb{R}^m$, while the other factors are irreducible symmetric Hadamard manifolds. Such decomposition is unique up to isometries and permutations of its factors (see for instance [17, Ch. IV, § 6]) and each factor is an *Einstein* manifold (see for instance [5, Note 10.83, p. 298]). The irreducible simply connected symmetric spaces are classified and the complete list is for instance in [5, pp. 311–317] and in [4, pp. 306–308].

In this paper we determine the De Rham decomposition of every such $(Fix(\Phi), g)$.

Remark 2.12 In [14, Rem. 4.6] we described geometrically the particular isometries δ, j and $j\circ\delta$ of $(\mathcal{P}_n, g)$ as follows:

- δ is the orthogonal symmetry with respect to the hypersurface $SL\mathcal{P}_n$;
- j is the symmetry with respect to I_n;
- $j\circ\delta = \delta\circ j$ is the orthogonal symmetry with respect to the geodesic $\mathcal{R} = \{tI_n : t \in \mathbb{R}, t > 0\}$ (i.e. the geodesic through I_n and orthogonal to $SL\mathcal{P}_n$). In particular $Fix(j) = \{I_n\}$, $Fix(\delta) = SL\mathcal{P}_n$ and $Fix(j\circ\delta) = \mathcal{R}$.

Hence we want to describe explicitly the fixed loci, when the previous isometries are composed with congruences.

Remark 2.13 Let $M \in GL_n$. Then $Fix(\Gamma_M\circ\delta)$ and $Fix(\Gamma_M\circ j)$ are both contained in $\{P \in \mathcal{P}_n : det(P) = |det(M)|\}$.

This follows by computing the determinants from the equalities: $\dfrac{MPM^T}{det(P)^{2/n}} = P$ and $MP^{-1}M^T = P$.

3 The fixed points of the isometries Γ_M

Proposition 3.1 *Let $M \in GL_n$ and let us consider the isometry Γ_M of $\mathcal{P}_n$. The following facts are equivalent*:

a) *Γ_M is elliptic*;
b) *M is similar to an orthogonal matrix*;
c) *M is semi-simple and its eigenvalues have modulus* 1.

If this is the case, the RJS form of M is of the type:

$J_M = I_p \oplus E_{\theta_1}^{\oplus m_1} \oplus \cdots \oplus E_{\theta_r}^{\oplus m_r} \oplus (-I_q)$ (*with* $p, q, r \geq 0$, $m_j > 0$ *for every possible* j, $p + q + 2m_1 + \cdots + 2m_r = n$ and $0 < \theta_1 < \theta_2 < \cdots < \theta_r < \pi$).

Proof The equivalence between (b) and (c) and the assertion about the RJS form are in Remarks–Definitions 1.7.

Hence it suffices to prove the equivalence between (a) and (b).

If $P \in \mathcal{P}_n$ verifies: $MPM^T = P$, then, for any $C \in GL_n$ satisfying $P = CC^T$, we get: $(C^{-1}MC)(C^{-1}MC)^T = (C^{-1}MC)(C^TM^TC^{-T}) = I_n$, i.e. $C^{-1}MC \in \mathcal{O}_n$.

For the converse, let C be any non-singular matrix such that $C^{-1}MC \in \mathcal{O}_n$, then: $(C^{-1}MC)(C^TM^TC^{-T}) = I_n$ and finally: $M(CC^T)M^T = CC^T$. This allows to conclude.

Proposition 3.2 *Let $M \in GL_n$ and assume that the isometry Γ_M of $\mathcal{P}_n$ is elliptic. Then*

i) $Fix(\Gamma_M) = \{CC^T : C \in GL_n, C^{-1}MC \in \mathcal{O}_n\} = \{FF^T : F \in GL_n, FJ_MF^{-1} = M\}$;
ii) *for any fixed matrix* $F_0 \in GL_n$ such that $F_0J_MF_0^{-1} = M$, *we have*:

$Fix(\Gamma_M) = \{F_0AA^TF_0^T \,:\, A \in \mathcal{C}_{J_M}\}$.

Proof The first equality of (i) has been essentially obtained in verifying the equivalence between (a) and (b) of Proposition 3.1. For the second equality of (i), it suffices to note that J_M is orthogonal and that if $C^{-1}MC \in \mathcal{O}_n$ then $C^{-1}MC = QJ_MQ^T$ for some $Q \in \mathcal{O}_n$, hence $F = CQ$ satisfies $FF^T = CC^T$ and $F^{-1}MF = J_M$.

Now we prove (ii). Let $P \in \mathcal{P}_n$ be a fixed point of Γ_M. By (i), we can choose a matrix $F_0 \in GL_n$ such that $F_0J_MF_0^{-1} = M$ and $F_0F_0^T = P$. Let $F \in GL_n$ any other matrix such that $FJ_MF^{-1} = M$ and $FF^T = P$, then an easy computation shows that the matrix $A := F_0^{-1}F$ satisfies $AJ_M = J_MA$, i. e. $A \in \mathcal{C}_{J_M}$. Therefore $P = FF^T = (F_0A)(F_0A)^T = F_0AA^TF_0^T$ with $A = F_0^{-1}F \in \mathcal{C}_{J_M}$.

Conversely let $F = F_0A$ with $A \in \mathcal{C}_{J_M}$. Then $FJ_MF^{-1} = F_0AJ_MA^{-1}F_0^{-1} = F_0J_MF_0^{-1} = M$. Then, by assertion (i), $FF^T = (F_0A)(F_0A)^T = F_0AA^TF_0^T$ is a fixed point of Γ_M in $\mathcal{P}_n$.

Lemma 3.3 *Let* $0 < \theta_1 < \theta_2 < \cdots < \theta_r < \pi$ *be real numbers and let* $J := I_p \oplus E_{\theta_1}^{\oplus m_1} \cdots \oplus E_{\theta_r}^{\oplus m_r} \oplus (-I_q)$ with $p, q, r \geq 0$, $m_j > 0$ for every possible j, $p + q + 2m_1 + \cdots + 2m_r = n$.

Then the set of matrices of M_n, *commuting with* J, *is the vector space*

$$M_p \oplus \rho(M_{m_1}(\mathbb{C})) \oplus \cdots \oplus \rho(M_{m_r}(\mathbb{C})) \oplus M_q.$$

In particular the Lie group of non-singular matrices, commuting with J, is

$$GL_p \oplus \rho(GL_{m_1}(\mathbb{C})) \oplus \cdots \oplus \rho(GL_{m_r}(\mathbb{C})) \oplus GL_q,$$

which is an algebraic reductive subgroup of GL_n.

Proof In order to simplify the next computations we denote $\sigma = m_1 + \cdots + m_r$, $F_0 = I_p$, $F_{\sigma+1} = -I_q$, $F_1 = F_2 = \cdots = F_{m_1} = E_{\theta_1}$, $F_{m_1+1} = \cdots = F_{m_1+m_2} = E_{\theta_2}, \ldots, F_{m_1+\cdots+m_{r-1}+1} = \cdots = F_\sigma = E_{\theta\,r}$, so that $J = F_0 \oplus F_1 \oplus \cdots \oplus F_\sigma \oplus F_{\sigma+1}$.

Let $A \in M_n$. We write A in blocks: $A = (A_{ij})$ with $i,j = 0, \ldots, \sigma + 1$, $A_{ij} \in M_2$ for every $1 \leq i,j \leq \sigma$, $A_{00} \in M_p$, $A_{\sigma+1,\sigma+1} \in M_q$ and with the remaining matrices of obvious orders, in line with the decomposition in blocks.

The condition $AJ = JA$ is equivalent to

(**)$A_{ij}F_j = F_iA_{ij}$, for every $i,j = 0, \ldots, \sigma + 1$.

Easy computations show directly that $A_{0,\sigma+1} = A_{\sigma+1,0} = 0$ and that A_{00} and $A_{\sigma+1,\sigma+1}$ are generic matrices in M_p and M_q respectively.

When $i \in \{0, \sigma + 1\}$ and $1 \leq j \leq \sigma$ or $j \in \{0, \sigma + 1\}$ and $1 \leq i \leq \sigma$, the condition (**) implies that $A_{ij} = 0$. Indeed, when $i = 0$ and $1 \leq j \leq \sigma$, (**) gives: $A_{0j}(F_j - I_2) = 0$ and we conclude since $det(F_j - I_2) > 0$. Analogously we can conclude in the other three cases.

Now, for $1 \leq i,j \leq \sigma$, (**) can be written as

$A_{ij}\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}A_{ij}$, with $\varphi, \psi \in \{\theta_1, \ldots, \theta_r\}$.

This gives a homogeneous linear system 4×4 with unknowns the entries of the matrix A_{ij}, whose determinant is $[(\cos\psi - \cos\varphi)^2 - \sin^2\psi + \sin^2\varphi]^2 + 4(\cos\psi - \cos\varphi)^2\sin^2\psi$. This expression is non-zero except for $\varphi = \psi$.

Hence, if $\varphi \neq \psi$, then $A_{ij} = 0$.

If $\varphi = \psi$, then the rank of the matrix associated to the system is 2 and, so, the space of its solutions is the $\mathbb{R}$-vector space spanned by I_2 and E, i.e. it is $\rho(\mathbb{C})$.

This concludes the first part of the statement. The second part follows easily from arguments about the non-singularity of the matrices.

Proposition 3.4 *Let $M \in GL_n$ such that Γ_M is elliptic with eigenvalues* 1 *of multiplicity p, -1 of multiplicity q , $e^{\pm i\theta_1}$ both with multiplicity $m_1, \cdots$, up to $e^{\pm i\theta_r}$ both with multiplicity m_r,* with $p, q, r \geq 0$, $m_j > 0$ for $1 \leq j \leq r$, $p + q + 2m_1 + \cdots + 2m_r = n$ *and* $0 < \theta_1 < \cdots < \theta_r < \pi$. Fix $F_0 \in GL_n$ *such that $F_0 J_M F_0^{-1} = M$. Then*

$$\Gamma_{F_0^{-1}}(Fix(\Gamma_M)) = \mathcal{P}_p \oplus \rho(\mathcal{H}_{m_1}) \oplus \cdots \oplus \rho(\mathcal{H}_{m_r}) \oplus \mathcal{P}_q.$$

Hence $\big(Fix(\Gamma_M), g\big)$ is a closed simply connected totally geodesic symmetric Riemannian submanifold of $(\mathcal{P}_n, g)$ isometric to the Riemannian product

$$(\mathcal{P}_p, g) \times (\mathcal{H}_{m_1}, 2\gamma) \times \cdots \times (\mathcal{H}_{m_r}, 2\gamma) \times (\mathcal{P}_q, g).$$

Proof From the previous Lemma 3.3, the set of matrices AA^T with $A \in \mathcal{C}_{J_M}$ is: $\mathcal{P}_p \oplus \rho(\mathcal{H}_{m_1}) \oplus \cdots \oplus \rho(\mathcal{H}_{m_r}) \oplus \mathcal{P}_q$. Hence, from the Proposition 3.2, $Fix(\Gamma_M) = \Gamma_{F_0}(\mathcal{P}_p \oplus \rho(\mathcal{H}_{m_1}) \oplus \cdots \oplus \rho(\mathcal{H}_{m_r}) \oplus \mathcal{P}_q)$. Since Γ_{F_0} is an isometry of $(\mathcal{P}_n, g)$ and by remarking that the Riemannian manifold $(\mathcal{P}_p \oplus \rho(\mathcal{H}_{m_1}) \oplus \cdots \oplus \rho(\mathcal{H}_{m_r}) \oplus \mathcal{P}_q, g)$ is canonically isometric to $(\mathcal{P}_p, g) \times (\mathcal{H}_{m_1}, 2\gamma) \times \cdots \times (\mathcal{H}_{m_r}, 2\gamma) \times (\mathcal{P}_q, g)$, we can conclude by Proposition 2.6, taking into account that $\mathcal{P}_p \oplus \rho(\mathcal{H}_{m_1}) \oplus \cdots \oplus \rho(\mathcal{H}_{m_r}) \oplus \mathcal{P}_q = G \cap \mathcal{P}_n$, where G is the algebraic reductive subgroup of GL_n defined by $G := GL_p \oplus \rho(GL_{m_1}(\mathbb{C})) \oplus \cdots \oplus \rho(GL_{m_r}(\mathbb{C})) \oplus GL_q$.

Remark 3.5 The values p, q, r in Proposition 3.4 are non-negative and not all zero. When some of them vanishes, the Riemannian product in this Proposition must be intended in a suitable (but obvious) way. For instance, if $p = 0$ (or $q = 0$), the factor $(\mathcal{P}_p, g)$ (or $(\mathcal{P}_q, g)$) does not appear and if $r = 0$, no factor $(\mathcal{H}_{m_j}, 2\gamma)$ appears. Analogous remarks can be done about next Propositions 4.4, 5.9 and 6.2.

Remark 3.6 For every $m \geq 1$, it is well-known that $(\mathcal{P}_m, g)$ is isometric to the Riemannian product of $(SL\mathcal{P}_m, g)$ with $\mathbb{R}$ and that $SL\mathcal{P}_m$ is diffeomorphic to SL_m/SO_m (see for instance [7, p. 325]). Hence, remembering that $SL\mathcal{P}_1$ is a point, we get that the De Rham factors of $(\mathcal{P}_m, g)$ are SL_m/SO_m and $\mathbb{R}$ when $m \geq 2$, while, when $m = 1$, $\mathbb{R}$ is the unique factor (see for instance [4, p. 306]).

Analogously it is easy to show that $(\mathcal{H}_m, 2\gamma)$ is isometric to the Riemannian product of $(SL\mathcal{H}_m, 2\gamma)$ with $\mathbb{R}$ and that $SL\mathcal{H}_m$ is diffeomorphic to $SL_m(\mathbb{C})/SU_m$. Hence, as above, we get that the De Rham factors of $(\mathcal{H}_m, 2\gamma)$ are $SL_m(\mathbb{C})/SU_m$ and $\mathbb{R}$ when $m \geq 2$, while, when $m = 1$, $\mathbb{R}$ is again the unique factor (see again for instance [4, p.308]).

We denote by $r' = r'(p, q, r)$ the quantity: $r' = r$ if $p = q = 0$, $r' = r + 1$ if either $p = 0$ or $q = 0$ (but not both zero) and $r' = r + 2$ if p and q are both non-zero. Note that $r' \geq 1$. Now we can conclude with the following result

Proposition 3.7 *Let $M \in GL_n$ such that Γ_M is elliptic, let $p, q, r, m_1, \ldots, m_r$ be as in Proposition* 3.4 *and r' be as in Remark* 3.6*. Then, up to isometries, the De Rham factors of $\big(Fix(\Gamma_M), g\big)$ are*:

a) $\mathbb{R}^{r'}$;
b) SL_p/SO_p, *if* $p \geq 2$;

c) SL_q/SO_q, *if* $q \geq 2$;
d) $SL_{m_j}(\mathbb{C})/SU_{m_j}$ *for all indices* $j = 1, \ldots, r$ such that $m_j \geq 2$, *if* $r \geq 1$.

4 The fixed points of the isometries $\Gamma_M \circ \delta$

Proposition 4.1 *Let* $M \in GL_n$ *and let us consider the isometry* $\Gamma_M \circ \delta$ *of* $\mathcal{P}_n$.

1) *The following facts are equivalent*:

a) $\Gamma_M \circ \delta$ *is elliptic*;
b) *M is semi-simple and all of its eigenvalues have the same modulus. If this is the case, the RJS form of M is of the type*

$$J_M = |det(M)|^{1/n}[I_p \oplus E_{\theta_1}^{\oplus m_1} \oplus \cdots \oplus E_{\theta_r}^{\oplus m_r} \oplus (-I_q)]$$

(*with* $p, q, r \geq 0$, $m_j > 0$ *for every possible* j, $p + q + 2m_1 + \cdots + 2m_r = n$ *and* $0 < \theta_1 < \theta_2 < \cdots < \theta_r < \pi$).

2) *If* $\Gamma_M \circ \delta$ *is elliptic, then*

$$Fix(\Gamma_M \circ \delta) = Fix(\Gamma_{[\frac{M}{|det(M)|^{1/n}}]}) \cap \{P \in \mathcal{P}_n \, : \, det(P) = |det(M)|\}.$$

Proof We first prove part (2). Assume that $\dfrac{MPM^T}{det(P)^{2/n}} = P$. By Remark 2.13, we get: $|det(M)| = det(P)$ and therefore $(\dfrac{M}{|det(M)|^{1/n}})P\big(\dfrac{M}{|det(M)|^{1/n}}\big)^T = P$.

The other inclusion follows easily in a similar way.

Part (1) follows from (2) and from Proposition 3.1, since $Fix(\Gamma_{[\frac{M}{|det(M)|^{1/n}}]}) \neq \emptyset$ implies $Fix(\Gamma_{[\frac{M}{|det(M)|^{1/n}}]}) \cap \{P \in \mathcal{P}_n \, : \, det(P) = |det(M)|\} \neq \emptyset$ too, because every congruence is linear.

Proposition 4.2 *Let* $M \in GL_n$, *assume that the isometry* $\Gamma_M \circ \delta$ *of* $\mathcal{P}_n$ *is elliptic and fix a matrix* $F_0 \in GL_n$ *such that* $F_0 J_M F_0^{-1} = M$ *and such that* $|det(F_0)| = \sqrt{|det(M)|}$. *Then*

$$Fix(\Gamma_M \circ \delta) = \{F_0 A A^T F_0^T \, : \, A \in \mathcal{C}_{J_M}, \, det(AA^T) = 1\}.$$

Proof If $M = F_0 J_M F_0^{-1}$, then $\frac{M}{|det(M)|^{1/n}} = F_0 J_{[\frac{M}{|det(M)|^{1/n}}]} F_0^{-1}$. From the Proposition 3.2, we get that $F_0 F_0^T$ is a fixed point of $\Gamma_{[\frac{M}{|det(M)|^{1/n}}]}$ with $det(F_0 F_0^T) = |det(M)|$. Hence, again by Propositions 3.2 and 4.1, we obtain:

$Fix(\Gamma_M \circ \delta) = \{F_0 A A^T F_0^T \, : \, A \in \mathcal{C}_{J_M}, \, det(AA^T) = 1\}$, since the matrices commuting with J_M are precisely the matrices commuting with $J_{[\frac{M}{|det(M)|^{1/n}}]}$.

Notation 4.3 We denote by $\mathbf{S}\big((\mathcal{P}_p, g) \times (\mathcal{H}_{m_1}, 2\gamma) \times \cdots \times (\mathcal{H}_{m_r}, 2\gamma) \times (\mathcal{P}_q, g)\big)$ the Riemannian submanifold of the Riemannian product $(\mathcal{P}_p, g) \times (\mathcal{H}_{m_1}, 2\gamma) \times \cdots \times (\mathcal{H}_{m_r}, 2\gamma) \times (\mathcal{P}_q, g)$, consisting in elements $(A, B_1, \ldots, B_r, C) \in \big(\mathcal{P}_p \times \mathcal{H}_{m_1} \times \cdots \times \mathcal{H}_{m_r} \times \mathcal{P}_q\big)$ such that $det(A) \, [det(B_1)]^2 \, \ldots \, [det(B_r)]^2 \, det(C) = 1$.

Proposition 4.4 *Let* $M \in GL_n$ *such that* $\Gamma_M \circ \delta$ *is elliptic and set* $\eta = |det(M)|^{1/n}$ *so the eigenvalues of* M *are*: η *of multiplicity* p, $-\eta$ *of multiplicity* q , $\eta\, e^{\pm i\theta_1}$ *both with multiplicity* m_1. $\cdots$, *up to* $\eta\, e^{\pm i\theta_r}$ *both with multiplicity* m_r, *with* $p, q, r \geq 0$, $m_j > 0$ for $1 \leq j \leq r$, $p + q + 2m_1 + \cdots + 2m_r = n$ *and* $0 < \theta_1 < \cdots < \theta_r < \pi$. Fix $F_0 \in GL_n$ *such that* $F_0 J_M F_0^{-1} = M$ *and such that* $|det(F_0)| = \sqrt{|det(M)|}$. *Then*

$$\Gamma_{F_0^{-1}}(Fix(\Gamma_M \circ \delta)) = (\mathcal{P}_p \oplus \rho(\mathcal{H}_{m_1}) \oplus \cdots \oplus \rho(\mathcal{H}_{m_r}) \oplus \mathcal{P}_q) \cap SL\mathcal{P}_n.$$

Hence $(Fix(\Gamma_M \circ \delta), g)$ *is a closed simply connected totally geodesic symmetric Riemannian submanifold of* $(\mathcal{P}_n, g)$ *isometric to*

$$\mathbf{S}\big((\mathcal{P}_p, g) \times (\mathcal{H}_{m_1}, 2\gamma) \times \cdots \times (\mathcal{H}_{m_r}, 2\gamma) \times (\mathcal{P}_q, g)\big).$$

Proof It is analogous to the proof of Proposition 3.4 via Propositions 4.2 and 2.6, taking into account that $(\mathcal{P}_p \oplus \rho(\mathcal{H}_{m_1}) \oplus \cdots \oplus \rho(\mathcal{H}_{m_r}) \oplus \mathcal{P}_q) \cap SL\mathcal{P}_n = G' \cap \mathcal{P}_n$ where G' is the algebraic reductive subgroup of GL_n defined by $G' := \big(GL_p \oplus \rho(GL_{m_1}(\mathbb{C})) \oplus \cdots \oplus \rho(GL_{m_r}(\mathbb{C})) \oplus GL_q\big) \cap SL_n$ and that $(G' \cap \mathcal{P}_n, g)$ is isometric to $\mathbf{S}\big((\mathcal{P}_p, g) \times (\mathcal{H}_{m_1}, 2\gamma) \times \cdots \times (\mathcal{H}_{m_r}, 2\gamma) \times (\mathcal{P}_q, g)\big)$.

Proposition 4.5 *Let* $M \in GL_n$ *such that* $\Gamma_M \circ \delta$ *is elliptic, let* $p, q, r, m_1, \ldots, m_r$ *be as in Proposition* 4.4 *and* r' *be as in Remark* 3.6. *Then, up to isometries, the De Rham factors of* $\big(Fix(\Gamma_M \circ \delta), g\big)$ *are*:

a) $\mathbb{R}^{r'-1}$, *if* $r' \geq 2$;
b) $SL_p/S\mathcal{O}_p$, *if* $p \geq 2$;
c) $SL_q/S\mathcal{O}_q$, *if* $q \geq 2$;
d) $SL_{m_j}(\mathbb{C})/SU_{m_j}$ *for all indices* $j = 1, \ldots, r$ *such that* $m_j \geq 2$, *if* $r \geq 1$.

Proof We proof this result under the assumption that p, q, r are all non-zero; in this case $r' - 1 = r + 1$. The proof can be easily adapted to the cases when some of these values vanishes.

Let τ be the flat Riemannian metric on $\mathbb{R}^{r+1}$ defined by $\tau = \frac{dx_0^2}{p} + \sum_{i=1}^r \frac{dx_i^2}{2m_i} + \frac{(\sum_{i=0}^r dx_i)^2}{q}$.

Let F be the map from $(SL\mathcal{P}_p, g) \times \prod_{i=1}^r (SL\mathcal{H}_{m_i}, 2\gamma) \times (SL\mathcal{P}_q, g) \times (\mathbb{R}^{r+1}, \tau)$ onto $\mathbf{S}\big((\mathcal{P}_p, g) \times (\mathcal{H}_{m_1}, 2\gamma) \times \cdots \times (\mathcal{H}_{m_r}, 2\gamma) \times (\mathcal{P}_q, g)\big)$, defined by: $F(\alpha, \beta_1, \ldots, \beta_r, \delta, t_0, t_1, \ldots, t_r)) = \big(e^{t_0/p}\, \alpha, e^{t_1/2m_1}\, \beta_1, \ldots, e^{t_r/2m_r}\, \beta_r, e^{-(\sum_{i=0}^r t_i)/q}\, \delta\big)$.

It is easy to check that F is bijective with inverse:
$F^{-1}(A, B_1, \ldots, B_r, C) = \big(\frac{A}{det(A)^{1/p}}, \frac{B_1}{det(B_1)^{1/m_1}}, \ldots, \frac{B_r}{det(B_r)^{1/m_r}}, \frac{C}{det(C)^{1/q}}$, $\ln(det(A)), 2\ln(det(B_1)), \ldots, 2\ln(det(B_r))\big)$.

We want to prove that F is an isometry.

For, let ψ be the metric on $\mathbf{S}\big((\mathcal{P}_p, g) \times (\mathcal{H}_{m_1}, 2\gamma) \times \cdots \times (\mathcal{H}_{m_r}, 2\gamma) \times (\mathcal{P}_q, g)\big)$.

By direct computations, we get:

$$DF_{(\alpha,\beta_1,\ldots,\beta_r,\delta,t_0,t_1,\ldots,t_r)}(U, V_1, \ldots V_r, W, h_0, \ldots, h_r) = \big(e^{t_0/p}(U + \frac{h_0}{p}\alpha), e^{t_1/2m_1}(V_1 + \frac{h_1}{2m_1}\beta_1), \ldots,$$
$$e^{t_r/2m_r}(V_r + \frac{h_r}{2m_r}\beta_r),\ e^{-\sum_{j=0}^r t_j/q}(W - \frac{\sum_{j=0}^r h_j}{q}\delta)\big)$$

and

$$\begin{aligned}&[F^*\psi]_{(\alpha,\beta_1,\dots,\beta_r,\delta,t_0,\dots,t_r)}\big((U,V_1,\dots V_r,W,h_0,\dots,h_r),(U',V_1',\dots V_r',W',h_0',\dots,h_r')\big)\\ &= tr(\alpha^{-1}U\alpha^{-1}U')+\frac{h_0'}{p}tr(\alpha^{-1}U)+\frac{h_0}{p}tr(\alpha^{-1}U')+\frac{h_0h_0'}{p}+2\textstyle\sum_{j=1}^r[tr(\beta_j^{-1}V_j\beta_j^{-1}V_j')+\displaystyle\frac{h_j}{2m_j}tr(\beta_j^{-1}V_j')\\ &\quad+\frac{h_j'}{2m_j}tr(\beta_j^{-1}V_j)+\frac{h_jh_j'}{4m_j}]+tr(\delta^{-1}W\delta^{-1}W')-(\textstyle\sum_{j=0}^r h_j)\displaystyle\frac{tr(\delta^{-1}W')}{q}-(\textstyle\sum_{j=0}^r h_j')\displaystyle\frac{tr(\delta^{-1}W)}{q}+\frac{(\sum_{j=0}^r h_j)(\sum_{j=0}^r h_j')}{q}.\end{aligned}$$

Since $U, U' \in T_\alpha(SL\mathcal{P}_p)$, $V_j, V_j' \in T_{\beta_j}(SL\mathcal{H}_{m_j})$ and $W, W' \in T_\delta(SL\mathcal{P}_q)$, from Jacobi's formula, we get: $tr(\alpha^{-1}U) = tr(\alpha^{-1}U') = tr(\beta_j^{-1}V_j) = tr(\beta_j^{-1}V_j') = tr(\delta^{-1}W) = tr(\delta^{-1}W') = 0$, so the right side of the previous equality is $g_\alpha(U,U') + \sum_{j=1}^r 2\gamma_{\beta_j}(V_j,V_j') + g_\delta(W,W') + \tau\big((h_0,\dots,h_r),(h_0',\dots,h_r')\big)$.

Hence F is an isometry.

Since τ is a flat metric, $(\mathbb{R}^{r+1},\tau)$ is isometric to the Euclidean space $\mathbb{R}^{r+1}$.

This allows to conclude the proof of the Proposition, arguing as in Remark 3.6.

5 The fixed points of the isometries $\Gamma_M \circ j$

Proposition 5.1 *Let $M \in GL_n$ and let us consider the isometry $\Gamma_M \circ j$ of $\mathcal{P}_n$. The following facts are equivalent*:

a) *$\Gamma_M \circ j$ is elliptic*;
b) *M and M^{-T} are $\mathbb{R}$-congruent via a positive definite matrix*;
c) *M is $\mathbb{R}$-congruent to an orthogonal matrix*;
d) *M is $\mathbb{R}$-congruent to a normal matrix*;
e) *$\Gamma_{MM^{-T}}$ is elliptic*;
f) *MM^{-T} is semi-simple with eigenvalues of modulus* 1;
g) *MM^{-T} is similar to an orthogonal matrix.*

Proof The equivalence (a) ⇔ (b) follows by remarking that $M(CC^T)^{-1}M^T = CC^T$ is equivalent to $M = CC^TM^{-T}CC^T$.

The equivalence: (a) ⇔ (c) follows by remarking that $M(CC^T)^{-1}M^T = CC^T$ if and only if $(C^{-1}MC^{-T})(C^{-1}MC^{-T})^T = I_n$ if and only if $C^{-1}MC^{-T} \in \mathcal{O}_n$.

We get (d) ⇒ (c) from Remark 1.5, while (c) ⇒ (d) is trivial.

Now note that, if $H \in GL_n$, we have: $(HMH^T)(HM^TH^T) = (HM^TH^T)(HMH^T)$ if and only if $(M^{-T}M)H^TH(M^{-T}M)^T = H^TH$ if and only if $\Gamma_{M^{-T}M}$ has H^TH as fixed point in $\mathcal{P}_n$. Since $M^{-T}M$ and MM^{-T} are similar, from Proposition 3.1, we get the equivalence (d) ⇔ (e).

Finally (e) ⇔ (f) ⇔ (g) is proved in Proposition 3.1.

Lemma 5.2 *Let $M \in GL_n$ and assume that $\Gamma_M \circ j$ is elliptic. Then $Fix(\Gamma_M \circ j) \subseteq Fix(\Gamma_{MM^{-T}})$.*

Proof If $P \in Fix(\Gamma_M \circ j)$, then $MP^{-1}M^T = P$, i.e. $M^{-T}PM^{-1} = P^{-1}$. Hence we have: $MM^{-T}P(MM^{-T})^T = M(M^{-T}PM^{-1})M^T = MP^{-1}M^T = P$ and so $P \in Fix(\Gamma_{MM^{-T}})$.

Remarks 5.3

a) Let $M \in GL_n$, then M is normal if and only if MM^{-T} is orthogonal.
b) Let $M, S \in GL_n$. Then $S^{-1}MS^{-T}$ is normal if and only if $S^{-1}MM^{-T}S$ is orthogonal.

Indeed $M^TM = MM^T$ if and only if $(MM^{-T})^T(MM^{-T}) = I_n$ and this gives (a).

Part (b) follows from (a), since $S^{-1}MM^{-T}S = (S^{-1}MS^{-T})(S^{-1}MS^{-T})^{-T}$.

Remarks-Definitions 5.4 Let $M \in GL_n$, assume that $\Gamma_M \circ j$ is elliptic or, equivalently, that M is $\mathbb{R}$-congruent to an orthogonal matrix.

a) By Proposition 5.1, we denote by S a matrix in GL_n such that $S^{-1}MM^{-T}S \in \mathcal{O}_n$. By Remarks 5.3 (b), $S^{-1}MS^{-T}$ is normal and so, by Theorem 1.4 and Remark 1.5, $S^{-1}MS^{-T} = QU = UQ = \sqrt{Q}\,U\sqrt{Q}$ where $Q \in \mathcal{P}_n$ and $U \in \mathcal{O}_n$ are the components of the polar decomposition of $S^{-1}MS^{-T}$. This gives also that M is $\mathbb{R}$-congruent to U. Conversely, it is easy to verify that every $U \in \mathcal{O}_n$, which is $\mathbb{R}$-congruent to M, can be obtained in this way, starting from a matrix $S \in GL_n$ such that $S^{-1}MM^{-T}S$ is orthogonal. Fixed a matrix S as above, the explicit expression for U is the following: $U = (\sqrt{S^{-1}MS^{-T}S^{-1}M^TS^{-T}}\,)^{-1}S^{-1}MS^{-T}$. In particular, if M is normal, then we can choose $S = I_n$, so that $U = (\sqrt{MM^T})^{-1}M$.
b) Following Remark–Definition 1.8, we denote by $Z \in \mathcal{O}_n$ a matrix such that $U = Z\widetilde{J}_UZ^T$ and, so, for $R = S\sqrt{Q}Z$ we get: $M = R\widetilde{J}_UR^T$.

Proposition 5.5 *Let $M \in GL_n$, assume that $\Gamma_M \circ j$ is elliptic, and let U be an orthogonal matrix, $\mathbb{R}$-congruent to M, and $R \in GL_n$ such that $M = R\widetilde{J}_UR^T$ (remember Remarks-Definitions 5.4). Then*

$$Fix(\Gamma_M \circ j) = \Gamma_R(\{GG^T \,:\, G \in \mathcal{C}_{J_{MM^{-T}}}, GG^T \in \mathcal{K}_{\widetilde{J}_U}\}).$$

Proof By Lemma 5.2, $Fix(\Gamma_M \circ j) \subseteq Fix(\Gamma_{MM^{-T}})$.

Now $MM^{-T} = R\widetilde{J}_UR^TR^{-T}\widetilde{J}_UR^{-1} = R(\widetilde{J}_U)^2R^{-1} = RJ_{U^2}R^{-1} = RJ_{MM^{-T}}R^{-1}$. By Proposition 3.2 (ii), if $P \in Fix(\Gamma_M \circ j)$, then there exists $G \in \mathcal{C}_{J_{MM^{-T}}}$ such that $RGG^TR^T = P$. Moreover $MP^{-1}M^T = P$ is equivalent to $\widetilde{J}_U = (GG^T)\widetilde{J}_U(GG^T)^T$, i.e. to $GG^T \in \mathcal{K}_{\widetilde{J}_U}$; hence $Fix(\Gamma_M \circ j) \subseteq \{RGG^TR^T \,:\, G \in \mathcal{C}_{J_{MM^{-T}}}, GG^T \in \mathcal{K}_{\widetilde{J}_U}\} = \Gamma_R(\{GG^T \,:\, G \in \mathcal{C}_{J_{MM^{-T}}}, GG^T \in \mathcal{K}_{\widetilde{J}_U}\})$.

For the other inclusion, let $P = RGG^TR^T$ with $G \in \mathcal{C}_{J_{MM^{-T}}}$ and $GG^T \in \mathcal{K}_{\widetilde{J}_U}$. Then we have: $MP^{-1}M^T = R\widetilde{J}_U(GG^T)^{-1}(\widetilde{J}_U)^TR^T = RGG^T\widetilde{J}_U(\widetilde{J}_U)^TR^T = P$ and then $P \in Fix(\Gamma_M \circ j)$ and the Proposition is proved.

Example 5.6 Now let $M = \Omega_p := I_p \oplus (-I_{n-p})$ with $p = 0, \dots, n$.

Note that Ω_p is diagonal, orthogonal, $\Omega_p^2 = I_n$ and that Ω_p agrees with its RJA form $\widetilde{J}_{\Omega_p}$. Hence, by Proposition 5.1, $\Gamma_{\Omega_p} \circ j$ is elliptic and with the same notations of Remarks-Definitions 5.4, we can choose $S = Q = Z = I_n$ and $U = \Omega_p$.

Hence, by Proposition 5.5, $Fix(\Gamma_{\Omega_p} \circ j) = \mathcal{P}_n \cap \mathcal{O}(p, n-p) = \mathcal{P}_n \cap S\mathcal{O}_0(p, n-p)$.

Moreover, $S\mathcal{O}_0(p, n-p)$ is a reductive subgroup of GL_n, satisfying the condition (*) in Proposition 2.6 and we have also $S\mathcal{O}_0(p, n-p) \cap S\mathcal{O}_n = S\mathcal{O}_p \oplus S\mathcal{O}_{n-p}$ since the inclusion $S\mathcal{O}_0(p, n-p) \cap S\mathcal{O}_n \supseteq S\mathcal{O}_p \oplus S\mathcal{O}_{n-p}$ is trivial, while it is well-known that $S\mathcal{O}_p \oplus S\mathcal{O}_{n-p}$ is a maximal compact subgroup of $S\mathcal{O}_0(p, n-p)$ (see for instance [15, Ch. VI]). By Proposition 2.6, this allows to conclude that

$Fix(\Gamma_{\Omega_p} \circ j) = \mathcal{P}_n \cap S\mathcal{O}_0(p, n-p)$ is diffeomorphic to the irreducible symmetric space $S\mathcal{O}_0(p, n-p)/\big(S\mathcal{O}_p \oplus S\mathcal{O}_{n-p}\big)$, whose dimension is $p(n-p)$.

Example 5.7 Consider the case: $n = 2m$ and $M = \Lambda_m := E^{\oplus m}$.

Arguing as in Example 5.6, we get that Λ_m is orthogonal, skew-symmetric, $\Lambda_m^2 = -I_{2m}$ and Λ_m agrees with its RJA form $\widetilde{J}_{\Lambda_m}$: $\Lambda_m = \widetilde{J}_{\Lambda_m}$. Hence $\Gamma_{\Lambda_m} \circ j$ is elliptic and we can get: $S = Q = Z = I_{2m}$ and $U = \Lambda_m$.

Therefore $Fix(\Gamma_{\Lambda_m} \circ j) = \{GG^T : GG^T \in \mathcal{K}_{\Lambda_m}\}$.

Now let W be an orthogonal matrix such that $\Lambda_m = W \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} W^T$.

It is easy to verify that $\mathcal{K}_{\Lambda_m} = \Gamma_W(Sp_{2m})$. Then we get

$Fix(\Gamma_{\Lambda_m} \circ j) = \mathcal{P}_{2m} \cap \mathcal{K}_{\Lambda_m} = \mathcal{P}_{2m} \cap \Gamma_W(Sp_{2m}) = \Gamma_W(\mathcal{P}_{2m} \cap Sp_{2m})$.

Moreover, since Sp_{2m} is an algebraic reductive subgroup of GL_{2m} and $Sp_{2m} \cap \mathcal{O}_{2m} = \rho(U_m)$ (see for instance [11, Prop. 2.12, p. 33]), by Proposition 2.6 we get that $Fix(\Gamma_{\Lambda_m} \circ j) = \Gamma_W(\mathcal{P}_{2m} \cap Sp_{2m})$ is diffeomorphic to the irreducible symmetric space $Sp_{2m}/\rho(U_m)$, whose dimension is $m(m+1)$.

Example 5.8 Now let $\theta \in (0, \frac{\pi}{2})$, $\mu \geq 0$, $\nu \geq 0$, $\mu + \nu \geq 1$ and $n = 2(\mu + \nu)$. Let $M = \Theta_{\theta;\mu,\nu} := E_\theta^{\oplus \mu} \oplus (-E_\theta^{\oplus \nu})$.

Arguing again as in Example 5.6, we get that $\Theta_{\theta;\mu,\nu}$ is orthogonal, $\Theta_{\theta;\mu,\nu}^2 = E_{2\theta}^{\oplus(\mu+\nu)}$ and $\Theta_{\theta;\mu,\nu}$ agrees with its RJA form $\widetilde{J}_{\Theta_{\theta;\mu,\nu}}$. Hence $\Gamma_{\Theta_{\theta;\mu,\nu}} \circ j$ is elliptic and we can choose $S = Q = Z = I_{2(\mu+\nu)}, U = \Theta_{\theta;\mu,\nu}$ so that $Fix(\Gamma_{\Theta_{\theta;\mu,\nu}} \circ j) = \{GG^T : G \in \mathcal{C}_{E_{2\theta}^{\oplus(\mu+\nu)}}, GG^T \in \mathcal{K}_{\Theta_{\theta;\mu,\nu}}\}$.

By Lemma 3.3, $\mathcal{C}_{E_{2\theta}^{\oplus(\mu+\nu)}} = \rho(GL_n(\mathbb{C}))$, where $\rho : GL_{\mu+\nu}(\mathbb{C}) \to GL_{2(\mu+\nu)}$ is the monomorhism defined in Remark 2.8. Note that $\Theta_{\theta;\mu,\nu} = \rho(e^{i\theta}(I_\mu \oplus (-I_\nu)))$.

Since ρ preserves products and it is injective, we get that $Fix(\Gamma_{\Theta_{\theta;\mu,\nu}} \circ j) = \rho\big(\{HH^* : H \in GL_{\mu+\nu}(\mathbb{C}), HH^*(e^{i\theta}(I_\mu \oplus (-I_\nu)))HH^* = e^{i\theta}(I_\mu \oplus (-I_\nu))\}\big) = \rho\big(\{HH^* : H \in GL_{\mu+\nu}(\mathbb{C}), HH^*(I_\mu \oplus (-I_\nu)))HH^* = (I_\mu \oplus (-I_\nu))\}\big) = \rho(\mathcal{H}_{\mu+\nu} \cap U(\mu,\nu)) = \rho(\mathcal{H}_{\mu+\nu} \cap SU(\mu,\nu))$.

Now $\rho(\mathcal{H}_{\mu+\nu} \cap SU(\mu,\nu)) = \mathcal{P}_{2(\mu+\nu)} \cap \rho(SU(\mu,\nu))$ is an algebraic reductive subgroup of GL_n and, arguing as in Example 5.6, $\rho(SU(\mu,\nu)) \cap \mathcal{O}_{2(\mu+\nu)} = \rho(S(U_\mu \oplus U_\nu))$ (where $S(U_\mu \oplus U_\nu)$ consists in matrices of $U_\mu \oplus U_\nu$ with determinant 1); indeed $\rho(S(U_\mu \oplus U_\nu))$ is trivially included in $\rho(SU(\mu,\nu)) \cap \mathcal{O}_{2(\mu+\nu)}$ and $S(U_\mu \oplus U_\nu)$ is a maximal compact subgroup of $SU(\mu,\nu)$. By Proposition 2.6, we obtain that $Fix(\Gamma_{\Theta_{\theta;\mu,\nu}} \circ j) = \mathcal{P}_{2(\mu+\nu)} \cap \rho(SU(\mu,\nu)) = \rho(\mathcal{H}_{\mu+\nu} \cap SU(\mu,\nu))$ is diffeomorphic to the irreducible symmetric space $SU(\mu,\nu)/S(U_\mu \oplus U_\nu)$, whose dimension is $2\mu\nu$.

Proposition 5.9 *Let $M \in GL_n$, assume that $\Gamma_M \circ j$ is elliptic, and let U be an orthogonal matrix, $\mathbb{R}$-congruent to M (remember Remarks-Definitions 5.4).*

Let $\widetilde{J}_U = I_p \oplus (-I_q) \oplus E_{\phi_1}^{\oplus \mu_1} \oplus (-E_{\phi_1}^{\oplus \nu_1}) \oplus \cdots \oplus E_{\phi_h}^{\oplus \mu_h} \oplus (-E_{\phi_h}^{\oplus \nu_h}) \oplus E^{\oplus k}$ the RJA form of U. Then $Fix(\Gamma_M \circ j)$ and $(\mathcal{P}_{p+q} \cap S\mathcal{O}_0(p,q)) \oplus \rho(\mathcal{H}_{\mu_1+\nu_1} \cap U(\mu_1,\nu_1)) \oplus \cdots \oplus \rho(\mathcal{H}_{\mu_h+\nu_h} \cap U(\mu_h,\nu_h)) \oplus (\mathcal{P}_{2k} \cap Sp_{2k})$ are isometric (by congruence) as Riemannian submanifolds of $(\mathcal{P}_n, g)$.

In particular $(Fix(\Gamma_M \circ j), g)$ is a closed simply connected totally geodesic symmetric Riemannian submanifold of $(\mathcal{P}_n, g)$ of dimension $pq + 2\sum_{j=1}^h \mu_j\nu_j + k(k+1)$, *isometric to the Riemannian product*

$$(\mathcal{P}_{p+q} \cap S\mathcal{O}_0(p,q), g) \times \prod_{j=1}^{h} (\mathcal{H}_{\mu_j+\nu_j} \cap U(\mu_j,\nu_j), 2\gamma) \times (\mathcal{P}_{2k} \cap Sp_{2k}, g).$$

Proof We have: $J_{MM^{-T}} = J_{U^2} = (\widetilde{J}_U)^2 = I_{p+q} \oplus E_{2\phi_1}^{\oplus(\mu_1+\nu_1)} \oplus \cdots \oplus E_{2\phi_h}^{\oplus(\mu_h+\nu_h)} \oplus (-I_{2k})$, hence, by Lemma 3.3, $\mathcal{C}_{J_{MM^{-T}}} = GL_{p+q} \oplus \rho(GL_{\mu_1+\nu_1}(\mathbb{C})) \oplus \cdots \oplus \rho(GL_{\mu_h+\nu_h}(\mathbb{C})) \oplus GL_{2k}$.

Therefore: $\{GG^T : G \in \mathcal{C}_{J_{MM^{-T}}}\} = \mathcal{P}_{p+q} \oplus \rho(\mathcal{H}_{\mu_1+\nu_1}) \oplus \cdots \oplus \rho(\mathcal{H}_{\mu_h+\nu_h}) \oplus \mathcal{P}_{2k}$.

Now, arguing as in Examples 5.6, 5.7 and 5.8, we get $\{GG^T : G \in \mathcal{C}_{J_{MM^{-T}}}\} \cap \mathcal{K}_{\widetilde{J}_U} = (\mathcal{P}_{p+q} \cap SO_0(p,q)) \oplus \rho(\mathcal{H}_{\mu_1+\nu_1} \cap U(\mu_1,\nu_1)) \oplus \cdots \oplus \rho(\mathcal{H}_{\mu_h+\nu_h} \cap U(\mu_h,\nu_h)) \oplus \Gamma_W(\mathcal{P}_{2k} \cap Sp_{2k})$.

We conclude by Proposition 5.5.

Proposition 5.10 *Let $M \in GL_n$, assume that $\Gamma_M \circ j$ is elliptic and let U be an orthogonal matrix, $\mathbb{R}$-congruent to M, having 1 as eigenvalue with multiplicity $p \geq 0$, -1 as eigenvalue with multiplicity $q \geq 0$, i as eigenvalue with multiplicity $k \geq 0$ and call $\phi_1, \ldots, \phi_h$ $(h \geq 0)$, the set of mutually distinct possible values in $(0, \frac{\pi}{2})$ such that $e^{i\phi}$* or *$e^{i(\pi-\phi)}$ is an eigenvalue of U.*

Moreover, if $h > 0$, *for every $j = 1, \ldots, h$, we set to be μ_j the multiplicity of $e^{i\phi_j}$ and ν_j to be the multiplicity of $e^{i(\pi-\phi_j)}$ (note that $\mu_j \geq 0$, $\nu_j \geq 0$, $\mu_j + \nu_j \geq 1$ and p, q, k, h are not all zero).*

Then, up to isometries, the De Rham factors of $\big(Fix(\Gamma_M \circ j), g\big)$ are:

a) $\mathbb{R}$, *if $p = q = 1$;*
b) $SO_0(p,q)/(SO_p \oplus SO_q)$, if $p, q \geq 1$ *and* $p + q \geq 3$;
c) $Sp_{2k}/\rho(U_k)$, *if $k \geq 1$;*
d) $SU(\mu_j,\nu_j)/S(U_{\mu_j} \oplus U_{\nu_j})$, *if $h \geq 1$ for every $j = 1, \ldots, h$ such that $\mu_j, \nu_j \geq 1$.*

Proof It follows from Proposition 5.9 and from Examples 5.6, 5.7 and 5.8, by analogous arguments, developed in Remark 3.6.

6 The fixed points of the isometries $\Gamma_M \circ j \circ \delta$

Proposition 6.1 *Let $M \in GL_n$ and let us consider the isometry $\Gamma_M \circ j \circ \delta$ of $\mathcal{P}_n$. The following facts are equivalent:*

a) *$\Gamma_M \circ j \circ \delta$ is elliptic;*
b) *$\Gamma_M \circ j$ is elliptic and $det(M) = \pm 1$;*
c) *M and M^{-T} are $\mathbb{R}$-congruent via a positive definite matrix and $det(M) = \pm 1$;*
d) *M is $\mathbb{R}$-congruent to an orthogonal matrix and $det(M) = \pm 1$;*
e) *M is $\mathbb{R}$-congruent to a normal matrix and $det(M) = \pm 1$;*
f) *$\Gamma_{MM^{-T}}$ is elliptic and $det(M) = \pm 1$;*
g) *MM^{-T} is semi-simple with eigenvalues of modulus 1 and $det(M) = \pm 1$.*

Proof It suffices to prove the equivalence (a) ⇔ (d) and the other equivalences will follow from Proposition 5.1.

Assume first that $det(P)^{2/n} M P^{-1} M^T = P$, then we get $det(M) = \pm 1$ simply by computing the determinants. After setting $P = CC^T$, we get:

$det(CC^T)^{2/n} M (CC^T)^{-1} M^T = CC^T$, and thus

$$([\frac{C}{|det(C)|^{1/n}}]^{-1} M [\frac{C}{|det(C)|^{1/n}}]^{-T})([\frac{C}{|det(C)|^{1/n}}]^{-1} M [\frac{C}{|det(C)|^{1/n}}]^{-T})^T = I_n,$$

i.e. $[\frac{C}{|det(C)|^{1/n}}]^{-1}M[\frac{C}{|det(C)|^{1/n}}]^{-T} \in \mathcal{O}_n$ and we get (a) ⇒ (d).

For the converse, assume that $M = KUK^T$ with $K \in GL_n$, $U \in \mathcal{O}_n$ and $det(M) = \pm 1$. By computing the determinants we obtain: $det(U) = det(M)$ and $det(KK^T) = 1$. Hence: $det(KK^T)^{2/n}M(KK^T)^{-1}M^T = MK^{-T}K^{-1}M^T = KK^T$ (after replacing M with KUK^T), i.e. KK^T is a fixed point of $\Gamma_M \circ j \circ \delta$.

Proposition 6.2 *Let $M \in GL_n$ and assume that $\Gamma_M \circ j \circ \delta$ is elliptic. Then*

a) $\emptyset \neq Fix(\Gamma_M \circ j) \subseteq SL\mathcal{P}_n$;
b) $Fix(\Gamma_M \circ j \circ \delta) = \mathbb{R}^+ \cdot Fix(\Gamma_M \circ j) := \{tP : t > 0, P \in Fix(\Gamma_M \circ j)\}$;
c) $(Fix(\Gamma_M \circ j \circ \delta), g)$ is *a closed simply connected totally geodesic symmetric Riemannian submanifold of* $(\mathcal{P}_n, g)$, *isometric to the Riemannian product*

$(Fix(\Gamma_M \circ j), g) \times (\mathbb{R}, \varepsilon)$ *(ε is the ordinary euclidean metric) and hence, with the same notations as in Proposition 5.9, it is isometric to the Riemannian product* $(\mathcal{P}_{p+q} \cap SO_0(p,q), g) \times \prod_{j=1}^{h}(\mathcal{H}_{\mu_j+\nu_j} \cap U(\mu_j,\nu_j), 2\gamma) \times (\mathcal{P}_{2k} \cap Sp_{2k}, g) \times (\mathbb{R}, \varepsilon)$ *and its dimension is* $pq + 2\sum_{j=1}^{h}\mu_j\nu_j + k(k+1) + 1$.

Proof

a) It follows from Proposition 6.1 and from Remark 2.13.
b) If $P \in Fix(\Gamma_M \circ j)$, then $MP^{-1}M^T = P$ and $det(P) = 1$ from (a).

Hence, for every $t \in \mathbb{R}^+$, $det(tP)^{2/n}M(tP)^{-1}M^T = tMP^{-1}M^T = tP$,

i.e. $tP \in Fix(\Gamma_M \circ j \circ \delta)$.

For the other inclusion, it suffices to note that, if $P \in Fix(\Gamma_M \circ j \circ \delta)$, then $\frac{P}{det(P)^{1/n}} \in Fix(\Gamma_M \circ j)$.

Indeed we have: $M(\frac{P}{det(P)^{1/n}})^{-1}M^T = det(P)^{1/n}MP^{-1}M^T = \frac{P}{det(P)^{1/n}}$.

c) The mapping $(\mathcal{P}_n, g) \to (SL\mathcal{P}_n, g) \times (\mathbb{R}, \varepsilon)$, $P \mapsto (\frac{P}{det(P)^{1/n}}, \frac{\ln(det(P))}{\sqrt{n}})$, is an isometry, as proved in [14, Proof of Prop. 2.7]. By part (b), the restriction of this mapping to $Fix(\Gamma_M \circ j \circ \delta)$ is an isometry from $(Fix(\Gamma_M \circ j \circ \delta), g)$ onto $(Fix(\Gamma_M \circ j), g) \times (\mathbb{R}, \varepsilon)$.

Proposition 6.3 *Let $M \in GL_n$, assume that $\Gamma_M \circ j \circ \delta$ is elliptic, let U* be *an orthogonal matrix, $\mathbb{R}$-congruent to M, and let $p, q, k, h, \mu_j, \nu_j \geq 0$ as in Proposition 5.10.*

Then, up to isometries, the De Rham factors of $\big(Fix(\Gamma_M \circ j \circ \delta), g\big)$ are:

a) $\mathbb{R}^2$, *if $p = q = 1$;*
b) $\mathbb{R}$, *if $p \neq 1$ or $q \neq 1$;*
c) $SO_0(p,q)/(SO_p \oplus SO_q)$, *if $p, q \geq 1$ and $p + q \geq 3$;*
d) $Sp_{2k}/\rho(U_k)$, *if $k \geq 1$;*
e) $SU(\mu_j,\nu_j)/S(U_{\mu_j} \oplus U_{\nu_j})$, if $h \geq 1$, *for every $j = 1, \dots, h$ such that $\mu_j, \nu_j \geq 1$.*

Proof It follows directly from Propositions 6.2 and 5.10.

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