JOURNAL OF MATHEMATICAL PHYSICS **49**, 113102 (2008)

Analytical solutions to the Navier–Stokes equations

Yuen Manwai[a)]

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

(Received 1 August 2008; accepted 14 October 2008; published online 12 November 2008)

With the previous results for the analytical blowup solutions of the N-dimensional $(N\geq 2)$ Euler–Poisson equations, we extend the same structure to construct an analytical family of solutions for the isothermal Navier–Stokes equations and pressureless Navier–Stokes equations with density-dependent viscosity. © *2008 American Institute of Physics.* [DOI: 10.1063/1.3013805]

I. INTRODUCTION

The Navier–Stokes equations can be formulated in the following form:

$$\begin{aligned} \rho_t + \nabla\cdot(\rho u) &= 0, \\ (\rho u)_t + \nabla\cdot(\rho u\otimes u) + \delta\nabla P &= \mathrm{vis}(\rho,u). \end{aligned} \tag{1}$$

As usual, $\rho=\rho(x,t)$ and $u(x,t)$ are the density and the velocity, respectively. $P=P(\rho)$ is the pressure. We use a γ-law on the pressure, i.e.,

$$P(\rho) = K\rho^{\gamma}, \tag{2}$$

with $K>0$, which is a universal hypothesis. The constant $\gamma=c_P/c_v\geq 1$, where c_p and c_v are the specific heats per unit mass under constant pressure and constant volume, respectively, is the ratio of the specific heats. γ is the adiabatic exponent in (2). In particular, the fluid is called isothermal if $\gamma=1$. It can be used for constructing models with nondegenerate isothermal fluid. δ can be the constant 0 or 1. When $\delta=0$, we call the system pressureless; when $\delta=1$, we call that it is with pressure. Additionally, $\mathrm{vis}(\rho,u)$ is the viscosity function. When $\mathrm{vis}(\rho,u)=0$, the system (1) becomes the Euler equations. For the detailed study of the Euler and Navier–Stokes equations, see Refs. 1 and 4. In the first part of this article, we study the solutions of the N-dimensional $(N\geq 1)$ isothermal equations in radial symmetry,

$$\begin{aligned} \rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u &= 0, \\ \rho(u_t + uu_r) + \nabla K\rho &= \mathrm{vis}(\rho,u). \end{aligned} \tag{3}$$

Definition 1: (Blowup) *We say a solution blows up if one of the following conditions is satisfied.*

(1) *The solution becomes infinitely large at some point x and some finite time T.*
(2) *The derivative of the solution becomes infinitely large at some point x and some finite time T.*

For the formation of singularity in the three-dimensional case for the Euler equations, please refer to the paper of Sideris.[10] In this article, we extend the results from the study of the (blowup) analytical solutions in the N-dimensional $(N\geq 2)$ Euler–Poisson equations, which describes the

[a)]Electronic mail: nevetsyuen@hotmail.com.

evolution of the gaseous stars in astrophysics[2,3,7,12,13] to the Navier–Stokes equations. For the same kinds of blowup results in the nonisothermal case of the Euler or Navier–Stokes equations, please refer to Refs. 5 and 12.

Recently, in Yuen's results,[13] there exists a family of the blowup solution for the Euler–Poisson equations in the two-dimensional radial symmetry case,

$$\begin{aligned}
&\rho_t + u\rho_r + \rho u_r + \frac{1}{r}\rho u = 0,\\
&\rho(u_t + uu_r) + K\rho_r = -\frac{2\pi\rho}{r}\int_0^r \rho(t,s)s\,ds.
\end{aligned} \tag{4}$$

The solutions are

$$\begin{aligned}
&\rho(t,r) = \frac{1}{a(t)^2}e^{y(r/a(t))}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,\\
&\ddot{a}(t) = -\frac{\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,\\
&\ddot{y}(x) + \frac{1}{x}\dot{y}(x) + \frac{2\pi}{K}e^{y(x)} = \mu, \quad y(0) = \alpha, \quad \dot{y}(0) = 0,
\end{aligned} \tag{5}$$

where $K>0$, $\mu=2\lambda/K$ with a sufficiently small λ, and α are constants.

(1) When $\lambda>0$, the solutions blow up in a finite time T.
(2) When $\lambda=0$, if $a_1<0$, the solutions blow up at $t=-a_0/a_1$.

In this paper, we extend the above result to the isothermal Navier–Stokes equations in radial symmetry with the usual viscous function

$$\mathrm{vis}(\rho,u) = \upsilon\Delta u,$$

where υ is a positive constant,

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \tag{6a}$$

$$\rho(u_t + uu_r) + K\rho_r = \upsilon\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right). \tag{6b}$$

Theorem 2: *For the N-dimensional isothermal Navier–Stokes equations in radial symmetry (6a) and (6b), there exists a family of solutions; those are*

$$\begin{aligned}
&\rho(t,r) = \frac{1}{a(t)^N}e^{y(r/a(t))}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,\\
&\ddot{a}(t) = \frac{-\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,\\
&y(x) = \frac{\lambda}{2K}x^2 + \alpha,
\end{aligned} \tag{7}$$

where α and λ are arbitrary constants.

In particular, for $\lambda > 0$, the solutions blow up in finite time T.

In the last part, the corresponding solutions to the pressureless Navier–Stokes equations with density-dependent viscosity are also studied.

II. THE ISOTHERMAL ($\gamma=1$) CASES

Before we present the proof of Theorem 2, Lemma 6 of Ref. 13 could be needed to further extend to the N-dimensional space.

Lemma 3: (The Extension of Lemma 6 of Ref. 13) *For the equation of conservation of mass in radial symmetry,*

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \tag{8}$$

there exist solutions

$$\rho(t,r) = \frac{f(r/a(t))}{a(t)^N}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r, \tag{9}$$

with the form $f \geq 0 \in C^1$ and $a(t) > 0 \in C^1$.

Proof: We just plug (9) into (8). Then

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = \frac{-N\dot{a}(t)f(r/a(t))}{a(t)^{N+1}} - \frac{\dot{a}(t)r\dot{f}(r/a(t))}{a(t)^{N+2}} + \frac{\dot{a}(t)r}{a(t)}\frac{\dot{f}(r/a(t))}{a(t)^{N+1}} + \frac{f(r/a(t))}{a(t)^N}\frac{\dot{a}(t)}{a(t)} + \frac{N-1}{r}\frac{f(r/a(t))}{a(t)^N}\frac{\dot{a}(t)}{a(t)}r = 0.$$

The proof is completed. ■

Besides, Lemma 7 of Ref. 13 is also useful. For the better understanding of the lemma, the proof is given here.

Lemma 4: (Lemma 7 of Ref. 13) *For the Emden equation,*

$$\ddot{a}(t) = -\frac{\lambda}{a(t)}, \qquad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \tag{10}$$

we have that, if $\lambda > 0$, there exists a finite time $T_{-} < +\infty$ such that $a(T_{-})=0$.

Proof: By integrating (10), we have

$$0 \leq \tfrac{1}{2}\dot{a}(t)^2 = -\lambda \ln a(t) + \theta, \tag{11}$$

where $\theta = \lambda \ln a_0 + \frac{1}{2}a_1^2$.

From (11), we get

$$a(t) \leq e^{\theta/\lambda}.$$

If the statement is not true, we have

$$0 < a(t) \leq e^{\theta/\lambda} \quad \text{for all} \quad t \geq 0.$$

However, since

$$\ddot{a}(t) = -\frac{\lambda}{a(t)} \leq \frac{-\lambda}{e^{\theta/\lambda}},$$

we integrate this twice to deduce

$$a(t) \leq \int_0^t \int_0^\tau \frac{-\lambda}{e^{\theta/\lambda}} ds d\tau + C_1 t + C_0 = \frac{-\lambda t^2}{2e^{\theta/\lambda}} + C_1 t + C_0.$$

By taking t large enough, we get

$$a(t) < 0.$$

As a contradiction is met, the statement of the lemma is true. ■

By extending the structure of the solutions (5) to the two-dimensional isothermal Euler–Poisson equations (4) in Ref. 13, it is a natural result to get the proof of Theorem 2.

Proof of Theorem 2: By using Lemma 3, we can get that (7) satisfy (6a). For the momentum equation, we have

$$\rho(u_t + u \cdot u_r) + K\rho_r - \upsilon\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right) = \rho\frac{\ddot{a}(t)}{a(t)}r + \frac{K}{a(t)}\rho\dot{y}\left(\frac{r}{a(t)}\right)$$
$$= \frac{\rho}{a(t)}\left[-\frac{\lambda r}{a(t)} + K\dot{y}\left(\frac{r}{a(t)}\right)\right].$$

By choosing

$$y(x) = \frac{\lambda}{2K}x^2 + \alpha,$$

we have verified that (7) satisfies the above (6b). If $\lambda > 0$, by Lemma 4, there exists a finite time T for such that $a(T_-) = 0$. Thus, there exist blowup solutions in finite time T. The proof is completed. ■

With the assistance of the blowup rate results of the Euler–Poisson equations, i.e., Theorem 3 in Ref. 13, it is trivial to have the following theorem.

Theorem 5: *With $\lambda > 0$, the blowup rate of the solutions (7) is*

$$\lim_{t \to T_*} \rho(t,0)(T_* - t)^\alpha \geq O(1),$$

where the blowup time T_* and $\alpha < N$ are constants.

Remark 6: *If we are interested in the mass of the solutions, the mass of the solutions can be calculated by*

$$M(t) = \int_{R^N} \rho(t,s) ds = \alpha(N)\int_0^{+\infty} \rho(t,s)s^{N-1} ds,$$

where $\alpha(N)$ denotes some constant related to the unit ball in R^N: $\alpha(1)=1$; $\alpha(2)=2\pi$; for $N \geq 3$,

$$\alpha(N) = N(N-2)V(N) = N(N-2)\frac{\pi^{N/2}}{\Gamma(N/2+1)},$$

where $V(N)$ is the volume of the unit ball in R^N and Γ is the gamma function. We observe the following for the mass of the initial time 0:

(1) *For $\lambda \geq 0$,*

$$M(0) = \frac{\alpha(N)}{a_0^N}\int_0^{+\infty} e^{(\lambda/2K)s^2+\alpha} s^{N-1} ds.$$

The mass is infinitive. The very large density comes from the ends outside the origin O.

(2) *For* $\lambda < 0$,

$$M(0) = \frac{\alpha(N)}{a_0^N}\int_0^{+\infty} e^{(\lambda/2K)s^2+\alpha} s^{N-1} ds = \frac{\alpha(N)e^{\alpha}}{a_0^N}\int_0^{+\infty} e^{(\lambda/2K)s^2} s^{N-1} ds.$$

The mass of the solution can be arbitrarily small but without compact support if α *is taken to be a very small negative number.*

Remark 7: *Our results can be easily extended to the isothermal Euler/Navier–Stokes equations with frictional damping term with the assistance of Lemma 7 in Ref. 12,*

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0,$$

$$\rho(u_t + u \cdot u_r) + K\rho_r + \beta\rho u = v\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right),$$

where $\beta \geq 0$ *and* $v \geq 0$. *The solutions are*

$$\rho(t,r) = \frac{e^{y(r/a(t))}}{a(t)^N}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,$$

$$\ddot{a}(t) + \beta\dot{a}(t) = \frac{-\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

$$y(x) = \frac{\lambda}{2K}x^2 + \alpha.$$

Remark 8: *Our results can be easily extended to the isothermal Euler/Navier–Stokes equations with frictional damping term with the assistance of Lemma 7 in Ref. 12,*

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0,$$

$$\rho(u_t + u \cdot u_r) + K\rho_r + \beta\rho u = v\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right),$$

where $\beta \geq 0$ *and* $v \geq 0$. *The solutions are*

$$\rho(t,r) = \frac{e^{y(r/a(t))}}{a(t)^N}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,$$

$$\ddot{a}(t) + \beta\dot{a}(t) = \frac{-\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

$$y(x) = \frac{\lambda}{2K}x^2 + \alpha.$$

Remark 9: *The solutions (5) to the Euler–Poisson equations only work for the two-dimensional case. However, the solutions (7) to the Navier–Stokes equations work for the N-dimensional* $(N \geq 1)$ *case.*

Remark 10: *We may extend the solutions to the two-dimensional Euler/Navier–Stokes equations with a solid core,*[6]

$$\rho_t + u\rho_r + \rho u_r + \frac{1}{r}\rho u = 0,$$

$$\rho(u_t + uu_r) + K\rho_r + \beta\rho u = \frac{M_0}{r} + v\left(u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u\right),$$

where $M_0 > 0$*; there is a unit stationary solid core locating* $[0, r_0]$*, where* r_0 *is a positive constant, surrounded by the distribution density. The corresponding solutions are*

$$\rho(t,r) = \frac{e^{y(r/a(t))}}{a(t)^2}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r \quad \text{for } r > r_0,$$

$$\ddot{a}(t) + \beta\dot{a}(t) = \frac{-\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

$$y(x) = \frac{\lambda}{2K}x^2 + M_0 \ln x + \alpha,$$

where $\alpha > -\lambda/2K$ *is a constant.*

III. PRESSURELESS NAVIER–STOKES EQUATIONS WITH DENSITY-DEPENDENT VISCOSITY

Now we consider the pressureless Navier–Stokes equations with density-dependent viscosity,

$$\text{vis}(\rho,u) \doteq \nabla(\mu(\rho)\nabla \cdot u),$$

in radial symmetry,

$$\begin{aligned} &\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \\ &\rho(u_t + uu_r) = (\mu(\rho))_r u_r + \mu(\rho)\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right), \end{aligned} \tag{12}$$

where $\mu(\rho)$ is a density-dependent viscosity function, which is usually written as $\mu(\rho) \doteq \kappa\rho^\theta$ with the constants κ, $\theta > 0$. For the study of this kind of the above system, the readers may refer to Refs. 8, 9, and 11.

We can obtain the same estimate about Lemma 4 to the following ordinary differential equation (ODE):

$$\begin{aligned} &\ddot{a}(t) = \frac{\lambda\dot{a}(t)}{a(t)^2}, \\ &a(0) = a_0 > 0, \quad \dot{a}(0) = a_1 \leq \frac{\lambda}{a_0}. \end{aligned} \tag{13}$$

Lemma 11: *For the ODE (13), with $\lambda > 0$, there exists a finite time $T_- < +\infty$ such that $a(T_-)=0$.*

Proof: If $a(t) > 0$ and $\dot{a}(0)=a_1 \leq \lambda / a_0$ for all time t, by integrating (13), we have

$$\dot{a}(t) = -\frac{\lambda}{a(t)} - \frac{\lambda}{a_0} + a_1 \leq -\frac{\lambda}{a(t)}. \tag{14}$$

Take the integration for (14),

$$\int_0^t a(s)\dot{a}(s)ds \leq -\int_0^t \lambda ds,$$

$$\tfrac{1}{2}[a(t)]^2 \leq -\lambda t + \tfrac{1}{2}a_0^2.$$

When t is very large, we have

$$\tfrac{1}{2}[a(t)]^2 \leq -1.$$

A contradiction is met. The proof is completed. ∎

Here we present another lemma before proceeding to the next theorem.

Lemma 12: *For the ODE,*

$$\dot{y}(x)y(x)^n - \xi x = 0,$$
$$y(0) = \alpha > 0, \quad n \neq -1, \tag{15}$$

where ξ and n are constants, we have the solution

$$y(x) = \sqrt[n+1]{\tfrac{1}{2}(n+1)\xi x^2 + \alpha^{n+1}}.$$

Proof: The above ODE (15) may be solved by the separation method,

$$\dot{y}(x)y(x)^n - \xi x = 0,$$

$$\dot{y}(x)y(x)^n = \xi x.$$

By taking the integration with respect to x,

$$\int_0^x \dot{y}(x)y(x)^n dx = \int_0^x \xi x dx,$$

we have

$$\int_0^x y(x)^n d[y(x)] = \tfrac{1}{2}\xi x^2 + C_1, \tag{16}$$

where C_1 is a constant.

By integration by part, then the identity becomes

$$y(x)^{n+1} - n\int_0^x y(x)^{n-1}\dot{y}(x)y(x)dx = \frac{1}{2}\xi x^2 + C_1,$$

$$y(x)^{n+1} - n\int_0^x \dot{y}(x)y(x)^n dx = \frac{1}{2}\xi x^2 + C_1.$$

From Eq. (16), we can have the simple expression for $y(x)$,

$$y(x)^{n+1} - n\left(\tfrac{1}{2}\xi x^2 + C_1\right) = \tfrac{1}{2}\xi x^2 + C_1,$$

$$y(x)^{n+1} = \tfrac{1}{2}(n+1)\xi x^2 + C_2,$$

where $C_2=(n+1)C_1$.

By plugging into the initial condition for $y(0)$, we have

$$y(0)^{n+1} = \alpha^{n+1} = C_2.$$

Thus, the solution is

$$y(x) = \sqrt[n+1]{\tfrac{1}{2}(n+1)\xi x^2 + \alpha^{n+1}}.$$

The proof is completed. ∎

The family of the solution to the pressureless Navier–Stokes equations with density-dependent viscosity,

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \tag{17a}$$

$$\rho(u_t + uu_r) = (\kappa\rho^\theta)_r u_r + \kappa\rho^\theta\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right), \tag{17b}$$

is presented as the following.

Theorem 13: *For the pressureless Navier–Stokes equations with density-dependent viscosity (17a) and (17b) in radial symmetry, there exists a family of solutions.*

For $\theta=1$,

$$\rho(t,r) = \frac{e^{y(r/a(t))}}{a(t)^N}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,$$

$$\ddot{a}(t) = \frac{\lambda\dot{a}(t)}{a(t)^2}, \quad a(0)=a_0>0, \quad \dot{a}(0)=a_1,$$

$$y(x) = \frac{\lambda}{2\kappa}x^2 + \alpha,$$

where α *and* λ *are arbitrary constants. In particular, for* $\lambda>0$ *and* $a_1 \le \lambda/a_0$, *the solutions blow up in finite time. For* $\theta\neq 1$,

$$\rho(t,r) = \begin{cases} \dfrac{y(r/a(t))}{a(t)^N} & \text{for } y\left(\dfrac{r}{a(t)}\right) \ge 0; \\ 0 & \text{for } y\left(\dfrac{r}{a(t)}\right) < 0 \end{cases}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,$$

$$\ddot{a}(t) = \frac{-\lambda\dot{a}(t)}{a(t)^{N\theta-2N+2}}, \quad a(0)=a_0>0, \quad \dot{a}(0)=a_1, \tag{18}$$

$$y(x)=\sqrt[\theta-1]{\frac{1}{2}(\theta-1)\frac{-\lambda}{\kappa\theta}x^2+\alpha^{\theta-1}},$$

where $\alpha>0$.

Proof of Theorem 13: To (17a), we may use Lemma 3 to check it. For $\theta=1$, (17b) becomes

$$\rho(u_t+u\cdot u_r)-(\kappa\rho)_r u_r-\kappa\rho_r\left(u_{rr}+\frac{N-1}{r}u_r-\frac{N-1}{r^2}u\right)$$

$$=\rho(u_t+u\cdot u_r)-(\kappa\rho_r u_r)=\rho\frac{\ddot{a}(t)}{a(t)}r-\left(\frac{\kappa e^{y(r/a(t))}}{a(t)^N}\right)_r\frac{\dot{a}(t)}{a(t)}=\rho\left(\frac{\lambda\dot{a}(t)r}{a(t)^3}\right)-\frac{\kappa e^{y(r/a(t))}\dot{y}\left(\frac{r}{a(t)}\right)}{a(t)^{N+1}}\frac{\dot{a}(t)}{a(t)}$$

$$=\frac{\rho\dot{a}(t)}{a(t)^2}\left(\frac{\lambda r}{a(t)}-\kappa\dot{y}\left(\frac{r}{a(t)}\right)\right),\tag{19}$$

where we use

$$\ddot{a}(t)=\frac{\lambda\dot{a}(t)}{a(t)^2}.$$

By choosing

$$y\left(\frac{r}{a(t)}\right)\doteq y(x)=\frac{\lambda}{2\kappa}x^2+\alpha,$$

(19) is equal to zero.

For the case of $\theta\neq1$, (17b) can be calculated,

$$\rho(u_t+u\cdot u_r)-(\kappa\rho^{\theta})_r u_r-\kappa\rho^{\theta}\left(u_{rr}+\frac{N-1}{r}u_r-\frac{N-1}{r^2}u\right)$$

$$=\rho\frac{\ddot{a}(t)}{a(t)}r-\left[\left(\frac{\kappa y\left(\frac{r}{a(t)}\right)}{a(t)^N}\right)^{\theta}\right]_r\frac{\dot{a}(t)}{a(t)}$$

$$=\rho\left(-\frac{\lambda\dot{a}(t)r}{a(t)^{N\theta-2N+2}a(t)}\right)-\frac{\kappa\theta y\left(\frac{r}{a(t)}\right)^{\theta-1}\dot{y}\left(\frac{r}{a(t)}\right)}{a(t)^{N(\theta-1)}a(t)}\frac{\dot{a}(t)}{a(t)}$$

$$=\rho\left(-\frac{\lambda\dot{a}(t)r}{a(t)^{N\theta-2N+2}a(t)}\right)-\frac{\kappa\theta y\left(\frac{r}{a(t)}\right)y\left(\frac{r}{a(t)}\right)^{\theta-2}\dot{y}\left(\frac{r}{a(t)}\right)\dot{a}(t)}{a(t)^N a(t)^{N\theta-2N+2}}$$

$$=\rho\left(-\frac{\lambda\dot{a}(t)r}{a(t)^{N\theta-2N+2}a(t)}\right)-\frac{\kappa\theta\rho y\left(\frac{r}{a(t)}\right)^{\theta-2}\dot{y}\left(\frac{r}{a(t)}\right)\dot{a}(t)}{a(t)^{N\theta-2N+2}}\tag{20}$$

$$=\frac{-\rho\dot{a}(t)}{a(t)^{N\theta-2N+2}}\left(-\frac{\lambda r}{a(t)}+\kappa\theta y\left(\frac{r}{a(t)}\right)^{\theta-2}\dot{y}\left(\frac{r}{a(t)}\right)\right).\tag{21}$$

Define $x\doteq r/a(t)$, $n\doteq\theta-2$; it follows

$$=\frac{-\rho\dot{a}(t)}{a(t)^{N\theta-2N+2}}(\lambda x+\kappa\theta y(x)^n\dot{y}(x))\tag{22}$$

$$=\frac{-\lambda\rho\dot{a}(t)}{a(t)^{N\theta-2N+2}}\left(x+\frac{\kappa\theta}{\lambda}y(x)^n\dot{y}(x)\right),\tag{23}$$

and $\xi\doteq\lambda/\kappa\theta$ in Lemma 12, and choose

$$y\left(\frac{r}{a(t)}\right) \doteq y(x) = \sqrt[\theta-1]{\frac{1}{2}(\theta-1)\frac{-\lambda}{\kappa\theta}x^2 + \alpha^{\theta-1}}.$$

Moreover this is easy to check that

$$\dot{y}(0) = 0.$$

Equation (22) is equal to zero. The proof is completed. ■

Remark 14: *By controlling the initial conditions in some solutions* (18)*, we may get the blowup solutions. Additionally the modified solutions can be extended to the system in radial symmetry with frictional damping,*

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0,$$

$$\rho(u_t + uu_r) + \beta\rho u = (\mu(\rho))_r u_r + \mu(\rho)\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right),$$

where $\beta > 0$*. With the assistance of the ODE,*

$$\ddot{a}(t) + \beta\dot{a}(t) = \frac{-\lambda\dot{a}(t)}{a(t)^S},$$

$$a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

where S *is a constant.*

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