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SVD-CLOSED SUBGROUPS OF THE UNITARY GROUP: GENERALIZED PRINCIPAL LOGARITHMS AND MINIMIZING GEODESICS

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Abstract. We study the set of generalized principal $\mathfrak{g}$-logarithms of any matrix belonging to a connected SVD-closed subgroup G of U_n, with Lie algebra $\mathfrak{g}$. This set is a non-empty disjoint union of a finite number of subsets diffeomorphic to homogeneous spaces, and it is related to a suitable set of minimizing geodesics. Many particular cases for the group G are explicitly analysed.

Contents

Keywords. Generalized principal logarithm; SVD-decomposition, SVD-closed subgroup, Frobenius metric, minimizing geodesics, (symmetric) homogeneous space.

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Introduction

If M is a matrix belonging to a connected closed subgroup G of $GL_n(\mathbb{C})$, having $\mathfrak{g}$ as Lie algebra, we say that a matrix $L \in \mathfrak{g}$ is a *generalized principal* $\mathfrak{g}$-*logarithm* of M, if $\exp(L) = M$ and $-\pi \leq Im(\lambda) \leq \pi$, for every eigenvalue λ of L; the set of all generalized principal $\mathfrak{g}$-logarithms of M is denoted by $\mathfrak{g}$–$plog(M)$. Our definition relaxes the usual one of *principal logarithm*, which excludes the matrices $M \in GL_n(\mathbb{C})$ with negative eigenvalues (see, for instance, [Higham 2008, Thm. 1.31]). The usual definition implies both existence and uniqueness of a principal logarithm. In some relevant cases, matrices with negative eigenvalues and belonging to a closed subgroup G of $GL_n(\mathbb{C})$, have an infinite set of generalized principal $\mathfrak{g}$–logarithms, on which it is possible to define some natural geometric structures. We have already studied the sets $\mathfrak{so}_n$–*plog*(M), if $M \in SO_n$, and $gl_n(\mathbb{R})$–*plog*(M), if M is semi-simple (see [Dolcetti-Pertici 2018a] and [Pertici 2022]). Our interest in the set $\mathfrak{g}$–*plog*(M) is related to a differential-geometric setting, which we briefly describe. Denote by ϕ the *Frobenius* (or *Hilbert-Schmidt*) positive definite real scalar product on $\mathfrak{gl}_n(\mathbb{C})$, defined by $\phi(A, B) := Re(tr(AB^*))$. If G is a connected closed subgroup G of the unitary group U_n (with Lie algebra $\mathfrak{g}$), we still denote by ϕ the Riemannian metric on G, obtained by restriction of the Frobenius scalar product of $\mathfrak{gl}_n(\mathbb{C})$. This metric is bi-invariant on G and the corresponding geodesics are the curves $\gamma(t) = P\exp(tX)$, where $X \in \mathfrak{g}$ and $P \in G$. The set of minimizing geodesic segments of (G, ϕ) is a classical and relevant subject of investigation.

In this paper we also assume that the group G is *SVD-closed*: a condition satisfied by many closed subgroup of U_n. The reason is that, under this assumption, for every $P_0, P_1 \in G$, the set of minimizing geodesic segments of (G, ϕ) with endpoints P_0 and P_1, can be parametrized by the set of generalized principal $\mathfrak{g}$–logarithms of $P_0^*P_1$ (see Theorem 6.5). Therefore, a geometric structure on $\mathfrak{g}$–*plog*$(P_0^*P_1)$ induces a corresponding structure on the set of minimizing geodesic segments joining P_0 and P_1.

To fully illustrate the statements of the title and of the previous result, we must explain the meaning of *SVD-closure*. Any matrix $M \in \mathfrak{gl}_n(\mathbb{C})\setminus\{0\}$ has a unique decomposition (called *SVD-decomposition* of M) of the form $M = \sum_{i=1}^{p} \sigma_i A_i$, where $\sigma_1 > \sigma_2 > \cdots > \sigma_p > 0$ are the non-zero singular values of M, and $A_1, A_2, \cdots A_p$ are non-zero complex matrices (called *SVD-components* of M) such that $A_h^*A_j = A_hA_j^* = 0$, for every $h \neq j$, and $A_jA_j^*A_j = A_j$, for every j. We say that a real Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}_n(\mathbb{C})$ is *SVD-closed* if, for any matrix $M \in \mathfrak{g} \setminus \{0\}$, all SVD-components of M belong to $\mathfrak{g}$. A closed subgroup of $GL_n(\mathbb{C})$ is *SVD-closed* if its Lie algebra is SVD-closed in $\mathfrak{gl}_n(\mathbb{C})$.

Sections 1 and 2 are devoted to recall many general basic notions and preliminary facts on matrices. In Section 3 we discuss and determine a wide class of SVD-closed real Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$. The key result is that the sets of fixed points of all automorphisms of the real Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, commuting with the map $\eta : A \mapsto A^*$ and preserving the so-called *triple Jordan product*, are SVD-closed real Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$ (see Proposition 3.5). In Section 4, we prove that many classical groups of matrices are SVD-closed, as, for instance, the real general linear group $GL_n(\mathbb{R})$, the unitary group U_n, the special orthogonal complex group $SO_n(\mathbb{C})$, the symplectic groups $Sp_{2n}(\mathbb{C})$, $Sp_{2n}(\mathbb{R})$, the generalized unitary groups $U_{(p,n-p)}$ and all their intersections. In particular, we analyse the following families of SVD-closed subgroups of U_n:

$\langle V \rangle_{U_n} := \{X \in U_n : XV = VX\}$, where V is an arbitrary unitary matrix, $\preccurlyeq Q \succcurlyeq_{U_n} := \{X \in U_n : XQX^T = Q\}$ and $\preccurlyeq Q \succcurlyeq_{SU_n} := \preccurlyeq Q \succcurlyeq_{U_n} \cap SU_n$, where Q is an arbitrary real orthogonal matrix. Among them, we find many classical closed subgroups of U_n, as, for instance, SO_n, Sp_n, $U_{(p,n-p)} \cap U_n$ and $\big(SO_{(p,n-p)}(\mathbb{C})\big) \cap U_n$.

In Section 5 we study the set $\mathfrak{g}$–$plog(M)$ for a matrix M, belonging to a connected SVD-closed subgroup G of U_n, with Lie algebra $\mathfrak{g}$. In particular we prove that $\mathfrak{g}$–$plog(M)$ is non-empty (see Proposition 5.5) and that it is a disjoint union of a finite number of compact submanifolds of $\mathfrak{g}$, each of which is diffeomorphic to a homogeneous space (Theorem 5.7). In Section 6 we obtain some results about of the Riemannian manifold (G, ϕ), where G is any connected SVD-closed subgroup of U_n, and, among them, the already mentioned Theorem 6.5. In addition, we compute the diameter of all connected SVD-closed subgroups of U_n that we considered in Section 4 (see Proposition 6.7).

The main result of Section 7 is Theorem 7.2, in which we prove that, for every $V \in U_n$ and $M \in \langle V \rangle_{U_n}$, the set $\langle V \rangle_{\mathfrak{u}_n}$–$plog(M)$ has a finite number of components, each of which is a simply connected compact submanifold of $\mathfrak{u}_n$, diffeomorphic to the product of suitable complex Grassmannians. Finally, the main result of Section 8 is Theorem 8.5, which states that, for every $Q \in O_n$ and $M \in \preccurlyeq Q \succcurlyeq_{SU_n}$, the set $\preccurlyeq Q \succcurlyeq_{\mathfrak{su}_n}$–$plog(M)$ has a finite number of components, each of which is a simply connected compact submanifold of $\mathfrak{su}_n$, diffeomorphic to the product of suitable complex Grassmannians with the symmetric homogeneous spaces $\dfrac{SO_{2m}}{U_m}$ and $\dfrac{Sp_\mu}{U_\mu}$.

1. Basic notations and some preliminary facts.

1.1. **Notations.**

a) In this paper we will use many standard notations from the matrix theory and from the theory of Lie groups and algebras.

Among these, if $\mathbb{K}$ is either the field of real numbers $\mathbb{R}$, or the field of complex numbers $\mathbb{C}$, or the associative division algebra of quaternions $\mathbb{H}$, then $\mathfrak{gl}_n(\mathbb{K})$ denotes the real Lie algebra of square matrices of order n and $GL_n(\mathbb{K})$ the Lie group of invertible matrices of order n, both with coefficients in $\mathbb{K}$. In any case, the identity matrix and the null matrix of order n are denoted by I_n and by $\mathbf{0}_n$, respectively, and we define also $\mathbb{K}^0 = \{0\}$. As usual, $\mathbf{i}$ is the unit imaginary number of $\mathbb{C}$ and $\mathbf{j}$, $\mathbf{k}$ are the further standard imaginary unities of $\mathbb{H}$, so that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. Note that any $q \in \mathbb{H}$ can be written in a unique way as $q = z + w\mathbf{j}$ with $z, w \in \mathbb{C}$, so that the complex field $\mathbb{C}$ can be identified with the set of quaternions of the form $z + 0 \cdot \mathbf{j}$, with $z \in \mathbb{C}$. We denote by $e^z := \sum_{i=0}^{+\infty} \frac{z^i}{i!}$ the exponential of $z \in \mathbb{C}$ and, if $z \neq 0$, by $\log(z)$, the unique complex logarithm of z, whose imaginary part lies in the interval $(-\pi, \pi]$.

For every $A \in \mathfrak{gl}_n(\mathbb{H})$, A^T, $\overline{A}$, $A^* := \overline{A}^T$ and A^{-1} (provided that A is invertible) are respectively transpose, conjugate, adjoint and inverse of the matrix A and $tr(A)$ is its trace. If $A \in \mathfrak{gl}_n(\mathbb{C})$, $\det(A)$ denotes its determinant, while $\exp(A) := \sum_{i=0}^{+\infty} \frac{A^i}{i!} \in GL_n(\mathbb{C})$ denotes the exponential of the matrix A.

If $M_1, \cdots, M_h$ are square matrices of orders $r_1, \cdots, r_h$, respectively, then $M_1 \oplus \cdots \oplus M_h$ denotes the related block-diagonal square matrix of order $r_1 + \cdots + r_h$. Moreover, if B is a $p \times p$ matrix, then $B^{\oplus h}$ denotes the $ph \times ph$ block-diagonal matrix $\underbrace{B \oplus \cdots \oplus B}_{h \text{ times}}$.

If $\mathcal{S}_1, \ldots, \mathcal{S}_m$ are sets of square matrices, then $\mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_m$ denotes the set of all matrices $B_1 \oplus \cdots \oplus B_m$ with $B_j \in \mathcal{S}_j$, for every j. If the sets $\mathcal{S}_1, \ldots, \mathcal{S}_m$ are mutually disjoint, we write $\bigsqcup_{i=1}^{h} S_i$ to denote their (disjoint) union.

To give a full generality to the results of this paper (and to their proofs), it is necessary to establish agreements on the notations that we will use: if h is a non-negative integer parameter, whenever, in any formula, we write any term as $\sum_{i=1}^{h} (\cdots)$, $\bigoplus_{i=1}^{h} (\cdots)$ or $\prod_{i=1}^{h} (\cdots)$, we mean that, if $h = 0$, this sum, this direct sum or this product must not appear in the related formula. Moreover, if G_n (for $n \geq 1$) denotes any classical Lie groups of matrices of order n, having Lie algebra $\mathfrak{g}_n$, and if H_n is a closed subgroup of G_n, we also assign a meaning to the expressions G_0, $\mathfrak{g}_0$, $\frac{G_0}{H_0}$, defining them all equal to a single point $\mathcal{Q}$ which, conventionally, satisfies the following conditions:

$\lambda\mathcal{Q} = \mathcal{Q}$, for every $\lambda \in \mathbb{C}$; $\mathcal{Q} \oplus B = B \oplus \mathcal{Q} = B$, for any square matrix B; $\mathcal{Q} \oplus \mathcal{S} = \mathcal{S} \oplus \mathcal{Q} = \mathcal{S}$, for any set of square matrices $\mathcal{S}$.

It is also useful to define the zero-order identity matrix I_0 and $M^{\oplus 0}$ (for every square matrix M) both equal to this point $\mathcal{Q}$ and, to simplify the notations and some statements, the complex numbers, which are not eigenvalues of a matrix M, will be called *eigenvalues of multiplicity zero* of M. Furthermore, we denote:

$\Omega := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $\Omega_n := \begin{pmatrix} \mathbf{0}_n & -I_n \\ I_n & \mathbf{0}_n \end{pmatrix}$; hence $\Omega_1 = \Omega$, while, for $n \geq 2$, we have $\Omega_n \neq \Omega^{\oplus n}$;

$W_{(p,q)} := I_p \oplus \mathbf{i} I_q$, for every $p, q \geq 0$ such that $p+q \geq 1$ ($W_{(p,q)}$ is unitary and diagonal);

$E_\varphi := \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} = \cos(\varphi)\, I_2 + \sin(\varphi)\, \Omega$, with $\varphi \in \mathbb{R}$, so $\Omega = E_{\pi/2}$ and $E_\varphi^{\oplus h} = \cos(\varphi) I_{2h} + \sin(\varphi)\Omega^{\oplus h}$ for every $h \geq 1$;

moreover, for every $p, q \geq 0$ with $p+q \geq 1$, $E_\varphi^{(p,q)} := E_\varphi^{\oplus p} \oplus (-E_\varphi)^{\oplus q}$ (so $E_\varphi^{(n,0)} = E_\varphi^{\oplus n}$) and $J^{(p,q)} := I_p \oplus (-I_q) = E_0^{(p,q)}$ (so $J^{(p,0)} = I_p$ and $J^{(0,q)} = -I_q$).

b) As usual, $O_n := \{X \in gl_n(\mathbb{R}) : XX^T = I_n\}$ is the real orthogonal group;
$U_n := \{X \in \mathfrak{gl}_n(\mathbb{C}) : XX^* = I_n\}$ is the (complex) unitary group;
$SO_n := \{X \in O_n : \det(X) = 1\}$, $SU_n := \{X \in U_n : \det(X) = 1\}$ are their special subgroups; while $U_n(\mathbb{H}) := \{X \in \mathfrak{gl}_n(\mathbb{H}) : XX^* = I_n\}$ is the quaternionic unitary group. Note that the identification (recalled in (a)) of $\mathbb{C}$ as a subalgebra of $\mathbb{H}$, allows to identify U_n with a subgroup of $U_n(\mathbb{H})$. In this paper this identification is always implied and not explicitly indicated. Furthermore, for every $p, q \geq 0$, with $p + q \geq 1$,

$O_{(p,q)}(\mathbb{C}) := \{X \in \mathfrak{gl}_{(p+q)}(\mathbb{C}) : XJ^{(p,q)}X^T = J^{(p,q)}\}$,
$SO_{(p,q)}(\mathbb{C}) := \{X \in O_{(p,q)}(\mathbb{C}) : det(X) = 1\}$,
$O_{(p,q)} := O_{(p,q)}(\mathbb{C}) \cap \mathfrak{gl}_{(p+q)}(\mathbb{R})$, $\quad SO_{(p,q)} := SO_{(p,q)}(\mathbb{C}) \cap \mathfrak{gl}_{(p+q)}(\mathbb{R})$,
are the complex and real indefinite orthogonal groups, with their special subgroups; $U_{(p,q)} := \{X \in \mathfrak{gl}_{(p+q)}(\mathbb{C}) : XJ^{(p,q)}X^* = J^{(p,q)}\}$ is the indefinite unitary group. Finally $Sp_{2n}(\mathbb{C}) := \{X \in \mathfrak{gl}_{2n}(\mathbb{C}) : X\Omega_n X^T = \Omega_n\}$ and $Sp_{2n}(\mathbb{R}) := Sp_{2n}(\mathbb{C}) \cap \mathfrak{gl}_{2n}(\mathbb{R})$ are, respectively, the complex and real symplectic groups; while $Sp_n := Sp_{2n}(\mathbb{C}) \cap U_{2n}$ is the compact symplectic group. Of course, all the previous are real Lie groups of matrices.
We recall that a well-known Cartan theorem states that a subgroup H of a given Lie group G is closed if and only if it is an embedded real submanifold of G. Of course, if the Lie group G is compact, then every closed subgroup of G is compact too.
If G is any Lie group and $P \in G$, then $T_P(G)$ denotes the tangent space of G at P.

c) The Lie algebras related to the previous Lie groups are denoted by:
$\mathfrak{so}_n = \{A \in gl_n(\mathbb{R}) : A = -A^T\}$, the Lie algebra of both O_n and SO_n;
$\mathfrak{u}_n = \{A \in \mathfrak{gl}_n(\mathbb{C}) : A = -A^*\}$, the Lie algebra of U_n;
$\mathfrak{su}_n = \{A \in \mathfrak{gl}_n(\mathbb{C}) : A = -A^*,\ tr(A) = 0\}$, the Lie algebra of SU_n;
$\mathfrak{u}_n(\mathbb{H}) = \{A \in \mathfrak{gl}_n(\mathbb{H}) : A = -A^*\}$, the Lie algebra of $U_n(\mathbb{H})$.
The Lie algebras of the remaining Lie groups will be denoted by the corresponding small gothic letters: for instance, $\mathfrak{so}_{(p,q)}(\mathbb{C})$ and $\mathfrak{sp}_n$ are the Lie algebras of $SO_{(p,q)}(\mathbb{C})$ and of Sp_n, respectively.

d) If $B \in GL_n(\mathbb{C})$, we denote by Ad_B the map from $\mathfrak{gl}_n(\mathbb{C})$ onto itself, defined by $Ad_B : A \mapsto Ad_B(A) := BAB^{-1}$. Note that Ad_B commutes with the exponential map. In this paper, we will still denote by Ad_B the restriction of this map to any subset of $\mathfrak{gl}_n(\mathbb{C})$. We indicate with τ, μ and η the maps from $\mathfrak{gl}_n(\mathbb{C})$ onto itself, given by: $\tau : A \mapsto A^T$, $\mu : A \mapsto \overline{A}$, $\quad \eta : A \mapsto A^*$. The maps μ, $-\tau$, $-\eta$ and Ad_B (with $B \in GL_n(\mathbb{C})$) are automorphisms of the real Lie algebra $\mathfrak{gl}_n(\mathbb{C})$; furthermore, the automorphisms μ, $-\tau$, $-\eta$ are involutive, mutually commuting and the composition of any two of them is the third automorphism; hence the group generated by μ, $-\tau$, $-\eta$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

e) We denote by ϕ the *Frobenius* (or *Hilbert-Schmidt*) positive definite real scalar product on $\mathfrak{gl}_n(\mathbb{C})$, defined by $\phi(A,B) := Re(tr(AB^*))$, and we denote by $\|A\|_\phi := \sqrt{\phi(A,A)} = \sqrt{tr(AA^*)}$, the related *Frobenius norm.* Note that, if $A \in \mathfrak{u}_n$, then $\|A\|_\phi^2 = -tr(A^2)$. Since the eigenvalues of the skew-hermitian matrix A are purely imaginary, we also get $\|A\|_\phi = \sqrt{-tr(A^2)} = \sqrt{\sum_{j=1}^{n} |\lambda_j|^2}$, where $\lambda_1, \cdots, \lambda_n$ are the n eigenvalues of A.

1.2. **Remarks.** a) The map $\rho : \mathbb{C} \to \mathfrak{gl}_2(\mathbb{R})$, given by $\rho(z) := Re(z)I_2 + Im(z)\Omega = \begin{pmatrix} Re(z) & -Im(z) \\ Im(z) & Re(z) \end{pmatrix}$, is a monomorphism of $\mathbb{R}$-algebras, such that $\rho(\overline{z}) = \rho(z)^T$ and such that $\rho(z) \in GL_2(\mathbb{R})$ as soon as $z \neq 0$. More generally, for any $h \geq 1$, we denote again by ρ the mapping: $\mathfrak{gl}_h(\mathbb{C}) \to \mathfrak{gl}_{2h}(\mathbb{R})$, which maps the $h \times h$ complex matrix $Z = (z_{ij})$ to the block matrix $\rho(Z) = (\rho(z_{ij})) \in \mathfrak{gl}_{2h}(\mathbb{R})$, having h^2 blocks of order 2×2. We say that ρ is the *decomplexification* map. It is not hard to prove that, if $\lambda_1, \cdots, \lambda_h$ are the h eigenvalues of any matrix $Z \in \mathfrak{gl}_h(\mathbb{C})$, then $\lambda_1, \overline{\lambda}_1, \cdots, \lambda_h, \overline{\lambda}_h$ are the $2h$ eigenvalues of $\rho(Z) \in \mathfrak{gl}_{2h}(\mathbb{R})$ and that ρ is a monomorphism of $\mathbb{R}$-algebras, whose restriction to $GL_h(\mathbb{C})$ is a monomorphism of Lie groups, having as image $\rho\big(\mathfrak{gl}_h(\mathbb{C})\big) \cap GL_{2h}(\mathbb{R})$. We have also $\rho(Z^*) = \rho(Z)^T$; so, the restriction of ρ to U_h is a monomorphism of Lie groups and $\rho\big(U_h\big) = \rho\big(\mathfrak{gl}_h(\mathbb{C})\big) \cap SO_{2h}$. From now on, to simplify the notations, the map ρ will be omitted, hence we will regard the real Lie algebra $\mathfrak{gl}_h(\mathbb{C})$ as Lie subalgebra of $\mathfrak{gl}_{2h}(\mathbb{R})$, the Lie groups $GL_h(\mathbb{C})$ and U_h as closed subgroups of $GL_{2h}(\mathbb{R})$ and SO_{2h}, respectively; in particular we will write $U_h = \mathfrak{gl}_h(\mathbb{C}) \cap SO_{2h}$.

b) We denote by $\Psi : \mathbb{H} \to \mathfrak{gl}_2(\mathbb{C})$ the map: $z + w\mathbf{j} \mapsto \Psi(z + w\mathbf{j}) := \begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix}$, where $z, w \in \mathbb{C}$; this map is a monomorphism of $\mathbb{R}$-algebras. Note that, for every $q \in \mathbb{H}$, we have $\Psi(\overline{q}) = (\Psi(q))^*$. It is possible to extend this map to a monomorphism of $\mathbb{R}$-algebras (still denoted by the same symbol) $\Psi : \mathfrak{gl}_h(\mathbb{H}) \to \mathfrak{gl}_{2h}(\mathbb{C})$ $(h \geq 1)$, which maps the $h \times h$ quaternion matrix $Q = (q_{ij})$ to the block matrix $\Psi(Q) = \big(\Psi(q_{ij})\big) \in \mathfrak{gl}_{2h}(\mathbb{C})$, having h^2 blocks of order 2×2. It can be easily checked that we have $\Psi(A^*) = (\Psi(A))^*$ and $(\Omega^{\oplus h})\Psi(A^*)(\Omega^{\oplus h})^T = (\Psi(A))^T$, for every $A \in \mathfrak{gl}_h(\mathbb{H})$. Moreover, Ψ maps $GL_h(\mathbb{H})$ into $GL_{2h}(\mathbb{C})$ and $U_h(\mathbb{H})$ into U_{2h}; both restrictions $GL_h(\mathbb{H}) \to GL_{2h}(\mathbb{C})$ and $U_h(\mathbb{H}) \to U_{2h}$ are monomorphisms of Lie groups. Hence, up to the isomorphim Ψ, we will consider $\mathfrak{gl}_h(\mathbb{H})$ as real Lie subalgebra of $\mathfrak{gl}_{2h}(\mathbb{C})$, $GL_h(\mathbb{H})$ as closed subgroup of $GL_{2h}(\mathbb{C})$ and $U_h(\mathbb{H})$ as closed subgroup of U_{2h}.
Note also that the monomorphism Ψ maps the closed subgroup U_h of $U_h(\mathbb{H})$ onto a closed subgroup of $\Psi(U_h(\mathbb{H})) \subset U_{2h}$, so that the elements of $\Psi(U_h)$ are the $2h \times 2h$ complex unitary matrices, having h^2 blocks Z_{ij} of the form: $Z_{ij} = \begin{pmatrix} z_{ij} & 0 \\ 0 & \overline{z}_{ij} \end{pmatrix}$, with $z_{ij} \in \mathbb{C}$.
As in the case of the map ρ, from now on, to simplify the notations, we will omit to indicate the map Ψ and so, for instance, we will simply write $U_h(\mathbb{H}) = U_{2h} \cap \mathfrak{gl}_h(\mathbb{H})$ and $\mathfrak{u}_h(\mathbb{H}) = \mathfrak{u}_{2h} \cap \mathfrak{gl}_h(\mathbb{H})$. From this last equality, we easily get that every matrix of $\mathfrak{u}_h(\mathbb{H})$ has trace 0. Therefore, since $U_h(\mathbb{H}) = \exp\big(\mathfrak{u}_h(\mathbb{H})\big)$, the group $U_h(\mathbb{H})$ is contained in SU_{2h}, hence $U_h(\mathbb{H}) = SU_{2h} \cap \mathfrak{gl}_h(\mathbb{H})$ and $\mathfrak{u}_h(\mathbb{H}) = \mathfrak{su}_{2h} \cap \mathfrak{gl}_h(\mathbb{H})$.

c) Fixed $n \geq 1$, for any $i, j = 1, \cdots, 2n$, let $W(i,j)$ be the square matrix of order $2n$, having 1 at the entry (i,j) and 0 elsewhere, and let B be the $2n \times 2n$ real matrix defined by $B := \sum_{j=1}^{n} \big(W(j, 2j-1) + W(n+j, 2j)\big)$. Since $W(i,j)W(h,k) = \delta_{jh} W(i,k)$, it is easy to check that B is an orthogonal matrix such that $B^T \Omega_n B = \Omega^{\oplus n}$; from this, one can get that X belongs to $U_n(\mathbb{H})$ if and only if BXB^T belongs to Sp_n , i.e. $Ad_B\big(U_n(\mathbb{H})\big) = Sp_n$. It is also easy to check that Ad_B maps the closed subgroup U_n of $U_n(\mathbb{H})$ onto the closed subgroup of Sp_n of matrices of the form $A \oplus \overline{A}$ with $A \in U_n$. Hence U_n can be regarded as the closed subgroup of Sp_n of matrices of this form, and so, the simply connected compact symmetric homogeneous space $\dfrac{Sp_n}{U_n}$, obtained in this way , is diffeomorphic to $\dfrac{U_n(\mathbb{H})}{U_n}$.

d) Let Φ be the automorphism of $\mathbb{R}$-algebra $\mathbb{H}$, defined by $\Phi(t+x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) = t+y\mathbf{i}+x\mathbf{j}-z\mathbf{k}$, for every $t, x, y, z \in \mathbb{R}$. We have: $\Phi(\overline{q}) = \overline{\Phi(q)}$, for every $q \in \mathbb{H}$. Acting on each single entry of the matrix, this map induces an automorphism (still denoted by Φ) of the $\mathbb{R}$-algebra $\mathfrak{gl}_n(\mathbb{H})$. Since $\Phi(A^*) = \Phi(A)^*$, for every $A \in \mathfrak{gl}_n(\mathbb{H})$, the restriction of Φ to $U_n(\mathbb{H})$ is an automorphism of Lie group $U_n(\mathbb{H})$, which maps U_n onto a closed subgroup of $U_n(\mathbb{H})$. Hence the homogeneous space $\dfrac{U_n(\mathbb{H})}{\Phi(U_n)}$ is diffeomorphic to $\dfrac{U_n(\mathbb{H})}{U_n}$ and, by (c), also to $\dfrac{Sp_n}{U_n}$. Remembering (b), up to the map Ψ, the subgroup $\Phi(U_n)$ of $U_n(\mathbb{H})$ can be identified with the subgroup of U_{2n}, whose elements are the $2n \times 2n$ special orthogonal matrices, having n^2 real blocks U_{ij} of the form: $U_{ij} = \begin{pmatrix} x_{ij} & -y_{ij} \\ y_{ij} & x_{ij} \end{pmatrix}$. Note that, remembering (a), the restriction of Φ to U_n agrees with the restriction to U_n of the decomplexification map ρ.

2. Commuting matrices and SVD-systems

2.1. **Notation.** Let $\mathcal{S} \subseteq \mathfrak{gl}_n(\mathbb{C})$ and $M \in \mathfrak{gl}_n(\mathbb{C})$. We denote
$\langle M \rangle_{\mathcal{S}} := \{X \in \mathcal{S} : XM = MX\}$ and $\preccurlyeq M \succcurlyeq_{\mathcal{S}} := \{X \in \mathcal{S} : XM = M\overline{X}\}$.

2.2. **Remarks.** a) Let $A \in U_n$, $M \in \mathfrak{gl}_n(\mathbb{C})$ and $\mathcal{S} \subseteq \mathfrak{gl}_n(\mathbb{C})$. It is easy to check that $Ad_A\big(\preccurlyeq M \succcurlyeq_{\mathcal{S}}\big) = \preccurlyeq AMA^T \succcurlyeq_{Ad_A(\mathcal{S})}$.
In particular, if $A \in O_n$, we get $Ad_A\big(\preccurlyeq M \succcurlyeq_{\mathcal{S}}\big) = \preccurlyeq Ad_A(M) \succcurlyeq_{Ad_A(\mathcal{S})}$.
b) Let G be a closed subgroup of $GL_n(\mathbb{C})$, having $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{C})$ as Lie algebra and let M be any matrix in $\mathfrak{gl}_n(\mathbb{C})$. Then $\langle M \rangle_G$ and $\preccurlyeq M \succcurlyeq_G$ are closed subgroups of G, whose Lie algebras are $\langle M \rangle_{\mathfrak{g}}$ and $\preccurlyeq M \succcurlyeq_{\mathfrak{g}}$, respectively.

2.3. **Lemma.** *a) Let $\varphi \in \mathbb{R}$, $\varphi \neq k\pi$, $k \in \mathbb{Z}$. Any matrix of $\mathfrak{gl}_{2n}(\mathbb{C})$ commutes with $E_\varphi^{\oplus n}$ if and only if it commutes with $\Omega^{\oplus n}$, i.e. $\langle E_\varphi^{\oplus n} \rangle_{\mathfrak{gl}_{2n}(\mathbb{C})} = \langle \Omega^{\oplus n} \rangle_{\mathfrak{gl}_{2n}(\mathbb{C})}$.*
b) Let $\mathcal{S}$ be any subset of $\mathfrak{gl}_{2n}(\mathbb{C})$, then $\langle \Omega^{\oplus n} \rangle_{\mathcal{S}}$ consists of the matrices of $\mathcal{S}$, having n^2 blocks of the form: $X_{ij} = \begin{pmatrix} a_{ij} & -b_{ij} \\ b_{ij} & a_{ij} \end{pmatrix}$, with $a_{ij}, b_{ij} \in \mathbb{C}$.

Proof. Part (a) is trivial and follows from $E_\varphi^{\oplus n} = \cos(\varphi) I_{2n} + \sin(\varphi)\Omega^{\oplus n}$ and $\sin(\varphi) \neq 0$. For part (b), we can write an arbitrary matrix of $\mathcal{S}$ in n^2 blocks, X_{ij}, each of them of order 2. We easily get that such a matrix commutes with $\Omega^{\oplus n}$ if and only if each block commutes with Ω, i. e. if and only if each X_{ij} is of the form stated in (b). □

2.4. **Lemma.** *Let $D := \bigoplus_{j=1}^{s} D_j \in \mathfrak{gl}_n(\mathbb{C})$ be a block diagonal matrix, with $D_j \in \mathfrak{gl}_{n_j}(\mathbb{C})$ simisimple matrices. Denote by S_j and by $-S_j$ $(j = 1, \cdots, s)$, respectively, the set of the eigenvalues of D_j and the sets of their opposites.*

a) Assume that $S_i \cap (-S_j) = \emptyset$ *as soon as* $i \neq j$*. Then a matrix* $A \in \mathfrak{gl}_n(\mathbb{C})$ *anticommutes with* D *if and only if* $A = \bigoplus_{j=1}^{s} A_j$*, where each* A_j *belongs to* $\mathfrak{gl}_{n_j}(\mathbb{C})$ *and anticommutes with* D_j*.*

b) Assume that $S_i \cap S_j = \emptyset$ *as soon as* $i \neq j$*. Then a matrix* $A \in \mathfrak{gl}_n(\mathbb{C})$ *commutes with* D *if and only if* $A = \bigoplus_{j=1}^{s} A_j$*, where each* A_j *belongs to* $\mathfrak{gl}_{n_j}(\mathbb{C})$ *and commutes with* D_j*.*

Proof. We proof only part (a), being part (b) similar and easier.
We write the matrix A in blocks $A = (A_{ij})$, consistent with the block structure of D, so the condition $AD = -DA$ is equivalent to $A_{ij} D_j = -D_i A_{ij}$, for $i, j = 1, \cdots, n$. Assume $i \neq j$ and let $\mathcal{B}$ be a basis of $\mathbb{C}^{n_j}$, consisting of eigenvectors of D_j. If $v \in \mathcal{B}$, with associated eigenvalue λ, then $D_i(A_{ij} v) = -A_{ij} D_j v = -\lambda(A_{ij} v)$. This implies that $A_{ij} v = 0$, otherwise (against the assumptions made) $-\lambda$ would be eigenvalue of D_i. This holds for every $v \in \mathcal{B}$ and so, $A_{ij} = \mathbf{0}$, as soon as $i \neq j$. Therefore $A = \bigoplus_{j=1}^{s} A_{jj}$, where each A_{jj} anticommutes with D_j. The converse is trivial. □

2.5. **Remark-Definition.** If $M \in \mathfrak{gl}_n(\mathbb{C})$ and G is a closed subgroup of $GL_n(\mathbb{C})$, we call *$Ad(G)$-orbit* of M, denoted by $Ad\big(G\big)(M)$, the set $\{Ad_B(M) = BMB^{-1} : B \in G\}$.
It is well-known that each orbit $Ad\big(G\big)(M)$ is an immersed submanifold of $\mathfrak{gl}_n(\mathbb{C})$, diffeomorphic to the homogeneous space $\frac{G}{\langle M \rangle_G}$, being $\langle M \rangle_G$ the isotropy subgroup of M with respect to the action of G; furthermore, if G is compact, then $Ad\big(G\big)(M)$ is a compact (embedded) submanifold of $\mathfrak{gl}_n(\mathbb{C})$ (see, for instance, [EoM-Orbit]).

2.6. **Remarks-Definitions.** A non-empty family of matrices $A_1, \cdots, A_p \in \mathfrak{gl}_n(\mathbb{C}) \setminus \{0\}$ is said to be an *SVD-system*, if $A_h^* A_j = A_h A_j^* = 0$, for every $h \neq j$, and $A_j A_j^* A_j = A_j$, for every $j = 1, \cdots, p$. Note that, if $A_1, \cdots, A_p$ is an SVD-system, then
a) the matrices $A_1, \cdots, A_p$ are linearly independent over $\mathbb{C}$;
b) $c_1 A_1, c_2 A_2, \cdots, c_p A_p$ is still an SVD-system, if $c_j \in \mathbb{C}$ and $|c_j| = 1$, for $j = 1, \cdots, p$.
We call *SVD-decomposition* of $M \in \mathfrak{gl}_n(\mathbb{C}) \setminus \{0\}$, any decomposition $M = \sum_{j=1}^{p} \sigma_j A_j$, where $A_1, \cdots, A_p \in \mathfrak{gl}_n(\mathbb{C}) \setminus \{0\}$ form an SVD-system and $\sigma_1 > \sigma_2 > \cdots > \sigma_p > 0$ are positive real numbers. Any matrix $M \in \mathfrak{gl}_n(\mathbb{C}) \setminus \{0\}$ has an SVD-decomposition $M = \sum_{j=1}^{p} \sigma_j A_j$ and this decomposition is unique, i.e. if $M = \sum_{h=1}^{q} \tau_h B_h$ is another SVD-decomposition, then $p = q$, $\sigma_j = \tau_j$ and $A_j = B_j$ for every $j = 1, \cdots, p$. The positive numbers $\sigma_1, \sigma_2, \ldots, \sigma_p$ are the distinct square roots of the non-zero eigenvalues of $M^* M$; they are known as the *non-zero singular values* of M. We say that the matrices $A_1, \cdots A_p$ are the *SVD-components* of M. For more information, see for instance [Horn-Johnson 2013, Thm. 2.6.3], [Ottaviani-Paoletti 2015, Thm.3.4] and also [Dolcetti-Pertici 2017, § 4].

2.7. **Lemma.** *Let* $A_1, \cdots, A_p$ *be an SVD-system of skew-hermitian matrices of order* n*, let* $\theta_1 > \theta_2 > \cdots > \theta_p$ *be real numbers and denote* $M := \sum_{j=1}^{p} \theta_j A_j$*. Then*

a) the eigenvalues of A_j *are:* $\mathbf{i}$ *with multiplicity* $\mu_j \geq 0$, $-\mathbf{i}$ *with multiplicity* $\nu_j \geq 0$ *(where* $\mu_j + \nu_j \geq 1$*) and* 0 *with multiplicity* $n - (\mu_j + \nu_j) \geq 0$, *for every* $j = 1, \cdots, p$*;*
b) the distinct eigenvalues of M *are* $\mathbf{i}\theta_j$ *with multiplicity* $\mu_j \geq 0$, $-\mathbf{i}\theta_j$ *with multiplicity* $\nu_j \geq 0$ *(for* $j = 1, \cdots, p$ *and* $\sum_{j=1}^{p} (\mu_j + \nu_j) \geq p$*), and* 0 *with multiplicity* $n - \sum_{j=1}^{p} (\mu_j + \nu_j) \geq 0$.

Proof. Since $A_1, \cdots, A_p$ is an SVD-system of skew-hermitian matrices, each matrix A_j satisfies the matrix equation $X^3 + X = 0$. This allows to obtain (a).
We have $A_h A_j = -A_h A_j^* = 0$, for every $h \neq j$; these conditions imply that, if v is an eigenvector of A_j associated with the eigenvalue $\mathbf{i}$ or $-\mathbf{i}$, then $A_h v = 0$, for every $j \neq h$. Moreover the same conditions give, in particular, that the matrices A_h and A_j commute, hence $A_1, \cdots, A_p$ are simultaneously diagonalizable (together with M) by means of a unitary matrix (see for instance [Horn-Johnson 2013, Thm. 2.5.5 p. 135]). Using a common (orthonormal) basis of eigenvectors, we easily obtain (b). □

2.8. **Lemma.** *Let* $A_1, A_2, \cdots, A_p$ *be an SVD-system of skew-hermitian matrices of order* n *and let* $\alpha_1, \alpha_2, \cdots, \alpha_p$ *be complex numbers. Then*

$$\exp(\sum_{j=1}^{p} \alpha_j A_j) = I_n + \sum_{j=1}^{p} \big[\sin(\alpha_j) A_j + (1 - \cos(\alpha_j)) A_j^2\big].$$

Proof. Since $A_1, A_2, \cdots, A_p$ are skew-hermitian, as in the proof of Lemma 2.7, the properties of being an SVD-system give: $A_h A_j = 0$, for $h \neq j$ (so A_h and A_j commute), and $A_j^3 = -A_j$, for every j. Hence $(\alpha_j A_j)^{2k-1} = (-1)^{k-1} \alpha_j^{2k-1} A_j$ and $(\alpha_j A_j)^{2k} = (-1)^{k-1} \alpha_j^{2k} A_j^2$, for every $j = 1, \cdots, p$ and for every $k \geq 1$. Therefore: $\exp(\sum_{j=1}^{p} \alpha_j A_j) = \prod_{j=1}^{p} \exp(\alpha_j A_j) = \prod_{j=1}^{p} \big[I_n + \sin(\alpha_j) A_j + (1 - \cos(\alpha_j)) A_j^2\big] = I_n + \sum_{j=1}^{p} \big[\sin(\alpha_j) A_j + (1 - \cos(\alpha_j)) A_j^2\big]$. □

2.9. **Remark.** Lemma 2.8 gives one of the possible generalizations of the classical Rodrigues' formula (see [Gallier-Xu 2002, Thm. 2.2] and [Dolcetti-Pertici 2018b, Ex. 4.11]). Note also that, from this Lemma, we obtain $\exp(\alpha\Omega) = E_\alpha$, for every $\alpha \in \mathbb{R}$.

3. SVD-closed real Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$

3.1. **Remark-Definition.** We say that a real Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}_n(\mathbb{C})$ is *SVD-closed* if all SVD-components of every matrix of $\mathfrak{g} \setminus \{0\}$ belong to $\mathfrak{g}$.
Note that any intersection of SVD-closed real Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$ is an SVD-closed real Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$.

3.2. **Notation.** We denote by $\mathfrak{A}_n$ the group, whose elements are the automorphisms f of the real Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, such that
i) $f \circ \eta = \eta \circ f$ (i.e. $f(A^*) = f(A)^*$, for every $A \in \mathfrak{gl}_n(\mathbb{C})$);
ii) $f(ABA) = f(A)f(B)f(A)$, for every $A, B \in \mathfrak{gl}_n(\mathbb{C})$ (i.e. f preserves the so-called *Jordan triple product*).

3.3. **Lemma.** *The elements of* $\mathfrak{A}_n$ *are precisely the following maps:*

(1) $X \mapsto Ad_V(X) = VXV^*$, *(2)* $X \mapsto \big(Ad_V \circ \mu\big)(X) = V\overline{X}V^*$,
(3) $X \mapsto \big(Ad_V \circ (-\tau)\big)(X) = -VX^TV^*$, *(4)* $X \mapsto \big(Ad_V \circ (-\eta)\big)(X) = -VX^*V^*$,
for every $V \in U_n$.

Proof. It is easy to check that the previous maps are elements of $\mathfrak{A}_n$.
For the converse, consider the decomposition $\mathfrak{gl}_n(\mathbb{C}) = \mathcal{H}_n \oplus \mathfrak{u}_n$, where $\mathcal{H}_n$ is the real vector subspace of $\mathfrak{gl}_n(\mathbb{C})$ of hermitian matrices, so that every matrix $Z \in \mathfrak{gl}_n(\mathbb{C})$ can be uniquely written as $Z = \dfrac{Z+Z^*}{2} + \dfrac{Z-Z^*}{2}$, with $\dfrac{Z+Z^*}{2} \in \mathcal{H}_n$ and $\dfrac{Z-Z^*}{2} \in \mathfrak{u}_n$; let $f \in \mathfrak{A}_n$ and denote by f_1 and by f_2 the restrictions of f to $\mathcal{H}_n$ and to $\mathfrak{u}_n$, respectively. Since $f \circ \eta = \eta \circ f$, we have $f_1(\mathcal{H}_n) = \mathcal{H}_n$ and $f_2(\mathfrak{u}_n) = \mathfrak{u}_n$. By [An-Hou 2006, Thm. 2.1], there exists a unitary matrix $V \in U_n$ such that we have
either $f_1 = Ad_V$ or $f_1 = -Ad_V$ or $f_1 = Ad_V \circ \mu$ or $f_1 = -Ad_V \circ \mu$.
In particular, this implies $f(I_n) = \pm I_n$.
Now we denote $\mathcal{M} := \mathbf{i}I_n$ and $\mathcal{N} := I_n - \mathcal{M} = (1-\mathbf{i})I_n$, so that $\mathcal{N}Y\mathcal{N} = -2\mathbf{i}Y$, for every $Y \in \mathfrak{gl}_n(\mathbb{C})$. Since f is an automorphism of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ and $\mathcal{M}$ belongs to its center $\mathcal{Z}$, then also $f(\mathcal{M})$ belongs to $\mathcal{Z}$, i.e. $f(\mathcal{M}) = \lambda I_n$ for some $\lambda \in \mathbb{C}$. Since f preserves the Jordan triple product, we get: $-f(I_n) = f(\mathcal{M}I_n\mathcal{M}) = \lambda^2 f(I_n)$. Hence $\lambda = \pm\mathbf{i}$, so that $f(\mathcal{N}) = f(I_n) - f(\mathcal{M}) = (\varepsilon_1 + \varepsilon_2\mathbf{i})I_n$, where $\varepsilon_1, \varepsilon_2 = \pm 1$; from this we get $f(\mathcal{N})^2 = 2\varepsilon\mathbf{i}I_n$, where $\varepsilon = \pm 1$. Fixed $Y \in \mathfrak{u}_n$, we have $(\mathbf{i}Y)^* = \mathbf{i}Y$ and, so, $\mathcal{N}Y\mathcal{N} = -2\mathbf{i}Y \in \mathcal{H}_n$. Remembering that f preserves the Jordan triple product, we get $-2f_1(\mathbf{i}Y) = f_1(\mathcal{N}Y\mathcal{N}) = f(\mathcal{N})f_2(Y)f(\mathcal{N}) = 2\varepsilon\mathbf{i}f_2(Y)$ and this gives $f_2(Y) = \varepsilon\mathbf{i}f_1(\mathbf{i}Y)$. This last equality implies that $f(Z) = \dfrac{1}{2}\big[f_1(Z+Z^*) + \varepsilon\mathbf{i}f_1(\mathbf{i}Z - \mathbf{i}Z^*)\big]$, for every $Z \in \mathfrak{gl}_n(\mathbb{C})$. Taking into account the four possible expressions for f_1 (and the fact that $\varepsilon = \pm 1$), easy computations allow to obtain the following eight possible expressions for f:
$\pm Ad_V$, $\pm Ad_V \circ \mu$, $\pm Ad_V \circ \eta$, $\pm Ad_V \circ \tau$.
But $-Ad_V$, $-Ad_V \circ \mu$, $Ad_V \circ \eta$, $Ad_V \circ \tau$ are not automorphisms of the real Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, while the remaining four are the expressions for f in the statement. □

3.4. **Remark.** If $f \in \mathfrak{A}_n$, then either $f(XY) = f(X)f(Y)$ for every $X, Y \in \mathfrak{gl}_n(\mathbb{C})$ (in the cases (1) and (2) of Lemma 3.3) or $f(XY) = -f(Y)f(X)$ for every $X, Y \in \mathfrak{gl}_n(\mathbb{C})$ (in the remaining cases (3) and (4)).

3.5. **Proposition.** *For every* $f \in \mathfrak{A}_n$, *the set* $Fix(f) := \{M \in \mathfrak{gl}_n(\mathbb{C}) : f(M) = M\}$ *is an SVD-closed real Lie subalgebra of* $\mathfrak{gl}_n(\mathbb{C})$.

Proof. Choose an element f of $\mathfrak{A}_n$; $Fix(f)$ is a real Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, since f is an automorphism of the real Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. Hence it suffices to prove that $Fix(f)$ is SVD-closed. Let $M = \sum_{i=1}^{p} \sigma_i A_i$ be a matrix of $Fix(f) \setminus \{0\}$, with its SVD-decomposition; since f is $\mathbb{R}$-linear, we have $M = f(M) = \sum_{i=1}^{p} \sigma_i f(A_i)$. By conditions (i), (ii) of Notation 3.2, we have $f(A_i)f(A_i)^*f(A_i) = f(A_iA_i^*A_i) = f(A_i)$, for $i = 1, \cdots, p$. Furthermore, by Remark

3.4, $f(A_i)f(A_j)^*$ equals either $f(A_iA_j^*)$ or $-f(A_j^*A_i)$ and, in both cases, $f(A_i)f(A_j)^*=0$, if $i\neq j$. Similarly, we get $f(A_i)^*f(A_j)=0$, if $i\neq j$. Hence $\sum_{i=1}^{p}\sigma_i f(A_i)$ is another SVD-decomposition of M; by uniqueness, we get $f(A_i)=A_i$, so every $A_i\in Fix(f)$. □

3.6. **Examples.** From Proposition 3.5 and from Lemma 3.3, we obtain that, for every $V\in U_n$, the following are SVD-closed real Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$:

$Fix\big(Ad_V\big)=\langle V\rangle_{\mathfrak{gl}_n(\mathbb{C})}$; $Fix\big(Ad_V\circ\mu\big)=\preccurlyeq V\succcurlyeq_{\mathfrak{gl}_n(\mathbb{C})}$; $Fix\big(Ad_V\circ(-\tau)\big)$;

$Fix\big(Ad_V\circ(-\eta)\big)$ (note that, if $V=I_n$, we have $Fix(-\eta)=\mathfrak{u}_n$).

Taking into account Remark-Definition 3.1, we obtain that

$\langle V\rangle_{\mathfrak{g}}=\langle V\rangle_{\mathfrak{gl}_n(\mathbb{C})}\cap\,\mathfrak{g}$ and $\preccurlyeq V\succcurlyeq_{\mathfrak{g}}\,=\,\preccurlyeq V\succcurlyeq_{\mathfrak{gl}_n(\mathbb{C})}\cap\,\mathfrak{g}$

are SVD-closed real Lie subalgebras of $\mathfrak{g}$, for every $V\in U_n$, and for every SVD-closed real Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}_n(\mathbb{C})$. In particular, for $\mathfrak{g}=\mathfrak{u}_\mathfrak{n}$, we deduce that

$Fix\big(Ad_V\circ(-\eta)\big)\cap\mathfrak{u}_n=Fix\big(Ad_V\big)\cap\mathfrak{u}_n=\langle V\rangle_{\mathfrak{u}_n}$ and

$Fix\big(Ad_V\circ(-\tau)\big)\cap\mathfrak{u}_n=Fix\big(Ad_V\circ\mu\big)\cap\mathfrak{u}_n=\preccurlyeq V\succcurlyeq_{\mathfrak{u}_n}$

are SVD-closed Lie subalgebras of $\mathfrak{u}_n$, for every $V\in U_n$.

Other particular SVD-closed real Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$ are the following:

$gl_n(\mathbb{R})=Fix(\mu)$; $\mathfrak{so}_n(\mathbb{C})=Fix(-\tau)$; $\mathfrak{so}_n=\mathfrak{u}_n\cap gl_n(\mathbb{R})$;

$\mathfrak{sp}_{2n}(\mathbb{C})=Fix\big(Ad_{\Omega_n}\circ(-\tau)\big)$; $\mathfrak{sp}_n=\mathfrak{sp}_{2n}(\mathbb{C})\cap\mathfrak{u}_{2n}$; $\mathfrak{su}_2=\mathfrak{sp}_2(\mathbb{C})\cap\mathfrak{u}_2$;

$\mathfrak{sp}_{2n}(\mathbb{R})=\mathfrak{sp}_{2n}(\mathbb{C})\cap gl_n(\mathbb{R})$; $\mathfrak{u}_{(p,q)}=Fix\big(Ad_{J(p,q)}\circ(-\eta)\big)$;

$\mathfrak{so}_{(p,q)}(\mathbb{C})=Fix\big(Ad_{J(p,q)}\circ(-\tau)\big)$; $\mathfrak{so}_{(p,q)}=\mathfrak{so}_{(p,q)}(\mathbb{C})\cap\mathfrak{gl}_{(p+q)}(\mathbb{R})$.

3.7. **Remark.** If $n\geq 3$, the following are not SVD-closed real Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$:

$\mathfrak{su}_n$, $\mathfrak{sl}_n(\mathbb{C})=\{M\in\mathfrak{gl}_n(\mathbb{C}):tr(M)=0\}$, $\mathfrak{sl}_n(\mathbb{R}):=\mathfrak{sl}_n(\mathbb{C})\cap gl_n(\mathbb{R})$.

We check it only for $\mathfrak{su}_3$; the generalization to $n>3$ and the other cases go similarly.

The SVD-components of the matrix $D=\begin{pmatrix}\mathbf{i}&0&0\\0&\mathbf{i}&0\\0&0&-2\mathbf{i}\end{pmatrix}$ are $\begin{pmatrix}0&0&0\\0&0&0\\0&0&-\mathbf{i}\end{pmatrix}$ and $\begin{pmatrix}\mathbf{i}&0&0\\0&\mathbf{i}&0\\0&0&0\end{pmatrix}$

(being 1 and 2 the singular values of D); since $D\in\mathfrak{su}_3$, while its SVD-components do not belong to $\mathfrak{su}_3$, we can conclude that the Lie algebra $\mathfrak{su}_3$ is not SVD-closed.

3.8. **Proposition.** *Let $\mathfrak{g}$ be an SVD-closed real Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$.*

a) For every $W\in\mathfrak{u}_n$, we have that $\langle W\rangle_{\mathfrak{g}}$ is an SVD-closed Lie subalgebra of $\mathfrak{g}$.

b) If $\mathfrak{g}$ is the Lie algebra of a closed subgroup of U_n, then every Cartan subalgebra of $\mathfrak{g}$ is SVD-closed.

Proof. Clearly, if $YW=WY$ then $Ye^{sW}=e^{sW}Y$, for every $s\in\mathbb{R}$; conversely, if $Ye^{sW}=e^{sW}Y$ for every $s\in\mathbb{R}$, then, differentiating with respect to s and putting $s=0$, we get $YW=WY$. Hence $\langle W\rangle_{\mathfrak{g}}=\mathfrak{g}\cap\big[\bigcap_{s\in\mathbb{R}}Fix\big(Ad_{\exp(sW)}\big)\big]$. We get (a), since $\exp(sW)\in U_n$, for every $s\in\mathbb{R}$. Part (b) follows from part (a), via [Sepanski 2007, Lemma 5.7 p. 100]. □

4. SVD-closed subgroups of U_n

4.1. **Remark-Definition.** We say that any subgroup of $GL_n(\mathbb{C})$ is *SVD-closed* if it is closed in $GL_n(\mathbb{C})$ and its Lie algebra is an SVD-closed real Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$. Note that, by Examples 3.6 and Remarks 2.2 (b), the subgroups of U_n, defined by
$\preccurlyeq V \succcurlyeq_{U_n} = \{X \in U_n : XV = V\overline{X}\} = \{X \in U_n : XVX^T = V\}$ and
$\langle V\rangle_{U_n} = \{X \in U_n : XV = VX\}$, are SVD-closed, for every matrix $V \in U_n$.
By Remark-Definition 3.1, the intersection of SVD-closed subgroups of $GL_n(\mathbb{C})$ is an SVD-closed subgroup of $GL_n(\mathbb{C})$; indeed, it is known that its Lie algebra is the intersection of Lie algebras of all SVD-closed subgroups ([Bourbaki 1975, Cor. 3 p. 307]). In the Sections 7 and 8, we will study the sets of generalized principal logarithms of matrices of the groups $\langle V\rangle_{U_n}$, where $V \in U_n$, and $\preccurlyeq Q \succcurlyeq_{SU_n} = \preccurlyeq Q \succcurlyeq_{U_n} \cap SU_n$, where $Q \in O_n$.
Note that we can obtain some classical Lie groups as follows:
$U_n = \langle I_n\rangle_{U_n}$, $\quad SO_n = \preccurlyeq I_n \succcurlyeq_{SU_n}$, $\quad Sp_n = \preccurlyeq \Omega_n \succcurlyeq_{SU_{2n}}$,
$U_{(p,n-p)} \cap U_n = \langle J^{(p,n-p)}\rangle_{U_n}$, $\quad SO_{(p,n-p)}(\mathbb{C}) \cap U_n = \preccurlyeq J^{(p,n-p)} \succcurlyeq_{SU_n}$,
for $p = 0, \cdots, n$. We need some preliminary results.

4.2. **Proposition.** *Let $V \in U_n$; denote by λ_1 (with multiplicity n_1), $\cdots, \lambda_r$ (with multiplicity n_r) its distinct eigenvalues, and choose $R \in U_n$ such that $V = Ad_R\big(\bigoplus_{j=1}^{r} \lambda_j I_{n_j}\big)$. Then $\langle V\rangle_{U_n} = Ad_R\big(\bigoplus_{j=1}^{r} U_{n_j}\big)$ and it is a (compact) connected SVD-closed subgroup of U_n, whose Lie algebra is $\langle V\rangle_{\mathfrak{u}_n} = Ad_R\big(\bigoplus_{j=1}^{r} \mathfrak{u}_{n_j}\big)$.*

Proof. The equality $\langle V\rangle_{U_n} = Ad_R\big(\bigoplus_{j=1}^{r} U_{n_j}\big)$ easily follows from Lemma 2.4 (b). This implies that $\langle V\rangle_{U_n}$ is compact and connected. As noted in Remark-Definition 4.1, $\langle V\rangle_{U_n}$ is SVD-closed too. Clearly, its Lie algebra is $\langle V\rangle_{\mathfrak{u}_n} = Ad_R\big(\bigoplus_{j=1}^{r} \mathfrak{u}_{n_j}\big)$. □

4.3. **Lemma.** *Let V any matrix of U_n. Then $\preccurlyeq V \succcurlyeq_{SU_n}$ is an SVD-closed subgroup of U_n, whose Lie algebra is $\preccurlyeq V \succcurlyeq_{\mathfrak{u}_n} = \preccurlyeq V \succcurlyeq_{\mathfrak{su}_n}$.*

Proof. The Lie algebra of $\preccurlyeq V \succcurlyeq_{SU_n}$ is $\preccurlyeq V \succcurlyeq_{\mathfrak{su}_n} \subseteq \preccurlyeq V \succcurlyeq_{\mathfrak{u}_n}$ and this last is SVD-closed, so it suffices to prove the reverse inclusion. If $X \in \preccurlyeq V \succcurlyeq_{\mathfrak{u}_n}$, being $V^*XV = \overline{X}$, then X is similar to its complex conjugate $\overline{X}$ and so, by [Horn-Johnson 2013, Cor. 3.4.1.7 p. 202], X is similar to a real matrix; therefore X has real trace; since any skew-hermitian matrix has trace with zero real part, we conclude that the trace of X is zero, i.e. $X \in \preccurlyeq V \succcurlyeq_{\mathfrak{su}_n}$. □

In the next results, we will need the matrices $W_{(p,q)}$, $E_{\varphi}^{(p,q)}$ and $J^{(p,q)}$ defined in Notations 1.1(a).

4.4. **Lemma.** *If $p = 0, 1, \cdots, n$, we have $O_{(p,n-p)}(\mathbb{C}) \cap U_n = Ad_{W_{(p,n-p)}}(O_n)$ and $SO_{(p,n-p)}(\mathbb{C}) \cap U_n = Ad_{W_{(p,n-p)}}(SO_n)$.*

Proof. Let $W := W_{(p,n-p)}$. Then the statements follow from Remarks 2.2 (a), since
$\preccurlyeq I_n \succcurlyeq_{U_n} = O_n$, $\quad \preccurlyeq I_n \succcurlyeq_{SU_n} = SO_n$, $\quad \preccurlyeq J^{(p,n-p)} \succcurlyeq_{U_n} = O_{(p,n-p)}(\mathbb{C}) \cap U_n$,

$\preccurlyeq J^{(p,n-p)} \succcurlyeq_{SU_n} = SO_{(p,n-p)}(\mathbb{C}) \cap U_n$, $\quad W I_n W^T = J^{(p,n-p)}$ and the groups U_n, SU_n are Ad_W-invariant. $\square$

4.5. **Lemma.** *For every* $\varphi \in \mathbb{R}$ *and* $p = 0, 1, \cdots, n$, *we have*
$\preccurlyeq E_\varphi^{(p,n-p)} \succcurlyeq_{U_{2n}} = Ad_{W_{(2p,2n-2p)}}(\preccurlyeq E_\varphi^{\oplus n} \succcurlyeq_{U_{2n}})$ *and*
$\preccurlyeq E_\varphi^{(p,n-p)} \succcurlyeq_{SU_{2n}} = Ad_{W_{(2p,2n-2p)}}(\preccurlyeq E_\varphi^{\oplus n} \succcurlyeq_{SU_{2n}})$.

Proof. Let $W := W_{(2p,2n-2p)}$. The groups U_{2n} and SU_{2n} are Ad_W-invariant and $W\, E_\varphi^{\oplus n}\, W^T = E_\varphi^{(p,n-p)}$; hence, by Remarks 2.2 (a), we get the statements . $\square$

4.6. **Lemma.** *Fix* $\varphi \in [0, 2\pi)$, *with* $\varphi \neq \frac{\pi}{2}$ *and* $\varphi \neq \frac{3}{2}\pi$; *consider the matrix* $E_\varphi^{\oplus n}$. *Then a matrix* $A \in \mathfrak{gl}_{2n}(\mathbb{C})$ *anticommutes with* $E_\varphi^{\oplus n}$ *if and only if* $A = \mathbf{0}_{2n}$.

Proof. Assume first $n = 1$, so $E_\varphi^{\oplus n} = E_\varphi = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$. If a matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{C}) \text{ anticommutes with } E_\varphi \text{, then } \begin{cases} 2\alpha\cos(\varphi) = (\gamma - \beta)\sin(\varphi) \\ 2\delta\cos(\varphi) = (\gamma - \beta)\sin(\varphi) \\ 2\gamma\cos(\varphi) = -(\alpha + \delta)\sin(\varphi) \\ 2\beta\cos(\varphi) = (\alpha + \delta)\sin(\varphi) \end{cases}.$$

Since $\cos(\varphi) \neq 0$, the previous conditions give: $\alpha = \delta$ and $\beta = -\gamma$, i.e. $A = \alpha I_2 + \gamma\Omega$. But this last matrix also commutes with the nonsingular matrix E_φ and so, A must be the null matrix.

If $n \geq 2$, we write any matrix of $A \in \mathfrak{gl}_{2n}(\mathbb{C})$ as $A := (A_{ij})$, with n^2 square blocks A_{ij} of order 2. A direct computation shows that, if A anticommutes with $E_\varphi^{\oplus n}$, then each block A_{ij} anticommutes with E_φ; hence, the proof follows from the case $n = 1$. $\square$

4.7. **Lemma.** *Fix* $\varphi \in (0, 2\pi)$ *with* $\varphi \neq \frac{\pi}{2}$, $\varphi \neq \pi$ *and* $\varphi \neq \frac{3}{2}\pi$. *Then we have*
$\preccurlyeq E_\varphi^{(p,n-p)} \succcurlyeq_{SU_{2n}} = \preccurlyeq E_\varphi^{(p,n-p)} \succcurlyeq_{U_{2n}} = Ad_{W_{(2p,2n-2p)}}(U_n)$, *for every* $p = 0, \cdots, n$, *in which we put (consistently with Remarks 1.2 (a))* $U_n = \mathfrak{gl}_n(\mathbb{C}) \cap SO_{2n} \subset SU_{2n}$.

Proof. By Lemma 4.5, we have to prove that $\preccurlyeq E_\varphi^{\oplus n} \succcurlyeq_{SU_{2n}} = \preccurlyeq E_\varphi^{\oplus n} \succcurlyeq_{U_{2n}} = U_n$. For, a complex matrix $X = X_1 + \mathbf{i}X_2$ (X_1, X_2 real matrices) satisfies the condition $X\, E_\varphi^{\oplus n} = E_\varphi^{\oplus n}\overline{X}$ if and only if $X_1\, E_\varphi^{\oplus n} = E_\varphi^{\oplus n} X_1$ and $X_2\, E_\varphi^{\oplus n} = -\, E_\varphi^{\oplus n} X_2$ and, by Lemmas 4.6 and 2.3, this is equivalent to say that $X \in \mathfrak{gl}_n(\mathbb{C}) \subseteq \mathfrak{gl}_{2n}(\mathbb{R})$ (and, in this case, $det(X) \geq 0$). Hence, by Remarks 1.2 (a), we get $\preccurlyeq E_\varphi^{\oplus n} \succcurlyeq_{SU_{2n}} = \mathfrak{gl}_n(\mathbb{C}) \cap SU_{2n} = \mathfrak{gl}_n(\mathbb{C}) \cap SO_{2n} = U_n$ and similarly, $\preccurlyeq E_\varphi^{\oplus n} \succcurlyeq_{U_{2n}} = \mathfrak{gl}_n(\mathbb{C}) \cap U_{2n} = \mathfrak{gl}_n(\mathbb{C}) \cap SO_{2n} = U_n$. $\square$

4.8. **Lemma.** *Remembering Remarks 1.2 (b), we have*
$\preccurlyeq \Omega^{\oplus n} \succcurlyeq_{SU_{2n}} = \preccurlyeq \Omega^{\oplus n} \succcurlyeq_{U_{2n}} = U_n(\mathbb{H})$ *and* $\preccurlyeq \Omega^{\oplus n} \succcurlyeq_{\mathfrak{su}_{2n}} = \preccurlyeq \Omega^{\oplus n} \succcurlyeq_{\mathfrak{u}_{2n}} = \mathfrak{u}_n(\mathbb{H})$.

Proof. Any matrix $X = Y + \mathbf{i}Z \in \mathfrak{gl}_{2n}(\mathbb{C})$ (with $Y, Z \in \mathfrak{gl}_{2n}(\mathbb{R})$) satisfies the condition $X\,\Omega^{\oplus n} = \Omega^{\oplus n}\overline{X}$ if and only if $Y\Omega^{\oplus n} = \Omega^{\oplus n} Y$ and $Z\Omega^{\oplus n} = -\Omega^{\oplus n} Z$. A direct computation shows that these conditions on Y and Z are equivalent to say that $Y = (Y_{ij})$

and $Z = (Z_{ij})$ are block matrices, whose blocks Y_{ij}, Z_{ij} are 2×2 real matrices of the form: $Y_{ij} = \begin{pmatrix} a_{ij} & -b_{ij} \\ b_{ij} & a_{ij} \end{pmatrix}$, $Z_{ij} = \begin{pmatrix} c_{ij} & d_{ij} \\ d_{ij} & -c_{ij} \end{pmatrix}$, for $i, j = 1, \cdots, n$. These last conditions are equivalent to say that $X = (X_{ij})$ is a block matrix, with n^2 blocks of the form: $X_{ij} = \begin{pmatrix} z_{ij} & -w_{ij} \\ \overline{w}_{ij} & \overline{z}_{ij} \end{pmatrix}$, and, by Remarks 1.2 (b), this is equivalent to say that $X \in \mathfrak{gl}_n(\mathbb{H})$. Hence $\preccurlyeq \Omega^{\oplus n} \succcurlyeq_{SU_{2n}} = SU_{2n} \cap \mathfrak{gl}_n(\mathbb{H}) = U_n(\mathbb{H}) = U_{2n} \cap \mathfrak{gl}_n(\mathbb{H}) = \preccurlyeq \Omega^{\oplus n} \succcurlyeq_{U_{2n}}$ and, by Remarks 2.2 (b), we also get $\preccurlyeq \Omega^{\oplus n} \succcurlyeq_{\mathfrak{su}_{2n}} = \preccurlyeq \Omega^{\oplus n} \succcurlyeq_{\mathfrak{u}_{2n}} = \mathfrak{u}_n(\mathbb{H})$. □

4.9. **Remarks.** a) For any $Q \in O_n$, there exists a matrix $A \in O_n$ such that $Q = Ad_A(\mathcal{J}) = A\mathcal{J}A^T$, where $\mathcal{J}$ is a matrix of the form $\mathcal{J} := J^{(p,q)} \oplus \big(\bigoplus_{j=1}^{h} E_{\varphi_j}^{(\mu_j, \nu_j)}\big) \oplus \Omega^{\oplus k}$,

with $0 < \varphi_1 < \varphi_2 < \cdots < \varphi_h < \dfrac{\pi}{2}$; $p + q + 2\sum_{j=1}^{h}(\mu_j + \nu_j) + 2k = n$; $p, q, k, \mu_j, \nu_j \geq 0$; $\mu_j + \nu_j \geq 1$ (see for instance [Dolcetti-Pertici 2021, Rem.-Def. 1.8], where we called $\mathcal{J}$ the *real Jordan auxiliary form* of Q). Hence the (possible) eigenvalues of Q and their multiplicities are the following: 1 of multiplicity $p \geq 0$; -1 of multiplicity $q \geq 0$; $\pm\mathbf{i}$ both of multiplicity $k \geq 0$; when $h > 0$, $e^{\pm\mathbf{i}\varphi_j}$ both of multiplicity $\mu_j \geq 0$ and $e^{\pm\mathbf{i}(\pi-\varphi_j)} = -e^{\mp\mathbf{i}\varphi_j}$ both of multiplicity $\nu_j \geq 0$, for every $j = 1, \cdots, h$. The condition $\mu_j + \nu_j \geq 1$ is equivalent to say that $e^{\pm\mathbf{i}\varphi_j}$ or $e^{\pm\mathbf{i}(\pi-\varphi_j)}$ (and possibly both) are effective eigenvalues of Q.

b) If $Q, A, \mathcal{J} \in O_n$ are as in (a), we have $Ad_A\big(I_1 \oplus (-I_{(n-1)})\big) \in \preccurlyeq Q \succcurlyeq_{SU_n}$ if and only if n is odd. Indeed, if n is odd, the real matrix Q has at least one real eigenvalue.

4.10. **Proposition.** *Let $Q \in O_n$; denote its eigenvalues (with their multiplicities) and the matrices $A, \mathcal{J} \in O_n$ as in Remarks 4.9 (a). If Z is the $n \times n$ unitary matrix defined by* $Z := A\Big(W_{(p,q)} \oplus \Big[\bigoplus_{j=1}^{h} W_{(2\mu_j, 2\nu_j)}\Big] \oplus I_{2k}\Big)$, *then*

$$\preccurlyeq Q \succcurlyeq_{U_n} = Ad_Z\Big(O_{(p+q)} \oplus \Big[\bigoplus_{j=1}^{h} U_{(\mu_j+\nu_j)}\Big] \oplus U_k(\mathbb{H})\Big),$$

$$\preccurlyeq Q \succcurlyeq_{SU_n} = Ad_Z\Big(SO_{(p+q)} \oplus \Big[\bigoplus_{j=1}^{h} U_{(\mu_j+\nu_j)}\Big] \oplus U_k(\mathbb{H})\Big),$$

and they are (compact) SVD-closed subgroups of U_n, whose common Lie algebra is

$$\preccurlyeq Q \succcurlyeq_{\mathfrak{su}_n} = \preccurlyeq Q \succcurlyeq_{\mathfrak{u}_n} = Ad_Z\Big(\mathfrak{so}_{(p+q)} \oplus \Big[\bigoplus_{j=1}^{h} \mathfrak{u}_{(\mu_j+\nu_j)}\Big] \oplus \mathfrak{u}_k(\mathbb{H})\Big).$$

The group $\preccurlyeq Q \succcurlyeq_{U_n}$ is connected if Q has no real eigenvalues, otherwise it has two connected components. In any case, $\preccurlyeq Q \succcurlyeq_{SU_n}$ is the connected component of $\preccurlyeq Q \succcurlyeq_{U_n}$ containing the identity I_n.

Proof. From Remark-Definition 4.1 and Lemma 4.3, it follows that the groups $\preccurlyeq Q \succcurlyeq_{U_n}$ and $\preccurlyeq Q \succcurlyeq_{SU_n}$ are SVD-closed and their common Lie algebras is $\preccurlyeq Q \succcurlyeq_{\mathfrak{u}_n} = \preccurlyeq Q \succcurlyeq_{\mathfrak{su}_n}$. By Remarks 2.2 (a), we have $\preccurlyeq Q \succcurlyeq_{U_n} = Ad_A(\preccurlyeq \mathcal{J} \succcurlyeq_{U_n})$, $\preccurlyeq Q \succcurlyeq_{SU_n} = Ad_A(\preccurlyeq \mathcal{J} \succcurlyeq_{SU_n})$. Now we determine the groups $\preccurlyeq \mathcal{J} \succcurlyeq_{U_n}$ and $\preccurlyeq \mathcal{J} \succcurlyeq_{SU_n}$. A matrix $X = X_1 + \mathbf{i}X_2 \in \mathfrak{gl}_n(\mathbb{C})$ (with $X_1, X_2 \in \mathfrak{gl}_n(\mathbb{R})$) satisfies the condition $X\mathcal{J} = \mathcal{J}\overline{X}$ if and only if $X_1\mathcal{J} = \mathcal{J}X_1$ and $X_2\mathcal{J} = -\mathcal{J}X_2$. By Lemma 2.4 (b), the condition $X_1\mathcal{J} = \mathcal{J}X_1$ implies that

$X_1 = Y_0 \oplus \left[\bigoplus_{j=1}^h Y_j\right] \oplus Y_{(h+1)}$, where $Y_0 \in \mathfrak{gl}_{(p+q)}(\mathbb{R})$, $Y_j \in \mathfrak{gl}_{(2\mu_j+2\nu_j)}(\mathbb{R})$ for every $j = 1, \cdots, h$ and $Y_{(h+1)} \in \mathfrak{gl}_{2k}(\mathbb{R})$. By Lemma 2.4 (a), the condition $X_2\mathcal{J} = -\mathcal{J}X_2$ implies that also the matrix X_2 must be block-diagonal, with blocks of the same type as the blocks of X_1. Therefore, if X satisfies the condition $X\mathcal{J} = \mathcal{J}\overline{X}$, then X is block-diagonal with similar blocks, this time complex instead of real. Of course, X is unitary if and only if each single block is unitary too. Then, setting $U = W_{(p,q)} \oplus \left[\bigoplus_{j=1}^h W_{(2\mu_j,2\nu_j)}\right] \oplus I_{2k}$ and taking into account also Lemmas 4.4, 4.7 and 4.8 , we obtain

$$\preccurlyeq \mathcal{J} \succcurlyeq_{U_n} = \preccurlyeq J^{(p,q)} \succcurlyeq_{U_{(p+q)}} \oplus \left[\bigoplus_{j=1}^h \preccurlyeq E_{\varphi_j}^{(\mu_j,\nu_j)} \succcurlyeq_{U_{(2\mu_j+2\nu_j)}}\right] \oplus \preccurlyeq \Omega^{\oplus k} \succcurlyeq_{U_{2k}} =$$
$$\preccurlyeq J^{(p,q)} \succcurlyeq_{U_{(p+q)}} \oplus \left[\bigoplus_{j=1}^h \preccurlyeq E_{\varphi_j}^{(\mu_j,\nu_j)} \succcurlyeq_{SU_{(2\mu_j+2\nu_j)}}\right] \oplus \preccurlyeq \Omega^{\oplus k} \succcurlyeq_{SU_{2k}} =$$
$$Ad_U\left(O_{(p+q)} \oplus \left[\bigoplus_{j=1}^h U_{(\mu_j+\nu_j)}\right] \oplus U_k(\mathbb{H})\right);$$
$$\preccurlyeq \mathcal{J} \succcurlyeq_{SU_n} = \preccurlyeq J^{(p,q)} \succcurlyeq_{SU_{(p+q)}} \oplus \left[\bigoplus_{j=1}^h \preccurlyeq E_{\varphi_j}^{(\mu_j,\nu_j)} \succcurlyeq_{SU_{(2\mu_j+2\nu_j)}}\right] \oplus \preccurlyeq \Omega^{\oplus k} \succcurlyeq_{SU_{2k}} =$$
$$Ad_U\left(SO_{(p+q)} \oplus \left[\bigoplus_{j=1}^h U_{(\mu_j+\nu_j)}\right] \oplus U_k(\mathbb{H})\right).$$

From these equalities, easily follow the statements that still remain to be proved. □

5. Generalized principal $\mathfrak{g}$-logarithms

5.1. **Definition.** Let G be a connected closed subgroup of $GL_n(\mathbb{C})$, whose Lie algebra is $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{C})$. If $M \in G$, we say that a matrix $L \in \mathfrak{g}$ is a *generalized principal $\mathfrak{g}$-logarithm* of M, if $\exp(L) = M$ and $-\pi \leq Im(\lambda) \leq \pi$, for every eigenvalue λ of L.
We denote by $\mathfrak{g}$–*plog*(M) the set of all generalized principal $\mathfrak{g}$-logarithms of any $M \in G$.

5.2. **Remarks.** a) In Introduction, we compared the previous definition with the usual definition of *principal logarithm* of a matrix $M \in GL_n(\mathbb{C})$ without negative eigenvalues, in which case the set $\mathfrak{gl}_n(\mathbb{C})$–*plog*$(M)$ consists of a unique matrix ([Higham 2008, Thm. 1.31]).
b) If G is any connected closed subgroup of $GL_n(\mathbb{C})$, with Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{C})$, then $\rho(G)$ is a connected closed subgroup of $GL_{2n}(\mathbb{R}) \subset GL_{2n}(\mathbb{C})$, having $\rho(\mathfrak{g}) \subset \mathfrak{gl}_{2n}(\mathbb{R}) \subset \mathfrak{gl}_{2n}(\mathbb{C})$ as Lie algebra, where ρ is the decomplexification map. Remembering the relationship between the eigenvalues of Z and $\rho(Z)$ (see Remarks 1.2 (a)), we easily get that
$\rho\big(\mathfrak{g}\text{–}plog(M)\big) = \rho(\mathfrak{g})\text{–}plog(\rho(M))$, for every $M \in G$.

5.3. **Lemma.** *Let G, H be connected closed subgroups of $GL_n(\mathbb{C})$ such that $G = Ad_A(H)$, for some $A \in GL_n(\mathbb{C})$, and let $\mathfrak{g}, \mathfrak{h} \subseteq \mathfrak{gl}_n(\mathbb{C})$ be their Lie algebras, respectively. Then*
$Ad_A(\mathfrak{h}\text{–}plog(M)) = \mathfrak{g}\text{–}plog(Ad_A(M))$, *for every $M \in H$.*
In particular, if G is any connected closed subgroup of $GL_n(\mathbb{C})$, we have
$Ad_A(\mathfrak{g}\text{–}plog(M)) = \mathfrak{g}\text{–}plog(Ad_A(M))$, *for every $A, M \in G$.*

Proof. Note that $G = Ad_A(H)$ implies that $\mathfrak{g} = Ad_A(\mathfrak{h})$. Hence $B \in \mathfrak{g}$ if and only if $A^{-1}BA \in \mathfrak{h}$. Since B and $A^{-1}BA$ are similar and $\exp(B) = AMA^{-1}$ if and only if $\exp(A^{-1}BA) = M$, we get: $B \in \mathfrak{g}$–*plog*$(Ad_A(M))$ if and only if $A^{-1}BA \in \mathfrak{h}$–*plog*$(M)$. □

5.4. **Remark.** The eigenvalues of any skew-hermitian matrix A are purely imaginary; so, the generalized principal $\mathfrak{u}_n$-logarithms of any $M \in U_n$ are the skew-hermitian logarithms of M, whose eigenvalues all have modulus in $[0, \pi]$. Note that, since all the eigenvalues of any $M \in U_n$ have modulus 1, the only possible negative eigenvalue of such M is -1.
In this Section, given any unitary matrix M of order n, we will denote its eigenvalues by $e^{\mathbf{i}\theta_1}$ with multiplicity m_1, $e^{\mathbf{i}\theta_2}$ with multiplicity m_2, up to $e^{\mathbf{i}\theta_p}$ with multiplicity m_p, where $\pi \geq \theta_1 > \theta_2 > \cdots > \theta_p > -\pi$ and $n = \sum_{j=1}^{p} m_j$. If -1 is not an eigenvalue of M (i.e. if $\theta_1 < \pi$), then the eigenvalues of the unique generalized principal $\mathfrak{gl}_n(\mathbb{C})$-logarithm of M are exactly: $\mathbf{i}\theta_1$ with multiplicity m_1, $\mathbf{i}\theta_2$ with multiplicity m_2, up to $\mathbf{i}\theta_p$ with multiplicity m_p. Instead, if -1 is an eigenvalue of M (i.e. if $\theta_1 = \pi$), then the eigenvalues of any generalized principal $\mathfrak{gl}_n(\mathbb{C})$-logarithm Y of M are exactly: $\mathbf{i}\pi$ of multiplicity h, $-\mathbf{i}\pi$ of multiplicity $m_1 - h$ (for some $h \in \{0, 1, \cdots, m_1\}$ depending on Y), $\mathbf{i}\theta_2$ with multiplicity m_2, up to $\mathbf{i}\theta_p$ with multiplicity m_p. Note that, if Y is any generalized principal $\mathfrak{u}_n$-logarithm of M, in any case we have $\|Y\|_\phi^2 = -tr(Y^2) = \sum_{j=1}^{n} m_j \theta_j^2 = \sum_{j=1}^{n} m_j |\log(e^{\mathbf{i}\theta_j})|^2$.

5.5. **Proposition.** *Let G be a connected SVD-closed subgroup of U_n, whose Lie algebra is $\mathfrak{g} \subseteq \mathfrak{u}_n$. Then*
a) $\mathfrak{g}$–plog$(M) \neq \emptyset$, for every $M \in G$ and, furthermore, if -1 is not an eigenvalue of M, then $\mathfrak{g}$–plog(M) consists of a single element;
b) If $Y \in \mathfrak{g}$–plog(M), then $\|Y\|_\phi \leq \|X\|_\phi$, for every $X \in \mathfrak{g}$ such that $\exp(X) = M$*; moreover the equality holds if and only if $X \in \mathfrak{g}$–plog(M).*

Proof. a) If $M = I_n$, it is clear that $\mathfrak{g}$–*plog*$(M) = \{\mathbf{0}_n\}$ and the statement holds true.
Fix $M \in G \setminus \{I_n\}$ and denote its eigenvalues as in Remark 5.4. Since G is compact and connected, we can choose a skew-hermitian matrix $X \in \mathfrak{g} \setminus \{\mathbf{0}_n\}$ such that $\exp(X) = M$ (see, for instance, [Bröcker-tomDieck 1985, Ch. IV Thm. 2.2]). Then, the n eigenvalues of X are $\mathbf{i}(\theta_1 + 2k_{1,1}\pi)$, $\mathbf{i}(\theta_1 + 2k_{1,2}\pi)$, $\cdots$, $\mathbf{i}(\theta_1 + 2k_{1,m_1}\pi)$; $\mathbf{i}(\theta_2 + 2k_{2,1}\pi)$, $\cdots$, $\mathbf{i}(\theta_2 + 2k_{2,m_2}\pi)$; $\cdots$; up to $\mathbf{i}(\theta_p + 2k_{p,1}\pi)$, $\cdots$, $\mathbf{i}(\theta_p + 2k_{p,m_p}\pi)$, where $k_{h,j} \in \mathbb{Z}$, for every h, j. We also denote by $\sigma_1 > \sigma_2 > \cdots > \sigma_s > 0$ the distinct non-zero singular values of X. Since $X \in \mathfrak{u}_n$, there exist $\psi_h \in \{\theta_1, \cdots, \theta_p\}$ and $t_h \in \mathbb{Z}$ such that $\sigma_h = |\psi_h + 2t_h\pi|$, for every $h = 1, \cdots, s$. If $X = \sum_{h=1}^{s} |\psi_h + 2t_h\pi| X_h$ is the SVD-decomposition of X, then every SVD-component X_h of X belongs to $\mathfrak{g}$, because G is SVD-closed. Of course, for $h = 1, \cdots, s$, we have $|\psi_h + 2t_h\pi| = \pm(\psi_h + 2t_h\pi)$, and so $X = \sum_{h=1}^{s} (\psi_h + 2t_h\pi) Y_h = \sum_{i=1}^{s} \psi_h Y_h + \sum_{i=1}^{s} 2\pi t_h Y_h$, where $Y_h = \pm X_h$. Note that, by Remarks-Definitions 2.6 (b), $\{Y_h\}_{1 \leq h \leq s}$ is still an SVD-system of elements of $\mathfrak{g}$. Taking into account Lemma 2.8 and the mutual commutativity of the Y_h's, we have: $M = \exp(X) = \exp(\sum_{h=1}^{s} \psi_h Y_h) \exp(\sum_{i=1}^{s} 2\pi l_h Y_h) = \exp(\sum_{h=1}^{s} \psi_h Y_h)$. So, if we denote $Y := \sum_{h=1}^{s} \psi_h Y_h$, we have $Y \in \mathfrak{g}$ and $M = \exp(Y)$. By Lemma 2.7, every non-zero eigenvalue of Y is of the form $\pm\mathbf{i}\theta_{\mathbf{h}}$, for some $h = 1, \cdots, p$; hence Y is a

generalized principal $\mathfrak{g}$-logarithm of M. By Remarks 5.2 (a), if -1 is not an eigenvalue of M, the set $\mathfrak{g}$–$plog(M)$ necessarily reduces to the single matrix Y.

b) Let $X \in \mathfrak{g}$ any logarithm of M, with eigenvalues as in (a), and let $Y \in \mathfrak{g}$–$plog(M)$. Then, $\|X\|_\phi^2 = -tr(X^2) = \sum_{j=1}^{p}\sum_{r=1}^{m_j}(\theta_j + 2k_{j,r}\pi)^2 = \sum_{j=1}^{p} m_j\theta_j^2 + 4\pi\sum_{j=1}^{p}\sum_{r=1}^{m_j} k_{j,r}(\theta_j + k_{j,r}\pi) =$ $-tr(Y^2) + 4\pi\sum_{j=1}^{p}\sum_{r=1}^{m_j} k_{j,r}(\theta_j + k_{j,r}\pi) = \|Y\|_\phi^2 + 4\pi\sum_{j=1}^{p}\sum_{r=1}^{m_j} k_{j,r}(\theta_j + k_{j,r}\pi)$ (with $k_{j,r} \in \mathbb{Z}$).

If $\theta_j \in (-\pi,\pi)$, we easily get $k_{j,r}(\theta_j + k_{j,r}\pi) \geq 0$, with equality if and only if $k_{j,r} = 0$.

If $\theta_1 = \pi$, clearly we get $k_{1,r}(\theta_1 + k_{1,r}\pi) = \pi k_{1,r}(1 + k_{1,r}) \geq 0$, with equality if and only if either $k_{1,r} = -1$ or $k_{1,r} = 0$. Since the case $k_{1,r} = -1$ gives $-\mathbf{i}\pi$ as eigenvalue of X, we can conclude that $\|X\|_\phi^2 \geq \|Y\|_\phi^2$, and the equality holds if and only if the possible eigenvalues of X are only $-\mathbf{i}\pi$ and $\mathbf{i}\theta_j$ $(1 \leq j \leq p)$, i.e. if and only if $X \in G \in \mathfrak{g}$–$plog(M)$. □

5.6. **Remark.** Assume that $n \geq 3$. As noted in Remark 3.7, SU_n is not SVD-closed. Moreover there are matrices $M \in SU_n$ such that $\mathfrak{su}_n$–$plog(M) = \emptyset$. This is the case of $M = e^{2\pi\mathbf{i}/n}I_n$. Indeed, -1 is not an eigenvalue of M (since $n \geq 3$), and hence, the unique generalized principal $\mathfrak{gl}_n(\mathbb{C})$–logarithm of M is $L := \dfrac{2\pi\mathbf{i}}{n}I_n$, whose trace is $2\pi\mathbf{i} \neq 0$, so $L \notin \mathfrak{su}_n$. Hence, the SVD-closure condition in Proposition 5.5 cannot be removed.

5.7. **Theorem.** *Let G be a connected SVD-closed subgroup of U_n, whose Lie algebra is $\mathfrak{g} \subseteq \mathfrak{u}_n$; let $M \in G$ and let T be a maximal torus of G containing M, with Lie algebra $\mathfrak{t}$. Then there are $L_1, \cdots, L_s \in \mathfrak{t}$–$plog(M)$ $(s \geq 1)$ such that $\mathfrak{g}$–$plog(M) = \bigsqcup_{j=1}^{s} Ad\big(\langle M\rangle_G\big)(L_j)$. Furthermore, each set $Ad\big(\langle M\rangle_G\big)(L_j)$ is a compact submanifold of $\mathfrak{g}$, diffeomorphic to the homogeneous space $\dfrac{\langle M\rangle_G}{\langle L_j\rangle_G}$.*

Proof. By Proposition 3.8 (b), T is SVD-closed, being $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$; so, by Proposition 5.5 (a), there exists a matrix $L \in \mathfrak{t}$–$plog(M)$. Furthermore, the exponential map $\exp : \mathfrak{t} \to T$ is a Lie group homomorphism (considering $\mathfrak{t}$ as an additive Lie group), so it is a covering map (see, for instance, [Alexandrino-Bettiol 2015, Prop. 1.24]) and the fiber $\exp^{-1}(M)$ is discrete. By Proposition 5.5 (b), the set $\mathfrak{t}$–$plog(M)$ is the intersection between $\exp^{-1}(M)$ and the sphere $\{W \in \mathfrak{t} : \|W\|_\phi = \|L\|_\phi\}$, therefore it is finite. We can choose a non-empty subset $\{L_1, \cdots, L_s\}$ of $\mathfrak{t}$–$plog(M)$ such that $L_h \notin Ad\big(\langle M\rangle_G\big)(L_i)$, if $h \neq i$, and such that every $L \in \mathfrak{t}$–$plog(M)$ belongs to $Ad\big(\langle M\rangle_G\big)(L_j)$, for some $j \in \{1, \cdots, s\}$; it is clear that $Ad\big(\langle M\rangle_G\big)(L_h) \bigcap Ad\big(\langle M\rangle_G\big)(L_i) = \emptyset$, for every $h \neq i$.

We now prove the set equality of the statement.

If $X = Ad_K(L_h)$, with $K \in \langle M\rangle_G$, for some $h \in \{1, \cdots, s\}$, then clearly $X \in \mathfrak{g}$–$plog(M)$. Conversely, let $Y \in \mathfrak{g}$–$plog(M)$. By [Sepanski 2007, Thm. 5.9 p. 101], there exists $Q \in G$ such that $Ad_Q(Y) \in \mathfrak{t}$, so that $\exp(Ad_Q(Y)) = Ad_Q(M) \in T$. By [Bröcker-tomDieck 1985, Lemma 2.5 p. 166], there exists H in the *normalizer* of T in G such that $Ad_H\big(Ad_Q(M)\big) = M$. Since $Ad_H(\mathfrak{t}) = \mathfrak{t}$, we have $Ad_H\big(Ad_Q(Y)\big) \in \mathfrak{t}$, with $\exp\big[Ad_H\big(Ad_Q(Y)\big)\big] = M$; so $Ad_H\big(Ad_Q(Y)\big) \in \mathfrak{t}$–$plog(M)$. Hence, there exist $j \in \{1, \cdots, s\}$ and $P \in \langle M\rangle_G$ such that

$Ad_H\big(Ad_Q(Y)\big) = Ad_P(L_j)$, and so, $Y = Ad_K(L_j)$, with $K := Q^*H^*P \in G$. Since $M = exp(Y) = exp(L_j)$, we get $M = Ad_K(M)$, i.e. $K \in \langle M\rangle_G$, and hence $Y \in Ad\big(\langle M\rangle_G\big)(L_j)$. We conclude by Remark-Definition 2.5, since $\langle M\rangle_G$ is compact and $\langle L_j\rangle_G \subseteq \langle M\rangle_G$. □

6. Closed subgroups of U_n endowed with the Frobenius metric

6.1. **Remark-Definition.** In this Section we consider an arbitrary closed subgroup G of U_n and we still denote by ϕ the Riemannian metric on G, obtained by restriction of the Frobenius scalar product of $\mathfrak{gl}_n(\mathbb{C})$ (remember Notations 1.1 (e)). It is easy to check that the metric ϕ (called the *Frobenius metric* of G) is bi-invariant on G and that we have $\phi_A(X,Y) = -tr(A^*XA^*Y)$, for every $A \in G$ and for every $X, Y \in T_A(G)$. We denote by $d := d_{(G,\phi)}$ the distance on G induced by ϕ and by $\delta(G,\phi)$ the *diameter* of G with respect to d. Of course $\delta(G,\phi) < +\infty$, because G is compact.

6.2. **Proposition.** *Let G be a closed subgroup of U_n and let $\mathfrak{g} \subseteq \mathfrak{u}_n$ be its Lie algebra. Then (G,ϕ) is a globally symmetric Riemannian manifold with non-negative sectional curvature, whose Levi-Civita connection agrees with the 0-connection of Cartan-Schouten of G. The geodesics of (G,ϕ) are the curves $\gamma(t) = P\exp(tX)$, for every $X \in \mathfrak{g}$ and $P \in G$; furthermore (G,ϕ) is a totally geodesic submanifold of (U_n,ϕ).*

For a proof of Proposition 6.2, we refer, for instance, to [Alexandrino-Bettiol 2015, § 2.2].

6.3. **Proposition.** *Let G be a connected closed subgroup of U_n and let $\mathfrak{g} \subseteq \mathfrak{u}_n$ be its Lie algebra. Then, for every $P_0, P_1 \in G$, the distance $d(P_0,P_1)$ is equal to the minimum of the set $\{\ \|X\|_\phi\ :\ X \in \mathfrak{g}$ and $\exp(X) = P_0^*P_1\ \}$.*

Proof. Any geodesic segment γ joining P_0 and P_1 can be parametrized by $\gamma(t) = P_0\exp(tX)$ ($t \in [0,1]$), with $X \in \mathfrak{g}$, $\exp(X) = P_0^*P_1$, and its length is $\sqrt{-tr(X^2)} = \|X\|_\phi$; so, we conclude by the Hopf-Rinow theorem (see, for instance, [Alexandrino-Bettiol 2015, p. 31]). □

6.4. **Remark.** Let G be a connected closed subgroup of U_n such that $-I_n \in G$. Then $\delta(G,\phi) \geq \sqrt{n}\,\pi$. Indeed, if $\exp(X) = -I_n$, with $X \in \mathfrak{g} \subseteq \mathfrak{u}_n$, the eigenvalues of X are of the form $(2k_j+1)\pi\mathbf{i}$, with $k_j \in \mathbb{Z}$, so $\|X\|_\phi = \sqrt{-tr(X^2)} = \sqrt{\sum_{j=1}^{n}(2k_j+1)^2}\cdot\pi \geq \sqrt{n}\,\pi$. Hence, by Proposition 6.3, we have $\delta(G,\phi) \geq d(I_n,-I_n) \geq \sqrt{n}\,\pi$.

6.5. **Theorem.** *Let G be a connected SVD-closed subgroup of U_n with Lie algebra $\mathfrak{g} \subseteq \mathfrak{u}_n$. Let $P_0, P_1 \in G$ and let $\mu_1, \cdots, \mu_n$ be the n eigenvalues of $P_0^*P_1$. Then*

a) $d(P_0,P_1) = \sqrt{\sum_{j=1}^{n} |\log(\mu_j)|^2}$;

b) the map: $X \mapsto \gamma(t) := P_0\exp(tX)$ $(0 \leq t \leq 1)$ is a bijection from $\mathfrak{g}$–plog$(P_0^*P_1)$ onto the set of minimizing geodesic segments of (G,ϕ), with endpoints P_0 and P_1.

Proof. Part (a) follows from Propositions 6.3, 5.5 and Remark 5.4; we also get (b), since the geodesic path: $t \mapsto P_0\exp(tX)$ is minimizing if and only if $X \in \mathfrak{g}$–$plog(P_0^*P_1)$. □

6.6. **Corollary.** *Let G be a connected SVD-closed subgroup of U_n. Then*
a) $\delta(G,\phi) \leq \sqrt{n}\,\pi$ *and the equality holds if and only if* $-I_n \in G$;
b) if $-I_n \in G$, *we have* $d(P_0, P_1) = \delta(G,\phi)$ *(with* $P_0, P_1 \in G$*) if and only if* $P_1 = -P_0$.

Proof. By Theorem 6.5 (a), we easily get the inequality in (a), while, if $-I_n \in G$, the equality follows from Remark 6.4. Conversely, assume that the equality holds. Since G is compact, by Theorem 6.5, there exist $P_0, P_1 \in G$ such that $\sqrt{n}\,\pi = d(P_0,P_1) = \sqrt{\sum_{j=1}^{n} |\log(\mu_j)|^2}$, where $\mu_1, \cdots, \mu_n$ are the eigenvalues of $P_0^* P_1 \in G \subseteq U_n$. Hence, for every $j = 1, \cdots, n$, we have $|\mu_j| = 1$, and so, $\log(\mu_j) = \mathbf{i}\theta$, with $\theta \in (-\pi, \pi]$. The above equality implies: $\log(\mu_j) = \mathbf{i}\pi$, so $\mu_j = -1$, for every j, and from this: $P_0^* P_1 = -I_n \in G$. From these arguments, we also easily obtain part (b). □

6.7. **Proposition.** *a)* $\delta(\langle V \rangle_{U_n}, \phi) = \sqrt{n}\,\pi$, *for every* $V \in U_n$ *and for every integer* $n \geq 1$;
b) $\delta\big(\preccurlyeq Q \succcurlyeq_{SU_n}, \phi\big) = \sqrt{n}\,\pi$, *for every* $Q \in O_n$ *and for every even integer* $n \geq 2$;
c) $\delta\big(\preccurlyeq Q \succcurlyeq_{SU_n}, \phi\big) = \sqrt{n-1}\,\pi$, *for every* $Q \in O_n$ *and for every odd integer* $n \geq 1$.

Proof. Parts a) and b) follow from Corollary 6.6 (a) (taking into account also Propositions 4.2 and 4.10), since, in both cases, the groups are connected, SVD-closed and contain $-I_n$.
c) If n is odd, by Remarks 4.9 (b), we have $P = Ad_A\big(I_1 \oplus (-I_{(n-1)})\big) \in \preccurlyeq Q \succcurlyeq_{SU_n}$ (with $A \in O_n$); hence, from Theorem 6.5 (a), we get $\delta\big(\preccurlyeq Q \succcurlyeq_{SU_n}, \phi\big) \geq d(I_n, P) = \sqrt{n-1}\,\pi$. Now let P_0, P_1 be arbitrary elements of $\preccurlyeq Q \succcurlyeq_{SU_n}$. Since n is odd, by Proposition 4.10, the matrix $P_0^* P_1 \in \preccurlyeq Q \succcurlyeq_{SU_n}$ has 1 as eigenvalue; so, from Theorem 6.5 (a), we get $d(P_0, P_1) \leq \sqrt{n-1}\,\pi$ and then (c) holds. □

6.8. **Remarks.** a) Remembering Remark-Definition 4.1 and Lemma 4.8, from Proposition 6.7, we deduce the following facts: the diameter of the groups U_n and $U_{(p,n-p)} \cap U_n$ ($p = 0, \cdots, n$) is $\sqrt{n}\,\pi$ (for $n \geq 1$); the diameter of Sp_n and $U_n(\mathbb{H})$ is $\sqrt{2n}\,\pi$ (for $n \geq 1$); the diameter of SO_n and $SO_{(p,n-p)}(\mathbb{C}) \cap U_n$ ($p = 0, \cdots, n$) is $\sqrt{n}\,\pi$, for every even integer $n \geq 2$; while the diameter of the groups SO_n, $SO_{(p,n-p)}(\mathbb{C}) \cap U_n$ ($p = 0, \cdots, n$), is equal to $\sqrt{n-1}\,\pi$, when the integer $n \geq 1$ is odd (see also [Dolcetti-Pertici 2018a, Cor. 4.12]).
b) There are examples of connected closed subgroups G of U_n such that $-I_n \in G$ and $\delta(G,\phi) > \sqrt{n}\,\pi$. For instance, denoted by G the one-parameter subgroup of U_2, given by $\exp(t\Delta)$ ($t \in \mathbb{R}$), where Δ is the diagonal matrix with eigenvalues $\pi\mathbf{i}$ and $3\pi\mathbf{i}$, it is easy to check that G is compact, not SVD-closed, $-I_2 \in G$ and $\delta(G,\phi) = d(I_2, -I_2) = \sqrt{10}\,\pi$.

7. Generalized principal $\langle V \rangle_{\mathfrak{u}_n}$–logarithms, with $V \in U_n$

7.1. **Proposition.** *Let* $M \in U_n$ *and* $\zeta \geq 0$ *be the multiplicity of* -1 *as eigenvalue of* M. *Then* $\mathfrak{u}_n$*–plog*(M) *is disjoint union of* $\zeta+1$ *compact submanifolds of* $\mathfrak{u}_n$, *called* $\mathcal{W}_0, \cdots, \mathcal{W}_\zeta$, *such that* $\mathcal{W}_j$ *is diffeomorphic to the complex Grassmannian* $\mathbf{Gr}(j; \mathbb{C}^\zeta)$, *for* $j = 0, \cdots, \zeta$.

Proof. If $\zeta = 0$, the statement is true, since $\mathfrak{u}_n$–*plog*(M) and $\mathbf{Gr}(0; \mathbb{C}^0)$ reduce to a point.

Assume now $\zeta \geq 1$. Let us denote the eigenvalues of M as in Remark 5.4, with $\theta_1 = \pi$ and $\zeta = m_1$. It is well-known that M can be diagonalized by means of a unitary matrix; hence, by Lemma 5.3, we can assume $M = (-I_\zeta) \oplus (\bigoplus_{j=2}^{p} e^{\mathbf{i}\theta_j} I_{m_j})$, so that, by Lemma 2.4 (b), we have $\langle M \rangle_{U_n} = U_\zeta \oplus (\bigoplus_{j=2}^{p} U_{m_j})$. Let T denote the maximal torus of U_n, passing through M, consisting of all unitary diagonal matrices, whose Lie algebra is the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{u}_n$, consisting of all skew-hermitian diagonal matrices (see, for instance, [Sepanski 2007, p. 98]). Since $|\theta_j| < \pi$, for every $j \geq 2$, we have that $\mathfrak{t}$–*plog*(M) is the set of the 2^ζ elements of the form $D \oplus (\bigoplus_{j=2}^{p} \mathbf{i}\theta_j I_{m_j})$, where D is any diagonal matrix of order ζ, having each diagonal element equal to either $\mathbf{i}\pi$ or $-\mathbf{i}\pi$. We denote $D_j := (\mathbf{i}\pi I_j) \oplus (-\mathbf{i}\pi I_{(\zeta-j)})$ and $L_j := D_j \oplus (\bigoplus_{j=2}^{p} \mathbf{i}\theta_j I_{m_j})$, so that $\langle L_j \rangle_{U_n} = U_j \oplus U_{(\zeta-j)} \oplus (\bigoplus_{j=2}^{p} U_{m_j})$, for $j = 0, \cdots, \zeta$. Clearly, each matrix of $\mathfrak{t}$–*plog*(M) belongs to the $Ad(\langle M \rangle_{U_n})$-orbit of a unique L_j. Denoted $\mathcal{W}_j := Ad(\langle M \rangle_{U_n})(L_j)$, by Theorem 5.7 we get: $\mathfrak{u}_n$–*plog*$(M) = \bigsqcup_{j=0}^{\zeta} \mathcal{W}_j$, with $\mathcal{W}_j$ compact submanifolds of $\mathfrak{u}_n$, diffeomorphic to $\dfrac{\langle M \rangle_{U_n}}{\langle L_j \rangle_{U_n}} = \dfrac{U_\zeta \oplus (\bigoplus_{j=2}^{p} U_{m_j})}{U_j \oplus U_{(\zeta-j)} \oplus (\bigoplus_{j=2}^{p} U_{m_j})} \simeq \dfrac{U_\zeta}{U_j \oplus U_{(\zeta-j)}}$, and it is well-known that this last homogeneous space is diffeomorphic to the complex Grassmannian $\mathbf{Gr}(j; \mathbb{C}^\zeta)$, for $j = 0, \cdots, \zeta$. $\square$

7.2. **Theorem.** *Let $V \in U_n$; denote by λ_1 (with multiplicity n_1), $\cdots, \lambda_r$ (with multiplicity n_r) its distinct eigenvalues, and choose $R \in U_n$ such that $V = Ad_R(\bigoplus_{j=1}^{r} \lambda_j I_{n_j})$. Then*

a) $M \in \langle V \rangle_{U_n}$ if and only if $M = Ad_R(\bigoplus_{j=1}^{r} M_j)$, with $M_j \in U_{n_j}$, for $j = 1, \cdots, r$;

b) if $M = Ad_R(\bigoplus_{j=1}^{r} M_j) \in \langle V \rangle_{U_n}$ (with $M_j \in U_{n_j}$), and $\zeta_j \geq 0$ is the multiplicity of -1 as eigenvalue of M_j $(1 \leq j \leq r)$, then the set $\langle V \rangle_{\mathfrak{u}_n}$–plog$(M)$ has $\prod_{j=1}^{r} (\zeta_j + 1)$ connected components, called $\mathcal{Z}(k_1, \cdots, k_r)$ (for $k_j = 0, 1, \cdots, \zeta_j$ and $j = 1, \cdots, r$); each component $\mathcal{Z}(k_1, \cdots, k_r)$ is a simply connected compact submanifold of $\mathfrak{u}_n$, diffeomorphic to the product of complex Grassmannians $\prod_{j=1}^{r} \mathbf{Gr}(k_j; \mathbb{C}^{\zeta_j})$.

Proof. Part (a) follows directly from Proposition 4.2. We now prove part (b). By Lemma 5.3, we can assume $V = \bigoplus_{j=1}^{r} \lambda_j I_{n_j}$ (i.e. $R = I_n$) and, so, again by Proposition 4.2, we have $\langle V \rangle_{U_n} = \bigoplus_{j=1}^{r} U_{n_j}$, $\langle V \rangle_{\mathfrak{u}_n} = \bigoplus_{j=1}^{r} \mathfrak{u}_{n_j}$ and $M = \bigoplus_{j=1}^{r} M_j$. From this, it follows that $L \in \langle V \rangle_{\mathfrak{u}_n}$–*plog*$(M)$ if and only if $L = L_1 \oplus \cdots \oplus L_r$, where $L_j \in \mathfrak{u}_{n_j}$–*plog*$(M_j)$, for every $j = 1, \cdots, r$. This implies that $\langle V \rangle_{\mathfrak{u}_n}$–*plog*$(M) = \bigoplus_{j=1}^{r} \mathfrak{u}_{n_j}$–*plog*$(M_j)$.
From Proposition 7.1, we get that the set $\mathfrak{u}_{n_j}$–*plog*(M_j) is disjoint union of $\zeta_j + 1$ compact submanifolds of $\mathfrak{u}_{n_j}$, called $\mathcal{W}_{j0}, \cdots, \mathcal{W}_{j\zeta_j}$, where $\mathcal{W}_{jk}$ is diffeomorphic to the complex Grassmannian $\mathbf{Gr}(k; \mathbb{C}^{\zeta_j})$, for every $k = 0, \cdots, \zeta_j$ and $j = 1, \cdots, r$. Hence:
$\langle V \rangle_{\mathfrak{u}_n}$–*plog*$(M) = \bigoplus_{j=1}^{r} (\bigsqcup_{k_j=0}^{\zeta_j} \mathcal{W}_{jk_j}) = \bigsqcup_{0 \leq k_1 \leq \zeta_1, \cdots, 0 \leq k_r \leq \zeta_r} \bigoplus_{j=1}^{r} \mathcal{W}_{jk_j}$, where each $\bigoplus_{j=1}^{r} \mathcal{W}_{jk_j}$

is a connected component of $\langle V\rangle_{\mathfrak{u}_n}$–$plog(M)$ and a compact submanifold of $\mathfrak{u}_n$, diffeomorphic to the product $\prod_{j=1}^{r}\mathbf{Gr}(k_j;\mathbb{C}^{\zeta_j})$. The total number of these components is $\prod_{j=1}^{r}(\zeta_j+1)$. Setting $\mathcal{Z}(k_1,\cdots,k_r):=\bigoplus_{j=1}^{r}\mathcal{W}_{jk_j}$ (for all possible indices), we obtain (b). □

8. Generalized principal $\preccurlyeq Q \succcurlyeq_{\mathfrak{su}_n}$–logarithms, with $Q\in O_n$

8.1. **Remark.** By Lemma 4.8, we have $U_n(\mathbb{H})=\preccurlyeq \Omega^{\oplus n}\succcurlyeq_{SU_{2n}}$. Then, arguing as in the proof of Lemma 4.3, it is easy to show that any matrix $M\in U_n(\mathbb{H})$ is similar to a real matrix; so, if -1 is an eigenvalue of $M\in U_n(\mathbb{H})$, its multiplicity is even and the eigenvalues of M can be listed as follows: -1 with multiplicity $2\mu\ge 2$, $e^{\pm\mathbf{i}\eta_1}$ both with multiplicity μ_1, $e^{\pm\mathbf{i}\eta_2}$ both with multiplicity μ_2, $\cdots$, up to $e^{\pm\mathbf{i}\eta_q}$ both with multiplicity μ_q $(q\ge 0)$, where $\pi>\eta_1>\eta_2>\cdots>\eta_q\ge 0$, with the agreement that, if $\eta_q=0$, the multiplicity of the corresponding eigenvalue 1 is $2\mu_q$. In any case we have: $\mu+\sum_{j=1}^{q}\mu_j=n$.

8.2. **Proposition.** *Let $M\in U_n(\mathbb{H})$; denote by $2\mu\ge 0$ the multiplicity of -1 as eigenvalue of M. Then $\mathfrak{u}_n(\mathbb{H})$–$plog(M)$ is a simply connected compact submanifold of $\mathfrak{u}_n(\mathbb{H})$, diffeomorphic to the symmetric homogeneous space $\dfrac{U_\mu(\mathbb{H})}{U_\mu}\simeq\dfrac{Sp_\mu}{U_\mu}$.*

Proof. If $\mu=0$ (i.e. if -1 is not an eigenvalue of M), the statement is true, remembering Notations 1.1 (a) and Proposition 5.5 (a). Assume now $\mu\ge 1$. It is easy to show that the group $T=\{\bigoplus_{j=1}^{n}E_{\theta_j}:\theta_1,\cdots\theta_n\in\mathbb{R}\}$ is a maximal torus of $U_n(\mathbb{H})$, whose Lie algebra is $\mathfrak{t}=\{\bigoplus_{j=1}^{n}\theta_j\Omega:\theta_1,\cdots\theta_n\in\mathbb{R}\}$. We denote the eigenvalues of M and their multiplicities as in Remark 8.1; then, by [Sepanski 2007, Thm. 5.12 (a)], there exists $K\in U_n(\mathbb{H})$ such that $M=Ad_K\big((-I_{2\mu})\oplus(\bigoplus_{j=1}^{q}E_{\eta_j}^{\oplus\mu_j})\big)$. By Lemma 5.3, we can assume $K=I_{2n}$; hence, by Remark 2.9, the set $\mathfrak{t}$–$plog(M)$ consists of the 2^{μ} elements of the form $\big(\bigoplus_{h=1}^{\mu}(\epsilon_h\pi\Omega)\big)\oplus\big(\bigoplus_{j=1}^{q}(\eta_j\Omega)^{\oplus\mu_j}\big)$, where each ϵ_h is either 1 or -1. All these elements belong to the same $Ad\big(\langle M\rangle_{U_n(\mathbb{H})}\big)$-orbit. Indeed, it suffices to remark that the matrix $\Psi(\mathbf{k})=\begin{pmatrix}0&-\mathbf{i}\\-\mathbf{i}&0\end{pmatrix}$ satisfies $\Psi(\mathbf{k})\,\Omega\,\Psi(\mathbf{k})^*=-\Omega$. Hence, by Theorem 5.7, $\mathfrak{u}_n(\mathbb{H})$–$plog(M)$ is a compact submanifold of $\mathfrak{u}_n(\mathbb{H})$, diffeomorphic to the homogeneous space $\dfrac{\langle M\rangle_{U_n(\mathbb{H})}}{\langle L\rangle_{U_n(\mathbb{H})}}$, where $L:=(\pi\Omega)^{\oplus\mu}\oplus\big(\bigoplus_{j=1}^{q}(\eta_j\Omega)^{\oplus\mu_j}\big)$. Recalling Remarks 1.2 (c), (d), we get the statement, since we have $\langle M\rangle_{U_n(\mathbb{H})}=U_\mu(\mathbb{H})\oplus\big(\bigoplus_{j=1}^{q}\Phi(U_{\mu_j})\big)$ and $\langle L\rangle_{U_n(\mathbb{H})}=\Phi(U_\mu)\oplus\big(\bigoplus_{j=1}^{q}\Phi(U_{\mu_j})\big)$. □

8.3. **Remark.** In Remarks 1.2 (c), we have seen that we have $Ad_B\big(U_n(\mathbb{H})\big)=Sp_n$, with $B\in O_{2n}$; so, by Lemma 5.3, we obtain $\mathfrak{sp}_n$–$plog(M)=Ad_B\big[\mathfrak{u}_n(\mathbb{H})$–$plog(Ad_{B^T}(M))\big]$, for every $M\in Sp_n$. Hence, by Proposition 8.2, we conclude that the set $\mathfrak{sp}_n$–$plog(M)$ is a simply connected compact submanifold of $\mathfrak{sp}_n$, diffeomorphic to the symmetric space $\dfrac{Sp_\mu}{U_\mu}$, where $2\mu\ge 0$ is the multiplicity of -1 as eigenvalue of M, for every $M\in Sp_n$.

8.4. **Proposition.** *Let $M \in SO_{(p,n-p)}(\mathbb{C}) \cap U_n$ $(p = 0, \cdots, n)$ and denote by $2m \geq 0$ the multiplicity of -1 as eigenvalue of M. Then the set $\big(\mathfrak{so}_{(p,n-p)}(\mathbb{C}) \cap \mathfrak{u}_n\big)$–plog$(M)$ is a compact submanifold of $\mathfrak{su}_n$, diffeomorphic to the homogeneous space $\dfrac{O_{2m}}{U_m}$; hence, if $m \geq 1$, this set has two connected components, both diffeomorphic to the simply connected compact symmetric homogeneous space $\dfrac{SO_{2m}}{U_m}$.*

Proof. By Lemmas 4.4 and 5.3, we can assume $p = n$, so that $SO_{(p,n-p)}(\mathbb{C}) \cap U_n = SO_n$, and, in this case, the Proposition has already been proved in [Dolcetti-Pertici 2018a, §3] and in [Pertici 2022, Thm. 4.7]. A further proof can be deduced from Theorem 5.7, but, for the sake of brevity, we omit it. □

8.5. **Theorem.** *Let $Q \in O_n$, and assume that Q has, as real Jordan form, the matrix $\mathcal{J} := J^{(p,q)} \oplus \big(\bigoplus_{j=1}^{h} E_{\varphi_j}^{(\mu_j,\nu_j)}\big) \oplus \Omega^{\oplus k}$, with $0 < \varphi_1 < \varphi_2 < \cdots < \varphi_h < \dfrac{\pi}{2}$, $p + q + 2\sum_{j=1}^{h}(\mu_j + \nu_j) + 2k = n$, $\quad p, q, k, \mu_j, \nu_j \geq 0$, $\quad \mu_j + \nu_j \geq 1$, and choose $A \in O_n$ such that $Q = Ad_A(\mathcal{J}) = A\mathcal{J}A^T$. Let Z be the $n \times n$ unitary matrix defined by $Z := A\left(W_{(p,q)} \oplus \left[\bigoplus_{j=1}^{h} W_{(2\mu_j, 2\nu_j)}\right] \oplus I_{2k}\right)$. Then*

a) $M \in\, \preccurlyeq Q \succcurlyeq_{SU_n}$ if and only if $M = Ad_Z\left[N \oplus \left(\bigoplus_{j=1}^{h} M_j\right) \oplus R\right]$, where $N \in SO_{(p+q)}$, $R \in U_k(\mathbb{H})$ and $M_j \in U_{(\mu_j+\nu_j)}$, for $j = 1, \cdots, h$.

b) If $M = Ad_Z\left[N \oplus \left(\bigoplus_{j=1}^{h} M_j\right) \oplus R\right] \in\, \preccurlyeq Q \succcurlyeq_{SU_n}$, denote by $2m \geq 0$ the multiplicity of -1 as eigenvalue of N, by $\zeta_j \geq 0$ the multiplicity of -1 as eigenvalue of M_j (for $1 \leq j \leq h$) and by $2\mu \geq 0$ the multiplicity of -1 as eigenvalue of R. Then we have

$$\preccurlyeq Q \succcurlyeq_{\mathfrak{su}_n}\text{–}plog(M) = \bigsqcup_{0 \leq l_1 \leq \zeta_1, \cdots, 0 \leq l_h \leq \zeta_h} \mathcal{V}(l_1, \cdots, l_h),$$

where each $\mathcal{V}(l_1, \cdots, l_h)$ is a compact submanifold of $\mathfrak{su}_n$, diffeomorphic to the product $\dfrac{O_{2m}}{U_m} \times \left[\prod_{j=1}^{h} \mathbf{Gr}\big(l_j; \mathbb{C}^{\zeta_j}\big)\right] \times \dfrac{Sp_\mu}{U_\mu}$.
If -1 is not an eigenvalue of N (i.e. if $m = 0$), then each $\mathcal{V}(l_1, \cdots, l_h)$ is connected and $\preccurlyeq Q \succcurlyeq_{\mathfrak{su}_n}$–plog$(M)$ has $\prod_{j=1}^{h}(\zeta_j + 1)$ components; while, if -1 is an eigenvalue of N (i.e. if $m \geq 1$), then each $\mathcal{V}(l_1, \cdots, l_h)$ has two connected components, both diffeomorphic to $\dfrac{SO_{2m}}{U_m} \times \left[\prod_{j=1}^{h} \mathbf{Gr}\big(l_j; \mathbb{C}^{\zeta_j}\big)\right] \times \dfrac{Sp_\mu}{U_\mu}$, so $\preccurlyeq Q \succcurlyeq_{\mathfrak{su}_n}$–plog$(M)$ has $2\prod_{j=1}^{h}(\zeta_j + 1)$ components.
In any case, all components of $\preccurlyeq Q \succcurlyeq_{\mathfrak{su}_n}$–plog$(M)$ are simply connected, compact and diffeomorphic to a symmetric homogeneous space.

Proof. Part (a) follows directly from Proposition 4.10. By Lemma 5.3, we can assume $\preccurlyeq Q \succcurlyeq_{SU_n} = SO_{(p+q)} \oplus \left[\bigoplus_{j=1}^{h} U_{(\mu_j+\nu_j)}\right] \oplus U_k(\mathbb{H})$ and $M = N \oplus \left(\bigoplus_{j=1}^{h} M_j\right) \oplus R$.
Therefore, arguing as in the proof of Theorem 7.2, we get $\preccurlyeq Q \succcurlyeq_{\mathfrak{su}_n}$–plog$(M) =$

$$\Big[\mathfrak{so}_{(p+q)}\text{–}plog(N)\Big] \oplus \left[\bigoplus_{j=1}^{h} \mathfrak{u}_{(\mu_j+\nu_j)}\text{–}plog(M_j)\right] \oplus \Big[\mathfrak{u}_k(\mathbb{H})\text{–}plog(R)\Big].$$

Hence we get (b), by means of Propositions 8.2, 8.4 and 7.1, via Remarks 5.2 (b). □

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