Comparisons between Modal-Parameter-Based and Flexibility-Based Damage Identification Methods

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Comparisons between Modal-Parameter-Based and Flexibility-Based Damage Identification Methods

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Abstract: Modal parameters and modal flexibility obtained from the modal experiment have been used for damage identification in frequency domain. These two methods are compared in this paper numerically and experimentally. The optimization objective function of the former method is a weighted function of the frequency and mode shape residues, in which different weights can be assigned to the frequencies and mode shapes so that their degrees of accuracy can be taken into account. The optimization objective function of the latter method is the modal flexibility residue. Based on the same modal data, the two methods are compared from different aspects, including the damage configurations and uncertainties in the modal data. Results show that using the modal parameters directly can detect damage more accurately and reliably than using the modal flexibility.

Key words: damage identification, modal parameters, modal flexibility, uncertainty, statistical model updating.

1. INTRODUCTION

Civil, mechanical, and aerospace structures are inevitably subjected to deterioration and damage in their service life resulting from environmental erosion, overloading, fatigue, material deterioration and unexpected events. The vibration-based structural damage identification methods have been developed extensively. A detailed review on this topic was given by Doebling *et al*. (1998). Modal parameters (such as natural frequency, mode shape, and damping) and their variants including modal flexibility have been adopted for damage detection in frequency domain by many researchers.

The modal-parameter-based damage identification methods adopt the changes in natural frequencies and mode shapes to detect damage. Cawley and Adams performed the pioneer study to detect damage with frequency changes based on the forward method (Cawley and Adams 1979). Hassiotis and Jeong (1995) developed the inverse method named as the model updating method to identify damage so that the model predictions match the measured natural frequencies in an optimal way. A damage identification method using mode shape sensitivities to the changes in the mass or stiffness of a tested structure was presented by Parloo *et al*. (2003). In practice, measurement data are always limited and contain noises or errors. Researchers have addressed the issue of uncertainty, including the modelling error and measurement noise. For example, Papadopoulos and Garcia (1998) studied the influence of the uncertainty of the measured modal parameters on the damage identification results and proposed a probabilistic damage identification method. Xia and Hao (2003) developed a statistical damage identification algorithm accounting for the uncertainties in both the frequency measurement and the finite element (FE) modelling, and derived the probability of damage existence. They further extended the statistical

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approach to the case with combined frequency and mode shape data for structural damage identification (Xia *et al*. 2002). Yeo *et al*. (2000) presented a damage assessment algorithm for framed structures based on the static response with a regularization technique. Statistical distributions of the system parameters with a set of noise-polluted measured data were obtained by the data perturbation method, and then the damage was assessed by a statistical hypothesis test approach. The non-probabilistic interval analysis was also developed (Wang *et al*. 2010, 2012) for damage identification, in which the uncertainty bounds, rather than the probabilistic distributions, of the measurement data were employed. This method can be applied when the experimental data are insufficient.

As observed by many researchers, the first a few modal parameters are insensitive to the structural damage, whereas higher modal parameters are difficult to be measured accurately in practice (Doebling *et al*. 1998). Other damage indicators including modal flexibility have been proposed, based on the modal parameters. The main merit of the modal flexibility lies in the fact that the higher modes have less contributions to the flexibility matrix as compared with the lower modes and thus the unmeasured high modes have an insignificant effect on the accuracy of the flexibility matrix. Pandey and Biswas (1994) examined the changes in the flexibility matrix of a structure not only for identifying the presence of damage but also locating the damage. Numerical and experimental beams were used to demonstrate the effectiveness of the proposed method. The method worked best when the damage was located at a section where a high bending moment occurred. A sensitivity-based FE model updating using the modal flexibility residual was carried out for damage detection by Jaishi and Ren (2006). Robinson *et al.* (1996) detected the structural damage in an aircraft fuselage using the measured modal flexibility matrix. Some hybrid methods have also been developed by other researchers (Li and Smith 1995; Yan and Golinval 2005). In particular, Jaishi and Ren (2005) compared four different objective functions consisting of frequency residual only, mode shape related residue only, flexibility residual only, and their combination. Their results showed that the last performed the best for FE model updating.

As the modal flexibility is derived from the experimental modal parameters, using the modal data directly and using the derived modal flexibility may results in different damage identification results. However, the comparison between the two methods is rare in terms of accuracy and robustness. In this paper, the two methods will be employed to detect structural damage based on the same set of modal data. The damage detection results will be compared through numerical and experimental examples. Different damage configurations and the effect of uncertainty will be investigated.

2. FE MODEL UPDATING USING BOTH FREQUENCIES AND MODE SHAPES

2.1. The Deterministic Model Updating

The free vibration problem of an undamped structure with n degrees of freedom can be expressed as

$$\boldsymbol{K}\boldsymbol{\phi}_i = \lambda_i \boldsymbol{M}\boldsymbol{\phi}_i, \quad i = 1,2,...,n \tag{1}$$

where $\boldsymbol{M}$ is the $n \times n$ mass matrix, $\boldsymbol{K}$ is the $n \times n$ stiffness matrix, and λ_i and ϕ_i are the ith eigenvalue and mode shape vector, respectively. In practice, only a few lowest frequencies and mode shapes can be measured at limited points during the vibration testing. Here the number of available modes is n_m, and the number of measurement points is n_p.

For convenience, $\boldsymbol{K}$ can be expressed in the following non-negative parameter decomposition form:

$$\boldsymbol{K} = \sum_{i=1}^{m} \alpha_i \boldsymbol{K}_i = \alpha_1 \boldsymbol{K}_1 + \alpha_2 \boldsymbol{K}_2 + ... + \alpha_m \boldsymbol{K}_m \tag{2}$$

where m is the number of elements in the structure, α_i is the elemental stiffness parameter (ESP), and $\boldsymbol{K}_i$ is the ith element stiffness matrix divided by α_i.

The frequencies (or eigenvalues) and mode shapes from the experimental testing can be used as the reference for updating the ESPs of the analytical model of the structure. An objective function f reflects the difference between the analytical modal parameters and the experimental counterparts. The FE model updating can be posed as a minimization problem to find $\boldsymbol{\alpha}^*$ design set such that

$$\begin{gathered} f\left(\boldsymbol{\alpha}^*\right) \le f\left(\boldsymbol{\alpha}\right), \qquad \forall \boldsymbol{\alpha} \\ \underline{\alpha_i} \le \alpha_i \le \overline{\alpha_i}, \qquad \boldsymbol{i} = 1,2, \quad \boldsymbol{m} \end{gathered} \tag{3}$$

where underline and bar represent the upper and lower bounds of the design variables, respectively. The bounds are required to meet the physical meaning of the structure.

Because the degrees of accuracy of the measured frequency and mode shape are different, different weights can be assigned to the eigenvalues and mode shapes. Consequently, the model updating based on frequencies and mode shapes can be expressed as (Hao and Xia 2002)

$$\begin{aligned} &\text{find} && \boldsymbol{\alpha} \quad \left(\underline{\boldsymbol{\alpha}} \le \boldsymbol{\alpha} \le \bar{\boldsymbol{\alpha}}\right) \\ &\text{min} \quad error = && \sum_{i=1}^{n_m} \left[\frac{\lambda_i^A(\boldsymbol{\alpha}) - \lambda_i^E}{\lambda_i^E} \right]^2 \\ & && + \sum_{i=1}^{n_m} \beta_i^2 \sum_{j=1}^{n_p} \left[\phi_{ji}^A(\boldsymbol{\alpha}) - \phi_{ji}^E \right]^2 \end{aligned} \tag{4}$$

where superscripts E and A represent the modal data from the experiment and the analytical model, respectively; β is a weight vector for the mode shape, and vector $\boldsymbol{\alpha} = \left(\alpha_1, \alpha_2, ..., \alpha_m\right)^T$. Eqn 4 is a constrained least-square problem, and can be solved with standard solvers such as the trust-region-reflective algorithm (Coleman and Li 1996).

When the measured modal data in both undamaged and damaged states are available, the two-step model updating (Xia and Hao 2003) can be employed and ESPs in both states (α^u and α^d) can be respectively obtained. The elemental stiffness reduction factor (SRF) is defined as the ESP change to the undamaged value as

$$SRF_i = (\alpha_i^d - \alpha_i^u) / \alpha_i^u \tag{5}$$

where superscripts "u" and "d" represent the updated ESP values in the undamaged and damaged states, respectively.

2.2. Monte Carlo Simulation for the Probability-Based Model Updating

Because uncertainties (noises) inevitably exist in the measured vibration data, the updated ESP vector ($\boldsymbol{\alpha}$) is subjected to uncertainty as well. In the model updating procedure, the uncertainties in the measured modal data are assumed as independent normally distributed random variables with zero means and specific covariances. In this regard, the eigenvalues and mode shapes can be expressed as

$$\lambda_i^E = \lambda_{i,0}^E \left(1 + X_{\lambda i}\right), \quad i = 1,2,...,n_m \tag{6}$$

$$\boldsymbol{\phi}_i^E = \boldsymbol{\phi}_{i,0}^E \left(1 + \boldsymbol{X}_{\phi i}\right), \quad i = 1,2,...,n_m \tag{7}$$

where '0' represents the true values, and $X_{\lambda i}$ and $\boldsymbol{X}_{\phi i}$ are relative random noises in the measured frequencies and mode shapes, respectively. The mean value of vector $\boldsymbol{X}$ is zero and the standard deviation represents the noise level.

The statistics (mean value and standard deviation) of α can then be calculated by the perturbation method (Hua *et al.* 2008) or Monte Carlo simulation. The latter method can also give statistical samples of the updated ESPs, from which the statistical distribution can be obtained. Previous studies (Xia and Hao 2003) have demonstrated that the statistical distribution of the ESPs in the updated model is also normal, verified by the goodness-of-fit test (Kottegoda and Rosso 1997).

Again when the measured modal data in both undamaged and damaged states are available and the two-step model updating (Xia and Hao 2003) is employed, the statistics of ESPs in both states ($\boldsymbol{\alpha}^u$ and $\boldsymbol{\alpha}^d$) can be respectively calculated.

3. FE MODEL UPDATING USING MODAL FLEXIBILITY

3.1. The Deterministic Model Updating

The modal flexibility matrix of the structure can be formulated from the measured vibration properties as (Jaishi and Ren 2006)

$$\boldsymbol{G} = \boldsymbol{\Phi}\left(\boldsymbol{\Lambda}\right)^{-1} \boldsymbol{\Phi}^T \tag{8}$$

where $\boldsymbol{\Lambda} = \mathbf{diag}\left(\lambda_1, \lambda_2, ..., \lambda_{n_m}\right)$ is a diagonal matrix whose elements are the eigenvalues.

Similarly the model updating based on the modal flexibility can be expressed as

$$\begin{aligned} &\text{find} && \boldsymbol{\alpha} \quad \left(\underline{\boldsymbol{\alpha}} \le \boldsymbol{\alpha} \le \bar{\boldsymbol{\alpha}}\right) \\ &\text{min} && error = \frac{\left\| G^A(\boldsymbol{\alpha}) - G^E \right\|}{\left\| G^E \right\|} \end{aligned} \tag{9}$$

where $\|\cdot\|$ represents the Frobenius norm of the matrix.

The comparison between Eqns 4 and 9 shows that the modal-parameter-based model updating method can consider the different accuracy of the measurements via the weight factor, whereas the flexibility-based method cannot. The influence of the weight factor on the damage identification results will be investigated later.

3.2. Monte Carlo Simulation for the Probability-Based Model Updating

For the same reason, the derived modal flexibility is not accurate because of uncertainties in the modal parameters. Consequently, the modal-flexibility-based method results in inaccurate ESPs in the updated model. With the same assumption of noise in the modal data as Eqns 6 and 7, the statistics and distribution of the updated ESPs can be estimated by Monte Carlo simulation.

Subsequently, the statistics of ESPs in both undamaged and damaged states ($\boldsymbol{\alpha}^u$ and $\boldsymbol{\alpha}^d$) can be obtained using the measured modal data in the undamaged and damaged states, respectively.

4. NUMERICAL EXAMPLE

In this section, a one-story steel frame as shown in Figure 1 will be utilized to compare the modal-parameter-based and the flexibility-based damage identification methods using the identical modal data.

The cross section of the beam is 40.50 × 6.0 mm^2, and that of the columns is 50.50 × 6.0 mm^2. The connection between the beam and columns is rigid. The mass density is 7.67 × 10^3 kg/m^3. Figure 1 shows the FE model with 30 Euler–Bernoulli beam elements (m = 30).

4.1. Damage Identification without Uncertainty

To illustrate the identifiability of damages in spatial locations, two different damage configurations as listed in Table 1 will be analyzed. First, the lowest 12 pairs of noise free natural frequencies and mode shapes (n_m = 12) are used to identify the structural damage.

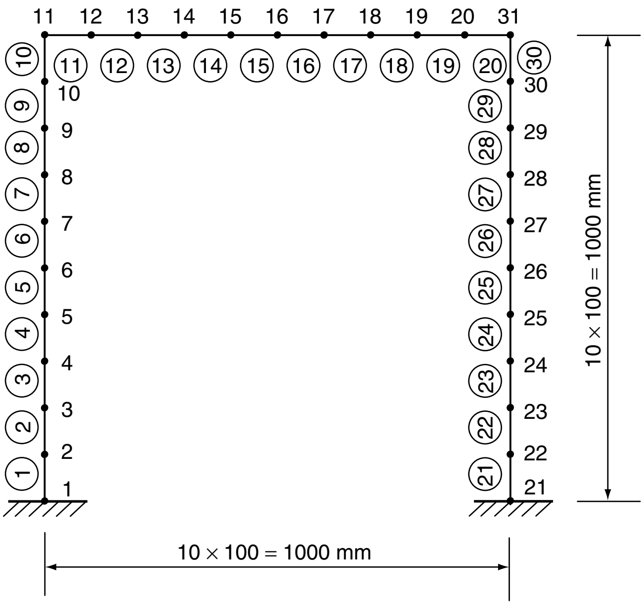

Figure 1. FE model of the frame

Table 1. Two damage configurations

Damage case	Locations of damage elements	SRF
1	4, 15	–20%, –30%
2	1, 4, 11, 15	–20%, –30%, –20%, –30%

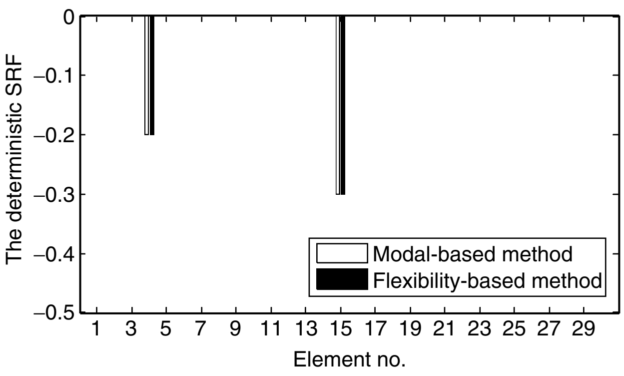

Figure 2. SRFs of the frame for damage case 1

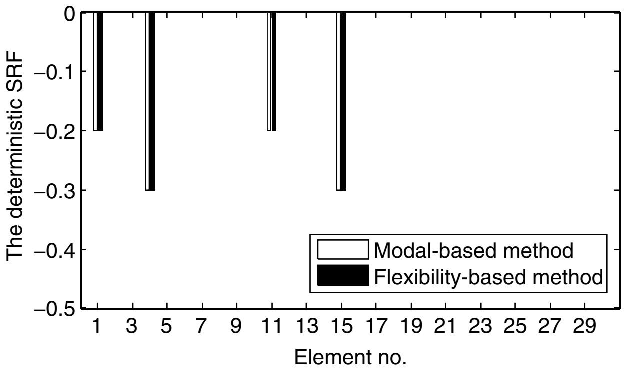

Figure 3. SRFs of the frame for damage case 2

Figure 2 and Figure 3 illustrate the SRFs obtained by the modal-parameter-based and flexibility-based damage identification methods, respectively. It can be seen that both methods can give the accurate results when the measured modal data are free of noise.

4.2. Influences of Uncertainty

Next the influence of the noise in the modal data on the identified results is investigated. Generally, the uncertainties of the measured mode shapes are larger than those of the frequencies in the modal testing. In this study, the uncertain frequencies and mode shapes are considered as the normal distributed random variables as Eqns 6 and 7. The mean values of the relative random noises $X_{\lambda i}$ and $\boldsymbol{X}_{\phi i}$ are zeros, and their standard deviations ξ_λ and ξ_ϕ) represent the noise level. With the Monte Carlo simulation, the mean values and standard deviations of SRFs are calculated from 500 samples. Figure 4 and Figure 5 respectively show the mean value and standard deviation of SRFs for damage case 1, in which the noise level are ξ_λ = 0 and ξ_ϕ = 1%. Figure 6 and Figure 7 show the statistics of SRFs for damage case 2. In both damage scenarios, the mean values of SRFs using the modal-based method are accurate whereas

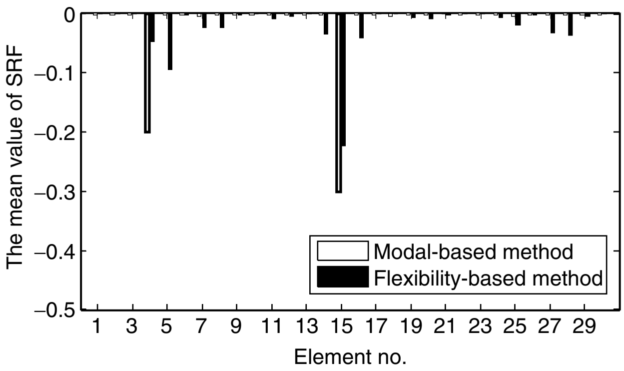

Figure 4. Mean value of SRF for damage case 1 ($\xi_\lambda = 0$ and $\xi_\phi = 1\%$)

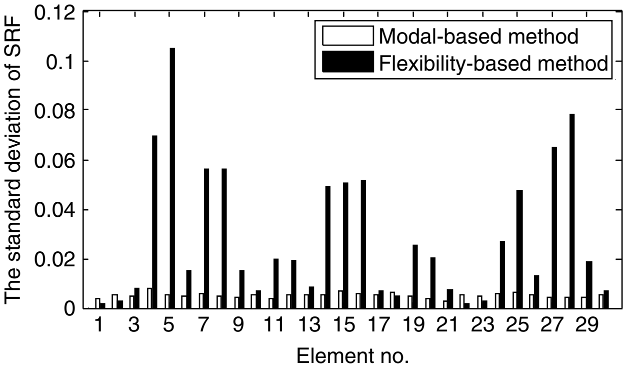

Figure 5. Standard deviation of SRF for damage case 1 ($\xi_\lambda = 0$ and $\xi_\phi = 1\%$)

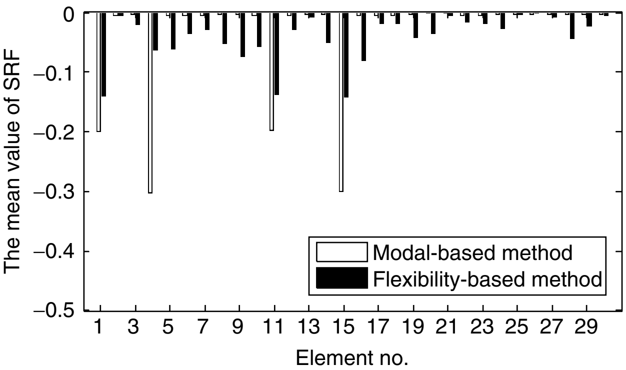

Figure 6. Mean value of SRF for damage case 2 ($\xi_\lambda = 0$ and $\xi_\phi = 1\%$)

those using the flexibility-based method not. Moreover, the standard deviations of SRFs using the modal-based method are small, indicating that the damaged elements can be detected reliably. On the contrary, the standard deviations of SRFs using the flexibility-based method are significant, although the noise level is low. Consequently, the SRFs have large variations and the damage cannot be identified reliably.

For a larger noise level, that is, $\xi_\lambda = 1\%$ and $\xi_\phi = 10\%$, the statistics of SRF for damage case 2 are drawn in

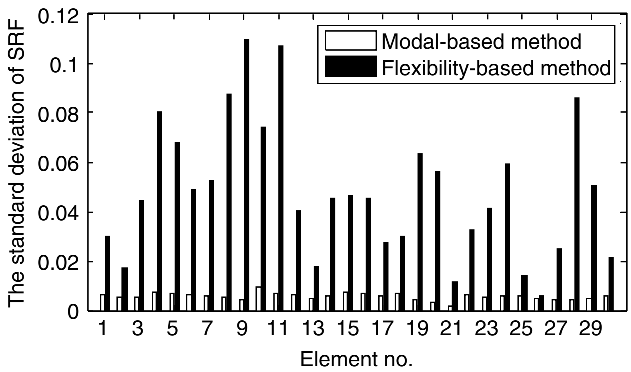

Figure 7. Standard deviation of SRF for damage case 2 ($\xi_\lambda = 0$ and $\xi_\phi = 1\%$)

Figure 8 and Figure 9. The modal-based damage identification method can still obtain accurate mean values and small standard deviations of SRFs, although having larger errors than those at the lower uncertainty level. With the flexibility-based damage identification method, however, the mean values of SRFs are completely incorrect and the standard deviations are very significant. All of the figures demonstrate that the modal-parameter-based damage identification method is

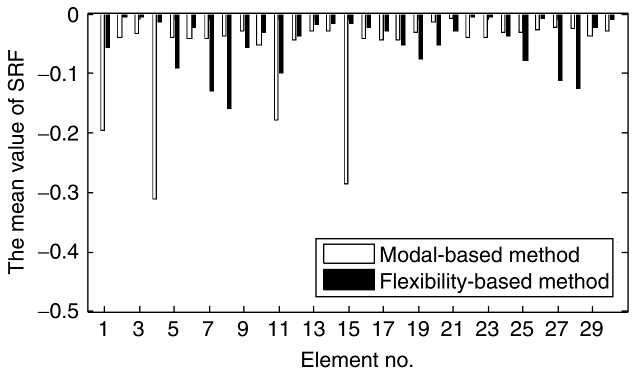

Figure 8. Mean value of SRF for damage case 2 ($\xi_\lambda = 1\%$ and $\xi_\phi = 10\%$)

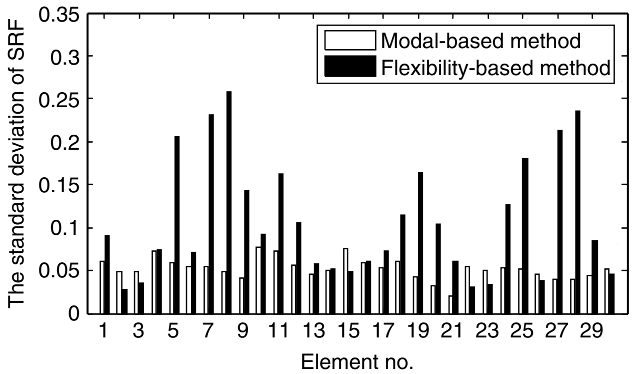

Figure 9. Standard deviation of SRF for damage case 2 ($\xi_\lambda = 1\%$ and $\xi_\phi = 10\%$)

more robust to the measurement noise than the flexibility-based damage identification method.

5. EXPERIMENTAL EXAMPLES

In this section, two laboratory-tested structures are used to compare the performance of the two damage identification methods.

5.1. One-Story Steel Frame

The one-story portal frame is shown in Figure 10. It has the same physical properties as the previous numerical example and the same FE model is used here. Four saw cuts with two different damage severities were made at element Nos. 1, 4, 11 and 15. The second damage configuration contains severer damage than the first. Details of the experiment can be found in Hao and Xia (2002). The first 12 frequencies and mode shapes were extracted by the non-linear least square method (Maia *et al*. 1997). Table 2 lists the measured frequencies in the undamaged and two damaged states.

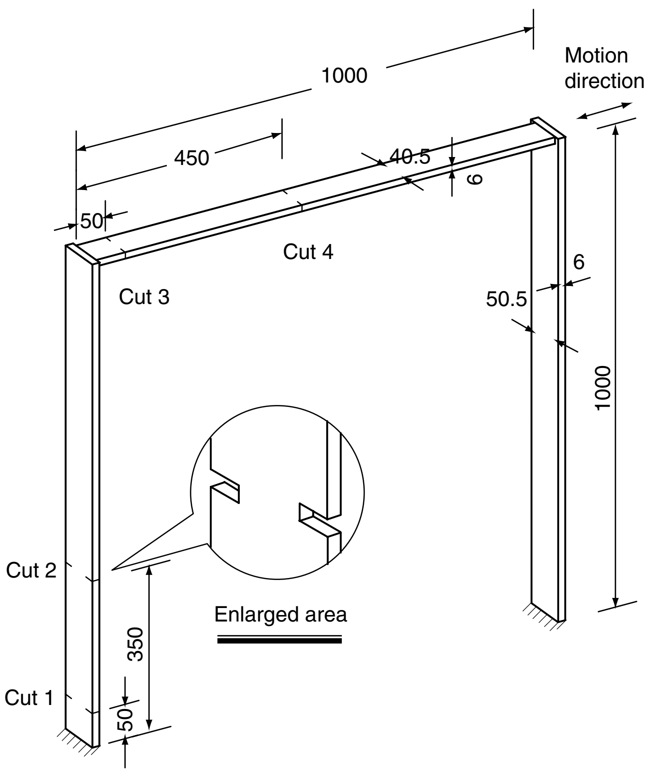

Figure 10. Configuration of the frame specimen (unit: mm)

Figure 11 and Figure 12 respectively show the identified SRF values of the first and second damage states using the two damage identification techniques. We can find that the modal-based identification method can detect damage at element Nos. 1, 4, 11, and 15 accurately in both damaged states, whereas the flexibility-based identification method cannot.

5.2. Three-Story Steel Frame

The dimensions of the three-story steel frame are shown in Figure 13; the cross section of the beam is 50.0×8.8 mm^2 and the columns are 50.0×4.4 mm^2. The mass density of the steel is 7.67×10^3 kg/m^3. The analytical model of the frame is composed of 44 nodes and 45 elements, as labeled in Figure 14.

Vibration tests were carried out on the structure in the undamaged state and two damaged states. The damage for the first damage configuration was located at element No. 2, and the damages for the second damage configuration were located at element Nos. 2 and 19. The output time history data were recorded at the lateral direction of the measured nodes (Nos. 1 ~ 44 as shown in Figure 14) to derive the frequency response functions. Based on the measurement data, 14 pairs of natural frequencies and the corresponding mass-normalized mode shapes are extracted by the rational fraction

Table 2. The first 12 measured natural frequencies in the undamaged and damaged states

Mode	Analytical frequency (Hz)	Measured frequency (Hz)		
		Undamaged state	First damaged state	Second damaged state
1	4.69	4.49	4.41	4.31
2	18.22	17.41	17.16	16.90
3	28.92	27.99	27.46	26.68
4	31.46	30.89	30.28	29.76
5	64.50	61.84	61.43	60.80
6	76.63	74.41	72.91	71.14
7	90.07	87.79	86.75	85.91
8	137.52	132.99	131.61	129.95
9	160.23	155.42	154.08	152.57
10	169.98	165.67	164.30	162.92
11	239.69	228.70	227.20	225.30
12	263.18	255.30	252.15	248.51

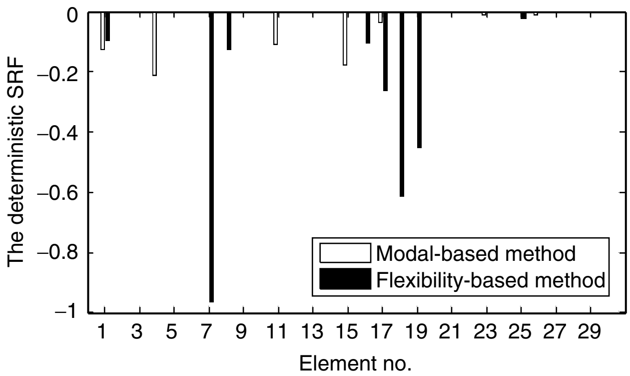

Figure 11. SRF of the one-story frame for the first damage state

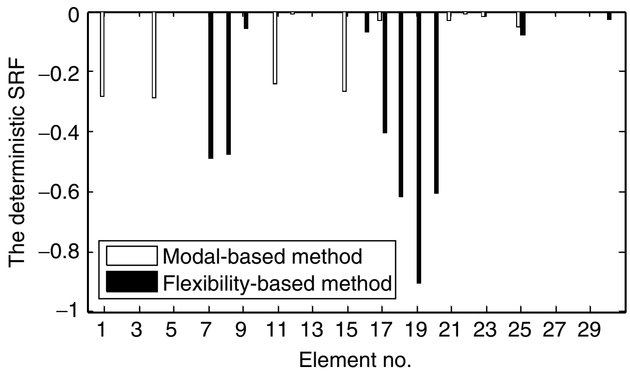

Figure 12. SRF of the one-story frame for the second damage state

Figure 13. The three-story frame structure

polynomial method (Richardson and Formenti 1982). Table 3 lists the measured 14 frequencies in the undamaged and damaged states.

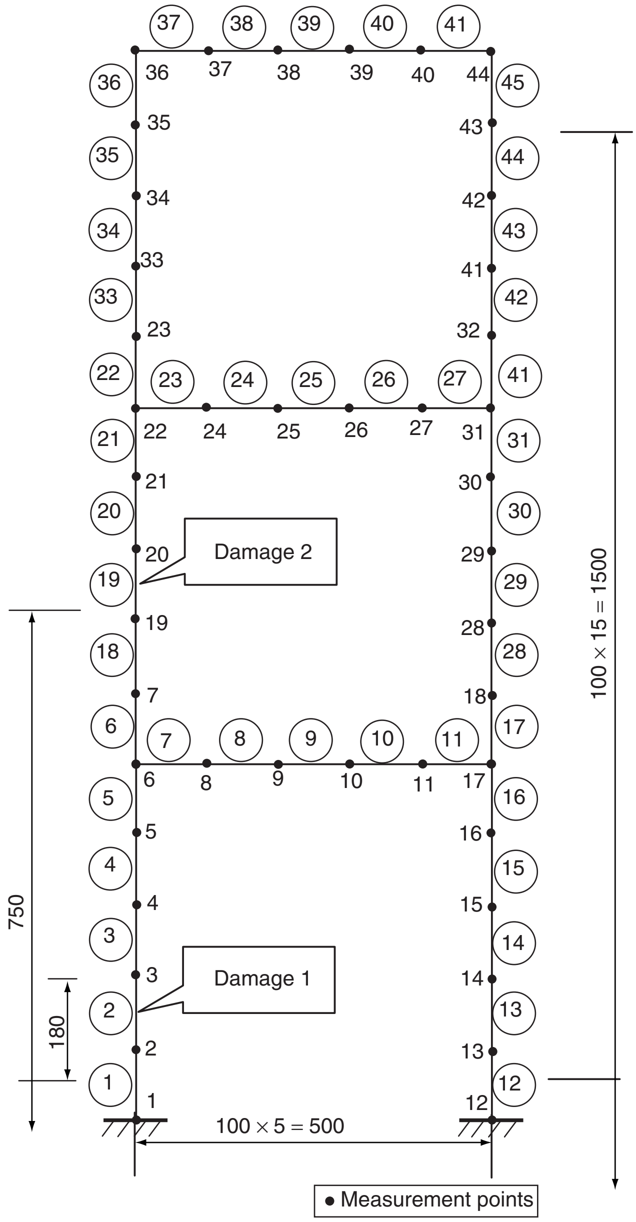

Figure 14. FE model of the three-story frame structure (unit: mm)

With the measured 14 modal parameters, SRFs of the structure in the first damage configuration are obtained using the two damage identification methods. The results are compared in Figure 15. We can find that both methods can give a higher SRF value at element No. 2 than other elements. In addition, the modal-based method presents less false identification on the undamaged elements than the flexibility-based method. Similarly Figure 16 compares the identified SRFs in the second damage configuration. Again both the modal-based and the flexibility-based identification methods give the higher SRF values at the damaged elements, whereas the former give lower SRF values at the undamaged elements than the latter.

Table 3. The measured 14 natural frequencies in the undamaged and damaged states

			Measured frequency (Hz)		
Mode	Analytical mode	Analytical frequency (Hz)	Undamaged state	First damage configuration	Second damage configuration
1	1	3.16	3.12	3.11	3.11
2	2	9.24	9.11	9.09	9.09
3	3	14.07	14.34	14.34	14.33
4	4	50.33	52.46	52.24	51.88
5	5	56.45	58.18	57.72	57.41
6	6	64.41	66.80	66.73	66.48
7	7	72.80	71.65	71.28	70.73
8	8	80.42	82.14	81.60	80.99
9	9	81.94	82.87	82.19	81.98
10	16	210.46	200.13	199.70	199.11
11	19	229.21	222.36	220.93	220.03
12	20	231.92	226.55	224.97	224.14
13	21	233.96	236.58	234.78	233.50
14	22	394.85	383.33	382.50	376.49

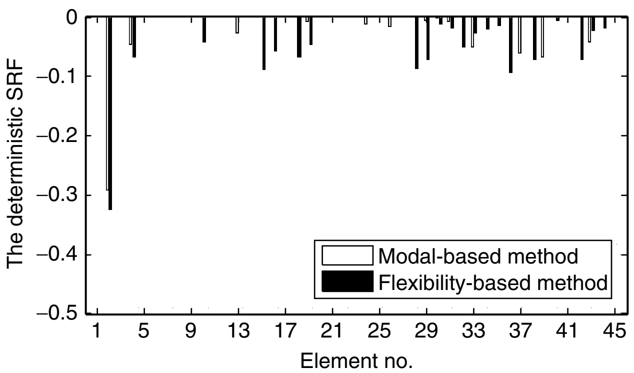

Figure 15. SRF of the three-story frame in the first damage configuration

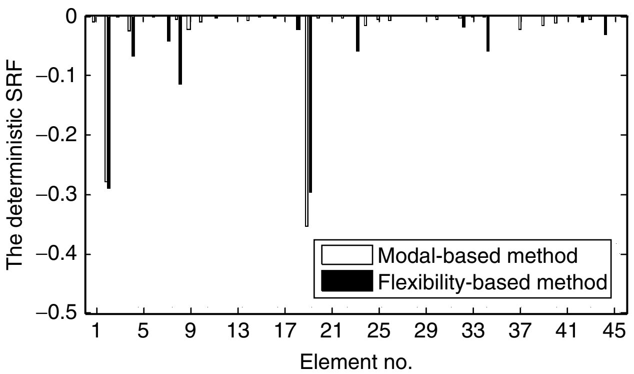

Figure 16. SRF of the three-story frame in the second damage configuration

6. CONCLUSIONS

Based on the same modal data, the modal-parameter-based and flexibility-based damage identification methods are extensively compared using one numerical and two experimental examples. Measurement noise is considered in the numerical example. It shows that the modal-parameter-based can obtain more accurate mean value and smaller standard deviation of the elemental stiffness parameters than the other, indicating that the former can provide robust damage identification results. The experimental study on the one-story and three-story frames also verifies that the flexibility-based method cannot always give the correct damage location and damage severity, as compared with the modal-parameter-based method. One inherent reason might be because the modal-parameter-based method can consider different degrees of accuracy of the measured frequency and mode shape via the weight factor, whereas the other method cannot. In practice, mode shapes always comprise larger noise than the frequencies. Introducing the weight factors in the modal-based method can downplay the noise effect on the model updating, causing more accurate results. This might be one disadvantage of the flexibility-based damage detection method and other modal variant methods, in which the damage indicators are derived from the modal parameter of the structure.

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