Calculation of eigenvalue and eigenvector derivatives with the improved Kron's substructuring method

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Abstract. For large-scale structures, the calculation of the eigensolution and the eigensensitivity is usually very time-consuming. This paper develops the Kron's substructuring method to compute the first-order derivatives of the eigenvalues and eigenvectors with respect to the structural parameters. The global structure is divided into several substructures. The eigensensitivity of the substructures are calculated via the conventional manner, and then assembled into the eigensensitivity of the global structure by performing some constraints on the derivative matrices of the substructures. With the proposed substructuring method, the eigenvalue and eigenvector derivatives with respect to an elemental parameter are computed within the substructure solely which contains the element, while the derivative matrices of all other substructures with respect to the parameter are zero. Consequently this can reduce the computation cost significantly. The proposed substructuring method is applied to the GARTEUR AG-11 frame and a highway bridge, which is proved to be computationally efficient and accurate for calculation of the eigensensitivity. The influence of the master modes and the division formations are also discussed.

Keywords: substructuring method; eigensolution; eigensensitivity; model updating.

1. Introduction

Finite element (FE) model updating technology has been extensively developed in aerospace, mechanical and civil engineering. It can serve for structural modification, model tuning, and damage identification (Friswell 1995). Model updating methods are usually classified into one-step methods and iterative methods (Brownjohn 2001). The one-step methods directly reconstruct the global stiffness matrix and the mass matrix, while the iterative methods modify the physical parameters in the FE model iteratively to realize an optimal match between the analytical modal properties (such as the frequencies and the modal shapes) and the measurements. The latter approach has been becoming more popular because they allow physical meaning of the obtained modifications and can

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preserve the symmetry, the positive-definiteness and the sparseness in the updated matrices. However, one drawback of the iterative methods lies in that the eigensolutions of the analytical model and their associated sensitivity matrices usually need to be calculated in each iteration (Bakir 2007).

Eigensensitivity is usually calculated in the global structure level. Fox and Kapoor (1968) firstly utilized the modal method to determine the eigenvalue and eigenvector derivatives by considering the changes of the physical parameters in the mass and stiffness matrices. The disadvantage of this method lies in that all modes of the system are required, which is computationally expensive for large-scale structures. Nelson (1976) proposed a more efficient method to calculate the eigenvector derivatives by using the modal parameters of that mode solely. Lin *et al.* (1995, 1996b) further improved the computation efficiency of the Nelson's method, by combining the inverse iteration technique, the singular value decomposition theory and the model reduction technique. The Nelson's method has also been developed to treat with the rigid body modes, the close or repeated modes by some researchers (Lin 1996a, Song 1996, Wu 2007).

Since calculation of the eigensensitivity usually dominants computation time during the iterative model updating process, how to calculate the eigensensitivity efficiently becomes a big challenge for the researchers. The substructuring technology can be a promising solution to accelerate the calculation of the eigensensitivity for large-scale structures. In general, the substructuring methods include three steps: first, the global structure is torn into some manageable substructures according to some division criteria; second, the substructures are analyzed independently to obtain the designated solutions (for example, the eigenpairs and eigensolution derivatives); finally, the solutions of the substructures are assembled to obtain the properties of the global structure by imposing constraints on the interface of the adjacent substructures (Yun *et al.* 1997). With the substructuring method, the eigensolutions and the eigensensitivity of the modified substructures are repeatedly analyzed, while the unmodified substructures are unchanged during the iterative model updating process. In addition, the substructuring method is expected to be more efficient when it is incorporated with the parallel computation (Fulton 1991) or the model reduction techniques (Choi *et al.* 2008, Xia and Lin 2004).

Hurty (1965) and Craig-Bampton (1968, 2000) developed a substructuring method based on the constraint modes with the fixed-interface condition of the substructures, while MacNeal (1971) and Rubin (1975) proposed a substructuring method based on the attachment modes with the free-interface condition. Qiu (1997) expressed the displacement of the substructures with the combination of the fixed interface modes and the free interface modes. Based on the different boundary conditions, Heo and Ehmann (1991) and Lallemand *et al*. (1999) derived the eigensolution derivatives by using the fixed-interface substructuring method and the free-interface substructuring method, respectively. The constraint modes or the linked force are required beforehand to construct the eigensensitivity formula. Gabriel Kron (1968) initiated a substructuring method to study the eigensolutions of the systems with large number of variables in a piece-wise manner. The Kron's method has a concise form, and has been developed by a few researchers (Simpson 1973, Simpson and Tabarrok 1968, Sehmi 1986). Recently, the Weng and Xia (2007) proposed a modal truncation technique to transform the original Kron's substructuring eigenequation into a simplified form, and improved the computation efficiency of the Kron's substructuring method. Only some lower eigenmodes of the substructures are retained as the master modes in the technique, while the higher modes are discarded and compensated with the residual flexibility. The method can achieve high precision with the second-order residual flexibility, or even high-order

residual flexibility.

The improved Kron's substructuring method is extended in this paper, to derive the first-order derivatives of the eigensolutions with respect to a structural parameter. With the proposed substructuring method, the derivatives matrices of the eigensolutions and the residual flexibility with respect to the elemental parameter are computed within a particular substructure, while the derivative matrices of the other substructures are zero. The eigensensitivity of the global structure with respect to the elemental parameter is recovered from the derivative matrices of the particular substructure. Since the residual flexibility is symmetric and directly related to the stiffness matrix, the first-order and high-order eigensensitivity can be calculated by directly re-differentiating the eigenequation with respect to the structural parameter. To verify the effectiveness and the accuracy of the proposed technique, the eigensensitivity formula is applied into the GARTEUR AG-11 structure and a highway bridge.

2. Basic theory

Generally, the global structure with N degree of freedoms (DOFs) is firstly divided into NS independent substructures. The jth ($j = 1, 2, \ldots, NS$) substructure with $n^{(j)}$ DOFs has the sub-matrices $\mathbf{K}^{(j)}$ and $\mathbf{M}^{(j)}$, and the associated $n^{(j)}$ eigenpairs as

$$\begin{gathered}\Lambda^{(j)} = \mathrm{Diag}[\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_{n^{(j)}}^{(j)}], \quad \Phi^{(j)} = [\phi_1^{(j)}, \phi_2^{(j)}, \ldots, \phi_{n^{(j)}}^{(j)}] \\ [\Phi^{(j)}]^T \mathbf{K}^{(j)} \Phi^{(j)} = \Lambda^{(j)}, \quad [\Phi^{(j)}]^T \mathbf{M}^{(j)} \Phi^{(j)} = \mathbf{I}^{(j)}\end{gathered} \tag{1}$$

The eigensolutions of the substructures are diagonally assembled into the primitive form as

$$\Lambda^p = \mathrm{Diag}[\Lambda^{(1)}, \Lambda^{(2)}, \ldots, \Lambda^{(NS)}], \quad \Phi^p = \mathrm{Diag}[\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(NS)}] \tag{2}$$

Hereinafter, the superscript 'p' represents the primitive matrices, which are diagonally assembled from the substructures directly. The divided substructures are then reconnected by the virtual work principle and the geometric compatibility. Kron's substructuring method makes full use of the orthogonality properties, and transforms the eigenequation of the assembled global structure into (Sehmi 1986)

$$\begin{bmatrix} \Lambda^p - \bar{\lambda}\mathbf{I} & -\Gamma \\ -\Gamma^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{z} \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{3}$$

in which $\Gamma = [\mathbf{C}\Phi^p]^T$, and $\mathbf{C}$ is a rectangular connection matrix, which constraints the interface DOFs to move jointly (Sehmi 1986, Turner 1983). In $\mathbf{C}$ matrix, each row contains two non-zero elements. For rigid connections the two elements will be 1 and −1. If the connected points x_1 and x_2 are not rigidly connected, which has the relationship $x_1 = rx_2$, the two elements in the corresponding row of matrix $\mathbf{C}$ will be 1 and $-r$. Kron's substructuring method considers the connection condition by the matrix $\mathbf{C}$, and has a concise form (Turner 1983). τ is the internal connection forces; $\bar{\lambda}$ is the eigenvalue of the global structure; $\mathbf{z}$ is regarded as the mode participation factor, which indicates the contribution of the eigenmodes of the substructures to the

eigenmodes of the global structure. The eigenvectors of the global structure $\boldsymbol{\Phi}$ can be recovered by $\overline{\boldsymbol{\Phi}} = \boldsymbol{\Phi}^p\{\mathbf{z}\}$ and removing the identical elements of $\overline{\boldsymbol{\Phi}}$ at the interfaces.

In Eq. (2), the primitive matrices $[\Lambda^p]$ and $[\boldsymbol{\Phi}^p]$ require calculating the complete eigensolutions of all substructures, which is time-consuming. The complete modes of each substructure are partitioned into the master part and the slave part (Weng *et al.* 2009). The first a few eigenmodes in each substructure are retained as the 'master' modes, while the residual higher eigenmodes are discarded as the 'slave' modes and compensated by the first-order residual flexibility. Assuming that the subscript 'm' and 's' represents the 'master' and 'slave' variables respectively, the jth substructure has $m^{(j)}$ 'master' eigenpairs and $s^{(j)}$ 'slave' eigenpairs as

$$\begin{gathered}\Lambda_m^{(j)} = \mathrm{Diag}[\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_{m^{(j)}}^{(j)}], \quad \boldsymbol{\Phi}_m^{(j)} = [\boldsymbol{\phi}_1^{(j)}, \boldsymbol{\phi}_2^{(j)}, \ldots, \boldsymbol{\phi}_{m^{(j)}}^{(j)}] \\ \Lambda_s^{(j)} = \mathrm{Diag}[\lambda_{m^{(j)}+1}^{(j)}, \lambda_{m^{(j)}+2}^{(j)}, \ldots, \lambda_{m^{(j)}+s^{(j)}}^{(j)}], \quad \boldsymbol{\Phi}_s^{(j)} = [\boldsymbol{\phi}_{m^{(j)}+1}^{(j)}, \boldsymbol{\phi}_{m^{(j)}+2}^{(j)}, \ldots, \boldsymbol{\phi}_{m^{(j)}+s^{(j)}}^{(j)}]\end{gathered} \tag{4}$$

Assembling the master eigenpairs and the slave eigenpairs respectively, one has

$$\begin{gathered}\Lambda_m^p = \mathrm{Diag}[\Lambda_m^{(1)}, \Lambda_m^{(2)}, \ldots, \Lambda_m^{(NS)}], \quad \boldsymbol{\Phi}_m^p = \mathrm{Diag}[\boldsymbol{\Phi}_m^{(1)}, \boldsymbol{\Phi}_m^{(2)}, \ldots, \boldsymbol{\Phi}_m^{(NS)}] \\ \Lambda_s^p = \mathrm{Diag}[\Lambda_s^{(1)}, \Lambda_s^{(2)}, \ldots, \Lambda_s^{(NS)}], \quad \boldsymbol{\Phi}_s^p = \mathrm{Diag}[\boldsymbol{\Phi}_s^{(1)}, \boldsymbol{\Phi}_s^{(2)}, \ldots, \boldsymbol{\Phi}_s^{(NS)}] \\ \boldsymbol{\Gamma}_m = [\mathbf{C}\boldsymbol{\Phi}_m^p]^T, \quad \boldsymbol{\Gamma}_s = [\mathbf{C}\boldsymbol{\Phi}_s^p]^T \\ m^p = \sum_{j=1}^{NS} m^{(j)}, \quad s^p = \sum_{j=1}^{NS} s^{(j)}, \quad m^{(j)} + s^{(j)} = n^{(j)}, \quad (j = 1, 2, \ldots, NS)\end{gathered} \tag{5}$$

Partitioning Eq. (3) according to the master and slave modes, Eq. (3) can be expanded as

$$\begin{bmatrix} \Lambda_m^p - \bar{\lambda}\mathbf{I} & \mathbf{0} & -\boldsymbol{\Gamma}_m \\ \mathbf{0} & \Lambda_s^p - \bar{\lambda}\mathbf{I} & -\boldsymbol{\Gamma}_s \\ -\boldsymbol{\Gamma}_m^T & -\boldsymbol{\Gamma}_s^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{z}_m \\ \mathbf{z}_s \\ \boldsymbol{\tau} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{6}$$

The second line of Eq. (6) gives

$$\mathbf{z}_s = (\Lambda_s^p - \bar{\lambda}\mathbf{I})^{-1}\boldsymbol{\Gamma}_s\boldsymbol{\tau} \tag{7}$$

Substituting Eq. (7) into Eq. (6) results in

$$\begin{bmatrix} \Lambda_m^p - \bar{\lambda}\mathbf{I} & -\boldsymbol{\Gamma}_m \\ -\boldsymbol{\Gamma}_m^T & -\boldsymbol{\Gamma}_s^T(\Lambda_s^p - \bar{\lambda}\mathbf{I})^{-1}\boldsymbol{\Gamma}_s \end{bmatrix} \begin{Bmatrix} \mathbf{z}_m \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{8}$$

The interested eigenvalues $\bar{\lambda}$ correspond to the lowest modes of the global structure, and are far less than the items in Λ_s^p when the master modes are properly chosen. Eq. (8) is approximated as

$$\begin{bmatrix} \Lambda_m^p - \bar{\lambda}\mathbf{I} & -\Gamma_m \\ -\Gamma_m^T & -\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s \end{bmatrix} \begin{Bmatrix} \mathbf{z}_m \\ \boldsymbol{\tau} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{9}$$

Representing $\boldsymbol{\tau}$ with $\mathbf{z}_m$ from the second line of Eq. (9) and substituting it into the first line, the eigenequation is simplified into

$$[\Lambda_m^p + \Gamma_m(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}\Gamma_m^T]\{\mathbf{z}_m\} = \bar{\lambda}\{\mathbf{z}_m\} \tag{10}$$

$$\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s = \mathbf{C}\Phi_s^p(\Lambda_s^p)^{-1}[\Phi_s^p]^T\mathbf{C}^T \tag{11}$$

where $\Phi_s^p(\Lambda_s^p)^{-1}[\Phi_s^p]^T$ is the first-order residual flexibility, which is represented by diagonally assembling the stiffness matrices and the master modes of the substructures as

$$\Phi_s^p(\Lambda_s^p)^{-1}[\Phi_s^p]^T = \mathrm{Diag}[((\mathbf{K}^{(1)})^{-1} - \Phi_m^{(1)}(\Lambda_m^{(1)})^{-1}[\Phi_m^{(1)}]^T), \ldots, ((\mathbf{K}^{(NS)})^{-1} - \Phi_m^{(NS)}(\Lambda_m^{(NS)})^{-1}[\Phi_m^{(NS)}]^T)] \tag{12}$$

In Eq. (10), the mode participation factor $\{\mathbf{z}_m\}$ leads to the eigenvectors of the global structure via the transform of $\bar{\Phi} = \Phi_m^p\{\mathbf{z}_m\}$. The reduced eigenequation (Eq. (10)) has the size of m^p, which is much smaller than that of the original one (Eq. (3)).

3. Eigenvalue derivatives with the substructuring method

For the ith mode, the eigenequation (Eq. (10)) can be rewritten as

$$[\Lambda_m^p + \Gamma_m(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}\Gamma_m^T]\{\mathbf{z}_i\} = \bar{\lambda}_i\{\mathbf{z}_i\} \tag{13}$$

Eq. (13) is differentiated with respect to a design parameter r as

$$[\Lambda_m^p + \Gamma_m(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}\Gamma_m^T - \bar{\lambda}_i\mathbf{I}]\left\{\frac{\partial \mathbf{z}_i}{\partial r}\right\} + \frac{\partial[\Lambda_m^p + \Gamma_m(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}\Gamma_m^T - \bar{\lambda}_i\mathbf{I}]}{\partial r}\{\mathbf{z}_i\} = \{\mathbf{0}\} \tag{14}$$

Pre-multiplying $\{\mathbf{z}_i\}^T$ on both sides of Eq. (14) gives

$$\{\mathbf{z}_i\}^T[\Lambda_m^p + \Gamma_m(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}\Gamma_m^T - \bar{\lambda}_i\mathbf{I}]\left\{\frac{\partial \mathbf{z}_i}{\partial r}\right\} + \{\mathbf{z}_i\}^T\frac{\partial[\Lambda_m^p + \Gamma_m(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}\Gamma_m^T - \bar{\lambda}_i\mathbf{I}]}{\partial r}\{\mathbf{z}_i\} = \mathbf{0} \tag{15}$$

Due to symmetry of $[\Lambda_m^p + \Gamma_m(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}\Gamma_m^T - \bar{\lambda}_i\mathbf{I}]$, the first item in the left hand side of Eq. (15) is zero. Arranging Eq. (15), the derivative of the eigenvalue $\bar{\lambda}_i$ with respect to the design parameter r is

$$\frac{\partial \bar{\lambda}_i}{\partial r} = \{\mathbf{z}_i\}^T\frac{\partial[\Lambda_m^p + \Gamma_m(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}\Gamma_m^T]}{\partial r}\{\mathbf{z}_i\} \tag{16}$$

in which

$$\frac{\partial[\Lambda_m^p+\Gamma_m(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}\Gamma_m^T]}{\partial r}$$

$$=\frac{\partial\Lambda_m^p}{\partial r}+\frac{\partial\Gamma_m}{\partial r}(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}\Gamma_m^T+\Gamma_m\frac{\partial[(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}]}{\partial r}\Gamma_m^T+\Gamma_m(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}\frac{\partial\Gamma_m^T}{\partial r}$$

In Eq. (16), $\partial\Lambda_m^p/\partial r$ is the diagonal assembly of the eigenvalue derivatives of the substructures, and $\frac{\partial\Gamma_m}{\partial r}=\frac{\partial[\Phi_m^p]^T}{\partial r}\mathbf{C}^T$ is associated with the diagonal assembly of the eigenvector derivatives of the substructures. $\partial[(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}]/\partial r$ is the derivative matrix of the first-order residual flexibility of the substructures.

Since the substructures are independent, the derivative matrices of the eigenvalues, the eigenvectors and the residual flexibility are only calculated in the particular substructure (for example, the *r*th substructure) which contains the elemental parameter *r*. These quantities in other substructures are zero. Within the *r*th substructure, the eigenvalue and eigenvector derivatives can be obtained by the traditional methods, such as Nelson's method (Nelson 1976). The derivative of the residual flexibility with respective to the structural parameter *r* is

$$\frac{\partial(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}}{\partial r}=(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1}\frac{\partial(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)}{\partial r}(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)^{-1} \tag{17}$$

and

$$\frac{\partial(\Gamma_s^T(\Lambda_s^p)^{-1}\Gamma_s)}{\partial r}=\mathbf{C}\frac{\partial(\Phi_s^p(\Lambda_s^p)^{-1}[\Phi_s^p]^T)}{\partial r}\mathbf{C}^T=\mathbf{C}\times\mathrm{Diag}\begin{bmatrix}\mathbf{0} & & \\ & \frac{\partial((\mathbf{K}^{(r)})^{-1}-\Phi_m^{(r)}(\Lambda_m^{(r)})^{-1}[\Phi_m^{(r)}]^T)}{\partial r} & \\ & & \mathbf{0}\end{bmatrix}\times\mathbf{C}^T$$

$$=\mathbf{C}\times\mathrm{Diag}\begin{bmatrix}\mathbf{0} & & \\ & \frac{\partial(\mathbf{K}^{(r)})^{-1}}{\partial r}-\frac{\partial\Phi_m^{(r)}}{\partial r}(\Lambda_m^{(r)})^{-1}[\Phi_m^{(r)}]^T+\Phi_m^{(r)}(\Lambda_m^{(r)})^{-1}\frac{\partial\Lambda_m^{(r)}}{\partial r}(\Lambda_m^{(r)})^{-1}[\Phi_m^{(r)}]^T-\Phi_m^{(r)}(\Lambda_m^{(r)})^{-1}\frac{\partial[\Phi_m^{(r)}]^T}{\partial r} & \\ & & \mathbf{0}\end{bmatrix}\times\mathbf{C}^T \tag{18}$$

It is noted that, if the substructure (for example, the *j*th substructure) is free, the stiffness matrix $\mathbf{K}^{(j)}$ is singular, and the inversion of $\mathbf{K}^{(j)}$ to form the residual flexibility does not exist. Consequently, the derivative of the residual flexibility of the free-free substructure is not available. In this situation, the rigid body modes, which contribute to the zero-frequency modes, are employed (Felippa *et al.* 1998). The detailed procedures on how to calculate the residual flexibility and the first-order derivatives of the free-free substructures can be found in Appendix.

It is noted that the eigenvalue derivatives of the global structure with respect to an elemental parameter solely rely on the particular substructure (the *r*th substructure) rather than the other

substructures. Since the size of the substructures is always smaller than that of the global structure, the computation efficiency is improved. This significant merit might be very attractive when it is applied to the iterative model updating. With the substructuring method, only the modified substructures are re-analyzed, while other substructures are untouched. In addition, due to the symmetric and simple form of the residual flexibility, there is no difficulty to derive the high-order derivatives of the eigenvalues by directly differentiating the eigensensitivity equation (Eq. (16)), which needs to calculate the eigenvector derivatives first.

4. Eigenvector derivatives with the substructuring method

Since the ith eigenvector of the global structure can be recovered by

$$\overline{\mathbf{\Phi}}_i = \mathbf{\Phi}_m^p \{\mathbf{z}_i\} \tag{19}$$

the eigenvector derivative of the ith mode with respect to the structural parameter r can be differentiated as

$$\frac{\partial \overline{\mathbf{\Phi}}_i}{\partial r} = \frac{\partial \mathbf{\Phi}_m^p}{\partial r}\{\mathbf{z}_i\} + \mathbf{\Phi}_m^p \left\{\frac{\partial \mathbf{z}_i}{\partial r}\right\} \tag{20}$$

In Eq. (20), $\mathbf{\Phi}_m^p$ are the eigenvectors of the master modes in the substructures, and $\partial \mathbf{\Phi}_m^p / \partial r$ are the associated eigenvector derivatives of the master modes of the rth substructure. $\{\mathbf{z}_i\}$ is the eigenvector of the reduced eigenequation (Eq. (10)). Once the item $\{\partial \mathbf{z}_i / \partial r\}$ is available, the eigenvector derivative of the ith mode of the global structure can be obtained.

Similar to the Nelson's method, $\{\partial \mathbf{z}_i / \partial r\}$ is separated into the sum of a particular part and a homogeneous part as

$$\left\{\frac{\partial \mathbf{z}_i}{\partial r}\right\} = \{v_i\} + c_i \{\mathbf{z}_i\} \tag{21}$$

where c_i is a participation factor. Substituting Eq. (21) into Eq. (14) gives

$$[\Lambda_m^p + \Gamma_m (\Gamma_s^T (\Lambda_s^p)^{-1} \Gamma_s)^{-1} \Gamma_m^T - \overline{\lambda}_i \mathbf{I}](\{v_i\} + c_i \{\mathbf{z}_i\}) = -\frac{\partial [\Lambda_m^p + \Gamma_m (\Gamma_s^T (\Lambda_s^p)^{-1} \Gamma_s)^{-1} \Gamma_m^T - \overline{\lambda}_i \mathbf{I}]}{\partial r}\{\mathbf{z}_i\} \tag{22}$$

Since $[\Lambda_m^p + \Gamma_m (\Gamma_s^T (\Lambda_s^p)^{-1} \Gamma_s)^{-1} \Gamma_m^T - \overline{\lambda}_i \mathbf{I}]\{\mathbf{z}_i\} = \{\mathbf{0}\}$, Eq. (22) is simplified as

$$\mathbf{\Psi}\{v_i\} = \{Y_i\} \tag{23}$$

in which

$$\mathbf{\Psi} = [\Lambda_m^p + \Gamma_m (\Gamma_s^T (\Lambda_s^p)^{-1} \Gamma_s)^{-1} \Gamma_m^T - \overline{\lambda}_i \mathbf{I}], \quad \{Y_i\} = -\frac{\partial [\Lambda_m^p + \Gamma_m (\Gamma_s^T (\Lambda_s^p)^{-1} \Gamma_s)^{-1} \Gamma_m^T - \overline{\lambda}_i \mathbf{I}]}{\partial r}\{\mathbf{z}_i\}$$

All the items in $\mathbf{\Psi}$ and $\{Y_i\}$ have been obtained during the calculation of the eigenvalue derivatives in the previous section.

If there is no repeated root, the reduced system matrix $\mathbf{\Psi}$ has the size of m^p and the rank of $(m^p - 1)$. To solve Eq. (23), one sets the kth item of $\{v_i\}$ to be zero, and eliminates the corresponding row

and column of $\mathbf{\Psi}$ and the corresponding item of $\{Y_i\}$. The full rank equation is

$$\begin{bmatrix} \mathbf{\Psi}_{11} & \mathbf{0} & \mathbf{\Psi}_{13} \\ \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{\Psi}_{31} & \mathbf{0} & \mathbf{\Psi}_{33} \end{bmatrix} \begin{Bmatrix} v_{i1} \\ v_{ik} \\ v_{i3} \end{Bmatrix} = \begin{Bmatrix} Y_{i1} \\ 0 \\ Y_{i3} \end{Bmatrix} \tag{24}$$

where the pivot, k, is chosen at the maximum entry in $\{\mathbf{z}_i\}$. The vector $\{v_i\}$ can be solved from Eq. (24).

Solution of c_i requires the orthogonal condition of the eigenvector as

$$\{\mathbf{z}_i\}^T\{\mathbf{z}_i\} = 1 \tag{25}$$

Differentiating Eq. (25) with respect to r gives

$$\frac{\partial\{\mathbf{z}_i\}^T}{\partial r}\{\mathbf{z}_i\} + \{\mathbf{z}_i\}^T\frac{\partial\{\mathbf{z}_i\}}{\partial r} = 0 \tag{26}$$

Substituting Eq. (21) into Eq. (26) results in

$$(\{v_i\}^T + c_i\{\mathbf{z}_i\}^T)\{\mathbf{z}_i\} + \{\mathbf{z}_i\}^T(\{v_i\} + c_i\{\mathbf{z}_i\}) = 0 \tag{27}$$

Therefore, the participation factor c_i is obtained as

$$c_i = -\frac{1}{2}(\{v_i\}^T\{\mathbf{z}_i\} + \{\mathbf{z}_i\}^T\{v_i\}) \tag{28}$$

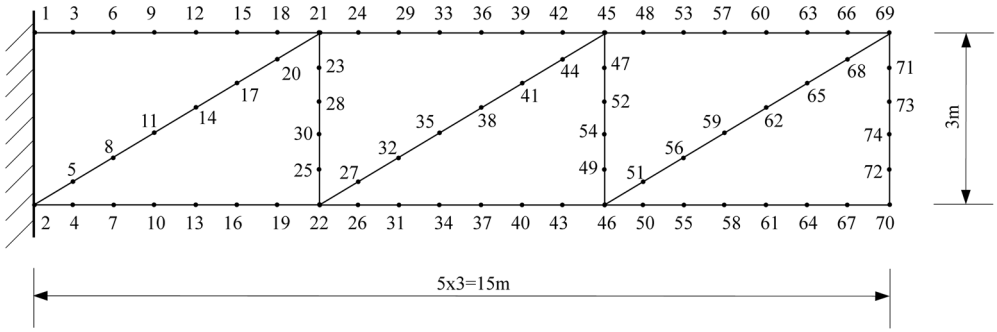
Finally, the first-order derivative of $\{\mathbf{z}_i\}$ with respect to the structural parameter r is

$$\left\{\frac{\partial\mathbf{z}_i}{\partial r}\right\} = \{v_i\} - \frac{1}{2}(\{v_i\}^T\{\mathbf{z}_i\} + \{\mathbf{z}_i\}^T\{v_i\})\{\mathbf{z}_i\} \tag{29}$$

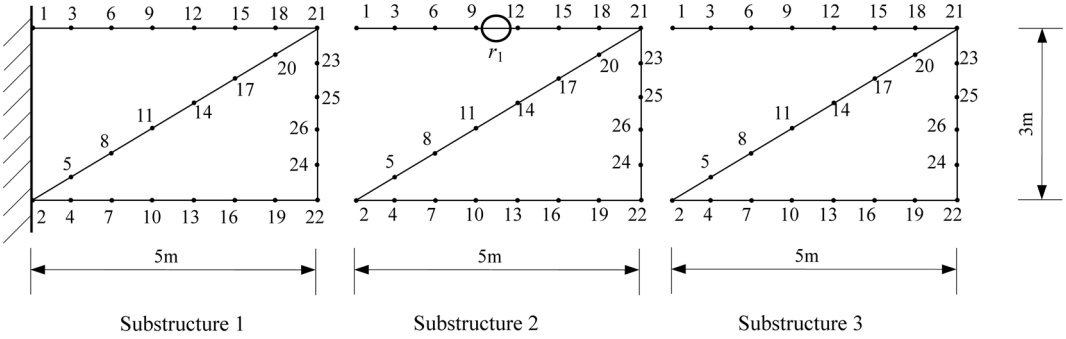
As far as Eq. (20) concerned, the eigenvector derivatives of the global structure can be regarded as the combination of the eigenvectors $\mathbf{\Phi}_m^p$ and the eigenvector derivatives $\partial\mathbf{\Phi}_m^p/\partial r$ of the substructures, while $\{\partial\mathbf{z}_i/\partial r\}$ and $\mathbf{z}$ act as the weight. Similar to the calculation of the eigenvalue derivatives, the calculation of the eigenvector derivatives of the global structure is equivalent to analyze the rth substructure and a reduced eigenequation. The procedure of the proposed substructuring method and its advantages will be demonstrated by two numerical examples.

5. Example 1: the GARTEUR structure

The first example presented here, GARTEUR AG-11 (as shown in Fig. 1(a)), serves to illustrate the procedures of the calculation of eigensensitivity with the proposed substructuring method. The frame is modeled by 78 Euler-Bernoulli beam elements and 74 nodes, as shown in Fig. 1(a). Each node has 3 DOFs, and there are 216 DOFs in total. The Young's modulus of each element is 75 GPa and the mass density is 2.80×10^3 kg/m^3. The moment of inertia of all members is 0.0756 m^4. The cross-section areas of the vertical, horizontal and diagonal bars are 0.006 m^2, 0.004 m^2 and 0.003 m^2, respectively.



(a) The global structure



(b) The configuration of the divided substructures

Fig. 1 The GARTEUR frame

The global structure is divided into 3 substructures ($NS = 3$) along the horizontal direction. To be an independent structure, each substructure has 26 elements and 26 nodes as shown in Fig. 1(b). The first 30 modes of each substructure are chosen as the master modes to calculate the eigensensitivity of the first 10 modes of the global structure.

Without losing generality, the Young's modulus of one element in Substructure 2 is chosen as the design parameter and denoted as r_1 in Fig. 1(b). The eigensensitivity of the first 10 modes of the global structure with respect to r_1 can be calculated with the proposed substructuring method:

(1) Calculate the eigensolutions of each substructure as: $\boldsymbol{\Lambda}_m^{(1)}, \boldsymbol{\Lambda}_m^{(2)}, \boldsymbol{\Lambda}_m^{(3)}, \boldsymbol{\Phi}_m^{(1)}, \boldsymbol{\Phi}_m^{(2)}, \boldsymbol{\Phi}_m^{(3)}$ ($m = 30$), and obtain the eigensolutions of the global structure with the reduced eigenequation Eq. (10) as: $\overline{\lambda}_i$, $\{\mathbf{z}_i\}, \overline{\boldsymbol{\Phi}}_i = \boldsymbol{\Phi}_m^p\{\mathbf{z}_i\}$ ($i = 1, 2, \ldots, 10$). In this structure, Substructure 2 and Substructure 3 have identical geometry and mechanical properties, and thus only one of them needs to be analyzed;

(2) Compute the eigenvalue and eigenvector derivatives of the first 30 modes of Substructure 2 with respect to the parameter r_1, $\partial\boldsymbol{\Lambda}_m^{(2)}/\partial r_1, \partial\boldsymbol{\Phi}_m^{(2)}/\partial r_1$, and calculate the derivative of the residual flexibility with respect to r_1, $\partial(\boldsymbol{\Phi}_s^{(2)}(\boldsymbol{\Lambda}_s^{(2)})^{-1}[\boldsymbol{\Phi}_s^{(2)}]^T)/\partial r_1$ from Eq. (17) and Eq. (18);

(3) Set the eigensolution derivatives and the residual flexibility derivatives of the other two substructures to be zero: $\partial\boldsymbol{\Lambda}_m^{(j)}/\partial r_1 = [\mathbf{0}]$, $\partial\boldsymbol{\Phi}_m^{(j)}/\partial r_1 = [\mathbf{0}]$, $\partial(\boldsymbol{\Phi}_s^{(j)}(\boldsymbol{\Lambda}_s^{(j)})^{-1}[\boldsymbol{\Phi}_s^{(j)}]^T)/\partial r_1 = [\mathbf{0}]$, ($j = 1, 3$), and construct the primitive form of the derivative matrices as:

$$\frac{\partial[\Lambda_m^p]}{\partial r_1}=\begin{bmatrix}\mathbf{0} & & \\ & \dfrac{\partial\Lambda_m^{(2)}}{\partial r_1} & \\ & & \mathbf{0}\end{bmatrix},\quad \frac{\partial[\Phi_m^p]}{\partial r_1}=\begin{bmatrix}\mathbf{0} & & \\ & \dfrac{\partial\Phi_m^{(2)}}{\partial r_1} & \\ & & \mathbf{0}\end{bmatrix}$$

$$\frac{\partial[(\Phi_s^p(\Lambda_s^p)^{-1}[\Phi_s^p]^T)]}{\partial r_1}=\begin{bmatrix}\mathbf{0} & & \\ & \dfrac{\partial(\Phi_s^{(2)}(\Lambda_s^{(2)})^{-1}[\Phi_s^{(2)}]^T)}{\partial r_1} & \\ & & \mathbf{0}\end{bmatrix}$$

(4) Obtain the first-order eigenvalue derivatives of the global structure $\partial\overline{\lambda}_i/\partial r_1$ ($i=1,\ 2,\ldots 10$) with Eq. (16).

(5) Calculate the first-order derivatives of $\{\mathbf{z}_i\}$ with respect to the parameter r_1 $\{\partial\mathbf{z}_i/\partial r_1\}$ from Eq. (29).

(6) Form the eigenvector derivatives of the global structure with respect to the parameter r_1 according to Eq. (20) and eliminate the identical values of $\partial\overline{\Phi}_i/\partial r$ at the tearing interfaces.

To verify the accuracy of the proposed substructuring method in calculation of the eigensensitivity, the traditional Nelson's method is directly employed to calculate the eigensensitivity of the global structure without dividing the global structure into individual substructures. The results from the proposed substructuring method and the global method are compared and shown in Table 1. The relative errors of the eigenvalue derivatives are less than 2%, which is sufficient for most of practical engineering applications.

Following the concept of modal assurance criterion (MAC) (Friswell and Mottershead 1995), the similarity of the eigenvector derivatives between the global method and the proposed substructuring method is denoted as Correlation of Eigenvector Derivatives (COED), and given by

$$\mathrm{COED}\left(\left\{\frac{\partial\boldsymbol{\phi}_i}{\partial r_1}\right\},\left\{\frac{\partial\tilde{\boldsymbol{\phi}}_i}{\partial r_1}\right\}\right)=\frac{\left|\left\{\dfrac{\partial\boldsymbol{\phi}_i}{\partial r_1}\right\}^T\cdot\left\{\dfrac{\partial\tilde{\boldsymbol{\phi}}_i}{\partial r_1}\right\}\right|^2}{\left(\left\{\dfrac{\partial\boldsymbol{\phi}_i}{\partial r_1}\right\}^T\cdot\left\{\dfrac{\partial\boldsymbol{\phi}_i}{\partial r_1}\right\}\right)\left(\left\{\dfrac{\partial\tilde{\boldsymbol{\phi}}_i}{\partial r_1}\right\}^T\cdot\left\{\dfrac{\partial\tilde{\boldsymbol{\phi}}_i}{\partial r_1}\right\}\right)} \tag{30}$$

where $\{\partial\tilde{\boldsymbol{\phi}}_i/\partial r_1\}$ represents the eigenvector derivative obtained by the global method, and $\{\partial\boldsymbol{\phi}_i/\partial r_1\}$ represents the eigenvector derivative by the proposed substructuring method. In this example, the COED values are above 0.995 for all modes as listed in Table 1, which indicates good accuracy in calculation of the eigenvector derivatives with the present method.

The master modes retained in the substructures affect the accuracy of the eigensensitivity calculated. Here 10 master modes and 20 master modes in each substructure are also employed to calculate the eigensensitivity. The relative errors of the eigenvalue derivatives are compared with the case of 30 master modes and illustrated in Fig. 2. It can be found that, more master modes can improve the accuracy of the eigensolution derivatives, as expected. The computational efficiency of the proposed method will be investigated in the following example with relatively large system matrices.

Table 1 The eigensensitivity of the GARTEUR structure

Mode	Eigenvalue derivatives			Correlation of Eigenvector Derivatives (COED)
	Nelson's method	Substructuring method	Relative error	
1	255.74	255.71	0.01%	0.996
2	3481.27	3480.95	0.01%	0.999
3	26363.72	26398.51	0.13%	0.999
4	82732.86	82851.69	0.14%	0.995
5	55230.69	55175.29	0.10%	0.999
6	5187.57	5159.04	0.55%	0.999
7	1646.92	1669.91	1.40%	0.999
8	51609.29	51665.04	0.11%	1.000
9	395796.88	396809.96	0.26%	0.999
10	100636.31	100607.13	0.03%	0.997

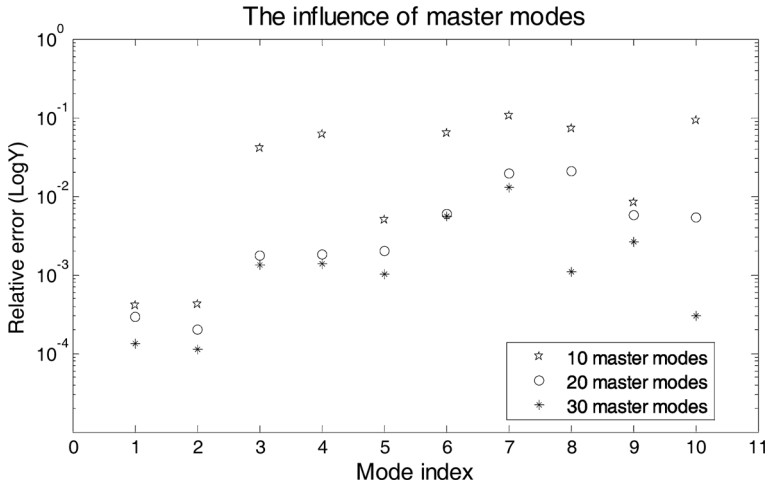


Fig. 2 The accuracy of the eigenvalue derivatives with different master modes

6. Example 2: a highway bridge

A practical bridge over the Balla Balla River in Western Australia is investigated here. The FE model of the bridge has 907 elements, 947 nodes each with 6 DOFs and 5420 DOFs in total, as shown in Fig. 3 (Xia *et al.* 2008).

The global structure is firstly divided into 11 substructures along the longitudinal direction as shown in Fig. 3. The detailed information of the 11 substructures is listed in Table 2. There are 50 master modes retained in each substructure to assemble the global structure.

The designed elemental parameters refer to the Young's modulus of the four shell elements,

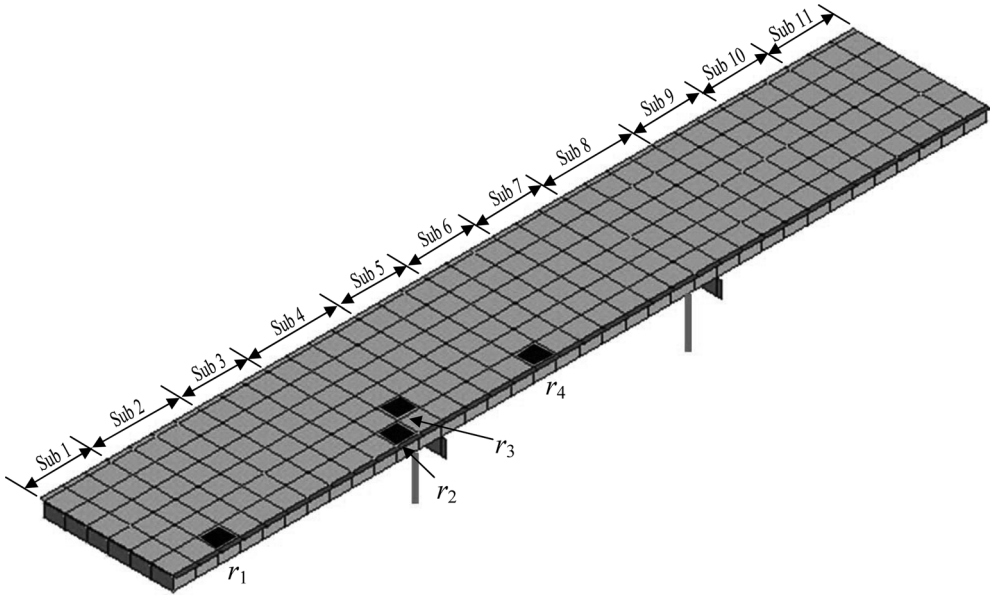


Fig. 3 The finite element model of the Balla Balla River Bridge

Table 2 The information on division formation with 11 substructures

Index of Substructure	Sub 1	Sub 2	Sub 3	Sub 4	Sub 5	Sub 6	Sub 7	Sub 8	Sub 9	Sub 10	Sub 11
Geometric range (m) in longitudinal direction	0~5	5~10	10~15	15~20	20~25	25~30	30~35	35~40	40~45	45~50	50~54
No. elements	99	88	66	116	66	66	66	116	66	66	99
No. nodes	113	115	92	143	92	92	92	143	92	92	113
No. tear nodes		23	23	23	23	23	23	23	23	23	23

denoted as r_1~r_4 in Fig. 3. The elemental parameters are intentionally located in different substructures while two parameters in one substructure. With the proposed substructuring method, the eigensensitivity of the first 20 modes of the global structure with respect to the four elemental parameters are calculated in Table 3. In addition, the eigensensitivity of the first 20 modes are directly calculated with the traditional Nelson's method based on the global structure for comparison.

Table 3 demonstrates that, when the global structure is divided into 11 substructures and the first 50 modes are retained as master modes in each substructure, the errors of eigenvalue derivatives of the first 20 modes are less than 3%, and the COED values are over 95%. These are acceptable for most of the engineering applications, such as model updating.

Here the computational efficiency is evaluated in terms of the computation time consumed by the CPU of computers during the calculation of the eigensensitivity with respect to the four design elemental parameters. The division formation of the substructures does affect the computation efficiency. To investigate the effect of the division formation, the bridge is also divided into 5 substructures, 8 substructures and 15 substructures, respectively. The information of different

Table 3 The eigensensitivity with respect to the four design structural parameters

	r_1				r_2				r_3				r_4			
	Eigenvalue derivatives				Eigenvalue derivatives				Eigenvalue derivatives				Eigenvalue derivatives			
Mode	Global method (10^{-2})	Present method (10^{-2})	Relative error	COED	Global method (10^{-2})	Present Method (10^{-2})	Relative error	COED	Global method (10^{-2})	Present method (10^{-2})	Relative error	COED	Global method (10^{-2})	Present method (10^{-2})	Relative error	COED
1	121.13	121.24	0.09%	0.989	223.79	223.99	0.09%	0.995	188.36	188.20	0.10%	0.997	378.76	378.89	0.03%	0.998
2	0.67	0.67	0.00%	1.000	0.18	0.18	0.00%	1.000	0.05	0.05	0.00%	1.000	0.02	0.02	0.00%	1.000
3	2.43	2.43	0.00%	1.000	7.35	7.34	0.14%	1.000	5.85	5.84	0.11%	1.000	6.91	6.89	0.28%	1.000
4	120.99	121.02	0.03%	0.973	329.45	329.46	0.00%	0.990	247.13	247.21	0.03%	0.997	374.17	373.96	0.06%	0.998
5	0.16	0.16	0.00%	0.997	0.48	0.48	0.00%	0.999	0.33	0.33	0.00%	0.999	0.37	0.37	0.00%	0.999
6	5.19	5.20	0.26%	0.995	9.17	9.19	0.26%	0.999	5.43	5.43	0.00%	0.999	14.31	14.33	0.14%	0.999
7	241.73	241.75	0.01%	0.960	140.42	140.44	0.02%	0.992	39.40	39.38	0.05%	0.997	4.21	4.20	0.26%	0.998
8	112.24	112.26	0.02%	0.976	468.14	468.18	0.02%	0.995	155.04	155.04	0.00%	0.999	1193.00	1194.77	0.15%	0.999
9	321.56	321.59	0.01%	0.990	566.79	566.83	0.01%	0.999	121.18	121.08	0.08%	0.999	8.72	8.73	0.13%	0.999
10	140.96	141.00	0.02%	0.997	23.86	23.84	0.09%	0.999	43.89	43.89	0.00%	0.999	559.87	559.77	0.02%	0.999
11	151.98	152.01	0.02%	0.969	73.59	73.62	0.03%	0.995	66.51	66.51	0.00%	0.999	314.84	314.73	0.03%	0.999
12	0.05	0.05	0.00%	1.000	0.06	0.06	0.00%	1.000	0.03	0.03	0.00%	1.000	0.00	0.00	0.00%	1.000
13	0.03	0.03	0.00%	1.000	0.05	0.05	0.00%	1.000	0.02	0.02	0.00%	1.000	0.18	0.18	0.00%	1.000
14	238.87	239.01	0.06%	0.993	179.95	179.80	0.22%	0.992	241.79	241.09	0.03%	0.996	754.21	745.73	1.12%	0.990
15	805.96	805.98	0.00%	0.998	874.12	873.25	0.10%	0.998	1090.01	1088.32	0.02%	0.998	76.32	76.66	0.39%	0.997
16	566.15	567.23	0.19%	0.998	760.02	760.64	0.08%	0.998	894.03	898.99	0.05%	0.999	227.23	226.81	0.02%	0.998
17	48.60	48.57	0.06%	0.970	678.54	677.02	0.22%	0.954	572.98	573.61	0.11%	0.948	4.78	4.74	0.82%	0.984
18	611.82	611.41	0.06%	0.979	1498.32	1496.84	0.11%	0.963	655.92	639.04	2.57%	0.969	1277.09	1281.57	0.36%	0.971
19	589.08	590.99	0.32%	0.984	11.90	11.77	1.06%	0.976	41.22	41.49	0.66%	0.997	170.75	169.89	0.51%	0.986
20	649.76	651.47	0.26%	0.987	792.71	790.86	0.29%	0.993	1265.73	1260.49	0.45%	0.996	3364.71	3360.68	0.16%	0.982

Table 4 The information on division formation with 5 substructures

Index of substructures	Sub 1	Sub 2	Sub 3	Sub 4	Sub 5
Geometric range (m) in longitudinal direction	0~10	10~20	20~30	30~40	40~54
No. elements	187	182	132	182	224
No. nodes	205	212	161	212	251
No. tear nodes	23	23	23	23	

Table 5 The information on division formation with 8 substructures

Index of substructure	Sub 1	Sub 2	Sub 3	Sub 4	Sub 5	Sub 6	Sub 7	Sub 8
Geometric range (m) in longitudinal direction	0~7	7~14	14~21	21~27	27~34	35~41	41~47	47~54
No. elements	143	110	116	88	110	116	88	136
No. nodes	159	138	143	115	138	143	115	159
No. tear nodes	23	23	23	23	23	23	23	

Table 6 The information on division formation with 15 substructures

Index of substructure	Sub 1	Sub 2	Sub 3	Sub 4	Sub 5	Sub 6	Sub 7	Sub 8	Sub 9	Sub 10	Sub 11	Sub 12	Sub 13	Sub 14	Sub 15
Geometric range in longitudinal direction (m)	0~3	3~7.5	7.5~10.5	10.5~13.5	13.5~16.5	16.5~19.5	19.5~22.5	22.5~27	27~30	30~34.5	34.5~37.5	37.5~40.5	40.5~45	45~49.5	49.5~54
No. elements	77	66	44	44	44	72	44	66	44	66	72	44	66	66	92
No. nodes	90	92	69	69	69	97	69	92	69	92	97	69	92	92	113
No. tear nodes	23	23	23	23	23	23	23	23	23	23	23	23	23	23	

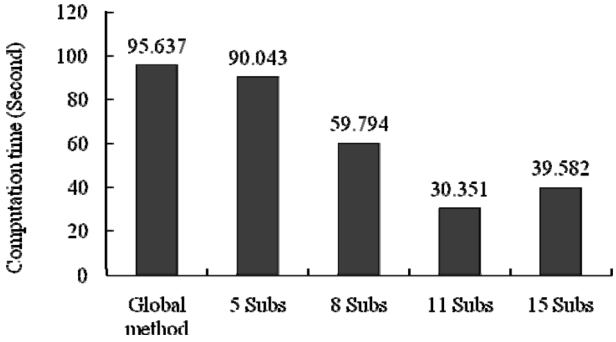


Fig. 4 The computation time with different division formations

division formations is given in Table 4 ~ Table 6.

For the four division formations, selecting different master modes will result in different precision. Based on the criteria that the relative errors of the eigenvalue derivatives for the first 20 modes of the global structure are less than 3%, there are 80 master modes retained in each substructure with the division formation of 5 substructures, 60 master modes in 8 substructures, 50 master modes in 11 substructures, and 50 master modes in 15 substructures. The computation time in calculation of the eigensensitivity, consumed by the global method and the proposed substructuring method with the four division schemes, are compared in Fig. 4.

From Fig. 4, it can be found that:

(1) Comparing with the traditional global method, the proposed substructuring method can reduce the computation time. This is because only a particular substructure and the reduced eigenequation need to be analyzed when forming the eigensensitivity of the global structure.

(2) The computational efficiency of the substructuring method heavily depends on the substructure division. For example, dividing the global structure into 5 substructures or 8 substructures takes longer computation time than that of 11 substructures. This is because handling large substructures takes longer time than handling smaller substructures. However, the division formation of 15 substructures is not as efficient as that of 11 substructures. The reason is that, excessive substructures lead to a large connection matrix $\mathbf{C}$ and the primitive matrices of the substructures. In that case, the transformation among these matrices will take more computation resources. This phenomenon has also been observed in calculation of the eigensolutions (Weng *et al.* 2009). The trade-off between the number of the substructures and the size of each substructure needs further studies. Nevertheless, the division formation can be tested in advance before applying the substructuring method to the iterative model updating process.

7. Conclusions

In this paper, the first-order eigenvalue and eigenvector derivatives have been derived based on the improved Kron's substructuring method. The eigensensitivity equation of the global structure is assembled from the eigensensitivity of particular substructures, and the eigensolution derivatives for

the reduced eigenequation are then calculated emulating the Nelson's method. Two numerical examples demonstrate that the proposed method can achieve a good accuracy when the proper master modes are retained.

The division formation of the global structure should be considered with caution. Too few substructures, which result in large-size substructures, might reduce the efficiency of the substructuring method. However, excessive substructures may introduce a large transformation matrix, and accordingly cause the assembly of the substructures to the global structure time-consuming. One should trade off the number of the substructures and the size of each substructure. Retaining more master modes in the substructures can achieve a better accuracy while cost more computation resource. The error estimation is required for the selection of the master modes in the substructures, which will be studied in the future. Moreover, further research will focus on how to improve the accuracy of the proposed substructuring method and implement it to the model updating process.

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Appendix: The residual flexibility and the derivatives for the free-free substructures

Without losing generality, here the residual flexibility and the derivative matrix are derived for an arbitrary structure with the stiffness and mass matrices of **K** and **M**, respectively.

The free-free structure has two kinds of eigenmodes: the n_r rigid body modes **R** and the n_d deformational modes $\mathbf{\Phi}_d$. The orthogonality of the rigid body modes satisfies

$$\mathbf{R}^T\mathbf{R} = \mathbf{I} \tag{A. 1}$$

Due to the fact that $(\mathbf{I}-\mathbf{R}\mathbf{R}^T)\mathbf{R}\mathbf{R}^T = \mathbf{0}$, an orthogonal projector associated with **R** can be constructed as

$$\mathbf{P} = \mathbf{I}-\mathbf{R}\mathbf{R}^T \tag{A. 2}$$

The orthogonality properties yield the spectral decompositions as

$$\mathbf{P} = \sum_{i=1}^{N_d}\mathbf{\Phi}_d\mathbf{\Phi}_d^T, \quad \mathbf{P}+\mathbf{R}\mathbf{R}^T = \sum_{i=1}^{N_d}\mathbf{\Phi}_d\mathbf{\Phi}_d^T+\mathbf{R}\mathbf{R}^T \tag{A. 3}$$

Since $\mathbf{K} = \sum_{i=1}^{N_d}\lambda_i\mathbf{\Phi}_d\mathbf{\Phi}_d^T$, including the rigid body modes into the free-free stiffness matrix **K** similarly gives

$$\mathbf{K}+\mathbf{R}\mathbf{R}^T = \sum_{i=1}^{N_d}\lambda_i\mathbf{\Phi}_d\mathbf{\Phi}_d^T+\mathbf{R}\mathbf{R}^T, \quad (\mathbf{K}+\mathbf{R}\mathbf{R}^T)^{-1} = \sum_{i=1}^{N_d}\frac{1}{\lambda_i}\mathbf{\Phi}_d\mathbf{\Phi}_d^T+\mathbf{R}\mathbf{R}^T \tag{A. 4}$$

The eigenvector of $\mathbf{K}+\mathbf{R}\mathbf{R}^T$ are identical to those of **K**, but including the rigid body modes and setting the eigenvalues of rigid body modes to be unity.

The complete eigenmodes are divided into n_m master modes $\mathbf{\Phi}_m$ and n_s slave modes $\mathbf{\Phi}_s$ according to the main sections of this paper. The master modes $\mathbf{\Phi}_m$ are composed by the n_r rigid body master modes **R** and the (n_m-n_r) normal master modes $\mathbf{\Phi}_{m\text{-}r}$. The deformational modes include the master modes $\mathbf{\Phi}_{m\text{-}r}$ and the slave modes $\mathbf{\Phi}_s$. Eq. (A.4) is equivalent to

$$(\mathbf{K}+\mathbf{R}\mathbf{R}^T)^{-1} = \sum_{i=1}^{n_m-n_r}\frac{1}{\lambda_i}\mathbf{\Phi}_{m\text{-}r}\mathbf{\Phi}_{m\text{-}r}^T+\sum_{j=1}^{n_s}\frac{1}{\lambda_j}\mathbf{\Phi}_s\mathbf{\Phi}_s^T+\mathbf{R}\mathbf{R}^T = \mathbf{\Phi}_{m\text{-}r}\mathbf{\Lambda}_{m\text{-}r}^{-1}\mathbf{\Phi}_{m\text{-}r}^T+\mathbf{\Phi}_s\mathbf{\Lambda}_s^{-1}\mathbf{\Phi}_s^T+\mathbf{R}\mathbf{R}^T \tag{A. 5}$$

The residual flexibility for the free-free structure is

$$\mathbf{\Phi}_s\mathbf{\Lambda}_s^{-1}\mathbf{\Phi}_s^T = (\mathbf{K}+\mathbf{R}\mathbf{R}^T)^{-1}-\mathbf{\Phi}_{m\text{-}r}\mathbf{\Lambda}_{m\text{-}r}^{-1}\mathbf{\Phi}_{m\text{-}r}^T-\mathbf{R}\mathbf{R}^T \tag{A. 6}$$

Accordingly, for the fixed structure without zero-frequency modes, the rigid-body modes are vanished, and the residual flexibility is simplified as usual form

$$\mathbf{\Phi}_s\mathbf{\Lambda}_s^{-1}\mathbf{\Phi}_s^T = \mathbf{K}^{-1}-\mathbf{\Phi}_m\mathbf{\Lambda}_m^{-1}\mathbf{\Phi}_m^T \tag{A. 7}$$

Considering the mass matrix, the orthogonal condition satisfies

$$\begin{aligned}&[\mathbf{\Phi}_r]^T\mathbf{K}\mathbf{\Phi}_r = \mathbf{0}, \ [\mathbf{\Phi}_{m\text{-}r}]^T\mathbf{K}\mathbf{\Phi}_{m\text{-}r} = \mathbf{\Lambda}_{m\text{-}r}, \ [\mathbf{\Phi}_s]^T\mathbf{K}\mathbf{\Phi}_s = \mathbf{\Lambda}_s\\ &[\mathbf{\Phi}_r]^T\mathbf{M}\mathbf{\Phi}_r = \mathbf{I}_r, \ [\mathbf{\Phi}_{m\text{-}r}]^T\mathbf{M}\mathbf{\Phi}_{m\text{-}r} = \mathbf{I}_{m\text{-}r}, \ [\mathbf{\Phi}_s]^T\mathbf{M}\mathbf{\Phi}_s = \mathbf{I}_s\end{aligned} \tag{A. 8}$$

Decomposing the mass matrix as $\mathbf{M} = \mathbf{M}^{1/2}\mathbf{M}^{1/2}$, and denoting

$$\tilde{\mathbf{\Phi}}_r = \mathbf{M}^{1/2}\mathbf{\Phi}_r, \ \tilde{\mathbf{\Phi}}_{m\text{-}r} = \mathbf{M}^{1/2}\mathbf{\Phi}_{m\text{-}r}, \ \tilde{\mathbf{\Phi}}_s = \mathbf{M}^{1/2}\mathbf{\Phi}_s, \ \tilde{\mathbf{K}} = \mathbf{M}^{1/2}\mathbf{K}\mathbf{M}^{1/2} \tag{A. 9}$$

the orthogonality satisfies

$$\tilde{\mathbf{\Phi}}_r^T \tilde{\mathbf{K}} \tilde{\mathbf{\Phi}}_r = \mathbf{0}\,,\ \ \tilde{\mathbf{\Phi}}_{m\text{-}r}^T \tilde{\mathbf{K}} \tilde{\mathbf{\Phi}}_{m\text{-}r} = \mathbf{\Lambda}_{m\text{-}r}\,,\ \ \tilde{\mathbf{\Phi}}_s^T \tilde{\mathbf{K}} \tilde{\mathbf{\Phi}}_s = \mathbf{\Lambda}_s$$

$$\tilde{\mathbf{\Phi}}_r^T \tilde{\mathbf{\Phi}}_r = \mathbf{I}_r\,,\ \ \tilde{\mathbf{\Phi}}_{m\text{-}r}^T \tilde{\mathbf{\Phi}}_{m\text{-}r} = \mathbf{I}_{m\text{-}r}\,,\ \ \tilde{\mathbf{\Phi}}_s^T \tilde{\mathbf{\Phi}}_s = \mathbf{I}_s$$

$$\tilde{\mathbf{\Phi}}_r^T \tilde{\mathbf{\Phi}}_{m\text{-}r} = \mathbf{0}\,,\ \ \tilde{\mathbf{\Phi}}_r^T \tilde{\mathbf{\Phi}}_s = \mathbf{0}\,,\ \ \tilde{\mathbf{\Phi}}_{m\text{-}r}^T \tilde{\mathbf{\Phi}}_s = \mathbf{0}$$

$$\tilde{\mathbf{\Phi}}_r^T \left(\tilde{\mathbf{K}} + \tilde{\mathbf{\Phi}}_r^T \tilde{\mathbf{\Phi}}_r \right) \tilde{\mathbf{\Phi}}_r = \mathbf{I}_r \tag{A. 10}$$

The first-order residual flexibility is represented by

$$\mathbf{\Phi}_s \mathbf{\Lambda}_s^{-1} \mathbf{\Phi}_s^T = \mathbf{M}^{-1/2} (\tilde{\mathbf{\Phi}}_s \mathbf{\Lambda}_s^{-1} \tilde{\mathbf{\Phi}}_s^T) \mathbf{M}^{-1/2} = \mathbf{M}^{-1/2} [(\tilde{\mathbf{K}} + \tilde{\mathbf{\Phi}}_r \tilde{\mathbf{\Phi}}_r^T)^{-1} - \tilde{\mathbf{\Phi}}_r \tilde{\mathbf{\Phi}}_r^T - \tilde{\mathbf{\Phi}}_{m\text{-}r} \mathbf{\Lambda}_{m\text{-}r}^{-1} \tilde{\mathbf{\Phi}}_{m\text{-}r}^T] \mathbf{M}^{-1/2} \tag{A. 11}$$

In Eq. (A.11), $(\tilde{\mathbf{K}} + \tilde{\mathbf{\Phi}}_r \tilde{\mathbf{\Phi}}_r^T)$ is nonsingular, and the first-order residual flexibility is obtainable.

For the free-free substructure, the derivative matrix of the first-order residual flexibility with respect to r is

$$\frac{\partial (\mathbf{\Phi}_s \mathbf{\Lambda}_s^{-1} \mathbf{\Phi}_s^T)}{\partial r} = \mathbf{M}^{1/2} \frac{\partial (\tilde{\mathbf{\Phi}}_s \mathbf{\Lambda}_s^{-1} \tilde{\mathbf{\Phi}}_s^T)}{\partial r} \mathbf{M}^{1/2} = \mathbf{M}^{-1/2} \frac{\partial ((\tilde{\mathbf{K}} + \tilde{\mathbf{\Phi}}_r \tilde{\mathbf{\Phi}}_r^T)^{-1} - \tilde{\mathbf{\Phi}}_r \tilde{\mathbf{\Phi}}_r^T - \tilde{\mathbf{\Phi}}_{m\text{-}r} \mathbf{\Lambda}_{m\text{-}r}^{-1} \tilde{\mathbf{\Phi}}_{m\text{-}r}^T)}{\partial r} \mathbf{M}^{-1/2} \tag{A. 12}$$

Since the rigid body modes are unchanged with the modification of the physical parameter r, the derivatives of the rigid body eigenvectors with respect to r are zero. Therefore, the derivative of the first-order residual flexibility with respect to r is

$$\frac{\partial (\mathbf{\Phi}_s \mathbf{\Lambda}_s^{-1} \mathbf{\Phi}_s^T)}{\partial r} = \mathbf{M}^{-1/2} \left[-(\tilde{\mathbf{K}} + \tilde{\mathbf{\Phi}}_r \tilde{\mathbf{\Phi}}_r^T)^{-1} \frac{\partial (\tilde{\mathbf{K}})}{\partial r} (\tilde{\mathbf{K}} + \tilde{\mathbf{\Phi}}_r \tilde{\mathbf{\Phi}}_r^T)^{-1} - \frac{\partial (\tilde{\mathbf{\Phi}}_{m\text{-}r} \mathbf{\Lambda}_{m\text{-}r}^{-1} \tilde{\mathbf{\Phi}}_{m\text{-}r}^T)}{\partial r} \right] \mathbf{M}^{-1/2} \tag{A. 13}$$

It should be noted here that, the rigid body modes are part of the master modes for the free-free substructures, when the master modes of the substructures are assembled to the reduced eigenequation (Eq. (10)). The rigid body modes are especially considered for the free-free substructures only when calculating the residual flexibility.

The design parameter r is considered as the stiffness elemental parameter in this research, and hence, the mass matrix $\mathbf{M}$ is assumed to be constant. If the design parameter r is mass elemental parameter, the eigensensitivity can be easily derived following the same procedures as described above but keeping the stiffness matrix $\mathbf{K}$ as constant.