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The classification of the Ricci and Plebański tensors in general relativity using Newman–Penrose formalism

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A list is given of a canonical set of the Newman–Penrose quantities Φ_{AB}, the tetrad components of the trace-free Ricci tensor, for each Plebański class according to Plebański's classification of this tensor. This comparative list can easily be extended to cover the classification in tetrad language of any second-order, trace-free, symmetric tensor in a space-time. A fourth-order tensor which is the product of two such tensors was defined by Plebański and used in his classification. This has the same symmetries as the Weyl tensor. The Petrov classification of this tensor, here called the Plebański tensor, is discussed along with the classification of the Ricci tensor. The use of the Plebański tensor in a couple of areas of general relativity is also briefly discussed.

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1. INTRODUCTION

Many authors have discussed the algebraic classification of the Ricci tensor in general relativity; see, for example, Refs. 1–9. Most of these authors have been concerned with obtaining canonical classifications of the trace-free Ricci tensor with components

$$S_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} R \tag{1.1}$$

(or, equivalently, any symmetric, rank-2, trace-free tensor in space-time). The canonical classification scheme of Plebański[2] will be used as the basic scheme in this paper.

There are two purposes in writing another paper in this area. The first one is to give a canonical list of the trace-free Ricci tensor components for each Plebański class in terms of the tetrad formalism of Newman and Penrose[10] (hereafter called NP), i.e., in terms of the NP quantities Φ_{AB} $(A = 0\text{–}2)$. Such a list does not seem to have been published before, although Hall[5] has an equivalent list in which he compares the Plebański scheme with a classification of the tetrad components of the trace-free Ricci tensor for a null basis. His scheme is thus equivalent to, but notationally different from, one using NP language. Ludwig and Scanlon[4] also list canonical types, but in a comparison with their classification scheme, which is very different from that of Plebański.

The second purpose is to emphasize a second-order tensor which Plebański makes some use of in his classification scheme but which is hardly used elsewhere. We call this the Plebański tensor. It is constructed from the product of two trace-free Ricci tensors and has the same symmetries as the Weyl tensor. This last fact is very important because it means that the Plebański tensor can be classified in the same way as the Weyl tensor; i.e., according to the usual Petrov (or Penrose–Petrov) scheme. Its use in a couple of areas of general relativity will be referred to later.

2. THE PLEBAŃSKI TENSOR

From the trace-free Ricci tensor S with components as in (1.1), form the fourth-order tensor with components

$$P^{\alpha\beta}{}_{\gamma\sigma} = S^{[\alpha}{}_{[\gamma} S^{\beta]}{}_{\sigma]} + \delta^{[\alpha}{}_{[\gamma} S_{\sigma]\lambda} S^{\beta]\lambda} - \tfrac{1}{6}\delta^{[\alpha}{}_{[\gamma}\delta^{\beta]}{}_{\sigma]} S_{\mu\nu} S^{\mu\nu}. \tag{2.1}$$

This is trace-free, i.e.,

$$P^{\alpha\beta}{}_{\alpha\sigma} = 0. \tag{2.2}$$

and has the same symmetries as the Weyl tensor. We shall call it the Plebański tensor. Its spinor equivalent is

$$\chi_{ABCD} = \tfrac{1}{4}\Phi_{(AB}{}^{E'F'}\Phi_{CD)E'F'}, \tag{2.3}$$

where the $\Phi_{ABC'D'}$ are the spinor equivalents of the $S_{\mu\nu}$ as defined by NP. χ_{ABCD} is the same as Plebański's V_{ABCD} [Ref. 2, Eq. (6.2)]. Then, by using the definitions $\Phi_{00} = \Phi_{000'0'}$, etc. as given by NP, (2.3) gives

$$\chi_0 \equiv \chi_{0000} = \tfrac{1}{2}(\Phi_{00}\Phi_{02} - \Phi_{01}{}^2), \tag{2.4a}$$

$$\chi_1 \equiv \chi_{0001} = \tfrac{1}{4}(\Phi_{00}\Phi_{12} + \Phi_{10}\Phi_{02} - 2\Phi_{01}\Phi_{11}), \tag{2.4b}$$

$$\chi_2 \equiv \chi_{0011} = \tfrac{1}{12}(\Phi_{00}\Phi_{22} - 4\Phi_{11}{}^2 + \Phi_{02}\Phi_{20} + 4\Phi_{10}\Phi_{12} - 2\Phi_{21}\Phi_{01}), \tag{2.4c}$$

$$\chi_3 \equiv \chi_{0111} = \tfrac{1}{4}(\Phi_{22}\Phi_{10} + \Phi_{12}\Phi_{20} - 2\Phi_{21}\Phi_{11}), \tag{2.4d}$$

$$\chi_4 \equiv \chi_{1111} = \tfrac{1}{2}(\Phi_{22}\Phi_{20} - \Phi_{21}{}^2). \tag{2.4e}$$

These χ_A now transform under the action of the homogeneous Lorentz group at a point in the same way as the tetrad components of the Weyl tensor [the NP Ψ_A $(A = 0\text{–}4)$]. For example, where **l**, **n**, **m**, $\bar{\mathbf{m}}$ are the four basis null vectors used by NP to span a given space-time (with **l** and **n** real and **m** complex), under the two-parameter group of null rotations leaving **l** fixed:

$$\begin{aligned} \tilde{l}_\mu &= l, \\ \tilde{n}_\mu &= z\bar{z} l_\mu + n_\mu + z\bar{m}_\mu + \bar{z} m_\mu, \\ \tilde{m}_\mu &= \bar{z} l_\mu + m_\mu, \end{aligned} \tag{2.5}$$

where z is an arbitrary complex scalar, the χ_A transform in an equivalent way to the Ψ_A:

TABLE. I Comparison of different classification schemes of the trace-free Ricci tensor, S. The table gives a canonical set of nonzero NP quantities Φ_{AB} which are the tetrad components of S for each Plebański class in the classification scheme in Ref. 2. Also given are corresponding Segré characteristics and the Petrov classification of the Plebański tensor constructed from S and defined in the text.

Plebański class	Segré characteristic	Canonical set of nonzero NP Φ_{AB}	Petrov classification of Plebański tensor	
$[T\text{-}S_1\text{-}S_2\text{-}S_3]_4$	[1111]	$\Phi_{00}=\Phi_{22},\Phi_{11},\Phi_{02}=\Phi_{20}$	I	I_a
$[2T\text{-}S_1\text{-}S_2]_3$	[(11)11]	$\Phi_{11},\Phi_{02}=\Phi_{20}$	D	D_{a1}
$[T\text{-}2S_1\text{-}S_2]_3$	[1(11)1]	$\Phi_{00}=\Phi_{22},\Phi_{11}$	D	D_{a2}
$[2T\text{-}2S]_2$	[(11)(11)]	Φ_{11}	D	D_{a3}
$[3T\text{-}S]_2$	[(111)1]	$2\Phi_{11}=\Phi_{02}$	O	O_{a1}
$[T\text{-}3S]_2$	[1(111)]	$\Phi_{00}=\Phi_{22}=2\Phi_{11}$	O	O_{a2}
$[4T]_1$	[(1111)]		O	O_{a3}
$[Z\text{-}\bar{Z}\text{-}S_1\text{-}S_2]_4$	$[Z\bar{Z}11]$	$\Phi_{00}=-\Phi_{22},\Phi_{11},\Phi_{02}=\Phi_{20}$	I	I_b
$[Z\text{-}\bar{Z}\text{-}2S]_3$	$[Z\bar{Z}(11)]$	$\Phi_{00}=-\Phi_{22},\Phi_{11}$	D	D_b
$[2N\text{-}S_1\text{-}S_2]_4$	[211]	$\Phi_{11},\Phi_{22},\Phi_{02}=\Phi_{20}$	II	II
$[2N\text{-}2S]_{(2-1)}$	[2(11)]	Φ_{11},Φ_{22}	D	D_2
$[3N\text{-}S]_3$	[(12)1]	$2\Phi_{11}=\Phi_{02},\Phi_{22}$	N	N_2
$[4N]_2$	[(112)]	Φ_{22}	O	O_2
$[3N\text{-}S]_4$	[31]	$2\Phi_{11}=\Phi_{02},\Phi_{01}\neq\Phi_{10}$	III	III
$[4N]_3$	[(13)]	Φ_{01}	N	N_3

$$\tilde{\chi}_0=\chi_0,$$
$$\tilde{\chi}_1=z\chi_0+\chi_1,$$
$$\tilde{\chi}_2=z^2\chi_0+2z\chi_1+\chi_2,$$
$$\tilde{\chi}_3=z^3\chi_0+3z^2\chi_1+3z\chi_2+\chi_3,$$
$$\tilde{\chi}_4=z^4\chi_0+4z^3\chi_1+6z^2\chi_2+4z\chi_3+\chi_4. \tag{2.6}$$

For other null notations, such as boosts in the **l**-**n** plane and rotations in the **m**-$\bar{\mathbf{m}}$ plane, the χ's transform as do the Ψ's and direct replacements of the Ψ's can be made in tables of transformations of the Ψ's, e.g., in the Tables E-3, 4, 5 in Ref. 11. These transformations can be used to find canonical forms of the χ's as for the Ψ's; e.g., χ is of Petrov (or Penrose–Petrov) type D iff rotations of the null vectors can be made such that χ_2 is the only nonzero χ_A.

The Φ_{AB}, of course, also change under the action of this group. A study of the changes of the quantity $\Phi_{22}\,\Phi_{20}-\Phi_{21}{}^2$ as in (2.4e) under (2.5) will lead to the change of χ_4 as in (2.6) and similarly with the other combinations. It was noticing this behavior that led one of us (A.W.-C.L.) to construct (2.4). MacCallum pointed out to one of us in a private communication that (2.4) was equivalent to (2.3) (and hence equivalent to Plebański's V_{ABCD}).

The relationship between $P_{\alpha\beta\gamma\delta}$ and the χ_A is

$$P_{\alpha\beta\gamma\delta}=16(Q_{\alpha\beta\gamma\delta}+\bar{Q}_{\alpha\beta\gamma\delta}), \tag{2.7}$$
$$\begin{aligned}Q_{\alpha\beta\gamma\delta}=\chi_0 Z^1{}_{\alpha\beta}Z^1{}_{\gamma\delta}&+\chi_1(Z^1{}_{\alpha\beta}Z^2{}_{\gamma\delta}+Z^2{}_{\alpha\beta}Z^1{}_{\gamma\delta})\\&+\chi_2(Z^2{}_{\alpha\beta}Z^2{}_{\gamma\delta}+Z^1{}_{\alpha\beta}Z^3{}_{\gamma\delta}+Z^3{}_{\alpha\beta}Z^1{}_{\gamma\delta})\\&-\chi_3(Z^3{}_{\alpha\beta}Z^2{}_{\gamma\delta}+Z^2{}_{\alpha\beta}Z^3{}_{\gamma\delta})+\chi_4 Z^3{}_{\alpha\beta}Z^3{}_{\gamma\delta},\end{aligned}$$

where the Z^i are defined as[12]

$$\begin{aligned}Z^1&\equiv Z^1{}_{\alpha\beta}dx^\alpha\wedge dx^\beta=\bar{\mathbf{m}}\wedge\mathbf{n},\\Z^2&\equiv Z^2{}_{\alpha\beta}dx^\alpha\wedge dx^\beta=\mathbf{n}\wedge\mathbf{l}-\bar{\mathbf{m}}\wedge\mathbf{m},\\Z^3&\equiv Z^3{}_{\alpha\beta}dx^\alpha\wedge dx^\beta=\mathbf{l}\wedge\mathbf{m}.\end{aligned} \tag{2.8}$$

3. CLASSIFICATION OF THE RICCI AND PLEBAŃSKI TENSORS

We present in Table I without derivation a list of a canonical set of nonzero NP quantities Φ_{AB} for each Plebański type of the trace-free Ricci tensor. Also in this table are corresponding lists of basic Petrov types [I (a or b), II, and III], Segré characteristics, and the Petrov classification of the corresponding Plebański tensor (2.1) [or spinor equivalent (2.3)]. A diagramatic representation of the degeneracies in the four main classes in Table I is given in Fig. 1.

A number of comments need to be made about the table. The entries of the nonzero Φ_{AB} are modulo a change $\mathbf{l}\leftrightarrow\mathbf{n}$. For example, in $[4N]_2$ the entry of Φ_{22} is equivalent to an entry of Φ_{00}. A given set of Φ_{AB} from some metric can be

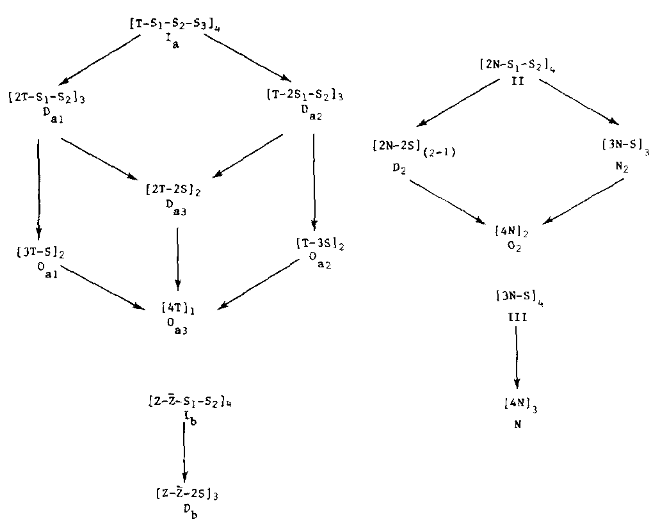

FIG. 1. Diagramatic representation of the degeneracies in the four main classes in the classification scheme in Table I. The Plebański class is given together with the Petrov classification of the Plebański tensor for each class.

put into the form of one of the cases listed in the table by means of the rotations of the base vectors of the type (2.5) or others discussed, for example in Ref. 11 (Tables E-3,4,5). The notation of the Plebański classes follows that in Ref. 2, e.g., $[T\text{-}S_1\text{-}S_2\text{-}S_3]_4$ becomes $[T\text{-}S_1\text{-}S_2\text{-}S_3]_{(1\text{-}1\text{-}1\text{-}1)}$ in Ref. 3. The classification of the tensor depends on the eigenvectors and character of the eigenspaces of that tensor, The symbol T, N, or S is used if the eigenspace of an eigenvalue contains a timelike, no timelike but a null eigenvector, or only spacelike eigenvectors, respectively, Z and $\bar{Z}$ refer to the eigenspace of a pair of complex conjugate eigenvalues. The numbers in round brackets specify the indices of nilpotency in the order in which the eigenvalues appear in the square bracket. In all but one case this is replaced by a single figure which is the order of the minimal equation. This together with the numbers of the kinds of eigenvectors determines uniquely the distribution of the nilpotent indices. (This would not be the case if $[2N\text{-}2S]_3$ was used). The Segré characteristic list should be self-evident. It is described by Hall[5] and used by a number of authors.

The classification of the Plebański tensor formed from the nonzero Φ_{AB} in each case is equivalent to the standard Penrose–Petrov classification of the Weyl tensor. Two columns are used in the table; one with the symbols I, II, III, D, N, and O which is all that could be used if the χ_A alone were given, and I_a,D_{a1} ... which can be used to distinguish between the types if more information, such as all the Φ_{AB}, is given. Note that the types D, N, and O here do not overlap with the types D, N and O as given by Goenner and Stachel[13] as they are classifying different quantities.

The classification used in Table I can be used to classify any trace-free symmetric rank-two tensor in a space-time; not just the Ricci one. It, is in fact, used in this way in a related paper.[14]

Other types of classification schemes as proposed by various authors could have been added to Table I, but were not in order to keep the table from becoming too cluttered.

4. CLASSIFICATION OF MORE GENERAL TENSORS

The classification of a second-order trace-free symmetric tensor such as the trace-free Ricci tensor **S** is straightforward and discussed in many places. Here the canonical forms of the NP quantities Φ_{AB} equivalent to the $S_{\mu\nu}$ are given for each Plebański class. However an entirely equivalent list holds for the tetrad components of any second-order symmetric tensor $X_{\mu\nu}$. Then X_{AB} ($A = 1$–4) can be defined as

$$\begin{aligned}
&X_{11} = X_{\mu\nu} l^\mu l^\nu, && X_{22} = X_{\mu\nu} n^\mu n^\mu,\\
&X_{33} = X_{\mu\nu} m^\mu m^\nu, && X_{44} = X_{\mu\nu} \bar{m}^\nu \bar{m}^\nu,\\
&X_{13} = X_{\mu\nu} l^\mu m^\nu, && X_{14} = X_{\mu\nu} l^\mu \bar{m}^\nu,\\
&X_{23} = X_{\mu\nu} n^\mu m^\nu, && X_{24} = X_{\mu\nu} n^\mu \bar{m}^\nu,\\
&X_{1234} \equiv X_{12} + X_{34} = X_{\mu\nu}(l^\mu n^\nu + m^\mu \bar{m}^\nu). &&
\end{aligned} \tag{4.1}$$

The trace of **X** is

$$X = X_{\mu\nu} g^{\mu\nu} = 2(X_{12} - X_{34}) = 2X_{\mu\nu}(l^\mu n^\nu - m^\mu \bar{m}^\nu). \tag{4.2}$$

If $X_{\mu\nu} = S_{\mu\nu}$, then

$$\begin{aligned}
&\Phi_{00} = -\tfrac{1}{2}X_{11}, && \Phi_{22} = -\tfrac{1}{2}X_{22},\\
&\Phi_{01} = -\tfrac{1}{2}X_{13}, && \Phi_{02} = -\tfrac{1}{2}X_{33},\\
&\Phi_{12} = -\tfrac{1}{2}X_{23}, && \Phi_{11} = -\tfrac{1}{4}X_{1234},
\end{aligned} \tag{4.3}$$

together with $X_{\cdot 4} = \bar{X}_{\cdot 3}$. The pattern of the indices A and B in the Φ_{AB} comes from that in their spinor equivalents: $\Phi_{01} = \Phi_{000'1'}$ etc., while in the X_{AB} it comes from vectors with which the inner products of the $X_{\mu\nu}$ are taken to form the X_{AB}.

In Table I, the column which contains the canonical set of Φ_{AB} could be replaced easily by one containing a canonical set of X_{AB} by using (4.3).

5. DISCUSSION

As stated in the Introduction, one purpose of this paper is to highlight the tensor **P** defined by (2.1) and which we have called the Plebański tensor. This tensor is little used in the literature but it seems to us that it should be used more often in the classification of second-order tensors. One of its main properties is that is has the same symmetries as the Weyl tensor.

McIntosh and Halford[14] (see also McIntosh[15] and McIntosh and van Leeuwen[16]) discuss solutions $x_{\mu\nu}$ of the equation

$$x_{\mu\nu} R^\mu{}_{\lambda\alpha\beta} + x_{\mu\lambda} R^\mu{}_{\nu\alpha\beta} = 0, \tag{5.1}$$

where $x_{\mu\nu}$ in not proportional to the $g_{\mu\nu}$, the components of the metric tensor g from which the $R^\mu{}_{\nu\alpha\beta}$ are constructed. This equation arises in particular in the study of curvature collineations and of the holonomy group in general relativity. It is shown there that the Petrov type of the Plebański tensor formed from such a nontrivial solution x of (5.1) is the same as the Petrov type of the Weyl tensor of the metric tensor for which such a solution exists. It is also almost always the same as the Petrov type of the Plebański tensor formed from the Ricci tensor for such a **g**. Likewise in a paper by MacCallum[17] on locally isotropic spacetimes with non-null homogeneous hypersurfaes, an examination of the Segré types (or, equivalently, Plebański classes) of the Ricci tensor of all those metrics as listed by MacCallum shows that once again the Petrov types of the Weyl and Plebański tensors are always the same. In both these cases it is not just that the Petrov types are the same but also that the repeated principal null directions of the Weyl and Plebański tensors align. MacCallum says in a private communication that these results are expected in his case because of the restrictions on the Weyl tensor and, in the same way, on the Plebański tensor of space-times which can admit isotropy groups (see, e.g., Theorem 2-2.6 by Ehlers and Kundt in Ref. 18). These properties will be discussed by one of us (C.McI) elsewhere.

Note added in proof: The components (2.4) of the Plebański tensor were published by Collinson and Shaw, Int. J. Theor. Phys. **6**, 347 (1972), in a discussion on the Rainich conditions for neutrino fields. They called the tensor (2.1) the Weyl Square. Another algebraic classification of the Ricci tensor, or, with the Einstein field equations holding, the matter tensor, in terms of NP quantities was published by Dozmorov, Sov. Phys. J. **16**, 1708 (1973)—English translation. This account, however, does not take into account all the degeneracies of the various classes in any sufficient way.

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