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Substructuring Approach to the Calculation of Higher-order Eigensensitivity

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Abstract

Calculation of eigensensitivity is usually time-consuming for a large-scale structure. This paper develops a substructuring method for computing the first, second and high order eigensensitivity. The local area of a structure is treated as an independent substructure to be analyzed. The eigensensitivity of global structure with respect to a design parameter are calculated from the eigensensitivity of a particular substructure that contains the design parameter, thus allowing a significant reduction in computational cost. The accuracy and efficiency of the substructuring method is proved by a frame structure and a highway bridge.

Keywords: eigensensitivity; eigensolution derivative; substructuring method; residual flexibility;

Nomenclature

K, M	Stiffness matrix, mass matrix
λ_i, $\mathbf{\Lambda}$	The i^{th} eigenvalue, matrix of eigenvalues
ϕ_i, $\mathbf{\Phi}$	The i^{th} eigenvector (mode shape), matrix of eigenvectors
$\bar{\mathbf{\Phi}}$	Matrix of expanded mode shapes
r	Elemental physical parameter
τ	Interface force along the boundaries of the substructures
$\mathbf{C}$	Connection matrix
N	Degrees of freedom of the global structure
N_S	Number of the independent substructures
NP	Size of the primitive matrix, the degrees of freedom of all substructures
$n^{(j)}$	Degrees of freedom of the *j*th substructure
$\mathbf{z}$	Participation factor of the substructural eigenmodes
$\mathbf{\Psi}$	Equivalent stiffness matrix

Superscripts

(j)	The *j*th substructure
p	Primitive matrix or vector
T	Transpose of matrix or vector

Subscripts

m	Master modes
s	Slave modes
i	The *i*th modes

1. Introduction

Structural health monitoring (SHM) systems identify potential structural damages by detecting the changes in structural parameters. The changes in structural parameters are usually identified by adjusting parameters of an a priori finite element (FE) model of a structure to reconcile its response with a set of measured modal data through an optimization process. To achieve this, the eigensolutions and eigensensitivity of the analytical model need to be calculated repeatedly [1, 2]. The eigensensitivity provides an estimate of the changes in the eigensolutions caused by the perturbations of design parameters of structural model. In optimization process, the eigensensitivity serves for indicating the searching direction of an optimization algorithm, which endows the more sensitive parameter (with respect to the objective function) a higher priority [3].

Eigensensitivity is most commonly calculated at the global structure level. Fox and Kapoor [4] firstly utilized the modal method to determine the first order eigenvalue and eigenvector derivatives by considering the changes of the physical parameters in the mass and stiffness matrices. The modal method requires the superposition of eigenmodes of the system to calculate the interested eigenvalue and eigenvector derivatives [5, 6]. Nelson [7] proposed an exact method in calculating the eigenvector derivatives of one mode by using the modal parameters of that mode only. Nelson's method has been further improved in terms of computational efficiency [8], and has also been generalized by taking into account the rigid body modes, and the close or repeated modes [9, 10]. Another approach for eigensensitivity computation is based on an algebraic formulation [11, 12]. The derivatives of each eigenvalue and its associated eigenvector are computed simultaneously by solving an algebraic system of equations.

Other than the computation of first order eigensensitivity, the second order or high order eigensensitivity is required when there are large changes in design parameters or when the systems has closely spaced natural frequencies. Brandon [13] calculated the second order eigensensitivity using the modal method. Friswell [14] extended Nelson's method to calculate the second and higher order eigenvector derivatives, by repeatedly differentiating Nelson's eigenequation. Based on the algebraic approach, Choi *et al.* [15] computed the first, second and high order derivatives of eigenvalues and eigenvectors associated with repeated eigenvalues. Guedria *et al.* [16] generalized Nelson's method to calculate the high order derivatives of damped and asymmetric systems, and Chouchane *et al*. [17] calculated the high order derivatives for damped and asymmetric systems using algebraic approach.

Nevertheless, a large-scale practical structure is usually represented by a complex model, involving a large number of degrees of freedom (DOFs) and undetermined structural parameters. In model updating or optimization process, structural parameters in the analytical model are iteratively modified to satisfy the objective functions in an optimal way. For a large-scale structure, it is computationally expensive to repeatedly extract the eigensensitivity from the large-size global system matrices.

Substructuring technology is preferable for the analysis of large complex structures [18-26]. Firstly, it is possible to analyze each substructure independently, or even concurrently with parallel computing. It is much easier and quicker to analyze the smaller substructures than a large global structure. While identical substructures exist, the computation load is reduced further by analyzing only one of the repeated substructures. Secondly, when local area of a structure is focused on, it is more efficient to adjusting one or more concerned substructures iteratively to reconcile the measured data. The substructuring methods allow the eigensolutions and eigensensitivity of the modified

substructures to be analyzed iteratively, whereas the unmodified substructures remain unchanged during the iterative model updating process. Thirdly, the number of parameters in each substructure is much smaller than that in the global structure. This improves the convergence of optimization process [23-26].

In general, substructuring methods include three steps: first, the global structure is separated into some manageable smaller substructures, each of which has far fewer DOFs and unknown parameters than the global structure; second, the substructures are analyzed independently to obtain designated solutions (for example, substructural eigenpairs and substructural derivative matrices); third, the solutions of the substructures are assembled to obtain the solutions of the global structure by imposing constraints on the interface of the adjacent substructures [26].

Kron [27] initiated a substructuring method for calculating the eigensolutions of systems with a large number of variables in a piece-wise manner. Kron's substructuring method has a concise form, and it has been further developed by a number of researchers [25-26, 28-29]. This paper extends the substructuring method to derive the eigenpair sensitivities with respect to structural parameters, including the first, second and high order eigensensitivity. The global structure is divided into manageable substructures. The derivative matrices of a few substructural eigensolutions with respect to a design structural parameters are computed for independent substructures. Afterwards, the substructural eigensensitivity matrices are assembled to recover the eigensensitivity of the global structure. As the substructural eigensensitivity matrices are zeros for those substructures that do not include the design parameter, the eigensensitivity of global structure is determined from the substructural eigensensitivity of a concerned substructure which contains the design parameter. Since it is much easier and quicker to analyze a small substructure than the large global structure, the

proposed substructuring method improves the computational efficiency significantly. The proposed substructuring method for eigensensitivity is applied to a three-span frame structure and a highway bridge to verify its efficiency and accuracy.

2. Substructuring method for eigensolution

In general, a global structure with N DOFs is divided into N_S independent substructures. The jth (j=1, 2, …, N_S) substructure with $n^{(j)}$ DOFs has the stiffness matrix $\mathbf{K}^{(j)}$ and mass matrix $\mathbf{M}^{(j)}$. The $n^{(j)}$ eigenpairs of the jth substructure are

$$\mathbf{\Lambda}^{(j)} = \mathrm{Diag}\left[\lambda_1^{(j)}, \lambda_2^{(j)}, ..., \lambda_{n^{(j)}}^{(j)}\right],\ \mathbf{\Phi}^{(j)} = \left[\phi_1^{(j)}, \phi_2^{(j)}, ..., \phi_{n^{(j)}}^{(j)}\right]$$

$$\left[\mathbf{\Phi}^{(j)}\right]^T \mathbf{K}^{(j)} \mathbf{\Phi}^{(j)} = \mathbf{\Lambda}^{(j)},\ \left[\mathbf{\Phi}^{(j)}\right]^T \mathbf{M}^{(j)} \mathbf{\Phi}^{(j)} = \mathbf{I}^{(j)} \tag{1}$$

The eigensolutions of the independent substructures are diagonally assembled into the primitive forms

$$\mathbf{\Lambda}^p = \mathrm{Diag}\left[\mathbf{\Lambda}^{(1)}, \mathbf{\Lambda}^{(2)}, ..., \mathbf{\Lambda}^{(N_S)}\right],\ \mathbf{\Phi}^p = \mathrm{Diag}\left[\mathbf{\Phi}^{(1)}, \mathbf{\Phi}^{(2)}, ..., \mathbf{\Phi}^{(N_S)}\right] \tag{2}$$

Hereinafter, superscript 'p' denotes the primitive matrices, which directly encompass the variables of the independent substructures without imposing any constraints on them. The partitioned substructures are then reconnected by the virtual work principle and geometric compatibility, and the eigenequation of the assembled global structure is written as [28].

$$\begin{bmatrix} \mathbf{\Lambda}^p - \bar{\lambda}\mathbf{I} & -\mathbf{\Gamma} \\ -\mathbf{\Gamma}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{z} \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{3}$$

where $\mathbf{\Gamma} = \left[\mathbf{C}\mathbf{\Phi}^p\right]^T$, τ represents the internal connecting forces between the adjacent substructures, $\bar{\lambda}$ is the eigenvalue of the global structure, and $\mathbf{z}$ is regarded as the mode participation factor that indicates the contribution of the substructural eigenmodes to the eigenmodes of the global structure.

After solving the reduced eigenequation, the eigenvectors of the global structure $\mathbf{\Phi}$ can be recovered by $\bar{\mathbf{\Phi}} = \mathbf{\Phi}^p \{\mathbf{z}\}$ and by removing the identical elements of $\bar{\mathbf{\Phi}}$ at the interfaces. $\mathbf{C}$ is the connection matrix, which constraints the interface DOFs of adjacent substructures to move jointly [28]. Each row of matrix $\mathbf{C}$ contains two non-zero elements, which are 1 and -1 for a rigid connection. Kron's substructuring method considers the connection condition by the matrix $\mathbf{C}$, and takes a concise form.

In Eq. (2), the primitive matrices $\left[\mathbf{\Lambda}^p\right]$ and $\left[\mathbf{\Phi}^p\right]$ are assembled from the complete eigensolutions of all substructures. It is time-consuming to calculate the complete eigensolutions of all substructures for a large-scale structure. Here the complete modes of each substructure are partitioned into a 'master' part and a 'slave' part [26]. The first few lower eigenmodes in each substructure are retained as the master modes, and the residual higher eigenmodes are discarded as the slave modes which will be compensated for with the first order residual flexibility. For example, the eigenmodes of the jth substructure (j=1, 2, …, N_S) are partitioned into $m^{(j)}$ master eigenpairs (represented by subscript 'm') and $s^{(j)}$ slave eigenpairs (represented by subscript 's') as

$$\mathbf{\Lambda}_m^{(j)} = \mathrm{Diag}\left[\lambda_1^{(j)}, \lambda_2^{(j)}, ..., \lambda_{m^{(j)}}^{(j)}\right], \quad \mathbf{\Phi}_m^{(j)} = \left[\phi_1^{(j)}, \phi_2^{(j)}, ..., \phi_{m^{(j)}}^{(j)}\right],$$

$$\mathbf{\Lambda}_s^{(j)} = \mathrm{Diag}\left[\lambda_{m^{(j)}+1}^{(j)}, \lambda_{m^{(j)}+2}^{(j)}, ..., \lambda_{m^{(j)}+s^{(j)}}^{(j)}\right], \quad \mathbf{\Phi}_s^{(j)} = \left[\phi_{m^{(j)}+1}^{(j)}, \phi_{m^{(j)}+2}^{(j)}, ..., \phi_{m^{(j)}+s^{(j)}}^{(j)}\right] \tag{4}$$

Assembling the master eigenpairs and the slave eigenpairs respectively, we have

$$\mathbf{\Lambda}_m^p = \mathrm{Diag}\left[\mathbf{\Lambda}_m^{(1)}, \mathbf{\Lambda}_m^{(2)}, ..., \mathbf{\Lambda}_m^{(N_S)}\right], \quad \mathbf{\Phi}_m^p = \mathrm{Diag}\left[\mathbf{\Phi}_m^{(1)}, \mathbf{\Phi}_m^{(2)}, ..., \mathbf{\Phi}_m^{(N_S)}\right],$$

$$\mathbf{\Lambda}_s^p = \mathrm{Diag}\left[\mathbf{\Lambda}_s^{(1)}, \mathbf{\Lambda}_s^{(2)}, ..., \mathbf{\Lambda}_s^{(N_S)}\right], \quad \mathbf{\Phi}_s^p = \mathrm{Diag}\left[\mathbf{\Phi}_s^{(1)}, \mathbf{\Phi}_s^{(2)}, ..., \mathbf{\Phi}_s^{(N_S)}\right],$$

$$\mathbf{\Gamma}_m = \left[\mathbf{C}\mathbf{\Phi}_m^p\right]^T, \ \mathbf{\Gamma}_s = \left[\mathbf{C}\mathbf{\Phi}_s^p\right]^T,$$

$$m^p = \sum_{j=1}^{N_S} m^{(j)}, \ s^p = \sum_{j=1}^{N_S} s^{(j)}, \ m^{(j)} + s^{(j)} = n^{(j)}, \ \left(j = 1, 2, ..., N_S\right) \tag{5}$$

Partitioning Eq. (3) according to the master and slave modes, it can be expanded to

$$\begin{bmatrix} \boldsymbol{\Lambda}_m^p - \bar{\lambda}\mathbf{I} & \mathbf{0} & -\boldsymbol{\Gamma}_m \\ \mathbf{0} & \boldsymbol{\Lambda}_s^p - \bar{\lambda}\mathbf{I} & -\boldsymbol{\Gamma}_s \\ -\boldsymbol{\Gamma}_m^T & -\boldsymbol{\Gamma}_s^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{z}_m \\ \mathbf{z}_s \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{6}$$

The second line of Eq. (6) gives

$$\mathbf{z}_s = \left(\boldsymbol{\Lambda}_s^p - \bar{\lambda}\mathbf{I}\right)^{-1} \boldsymbol{\Gamma}_s \tau \tag{7}$$

Substituting Eq. (7) into Eq. (6) leads to

$$\begin{bmatrix} \boldsymbol{\Lambda}_m^p - \bar{\lambda}\mathbf{I} & -\boldsymbol{\Gamma}_m \\ -\boldsymbol{\Gamma}_m^T & -\boldsymbol{\Gamma}_s^T \left(\boldsymbol{\Lambda}_s^p - \bar{\lambda}\mathbf{I}\right)^{-1} \boldsymbol{\Gamma}_s \end{bmatrix} \begin{Bmatrix} \mathbf{z}_m \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{8}$$

In general, the lower eigenmodes of the global structure are usually required, and the required eigenvalues $\bar{\lambda}$ is far less than the slave eigenvalues ($\boldsymbol{\Lambda}_s^p$) when the master modes are chosen properly [26]. In consequence, Eq. (8) is approximated as

$$\begin{bmatrix} \boldsymbol{\Lambda}_m^p - \bar{\lambda}\mathbf{I} & -\boldsymbol{\Gamma}_m \\ -\boldsymbol{\Gamma}_m^T & -\boldsymbol{\Gamma}_s^T \left(\boldsymbol{\Lambda}_s^p\right)^{-1} \boldsymbol{\Gamma}_s \end{bmatrix} \begin{Bmatrix} \mathbf{z}_m \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{9}$$

Representing τ with $\mathbf{z}_m$ from the second line of Eq. (9) and substituting it into the first line of Eq. (9), the eigenequation is simplified to

$$\boldsymbol{\Psi}\{\mathbf{z}_m\} = \bar{\lambda}\{\mathbf{z}_m\} \tag{10}$$

where $\boldsymbol{\Psi}$ is regarded as the equivalent stiffness matrix and is given by

$$\boldsymbol{\Psi} = \boldsymbol{\Lambda}_m^p + \boldsymbol{\Gamma}_m \varsigma^{-1} \boldsymbol{\Gamma}_m^T \quad \text{and} \quad \varsigma = \boldsymbol{\Gamma}_s^T \left(\boldsymbol{\Lambda}_s^p\right)^{-1} \boldsymbol{\Gamma}_s = \mathbf{C}\boldsymbol{\Phi}_s^p \left(\boldsymbol{\Lambda}_s^p\right)^{-1} \left[\boldsymbol{\Phi}_s^p\right]^T \mathbf{C}^T \tag{11}$$

In Eq. (11), $\boldsymbol{\Phi}_s^p \left(\boldsymbol{\Lambda}_s^p\right)^{-1} \left[\boldsymbol{\Phi}_s^p\right]^T$ is the first order residual flexibility, which is expressed by diagonally assembling the substructural stiffness matrices and master modes as follows.

$$\boldsymbol{\Phi}_s^p \left(\boldsymbol{\Lambda}_s^p\right)^{-1} \left[\boldsymbol{\Phi}_s^p\right]^T = \mathrm{Diag}\left[\left(\left(\mathbf{K}^{(1)}\right)^{-1} - \boldsymbol{\Phi}_m^{(1)} \left(\boldsymbol{\Lambda}_m^{(1)}\right)^{-1} \left[\boldsymbol{\Phi}_m^{(1)}\right]^T\right), \ldots, \left(\left(\mathbf{K}^{(N_S)}\right)^{-1} - \boldsymbol{\Phi}_m^{(N_S)} \left(\boldsymbol{\Lambda}_m^{(N_S)}\right)^{-1} \left[\boldsymbol{\Phi}_m^{(N_S)}\right]^T\right)\right] \tag{12}$$

It is noted that, if a substructure is free-free after partition, then the substructural stiffness matrix is singular, and the inverse of substructural stiffness matrix (in Eq. (12)) does not exist. In this case, rigid body modes are included in Eq. (12) to calculate the residual flexibility [30]. Detailed procedures for calculating the residual flexibility of free-free substructures can be found in the Appendix.

In Eq. (10), mode participation factor $\{\mathbf{z}_m\}$ leads to the eigenvectors of the global structure via the transform of $\bar{\mathbf{\Phi}} = \mathbf{\Phi}_m^p \{\mathbf{z}_m\}$. The reduced eigenequation (Eq. (10)) takes the size of m^p, which is much smaller than that of the original (Eq. (3)).

In the present substructuring method, only the master modes (some lower modes) of the substructures are employed to form the reduced eigenequation (Eq. (10)), whereas the slave modes (higher modes) are not calculated and are compensated with the first order residual flexibility. In Eq. (8), the nonlinear item $\left(\mathbf{\Lambda}_s^p - \bar{\lambda}\mathbf{I}\right)^{-1}$ is approximately represented by the first item of its Taylor expansion $\left(\mathbf{\Lambda}_s^p\right)^{-1}$ [26], and the error introduced by this approximation is

$$Error = \left(\mathbf{\Lambda}_s^p - \bar{\lambda}\mathbf{I}\right)^{-1} - \left(\mathbf{\Lambda}_s^p\right)^{-1} = \begin{bmatrix} \dfrac{1}{\left(\mathbf{\Lambda}_s^p\right)_1 - \bar{\lambda}} - \dfrac{1}{\left(\mathbf{\Lambda}_s^p\right)_1} & & \\ & \ddots & \\ & & \dfrac{1}{\left(\mathbf{\Lambda}_s^p\right)_{s^p} - \bar{\lambda}} - \dfrac{1}{\left(\mathbf{\Lambda}_s^p\right)_{s^p}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\bar{\lambda}}{\left(\left(\mathbf{\Lambda}_s^p\right)_1 - \bar{\lambda}\right)\left(\mathbf{\Lambda}_s^p\right)_1} & & \\ & \ddots & \\ & & \dfrac{\bar{\lambda}}{\left(\left(\mathbf{\Lambda}_s^p\right)_{s^p} - \bar{\lambda}\right)\left(\mathbf{\Lambda}_s^p\right)_{s^p}} \end{bmatrix} = \mathrm{Diag}\left(\dfrac{\bar{\lambda}}{\left(\left(\mathbf{\Lambda}_s^p\right)_i - \bar{\lambda}\right)\left(\mathbf{\Lambda}_s^p\right)_i} \right) \tag{13}$$

$$\text{Relative error} = \text{Diag}\left(\frac{\overline{\lambda}}{\frac{\left(\left(\boldsymbol{\Lambda}_s^p\right)_i - \overline{\lambda}\right)\left(\boldsymbol{\Lambda}_s^p\right)_i}{1}} \right) = \text{Diag}\left(\frac{\overline{\lambda}}{\left(\boldsymbol{\Lambda}_s^p\right)_i} \right) \quad (i=1, 2, \ldots, s^p) \tag{14}$$

Therefore,

$$\text{Largest relative error} = \frac{\overline{\lambda}}{\min\left(\boldsymbol{\Lambda}_s^p\right)} \tag{15}$$

It means that, the relative error of this substructuring method depends on $\frac{\overline{\lambda}}{\min\left(\boldsymbol{\Lambda}_s^p\right)}$. If the required eigenvalues of global structure ($\overline{\lambda}$) are far less than the minimum value of $\boldsymbol{\Lambda}_s^p$, the introduced error will be insignificant. That is to say, the minimum value of $\boldsymbol{\Lambda}_s^p$ controls the accuracy of the proposed substructuring method. As $\boldsymbol{\Lambda}_s^p$ includes the slave eigenvalues of the substructures, retaining more master modes in the substructures can increase $\min\left(\boldsymbol{\Lambda}_s^p\right)$, and thus improves the computation accuracy. The required number of master modes in each substructure depends on the accuracy requirement of a structure.

The division formation of the substructures influences the computational efficiency of the substructuring method. Dividing a structure into excessive or insufficient number of substructures are both undesirable. Too few substructures might introduce large-size substructures. Analysis of large-size independent substructures is a heavy work. However, excessive substructures might introduce a large transformation matrix, and accordingly cause the assembly of the substructures to the global structure time-consuming. One should trade off the number of the substructures and the size of each substructure. In Ref. 26, the authors has discussed the division of substructures and selection of master modes based on a numerical structure and a practical structure. The optimum number of

substructures is usually determined through an trial and error process. A few trials may be helpful before model updating and/or optimization process performed. As the eigensolutions and eigensensitivity are usually calculated with respect to a large number of undetermined parameters and they are iteratively calculated during optimization process, these trials will consume insignificant computational effort comparing with those consumed by model updating or optimization process. The optimum number of substructures determined by an elegant mathematical sense will be studied in future work.

3. First order eigenvalue and eigenvector derivatives

The first order eigensensitivity of the *i*th mode (*i*=1, 2, ..., *N*) with respect to a design structural parameter will be derived in this section. The design parameter is chosen to be the elemental stiffness parameter, for example the bending rigidity of an element, and is denoted as variable *r* in the *R*th substructure.

3.1 Eigenvalue derivative

For the *i*th mode, the eigenequation (Eq. (10)) can be rewritten as

$$\left[\mathbf{\Lambda}_m^p + \mathbf{\Gamma}_m \varsigma^{-1} \mathbf{\Gamma}_m^T\right]\{\mathbf{z}_i\} = \bar{\lambda}_i \{\mathbf{z}_i\} \tag{16}$$

Eq. (16) is differentiated with respect to the design parameter *r* as

$$\left[\mathbf{\Lambda}_m^p + \mathbf{\Gamma}_m \varsigma^{-1} \mathbf{\Gamma}_m^T - \bar{\lambda}_i \mathbf{I}\right]\left\{\frac{\partial \mathbf{z}_i}{\partial r}\right\} + \frac{\partial\left[\mathbf{\Lambda}_m^p + \mathbf{\Gamma}_m \varsigma^{-1} \mathbf{\Gamma}_m^T - \bar{\lambda}_i \mathbf{I}\right]}{\partial r}\{\mathbf{z}_i\} = \{\mathbf{0}\} \tag{17}$$

Pre-multiplying by $\{\mathbf{z}_i\}^T$ on both sides of Eq. (17) gives the first order derivative of eigenvalue $\bar{\lambda}_i$ with respect to design parameter *r*

$$\frac{\partial \bar{\lambda}_i}{\partial r} = \{\mathbf{z}_i\}^T \frac{\partial\left[\mathbf{\Lambda}_m^p + \mathbf{\Gamma}_m \varsigma^{-1} \mathbf{\Gamma}_m^T\right]}{\partial r}\{\mathbf{z}_i\} \tag{18}$$

where

$$\frac{\partial\left[\boldsymbol{\Lambda}_m^p+\boldsymbol{\Gamma}_m\varsigma^{-1}\boldsymbol{\Gamma}_m^T\right]}{\partial r}=\frac{\partial\boldsymbol{\Lambda}_m^p}{\partial r}+\frac{\partial\boldsymbol{\Gamma}_m}{\partial r}\varsigma^{-1}\boldsymbol{\Gamma}_m^T-\boldsymbol{\Gamma}_m\varsigma^{-1}\frac{\partial\varsigma}{\partial r}\varsigma^{-1}\boldsymbol{\Gamma}_m^T+\boldsymbol{\Gamma}_m\varsigma^{-1}\frac{\partial\boldsymbol{\Gamma}_m^T}{\partial r} \tag{19}$$

The derivative matrices $\frac{\partial\boldsymbol{\Lambda}_m^p}{\partial r}$, $\frac{\partial\boldsymbol{\Gamma}_m}{\partial r}$, and $\frac{\partial\varsigma}{\partial r}$ are formed from the eigenvalue derivatives, eigenvector derivatives, and residual flexibility derivatives of the substructures. Since the substructures are independent, these derivative matrices are calculated within the *R*th substructure solely, while those in other substructures are zeros, i.e.,

$$\frac{\partial\boldsymbol{\Lambda}_m^p}{\partial r}=\begin{bmatrix}\mathbf{0}&\cdots&\mathbf{0}&\cdots&\mathbf{0}\\ \vdots&\ddots&\vdots&\ddots&\vdots\\ \mathbf{0}&\cdots&\frac{\partial\boldsymbol{\Lambda}_m^{(R)}}{\partial r}&\cdots&\mathbf{0}\\ \vdots&\ddots&\vdots&\ddots&\vdots\\ \mathbf{0}&\cdots&\mathbf{0}&\cdots&\mathbf{0}\end{bmatrix},\quad \frac{\partial\boldsymbol{\Gamma}_m^T}{\partial r}=\mathbf{C}\frac{\partial\boldsymbol{\Phi}_m^p}{\partial r}=\mathbf{C}\begin{bmatrix}\mathbf{0}&\cdots&\mathbf{0}&\cdots&\mathbf{0}\\ \vdots&\ddots&\vdots&\ddots&\vdots\\ \mathbf{0}&\cdots&\frac{\partial\boldsymbol{\Phi}_m^{(R)}}{\partial r}&\cdots&\mathbf{0}\\ \vdots&\ddots&\vdots&\ddots&\vdots\\ \mathbf{0}&\cdots&\mathbf{0}&\cdots&\mathbf{0}\end{bmatrix}$$

$$\frac{\partial\varsigma}{\partial r}=\mathrm{C}\times\frac{\partial\left(\boldsymbol{\Phi}_s^p\left(\boldsymbol{\Lambda}_s^p\right)^{-1}\left[\boldsymbol{\Phi}_s^p\right]^T\right)}{\partial r}\times\mathrm{C}^T=\mathrm{C}\times\mathrm{Diag}\begin{bmatrix}\mathbf{0}&\cdots&\mathbf{0}&\cdots&\mathbf{0}\\ \vdots&\ddots&\vdots&\ddots&\vdots\\ \mathbf{0}&\cdots&\frac{\partial\left(\boldsymbol{\Phi}_s^{(R)}\left(\boldsymbol{\Lambda}_s^{(R)}\right)^{-1}\left[\boldsymbol{\Phi}_s^{(R)}\right]^T\right)}{\partial r}&\cdots&\mathbf{0}\\ \vdots&\ddots&\vdots&\ddots&\vdots\\ \mathbf{0}&\cdots&\mathbf{0}&\cdots&\mathbf{0}\end{bmatrix}\times\mathrm{C}^T$$

$$=\mathrm{C}\times\mathrm{Diag}\begin{bmatrix}\mathbf{0}&\cdots&\mathbf{0}&\cdots&\mathbf{0}\\ \vdots&\ddots&\vdots&\ddots&\vdots\\ \mathbf{0}&\cdots&\frac{\partial\left(\left(\mathbf{K}^{(R)}\right)^{-1}-\boldsymbol{\Phi}_m^{(R)}\left(\boldsymbol{\Lambda}_m^{(R)}\right)^{-1}\left[\boldsymbol{\Phi}_m^{(R)}\right]^T\right)}{\partial r}&\cdots&\mathbf{0}\\ \vdots&\ddots&\vdots&\ddots&\vdots\\ \mathbf{0}&\cdots&\mathbf{0}&\cdots&\mathbf{0}\end{bmatrix}\times\mathrm{C}^T \tag{20}$$

Treating the *R*th substructure as an independent structure, substructural eigenvalue derivatives $\frac{\partial\boldsymbol{\Lambda}_m^{(R)}}{\partial r}$ and substructural eigenvector derivatives $\frac{\partial\boldsymbol{\Phi}_m^{(R)}}{\partial r}$ can be calculated using traditional methods, such as modal method [4] or Nelson's method [7]. As the stiffness matrix of a free-free substructure is

singular, the rigid body modes are employed to form the derivative of free-free substructural residual flexibility in the Appendix.

3.2 Eigenvector derivative

As the *i*th eigenvector of the global structure can be recovered by

$$\bar{\boldsymbol{\Phi}}_i = \boldsymbol{\Phi}_m^p \{\mathbf{z}_i\} \tag{21}$$

the eigenvector derivative of the *i*th mode with respect to parameter *r* can be differentiated as

$$\frac{\partial \bar{\boldsymbol{\Phi}}_i}{\partial r} = \frac{\partial \boldsymbol{\Phi}_m^p}{\partial r}\{\mathbf{z}_i\} + \boldsymbol{\Phi}_m^p \left\{\frac{\partial \mathbf{z}_i}{\partial r}\right\} \tag{22}$$

In Eq. (22), the eigenvector derivatives of the global structure can be regarded as the superposition of substructural eigenvectors $\boldsymbol{\Phi}_m^p$ and substructural eigenvector derivatives $\frac{\partial \boldsymbol{\Phi}_m^p}{\partial r}$ of the independent substructures, and $\left\{\frac{\partial \mathbf{z}_i}{\partial r}\right\}$ and $\mathbf{z}$ act as the weights. Once $\left\{\frac{\partial \mathbf{z}_i}{\partial r}\right\}$ is available, the eigenvector derivative of the *i*th mode of the global structure can be obtained.

$\left\{\frac{\partial \mathbf{z}_i}{\partial r}\right\}$ is separated into the sum of a particular part and a homogeneous part as

$$\left\{\frac{\partial \mathbf{z}_i}{\partial r}\right\} = \{\mathbf{v}_i\} + c_i \{\mathbf{z}_i\} \tag{23}$$

where c_i is a participation factor. Substituting Eq. (23) into Eq. (17) leads to

$$\left[\boldsymbol{\Lambda}_m^p + \boldsymbol{\Gamma}_m \varsigma^{-1} \boldsymbol{\Gamma}_m^T - \bar{\lambda}_i \mathbf{I}\right]\left(\{\mathbf{v}_i\} + c_i \{\mathbf{z}_i\}\right) = -\frac{\partial\left[\boldsymbol{\Lambda}_m^p + \boldsymbol{\Gamma}_m \varsigma^{-1} \boldsymbol{\Gamma}_m^T - \bar{\lambda}_i \mathbf{I}\right]}{\partial r}\{\mathbf{z}_i\} \tag{24}$$

Given that $\left[\boldsymbol{\Lambda}_m^p + \boldsymbol{\Gamma}_m \varsigma^{-1} \boldsymbol{\Gamma}_m^T - \bar{\lambda}_i \mathbf{I}\right]\{\mathbf{z}_i\} = \{\mathbf{0}\}$, Eq. (24) can be simplified to

$$\left[\boldsymbol{\Psi} - \bar{\lambda}_i \mathbf{I}\right]\{\mathbf{v}_i\} = \{\mathbf{Y}_i\} \tag{25}$$

where

$$\{\mathbf{Y}_i\} = -\frac{\partial\left[\boldsymbol{\Lambda}_m^p + \boldsymbol{\Gamma}_m \varsigma^{-1} \boldsymbol{\Gamma}_m^T - \bar{\lambda}_i \mathbf{I}\right]}{\partial r}\{\mathbf{z}_i\}$$

In consequence, the vector $\{\mathbf{v}_i\}$ can be solved from Eq. (25).

The solution of c_i requires the orthogonal condition of eigenvector $\{\mathbf{z}_i\}^T\{\mathbf{z}_i\} = 1$, which is differentiated with respect to r as

$$\frac{\partial\{\mathbf{z}_i\}^T}{\partial r}\{\mathbf{z}_i\} + \{\mathbf{z}_i\}^T\frac{\partial\{\mathbf{z}_i\}}{\partial r} = 0 \tag{26}$$

Substituting Eq. (23) into Eq. (26), participation factor c_i is thus obtained as

$$c_i = -\frac{1}{2}\left(\{\mathbf{v}_i\}^T\{\mathbf{z}_i\} + \{\mathbf{z}_i\}^T\{\mathbf{v}_i\}\right) \tag{27}$$

Given the vector $\{\mathbf{v}_i\}$ and participation factor c_i, $\left\{\frac{\partial \mathbf{z}_i}{\partial r}\right\}$ can be formed from Eq. (23). Afterwards, the global eigenvector derivatives can be recovered from Eq. (22).

The eigenvector derivatives of the global structure is calculated by analyzing the Rth substructure and a reduced eigenequation. It is similar to the calculation of the eigenvalue derivatives. Using the proposed substructuring method, the calculation of first order eigenvalue and eigenvector derivatives of the global structure are equivalent to analyzing an independent substructure and a reduced eigenequation.

4. Second order eigenvalue and eigenvector derivatives

In this section, the second order eigenvalue and eigenvector derivatives are formulated using the substructuring method.

4.1 Eigenvalue derivative

Eq. (10) is differentiated with respect to two design parameters r_j and r_k as

$$\frac{\partial^2\left(\mathbf{\Psi}-\bar{\lambda}_i\mathbf{I}\right)}{\partial r_j\partial r_k}\{\mathbf{z}_i\}+2\frac{\partial\left(\mathbf{\Psi}-\bar{\lambda}_i\mathbf{I}\right)}{\partial r_j}\frac{\partial\{\mathbf{z}_i\}}{\partial r_k}+\left(\mathbf{\Psi}-\bar{\lambda}_i\mathbf{I}\right)\frac{\partial^2\{\mathbf{z}_i\}}{\partial r_j\partial r_k}=\mathbf{0} \tag{28}$$

Pre-multiplying by $\left\{\mathbf{z}_i^T\right\}$ on both sides of Eq. (28) gives the second order eigenvalue derivative

$$\frac{\partial^2\bar{\lambda}_i}{\partial r_j\partial r_k}=\left\{\mathbf{z}_i^T\right\}\frac{\partial^2\mathbf{\Psi}}{\partial r_j\partial r_k}\{\mathbf{z}_i\}+\left\{\mathbf{z}_i^T\right\}\frac{\partial\left(\mathbf{\Psi}-\bar{\lambda}_i\mathbf{I}\right)}{\partial r_j}\frac{\partial\{\mathbf{z}_i\}}{\partial r_k}+\left\{\mathbf{z}_i^T\right\}\frac{\partial\left(\mathbf{\Psi}-\bar{\lambda}_i\mathbf{I}\right)}{\partial r_k}\frac{\partial\{\mathbf{z}_i\}}{\partial r_j} \tag{29}$$

In Eq. (29), the first order derivatives (such as $\frac{\partial\left(\mathbf{\Psi}-\bar{\lambda}_i\mathbf{I}\right)}{\partial r_j}$ and $\frac{\partial\{\mathbf{z}_i\}}{\partial r_k}$) has been calculated in Section 3, and can be re-used here directly. The second order derivative $\frac{\partial^2\mathbf{\Psi}}{\partial r_j\partial r_k}$ is contributed by the second order eigenvalue derivatives, eigenvector derivatives and residual flexibility derivatives of the independent substructures as

$$\frac{\partial^2\mathbf{\Psi}}{\partial r_j\partial r_k}=\frac{\partial^2\left[\mathbf{\Lambda}_m^p+\mathbf{\Gamma}_m\varsigma^{-1}\mathbf{\Gamma}_m^T\right]}{\partial r_j\partial r_k}=\frac{\partial^2\mathbf{\Lambda}_m^p}{\partial r_j\partial r_k}+\frac{\partial^2\left(\mathbf{\Gamma}_m\varsigma^{-1}\mathbf{\Gamma}_m^T\right)}{\partial r_j\partial r_k} \tag{30}$$

where

$$\frac{\partial^2\left(\mathbf{\Gamma}_m\varsigma^{-1}\mathbf{\Gamma}_m^T\right)}{\partial r_j\partial r_k}=\left[\mathbf{\Gamma}_m\frac{\partial^2\left[\varsigma^{-1}\right]}{\partial r_j\partial r_k}\mathbf{\Gamma}_m^T+2\mathbf{\Gamma}_m\frac{\partial\left[\varsigma^{-1}\right]}{\partial r_j}\frac{\partial\mathbf{\Gamma}_m^T}{\partial r_k}\right]+2\left[\frac{\mathbf{\Gamma}_m}{\partial r_k}\varsigma^{-1}\frac{\partial\mathbf{\Gamma}_m^T}{\partial r_j}+\mathbf{\Gamma}_m\frac{\partial\left[\varsigma^{-1}\right]}{\partial r_k}\frac{\partial\mathbf{\Gamma}_m^T}{\partial r_j}+\mathbf{\Gamma}_m\varsigma^{-1}\frac{\partial^2\mathbf{\Gamma}_m^T}{\partial r_j\partial r_k}\right] \tag{31}$$

and

$$\frac{\partial^2\left[\varsigma^{-1}\right]}{\partial r_j\partial r_k}=-\varsigma^{-2}\frac{\partial^2\varsigma}{\partial r_j\partial r_k}+2\varsigma^{-3}\frac{\partial\varsigma}{\partial r_j}\frac{\partial\varsigma}{\partial r_k}=-\varsigma^{-2}\mathbf{C}\frac{\partial\left(\left(\mathbf{K}^p\right)^{-1}-\mathbf{\Phi}_m^p\left(\mathbf{\Lambda}_m^p\right)^{-1}\left[\mathbf{\Phi}_m^p\right]^T\right)}{\partial r_j\partial r_k}\mathbf{C}^T+2\varsigma^{-3}\frac{\partial\varsigma}{\partial r_j}\frac{\partial\varsigma}{\partial r_k} \tag{32}$$

Eqs. (30)-(32) show that, the derivative matrix $\frac{\partial^2\mathbf{\Psi}}{\partial r_j\partial r_k}$ is contributed by the second order derivatives of substructural master modes ($\frac{\partial^2\mathbf{\Lambda}_m^p}{\partial r_j\partial r_k}$, $\frac{\partial^2\mathbf{\Gamma}_m^T}{\partial r_j\partial r_k}$) and the multiplication of the first order derivative of

substructural master modes, for example, the multiplication of $\frac{\partial\left[\varsigma^{-1}\right]}{\partial r_j}$ and $\frac{\partial \boldsymbol{\Gamma}_m^T}{\partial r_k}$, and the multiplication involved $\frac{\partial \boldsymbol{\Gamma}_m^T}{\partial r_j}$ and $\frac{\partial \boldsymbol{\Gamma}_m^T}{\partial r_k}$.

If the design parameters r_j and r_k are located in the same substructure (for example, the Rth substructure), the second order derivatives of substructural eigenmodes are required for the Rth substructure solely, while those corresponding to the other substructures are zero, i.e.,

$$\frac{\partial^2 \boldsymbol{\Lambda}_m^p}{\partial r_j \partial r_k} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \dfrac{\partial^2 \boldsymbol{\Lambda}_m^{(R)}}{\partial r_j \partial r_k} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, \quad \frac{\partial^2 \boldsymbol{\Gamma}_m^T}{\partial r_j \partial r_k} = \mathbf{C}\frac{\partial^2 \boldsymbol{\Phi}_m^p}{\partial r_j \partial r_k} = \mathbf{C}\begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \dfrac{\partial^2 \boldsymbol{\Phi}_m^{(R)}}{\partial r_j \partial r_k} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix},$$

$$\frac{\partial^2 \left(\boldsymbol{\Phi}_s^p \left(\boldsymbol{\Lambda}_s^p \right)^{-1} \left[\boldsymbol{\Phi}_s^p \right]^T \right)}{\partial r_j \partial r_k} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \dfrac{\partial^2 \left(\boldsymbol{\Phi}_s^{(R)} \left(\boldsymbol{\Lambda}_s^{(R)} \right)^{-1} \left[\boldsymbol{\Phi}_s^{(R)} \right]^T \right)}{\partial r_j \partial r_k} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \tag{33}$$

In case that r_j and r_k are located in different substructures, the second order derivative matrices $\frac{\partial^2 \boldsymbol{\Lambda}_m^p}{\partial r_j \partial r_k}$, $\frac{\partial^2 \boldsymbol{\Phi}_m^p}{\partial r_j \partial r_k}$ and $\frac{\partial^2 \left(\boldsymbol{\Phi}_s^p \left(\boldsymbol{\Lambda}_s^p \right)^{-1} \left[\boldsymbol{\Phi}_s^p \right]^T \right)}{\partial r_j \partial r_k}$ are zero-matrix.

The primitive matrices of the substructural first order derivatives (for example, $\frac{\partial \varsigma}{\partial r_j}$ and $\frac{\partial \boldsymbol{\Gamma}_m^T}{\partial r_k}$) are diagonal assembly of the derivative matrices of the independent substructures as Eq.(20). They are non-zeros only in the sub-block corresponding to the substructure which includes the design

parameters r_j or r_k. It means that, if r_j and r_k are located in different substructures, the multiplication of the first order derivative matrices are zero-matrix as well.

In consequence, if the design parameters r_j and r_k are located in different substructures, all items in $\frac{\partial^2 \boldsymbol{\Psi}}{\partial r_j \partial r_k}$ are zeros, i.e., $\frac{\partial^2 \boldsymbol{\Psi}}{\partial r_j \partial r_k} = \frac{\partial^2 \boldsymbol{\Lambda}_m^p}{\partial r_j \partial r_k} + \frac{\partial^2 \left[\boldsymbol{\Gamma}_m \varsigma^{-1} \boldsymbol{\Gamma}_m^T \right]}{\partial r_j \partial r_k} = \mathbf{0}$. In this case, the second order derivative of the global eigenvalue is determined by the first order derivative matrices of Eq. (29) as

$$\frac{\partial^2 \bar{\lambda}_i}{\partial r_j \partial r_k} = 2 \{\mathbf{z}_i\}^T \frac{\partial \left(\boldsymbol{\Psi} - \bar{\lambda}_i \right)}{\partial r_j} \frac{\partial \{\mathbf{z}_i\}}{\partial r_k} \tag{34}$$

Those first order derivative matrices can be re-used directly from the calculation of lower order eigensensitivity in Section 3. If the design paremeters r_j and r_k are located in the same substructure (for example the *R*th substructure). The second order eigenvalue derivatives of the global structure (Eq. (29)) are calculated from the second order derivatives of the master modes of the *R*th substructure solely. $\frac{\partial^2 \boldsymbol{\Lambda}_m^p}{\partial r_j \partial r_k}$ and $\frac{\partial \boldsymbol{\Gamma}_m^T}{\partial r_j \partial r_k}$ are calculated within the *R*th substructure only by treating the *R*th substructure as an independent structure [14].

4.2 Eigenvector derivative

Differentiating Eq. (21) with respect to the two parameters r_j and r_k, the second order eigenvector derivative of the *i*th mode is acquired as

$$\frac{\partial^2 \bar{\boldsymbol{\Phi}}_i}{\partial r_j \partial r_k} = \frac{\partial^2 \boldsymbol{\Phi}_m^p}{\partial r_j \partial r_k} \{\mathbf{z}_i\} + \frac{\partial \boldsymbol{\Phi}_m^p}{\partial r_j} \frac{\partial \{\mathbf{z}_i\}}{\partial r_k} + \frac{\partial \boldsymbol{\Phi}_m^p}{\partial r_k} \frac{\partial \{\mathbf{z}_i\}}{\partial r_j} + \boldsymbol{\Phi}_m^p \frac{\partial^2 \{\mathbf{z}_i\}}{\partial r_j \partial r_k} \tag{35}$$

All items are available based on the previous analysis, except $\frac{\partial^2 \mathbf{z}_i}{\partial r_j \partial r_k}$. As before, $\frac{\partial^2 \{\mathbf{z}_i\}}{\partial r_j \partial r_k}$ is expressed as the sum of a particular part and a homogeneous part as

$$\frac{\partial^2 \{\mathbf{z}_i\}}{\partial r_j \partial r_k} = \left\{\mathbf{v}_{i(j,k)}\right\} + c_{i(j,k)} \{\mathbf{z}_i\} \tag{36}$$

The double-differentiated eigenequation (Eq. (28)) is rewritten as

$$\left(\mathbf{\Psi} - \bar{\lambda}_i \mathbf{I}\right) \frac{\partial^2 \{\mathbf{z}_i\}}{\partial r_j \partial r_k} = \left\{\mathbf{Y}_{i(j,k)}\right\} \tag{37}$$

where

$$\left\{\mathbf{Y}_{i(j,k)}\right\} = -\frac{\partial^2 \left(\mathbf{\Psi} - \bar{\lambda}_i \mathbf{I}\right)}{\partial r_j \partial r_k} \{\mathbf{z}_i\} - 2 \frac{\partial \left(\mathbf{\Psi} - \bar{\lambda}_i \mathbf{I}\right)}{\partial r_j} \frac{\partial \{\mathbf{z}_i\}}{\partial r_k} \tag{38}$$

$\left\{\mathbf{Y}_{i(j,k)}\right\}$ can be obtained directly using the interim results when calculating the eigenvalue derivatives.

Substituting Eq. (36) into Eq. (37) gives

$$\left(\mathbf{\Psi} - \bar{\lambda}_i \mathbf{I}\right) \left\{\mathbf{v}_{i(j,k)}\right\} = \left\{\mathbf{Y}_{i(j,k)}\right\} \tag{39}$$

from which $\left\{\mathbf{v}_{i(j,k)}\right\}$ can be calculated.

Differentiating the orthogonality equation $\{\mathbf{z}_i\}^T \{\mathbf{z}_i\} = 1$ with respect to r_j and r_k, we have

$$2\left(\frac{\partial \{\mathbf{z}_i\}^T}{\partial r_k} \frac{\partial \{\mathbf{z}_i\}}{\partial r_j} + \{\mathbf{z}_i\}^T \frac{\partial^2 \{\mathbf{z}_i\}}{\partial r_j \partial r_k} \right) = 0 \tag{40}$$

Substituting Eq. (36) into Eq. (40) leads to

$$2\left(\frac{\partial \{\mathbf{z}_i\}^T}{\partial r_k} \frac{\partial \{\mathbf{z}_i\}}{\partial r_j} + \{\mathbf{z}_i\}^T \left(\left\{\mathbf{v}_{i(j,k)}\right\} + c_{i(j,k)} \{\mathbf{z}_i\} \right) \right) = 0 \tag{41}$$

The participation factor $c_{i(j,k)}$ is obtained as

$$c_{i(j,k)} = -\frac{\partial \{\mathbf{z}_i\}^T}{\partial r_k}\frac{\partial \{\mathbf{z}_i\}}{\partial r_j} - \{\mathbf{z}_i\}^T \{\mathbf{v}_{i(j,k)}\} \tag{42}$$

Given $\{\mathbf{v}_{i(j,k)}\}$ and $c_{i(j,k)}$, the second order eigenvector derivative can be recovered from Eq. (35) and Eq. (36). The second order eigenvector derivative is obtained by analyzing the master modes of the *R*th substructure and the reduced eigenequation (Eq. (28)).

5. High order derivatives

Due to the symmetric property and simple form of the reduced eigenequation (Eq. (10)), it is easy to derive the high order derivatives of the eigensolutions by directly re-differentiating the reduced eigenequation.

5.1 Eigenvalue derivative

Eq. (16) is differentiated with respect to *k* design parameters (r_1 ... r_k) as

$$\frac{\partial^k \left(\mathbf{\Psi} - \bar{\lambda}_i \mathbf{I}\right)}{\partial r_1 \partial r_2 \cdots \partial r_k}\{\mathbf{z}_i\} + \frac{\partial^{k-1}\left(\mathbf{\Psi} - \bar{\lambda}_i \mathbf{I}\right)}{\partial r_1 \partial r_2 \cdots \partial r_{k-1}}\frac{\partial \{\mathbf{z}_i\}}{\partial r_k} + \cdots + \frac{\partial\left(\mathbf{\Psi} - \bar{\lambda}_i \mathbf{I}\right)}{\partial r_1}\frac{\partial^{k-1}\{\mathbf{z}_i\}}{\partial r_2 \partial r_3 \cdots \partial r_k} + \left(\mathbf{\Psi} - \bar{\lambda}_i \mathbf{I}\right)\frac{\partial^k \{\mathbf{z}_i\}}{\partial r_1 \partial r_2 \cdots \partial r_k} = \mathbf{0} \tag{43}$$

Premultiplying by $\{\mathbf{z}_i^T\}$ on both sides of Eq. (43), the *k*th order eigenvalue derivative is obtained as

$$\frac{\partial^k \bar{\lambda}_i}{\partial r_1 \partial r_2 \cdots \partial r_k} = \{\mathbf{z}_i^T\}\frac{\partial^k \mathbf{\Psi}}{\partial r_1 \partial r_2 \cdots \partial r_k}\{\mathbf{z}_i\} + \{\mathbf{z}_i^T\}\frac{\partial^{k-1}\left(\mathbf{\Psi} - \bar{\lambda}_i \mathbf{I}\right)}{\partial r_1 \partial r_2 \cdots \partial r_{k-1}}\frac{\partial \{\mathbf{z}_i\}}{\partial r_k} + \cdots + \{\mathbf{z}_i^T\}\frac{\partial\left(\mathbf{\Psi} - \bar{\lambda}_i \mathbf{I}\right)}{\partial r_1}\frac{\partial^{k-1}\{\mathbf{z}_i\}}{\partial r_2 \partial r_3 \cdots \partial r_k} \tag{44}$$

In Eq. (44), the *k*th order eigenvalue derivative comprises two parts: the *k*th order derivative $\frac{\partial^k \mathbf{\Psi}}{\partial r_1 \partial r_2 \cdots \partial r_k}$, and the multiplication of the derivatives of order (*k*-1) and lower which can be re-used directly from the calculation of lower order eigensensitivity.

Similar to the second order derivative, the *k*th order derivative $\frac{\partial^k \mathbf{\Psi}}{\partial r_1 \partial r_2 \cdots \partial r_k}$ is non-zero only if the *k* parameters (r_1, r_2 ,..., r_k) are located in the same substructure (for example, the *R*th substructure). In

this case, the *k*th order derivatives of substructural master modes of only the *R*th substructure are calculated to assemble the *k*th order eigensensitivity of the global structure. Otherwise, if the *k* parameters (r_1, r_2 ,..., r_k) are located in different substructures, the *k*th order eigensensitivity is determined by the derivatives of order (*k*-1) and lower of substructural master modes. It is seen that, if there are *m* design parameters among (r_1, r_2 ,..., r_k) located in the *M*th substructure, the substructural derivatives of order *m* are calculated within the *M*th substructure to form the *k*th order eigensensitivity of global structure.

5.2 Eigenvector derivative

Differentiating Eq. (21) with respect to *k* parameters (r_1, r_2 ,..., r_k), the *k*th order eigenvector derivative of the *i*th mode is acquired as

$$\frac{\partial^k \bar{\boldsymbol{\Phi}}_i}{\partial r_1 \partial r_2 \cdots \partial r_k} = \frac{\partial^k \boldsymbol{\Phi}_m^p}{\partial r_1 \partial r_2 \cdots \partial r_k}\{\mathbf{z}_i\} + \frac{\partial^{k-1} \boldsymbol{\Phi}_m^p}{\partial r_1 \partial r_2 \cdots \partial r_{k-1}} \frac{\partial \{\mathbf{z}_i\}}{\partial r_k} + \cdots + \frac{\partial \boldsymbol{\Phi}_m^p}{\partial r_1} \frac{\partial^{k-1}\{\mathbf{z}_i\}}{\partial r_2 \partial r_3 \cdots \partial r_k} + \boldsymbol{\Phi}_m^p \frac{\partial^k \{\mathbf{z}_i\}}{\partial r_1 \partial r_2 \cdots \partial r_k} \quad (45)$$

The item $\dfrac{\partial^k \{\mathbf{z}_i\}}{\partial r_1 \partial r_2 \cdots \partial r_k}$ are required to achieve the *k*th order eigenvector derivative of the global structure. $\dfrac{\partial^k \{\mathbf{z}_i\}}{\partial r_1 \partial r_2 \cdots \partial r_k}$ is written as the particular part and homogeneous part as

$$\frac{\partial^k \{\mathbf{z}_i\}}{\partial r_1 \partial r_2 \cdots \partial r_k} = \left\{\mathbf{v}_{i(r_1, r_2, \ldots r_k)}\right\} + c_{i(r_1, r_2, \ldots r_k)}\{\mathbf{z}_i\} \quad (46)$$

In Eq. (46), $\left\{\mathbf{v}_{i(r_1, r_2, \ldots r_k)}\right\}$ is calculated by substituting Eq. (46) into Eq. (43) and pre-multiplying by $\left\{\mathbf{z}_i^T\right\}$ on both sides of it as

$$\left(\boldsymbol{\Psi} - \bar{\lambda}_i \mathbf{I}\right)\left\{\mathbf{v}_{i(r_1, r_2, \ldots r_k)}\right\} = \left\{\mathbf{Y}_{i(r_1, r_2, \ldots r_k)}\right\} \quad (47)$$

where

$$\left\{\mathbf{Y}_{i(r_1,r_2,\ldots r_k)}\right\} = -\left(\frac{\partial^k \left(\boldsymbol{\Psi} - \bar{\lambda}_i \mathbf{I}\right)}{\partial r_1 \partial r_2 \cdots \partial r_k} \{\mathbf{z}_i\} + \frac{\partial^{k-1} \left(\boldsymbol{\Psi} - \bar{\lambda}_i \mathbf{I}\right)}{\partial r_1 \partial r_2 \cdots \partial r_{k-1}} \frac{\partial \{\mathbf{z}_i\}}{\partial r_k} + \cdots + \frac{\partial \left(\boldsymbol{\Psi} - \bar{\lambda}_i \mathbf{I}\right)}{\partial r_1} \frac{\partial^{k-1} \{\mathbf{z}_i\}}{\partial r_2 \partial r_3 \cdots \partial r_k} \right) \tag{48}$$

$c_{i(r_1,r_2,\ldots r_k)}$ is calculated from the *k*th order derivative of the orthogonal equation $\{\mathbf{z}_i\}^T \{\mathbf{z}_i\} = 1$ with respect to *k* parameters (r_1, r_2 ,..., r_k), which gives

$$\{\mathbf{z}_i\}^T \frac{\partial^k \{\mathbf{z}_i\}}{\partial r_1 \partial r_2 \cdots \partial r_k} = d_{i(r_1,r_2,\ldots r_k)} \tag{49}$$

and $d_{i(r_1,r_2,\ldots r_k)}$ contains terms involving derivatives of order (*k*-1) and lower as well. Thus substituting Eq. (46) into Eq. (49) gives the scalar

$$c_{i(r_1,r_2,\ldots r_k)} = d_{i(r_1,r_2,\ldots r_k)} - \{\mathbf{z}_i\}^T \left\{\mathbf{v}_{i(r_1,r_2,\ldots r_k)}\right\} \tag{50}$$

In consequence, the *k*th order eigenvector derivative of the global structure are formulated from $\left\{\mathbf{v}_{i(r_1,r_2,\ldots r_k)}\right\}$ and $c_{i(r_1,r_2,\ldots r_k)}$.

It is seen that, the proposed substructuring method calculates the first, second and high order eigenvector derivatives by analyzing the independent substructures that contains the design parameters and a reduced eigenequation. As the calculation of eigenvector derivatives usually consumes dominant computation resource in the common global method, this substructuring method improves the computational efficiency significantly, which will be demonstrated by numerical examples in Sections 6 and 7.

6. Case study 1: frame structure

A three-span frame structure is employed to illustrate the effectiveness of the proposed substructuring method in calculation of eigensensitivity. The frame structure comprises 160 two-dimensional beam elements as labeled in Fig. 1(a). There are 140 nodes and 408 DOFs in total. The

material constants of the beam elements are chosen as: bending rigidity (EI) =170×10^6 Nm^2, axial rigidity (EA) = 2500×10^6 N, mass per unit length (ρA) = 110 kg/m, and Poisson's ratio = 0.3. The frame is disassembled into three substructures (N_S = 3) when it is divided at 8 nodes as shown in Fig. 1(b). After division, there are 51, 55, 42 nodes in the three substructures with the DOFs of $n^{(1)}$=153, $n^{(2)}$=165, $n^{(3)}$=114, respectively.

The bending rigidity of one element is chosen as the design parameter, which is labeled Element 154 and denoted r_{154} in Fig. 1(a). The first 30 modes of each substructure are chosen as the master modes to calculate the eigensensitivity of the first 10 modes of the global structure with respect to r_{154}. The design parameter r_{154} is located in the second substructure. The substructural eigensensitivity of the master modes of the second substructure ($\frac{\partial \mathbf{\Lambda}_m^{(2)}}{\partial r_{154}}$ and $\frac{\partial \mathbf{\Phi}_m^{(2)}}{\partial r_{154}}$) are calculated, whilst the substructural eigensensitivity of other substructures are zeros ($\frac{\partial \mathbf{\Lambda}_m^{(1)}}{\partial r_{154}}=0$, $\frac{\partial \mathbf{\Lambda}_m^{(3)}}{\partial r_{154}}=0$, $\frac{\partial \mathbf{\Phi}_m^{(1)}}{\partial r_{154}}=0$, $\frac{\partial \mathbf{\Phi}_m^{(3)}}{\partial r_{154}}=0$). Afterwards, the substructural eigensensitivity are assembled to form the eigensensitivity of the global structure according to the proposed substructuring approach.

To verify the accuracy of the proposed substructuring method in calculation of eigensensitivity, the traditional Nelson's method [7] is employed to calculate the eigensensitivity of the global structure directly, that is, without division into individual substructures. The results from the proposed substructuring method and the traditional global method are compared in Table 1. It is seen that, the relative errors of all eigenvalue derivatives are less than 2%, which is accurate enough for most practical engineering applications.

Following Modal Assurance Criterion (MAC) [31], the similarity of the eigenvector derivatives obtained with the global method and the proposed substructuring method is denoted as the

Correlation of Eigenvector Derivatives (COED), and is given by

$$\mathrm{COED}\left(\left\{\frac{\partial \phi_i}{\partial r}\right\},\left\{\frac{\partial \tilde{\phi}_i}{\partial r}\right\}\right)=\frac{\left|\left\{\frac{\partial \phi_i}{\partial r}\right\}^T\left\{\frac{\partial \tilde{\phi}_i}{\partial r}\right\}\right|^2}{\left(\left\{\frac{\partial \phi_i}{\partial r}\right\}^T\left\{\frac{\partial \phi_i}{\partial r}\right\}\right)\left(\left\{\frac{\partial \tilde{\phi}_i}{\partial r}\right\}^T\left\{\frac{\partial \tilde{\phi}_i}{\partial r}\right\}\right)} \tag{51}$$

where $\left\{\frac{\partial \phi_i}{\partial r}\right\}$ represents the eigenvector derivative obtained with the traditional global method, and $\left\{\frac{\partial \tilde{\phi}_i}{\partial r}\right\}$ that with the proposed substructuring method. In this example, the COED values for all modes are above 0.99 as shown in Table 1, which indicates that the proposed method can achieve good accuracy of eigenvector derivative calculation.

Afterwards, the second order eigenvalue and eigenvector derivatives are calculated using substructuring method. The first 30 modes are retained in each substructure, to calculate the second order eigensensitivity for the first 10 modes of the global structure. The eigensensitivity with respect to the parameter pairs of (r_{146}, r_{146}), (r_{146}, r_{148}), and (r_{146}, r_{154}) in Fig. 1(a) are calculated in Table 2. The second order eigensolution derivatives with respect to (r_{146}, r_{146}) represent the case that the two design parameters are identical. The eigensolution derivatives with respect to (r_{146}, r_{148}) gives the results with respect to two different parameters in the same substructure, while (r_{146}, r_{154}) with the case that the two parameters are located in different substructures. When the parameter pairs of (r_{146}, r_{146}) and (r_{146}, r_{148}) are design parameters, the second order derivatives of substructural master modes of Substructure 1 are calculated to form the eigensensitivity of global structure, as the two design parameters are located in the same substructure. When the parameters (r_{146}, r_{154}) are design parameters, the first order derivatives of substructural master modes of Substructure 1 and Substructure 2 are used to form the global eigensensitivity. The results from previous step in calculation of first order eigensensitivity can be re-used here directly.

As before, the second order eigenvalue and eigenvector derivatives of the first 10 modes are calculated using traditional global method for comparison [14]. The second order eigensensitivity from the substructuring method and the global method are compared in Table 2. It is seen that, the relative differences of second order eigenvalue derivatives between the proposed substructuring method and the global method are less than 3%. Following the comparison of first order eigenvector derivative, the accuracy of the second order eigenvector derivative is denoted by COED as well, and is given by

$$\mathrm{COED}\left(\left\{\frac{\partial^2\phi_i}{\partial r_j\partial r_k}\right\},\left\{\frac{\partial^2\tilde{\phi}_i}{\partial r_j\partial r_k}\right\}\right)=\frac{\left|\left\{\frac{\partial^2\phi_i}{\partial r_j\partial r_k}\right\}^T\left\{\frac{\partial^2\tilde{\phi}_i}{\partial r_j\partial r_k}\right\}\right|^2}{\left(\left\{\frac{\partial^2\phi_i}{\partial r_j\partial r_k}\right\}^T\left\{\frac{\partial^2\phi_i}{\partial r_j\partial r_k}\right\}\right)\left(\left\{\frac{\partial^2\tilde{\phi}_i}{\partial r_j\partial r_k}\right\}^T\left\{\frac{\partial^2\tilde{\phi}_i}{\partial r_j\partial r_k}\right\}\right)} \tag{52}$$

where $\left\{\frac{\partial^2\phi_i}{\partial r_j\partial r_k}\right\}$ represents the second order eigenvector derivative from the global method, and $\left\{\frac{\partial^2\tilde{\phi}_i}{\partial r_j\partial r_k}\right\}$ that with the substructuring method. The COED values of most modes are above 0.98 in Table 3, which accuracy of the second order eigenvector derivatives are sufficient for practical application. The proposed substructuring method achieves good accuracy in calculation of first order and second order eigensensitivity.

7. Case study 2: practical bridge

To illustrate the computational efficiency of the proposed substructuring method in real structures, a practical bridge, the Balla Balla River Bridge in Western Australia is employed here [32]. An FE model based on design drawings was established. The FE model of this bridge has 907 elements, 947 nodes each has 6 DOFs, and 5420 DOFs in total, as shown in Fig. 2.

In this example, the second order eigensensitivity of the global structure are calculated for comparison, whilst the calculation of first order eigensensitivity is not presented here for clearance. The design parameters are chosen as the bending rigidity of slab elements denoted in Fig. 2. The second order eigenvalue and eigenvector derivatives with respect to parameter pairs of (r_1, r_1), (r_1, r_2), (r_1, r_3) are calculated using both the traditional global method and the proposed substructuring method for comparison.

First, the second order eigensensitivity are calculated by the proposed substructuring method. The global structure is divided into 11 substructures as Fig. 2. Parameters r_1 and r_2 are located in the first substructure, and r_3 in the fourth substructure. The first 50 substructural modes are retained as master, to calculate the eigensensitivity of first 20 modes of the global structure. The eigenvalue and eigenvector derivatives with respect to parameter pairs of (r_1, r_1), (r_1, r_2), (r_1, r_3) are calculated and listed in Table 3.

Afterwards, the second order eigenvalue and eigenvector derivatives of the first 20 modes are calculated using the traditional global method [14]. The results from the substructuring and the traditional global method are compared in Table 3. Table 3 demonstrates that the relative differences in the second order eigenvalue derivatives are less than 3% for most modes. The accuracy of the second order eigenvector derivative is denoted by COED in terms of Eq. (52). Table 3 reports the COED values of most modes are above 0.98, indicating a good accuracy of the second order eigenvector derivatives.

The computational efficiency of the substructuring method is evaluated in terms of the computation time in calculating the second order eigensensitivity with respect to the bending rigidity parameters

of slab elements located in the first substructure and the bending rigidity parameters of all slab elements. That is, the first elemental parameter r_j is designated to the bending rigidity of the 24 slab elements in the first substructure, and the second elemental parameter r_k is the bending rigidity of all 288 slab elements across the whole structure. Among the 288 parameters, 24 parameters are located in the first substructure, and the remainders are in the other substructures as listed in Table 4.

The computation time consumed by the substructuring method and traditional global method are compared in Table 4. The computation time of second order eigensensitivity includes the calculation of the first order eigensensitivity for both methods. The traditional global method totally takes up 13,427 seconds to calculate the second order eigensensitivity with respect to the 24 parameters r_j and the 288 parameters r_k, whereas the substructuring method takes 3,594 seconds. The computation time consumed by the proposed substructuring method is about 25% of that cost by the traditional global method. In particular, if parameters r_j and parameters r_k are both located in the first substructure, calculating the second order eigensensitivity with respect to the 24 parameters of r_j and 24 parameters of r_k takes 434.4 seconds. In the case that parameters r_j and r_k are located in different substructures, for example r_j is located in first substructure and r_k is located in the third, fifth, sixth, seventh or ninth substructure (The third, five, six, seven and nine substructures respectively contain 24 parameters r_k as well, and they have similar size with the first substructure), calculating the second order eigensensitivity costs about 280 seconds. As expected, the parameters r_j and r_k located in the same substructure takes longer time than they are in different substructures, since the former requires the item $\frac{\partial^2 \boldsymbol{\Psi}}{\partial r_j \partial r_k}$ within the first substructure which is zero in the latter.

Using the proposed substructuring method, only the substructures which contain the design parameters are analyzed to assemble the eigensensitivity of the global structure. This method is efficient in calculation of eigensensitivity of large-scale structure.

The more significant merit of the proposed method lies in the applications to model updating and damage identification. In general, model updating and damage identification need to re-calculate the eigensolutions and eigensensitivity of the entire structure when the parameters of some elements are changed. Using the substructuring method, only particular substructures need to be re-analyzed, while other substructures can be untouched.

8. Conclusions

This paper employs the substructuring method to derive the first, second and high order eigensensitivity with respect to structural parameters. The global structure is divided into manageable substructures. The derivative matrices of substructural master modes with respect to the design elemental parameters are computed for independent substructures. Afterwards, the substructural eigensensitivity matrices are assembled to recover the eigensensitivity of the global structure. As the substructural eigensensitivity are zeros for those substructures that do not include the design parameters, the eigensensitivity of global structure is determined from substructural eigensensitivity of particular substructures that contain the design parameters. As a result, the proposed substructuring method is efficient in calculating the eigensensitivity.

The first order eigensensitivity of the global structure is calculated from the first order derivative of master modes of one substructure which contain the design parameter. The second order eigensensitivity of the global structure is determined by the first order derivative of master modes of

two substructures that contain the two design parameters, if the design parameters are located in different substructures. In the case that the two design parameters are in the same substructure, the second order eigensensitivity of the global structure is recovered from the derivatives of substructural master modes of one substructure solely. In general, the *k*th order eigensensitivity of the global structure is recovered from the *k*th order derivative matrices of one substructure if the *k* design parameters are located in one substructure. Otherwise, the substructural derivative of order (*k*-1) and lower, which can be used directly from the lower order eigensensitivity calculation, are assembled for the eigensensitivity of the global structure. The first, second and high order eigenvector derivative are formed by analyzing the substructural master eigenmodes and a reduced eigenequation.

The application of the proposed substructuring method to a three-span frame structure and a highway bridge demonstrate that the substructuring method can achieve high accuracy in calculation of eigensensitivity. The computation time in calculation of eigensensitivity is far less than that of the traditional global method. The substructuring method divides the global structure into manageable substructures, and calculates the eigensensitivity of global structure by the analysis of independent substructures. It allows an easier and quicker analysis of large-scale structures which is sometimes difficult or even limited to be processed on ordinary personal computation. In addition, the substructuring method is promising to be used in iterative model updating process. Since the partitioned substructures are independent, the substructuring method allows the eigensolutions and eigensensitivity of the modified substructures to be analyzed repeatedly, whereas the unmodified substructures remain unchanged during the iterative optimization process.

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Appendix A: Residual flexibility matrix of free substructure

The stiffness matrix is singular for a free substructure due to the rigid body motion. The residual flexibility is not available from the inverse of singular stiffness matrix directly. In this section, the residual flexibility matrix and its derivative matrix are derived for a free structure by considering the rigid body modes.

The complete eigenmodes of a free substructure includes N_r rigid body modes $\mathbf{R}$ and N_d deformational modes $\mathbf{\Phi}_d$. According to the substructuring method of this paper, the eigenmodes of a substructure are divided into N_m master modes $\mathbf{\Phi}_m$ and N_s slave modes $\mathbf{\Phi}_s$. The master modes $\mathbf{\Phi}_m$ include the N_r rigid body modes $\mathbf{R}$ and the (N_m-N_r) deformational master modes $\mathbf{\Phi}_{m\text{-}r}$. The deformational modes include the deformational master modes $\mathbf{\Phi}_{m\text{-}r}$ and deformational slave modes $\mathbf{\Phi}_s$. Subscript m, s, r and d represent the master modes, slave modes, rigid body modes and deformational modes, respectively. The relation among master modes, slave modes, rigid body modes, and deformational modes is illustrated in Fig. A-1.

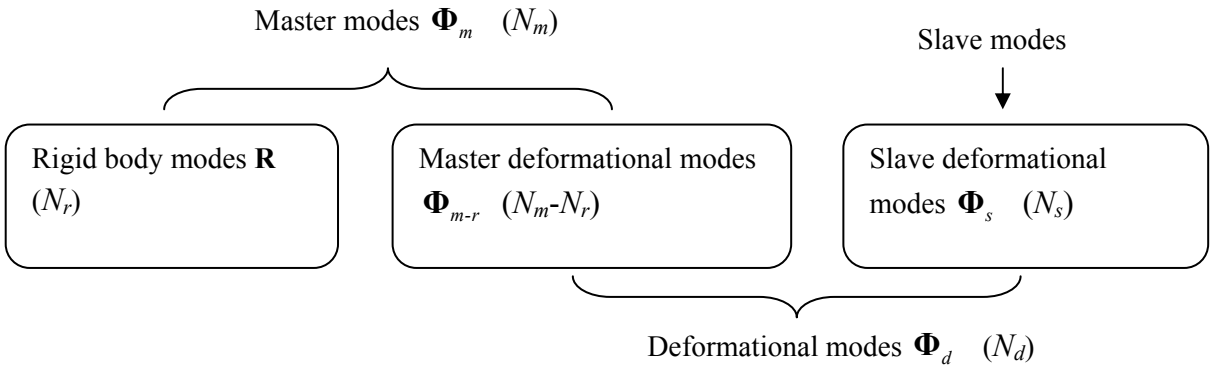


Figure A-1: Relation between different kinds of modal modes

A generalized stiffness matrix and a generalized flexibility matrix are defined as follows, which include the contribution made by both the rigid body modes and deformational modes.

$$\bar{\mathbf{K}} = \mathbf{K} + \mathbf{R}\mathbf{R}^T = \sum_{i=1}^{N_d} \lambda_i \phi_i \phi_i^T + \mathbf{R}\mathbf{R}^T \,,\; \bar{\mathbf{F}} = \mathbf{F} + \mathbf{R}\mathbf{R}^T = \sum_{i=1}^{N_d} \frac{1}{\lambda_i} \phi_i \phi_i^T + \mathbf{R}\mathbf{R}^T \qquad \text{(A-1)}$$

The eigenvectors of generalized stiffness matrix $\left(\mathbf{K} + \mathbf{R}\mathbf{R}^T\right)$ are identical to those of **K**, but only the eigenvalues of the rigid body modes are changed from 0 to 1. The generalized stiffness matrix $\bar{\mathbf{K}}$ and flexibility matrix $\bar{\mathbf{F}}$ are full-rank, and can be transformed with each other by inversion of

$$\left(\mathbf{K} + \mathbf{R}\mathbf{R}^T\right)^{-1} = \left(\sum_{i=1}^{N_d} \lambda_i \phi_i \phi_i^T + \sum_{i=1}^{N_r} 1\left(\phi_i \phi_i^T\right)\right)^{-1} = \left(\sum_{i=1}^{N_d+N_r} \lambda_i \phi_i \phi_i^T\right)^{-1} = \sum_{i=1}^{N_d+N_r} \frac{1}{\lambda_i} \phi_i \phi_i^T = \sum_{i=1}^{N_d} \frac{1}{\lambda_i} \phi_i \phi_i^T + \mathbf{R}\mathbf{R}^T = \mathbf{F} + \mathbf{R}\mathbf{R}^T \qquad \text{(A-2)}$$

$$\left(\mathbf{F} + \mathbf{R}\mathbf{R}^T\right)^{-1} = \sum_{i=1}^{N_d} \lambda_i \phi_i \phi_i^T + \mathbf{R}\mathbf{R}^T = \mathbf{K} + \mathbf{R}\mathbf{R}^T \qquad \text{(A-3)}$$

Here the generalized flexibility matrix can be expressed by the rigid body modes, master modes and slave modes as

$$\bar{\mathbf{F}} = \left(\mathbf{K} + \mathbf{R}\mathbf{R}^T\right)^{-1} = \sum_{\substack{i=1 \\ \phi \in \boldsymbol{\Phi}_{m\text{-}r}}}^{N_m - N_r} \frac{1}{\lambda_i} \phi_i \phi_i^T + \sum_{\substack{i=1 \\ \phi \in \boldsymbol{\Phi}_s}}^{N_s} \frac{1}{\lambda_i} \phi_i \phi_i^T + \mathbf{R}\mathbf{R}^T = \boldsymbol{\Phi}_{m\text{-}r} \boldsymbol{\Lambda}_{m\text{-}r}^{-1} \boldsymbol{\Phi}_{m\text{-}r}^T + \boldsymbol{\Phi}_s \boldsymbol{\Lambda}_s^{-1} \boldsymbol{\Phi}_s^T + \mathbf{R}\mathbf{R}^T \qquad \text{(A-4)}$$

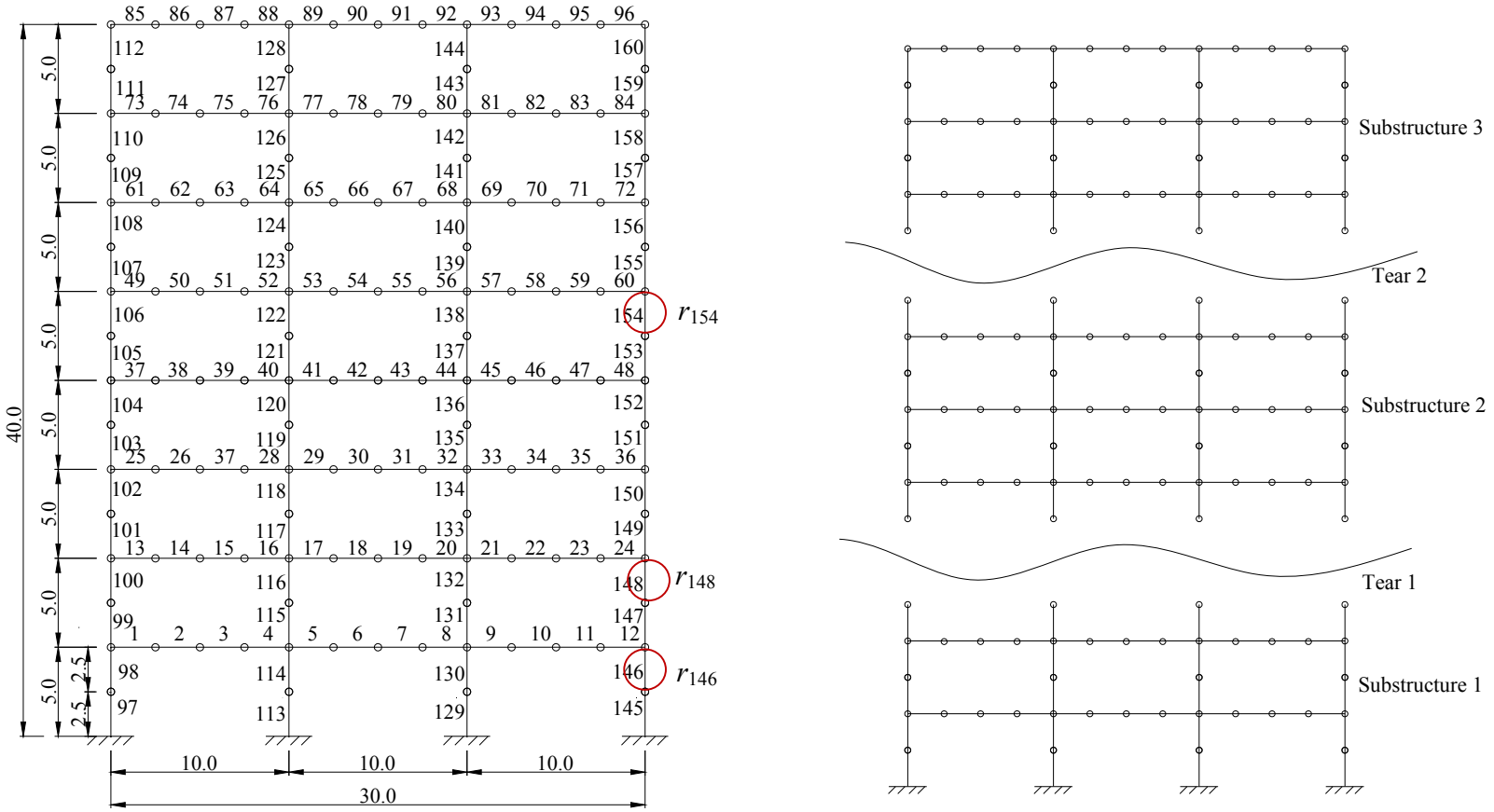
Therefore, the first-order residual flexibility matrix for the free structure can be expressed by the master modes as

$$\boldsymbol{\Phi}_s \boldsymbol{\Lambda}_s^{-1} \boldsymbol{\Phi}_s^T = \left(\mathbf{K} + \mathbf{R}\mathbf{R}^T\right)^{-1} - \boldsymbol{\Phi}_{m\text{-}r} \boldsymbol{\Lambda}_{m\text{-}r}^{-1} \boldsymbol{\Phi}_{m\text{-}r}^T - \mathbf{R}\mathbf{R}^T \qquad \text{(A-5)}$$

Differentiating the first-order residual flexibility (Eq. (A-5)) with respect to an elemental parameter r, the derivative matrix has the form of

$$\frac{\partial\left(\boldsymbol{\Phi}_s\boldsymbol{\Lambda}_s^{-1}\boldsymbol{\Phi}_s^T\right)}{\partial r}=\frac{\partial\left(\mathbf{K}+\mathbf{R}\mathbf{R}^T\right)^{-1}}{\partial r}-\frac{\partial\left(\boldsymbol{\Phi}_{m\text{-}r}\boldsymbol{\Lambda}_{m\text{-}r}^{-1}\boldsymbol{\Phi}_{m\text{-}r}^T\right)}{\partial r}$$

$$=-\left(\mathbf{K}+\mathbf{R}\mathbf{R}^T\right)^{-2}\frac{\partial\mathbf{K}}{\partial r}-\left(2\boldsymbol{\Phi}_{m\text{-}r}\boldsymbol{\Lambda}_{m\text{-}r}^{-1}\frac{\partial\left(\boldsymbol{\Phi}_{m\text{-}r}^T\right)}{\partial r}+\boldsymbol{\Phi}_{m\text{-}r}\frac{\partial\left(\boldsymbol{\Lambda}_{m\text{-}r}^{-1}\right)}{\partial r}\boldsymbol{\Phi}_{m\text{-}r}^T\right) \tag{A-6}$$



(a) global structure

(b) partitioned substructures

Fig. 1. Frame structure and the selected design parameters (unit: m)

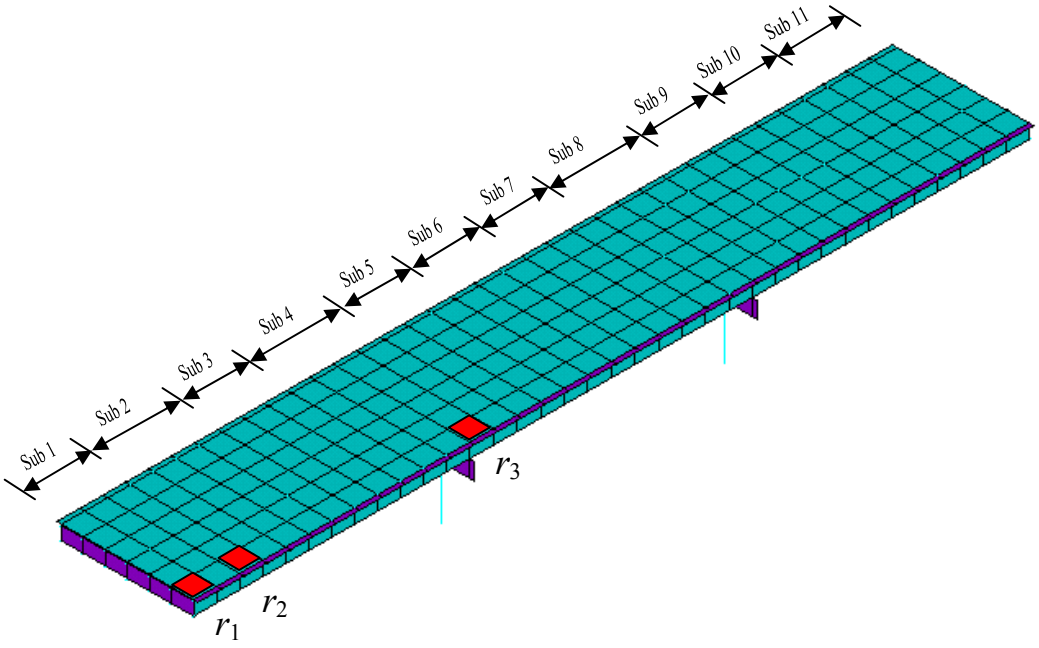


Fig. 2. FE model of the Balla Balla River Bridge and the selected design parameters

Table 1: Comparison of first-order eigensensitivity with respect to r_1

Mode	Eigenvalue derivatives			Correlation of eigenvector derivatives (COED)
	Nelson's method	Substructuring method	Difference (%)	
1	0.876	0.876	0.00%	0.999
2	3.621	3.622	0.02%	0.999
3	3.431	3.433	0.07%	0.992
4	49.478	49.567	0.18%	0.997
5	72.918	73.650	1.00%	0.997
6	292.125	294.986	0.98%	0.995
7	219.068	220.354	0.59%	0.999
8	742.183	756.540	1.93%	0.995
9	675.675	688.697	1.91%	0.990
10	526.273	535.207	1.70%	0.991

Table 2: Comparison of second-order eigenvalue derivative and eigenvector derivative (frame structure)

Mode	$\frac{\partial^2\lambda}{\partial r_{146}\partial r_{146}}$, $\frac{\partial^2\phi}{\partial r_{146}\partial r_{146}}$				$\frac{\partial^2\lambda}{\partial r_{146}\partial r_{148}}$, $\frac{\partial^2\phi}{\partial r_{146}\partial r_{148}}$				$\frac{\partial^2\lambda}{\partial r_{146}\partial r_{154}}$, $\frac{\partial^2\phi}{\partial r_{146}\partial r_{154}}$			
	Eigenvalue derivative			COED	Eigenvalue derivative			COED	Eigenvalue derivative			COED
	Global method	Substructuring method	Difference (%)		Global method	Substructuring method	Difference (%)		Global method	Substructuring method	Difference (%)	
1	-0.53	-0.53	0.00%	0.986	0.01	0.01	0.03%	0.991	0.00	0.00	0.08%	0.993
2	-2.62	-2.62	0.03%	0.993	0.03	0.03	0.34%	0.996	0.01	0.01	0.41%	0.987
3	-9.32	-9.33	0.12%	0.994	0.63	0.64	0.99%	0.993	0.14	0.14	0.00%	0.987
4	-34.57	-34.64	0.22%	0.990	3.20	3.25	1.52%	0.990	-0.19	-0.20	2.04%	0.990
5	-82.32	-82.33	0.01%	0.981	-9.17	-9.17	0.01%	0.997	-3.93	-3.94	0.27%	0.997
6	-316.52	-317.61	0.34%	0.994	-19.42	-19.83	2.11%	0.981	-5.55	-5.57	0.26%	0.990
7	-95.08	-95.32	0.26%	0.992	6.97	7.01	0.52%	0.985	5.00	5.11	2.26%	0.992
8	-762.82	-763.31	0.06%	0.995	1.09	1.09	0.10%	0.996	11.40	11.41	0.15%	0.993
9	-503.66	-503.17	0.10%	0.987	97.49	98.82	1.37%	0.981	56.76	57.29	0.93%	0.984
10	-277.40	-278.40	0.36%	0.997	54.77	53.64	2.07%	0.983	21.66	21.19	2.16%	0.974

Table 3: Comparison of second-order eigenvalue derivative and eigenvector derivative (bridge structure)

Mode	$\frac{\partial^2\lambda}{\partial r_1\partial r_1}$, $\frac{\partial^2\{\phi\}}{\partial r_1\partial r_1}$				$\frac{\partial^2\lambda}{\partial r_1\partial r_2}$, $\frac{\partial^2\{\phi\}}{\partial r_1\partial r_2}$				$\frac{\partial^2\lambda}{\partial r_1\partial r_3}$, $\frac{\partial^2\{\phi\}}{\partial r_1\partial r_3}$			
	Eigenvalue derivative			COED	Eigenvalue derivative			COED	Eigenvalue derivative			COED
	Global method	Substructuring method	Difference (%)		Global method	Substructuring method	Difference (%)		Global method	Substructuring method	Difference (%)	
1	-0.7010	-0.7012	0.03%	0.998	-0.0730	-0.0729	0.12%	0.996	-0.0295	-0.0296	0.21%	0.996
2	-0.0016	-0.0016	0.00%	0.999	0.0001	0.0001	0.00%	0.998	-0.0003	-0.0003	0.00%	0.993
3	-0.0030	-0.0031	0.66%	0.990	0.0008	0.0008	0.00%	0.983	0.0003	0.0003	0.00%	0.995
4	-0.1537	-0.1536	0.02%	0.985	0.0339	0.0340	0.29%	0.995	0.0036	0.0035	1.07%	0.990
5	-0.0008	-0.0008	0.00%	0.992	-0.0001	-0.0001	0.00%	0.994	0.0000	0.0000	0.00%	0.988
6	-0.0258	-0.0259	0.23%	0.992	-0.0025	-0.0025	0.00%	0.991	-0.0012	-0.0012	0.00%	0.994
7	-0.4020	-0.4022	0.05%	0.981	0.0938	0.0937	0.12%	0.986	0.0231	0.0230	0.55%	0.982
8	-0.0643	-0.0643	0.00%	0.987	0.0014	0.0014	0.00%	0.986	-0.0010	-0.0010	0.00%	0.985
9	-0.2160	-0.2168	0.36%	0.996	0.0161	0.0159	1.42%	0.985	-0.0062	-0.0063	1.50%	0.982
10	-0.1997	-0.1999	0.10%	0.974	0.0610	0.0609	0.16%	0.986	0.0502	0.0502	0.00%	0.992
11	-0.1860	-0.1864	0.22%	0.988	-0.0063	-0.0062	0.03%	0.985	0.0051	0.0052	1.92%	0.978
12	-0.0002	-0.0002	0.00%	0.981	0.0000	0.0000	0.00%	0.974	0.0000	0.0000	0.00%	0.969
13	-0.0001	-0.0001	0.00%	0.993	0.0000	0.0000	0.00%	0.970	0.0000	0.0000	0.00%	0.975
14	-1.1484	-1.1497	0.11%	0.991	-0.2129	-0.2139	0.48%	0.971	-0.1031	-0.1037	0.59%	0.985
15	-3.9496	-3.9901	1.03%	0.985	-0.7531	-0.7539	0.11%	0.980	-0.3398	-0.3401	0.08%	0.985
16	-2.4723	-2.4320	1.63%	0.986	-0.2371	-0.2343	1.17%	0.973	0.0102	0.0105	2.74%	0.987
17	-0.5327	-0.5337	0.20%	0.982	0.0462	0.0455	1.34%	0.977	-0.0066	-0.0065	1.67%	0.990
18	-4.9578	-4.9587	0.02%	0.977	-0.6186	-0.6166	0.32%	0.981	-0.1997	-0.1996	0.07%	0.980
19	-0.9347	-0.9247	1.07%	0.985	0.1271	0.1231	3.16%	0.983	-0.0093	-0.0094	1.15%	0.979
20	-1.3160	-1.3274	0.86%	0.988	-0.5541	-0.5642	1.82%	0.975	-0.3337	-0.3346	0.27%	0.974

Table 4: Computation time of the global and substructuring methods in calculation of second-order eigensensitivity

	Global method	Substructuring method										
		Location of parameter r_k										
		Sub 1	Sub 2	Sub 3	Sub 4	Sub 5	Sub 6	Sub 7	Sub 8	Sub 9	Sub 10	Sub 11
No. of parameters	288	24	32	24	32	24	24	24	32	24	32	16
Time (Second)	**13427**	434.4	382.3	286.3	398.7	286.7	283.6	283.0	390.3	282.9	377.6	188.8
		3594										