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Singular limits of a coupled elasto-plastic damage system as viscosity and hardening vanish

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Abstract

The paper studies the asymptotic analysis of a model coupling elastoplasticity and damage depending on three parameters—governing viscosity, plastic hardening, and convergence rate of plastic strain and displacement to equilibrium—as they vanish in different orders. The notion of limit evolution obtained is proven to coincide in any case with a notion introduced by Crismale and Rossi (SIAM J Math Anal 53(3):3420–3492, 2021), moreover, such solutions are closely related to those obtained in the vanishing-viscosity limit by Crismale and Lazzaroni (Calc Var Part Differ Equ 55(1):17, 2016), for the analogous model where only the viscosity parameter was present.

Keywords Rate-independent systems · Variational models · Vanishing viscosity and hardening · Balanced Viscosity solutions · Damage · Elasto-plasticity

Mathematics Subject Classification 35A15 · 35Q74 · 74C05

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1 Introduction

Rate-independent processes model evolutionary phenomena where the external loading is much slower than the internal oscillations of materials, while viscosities may be neglected. Despite a wide literature on the subject (see [24] and references therein), a further understanding is needed of the relations between the different notions of solution that have been proposed. In particular, a problem of interest in applications is to determine which kind of solution captures the limiting behavior of dynamical systems for small viscosity or inertia. For this reason, in this paper we compare different notions of solution, obtained with different approximation methods as viscosities tend to zero at different rates (but with no inertia).

We focus on a rate-independent system modeling damage in an elasto-plastic body occupying a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. The model was advanced and first studied in [5, 6], while the existence of *globally minimizing quasistatic evolutions* (or, equivalently, *Energetic solutions*) was first proved in [11]. In [8, 10], the *vanishing-viscosity* approach was instead exploited to find the so-called *Balanced Viscosity* solutions, obtaining the rate-independent system as the limit of a viscously perturbed system. Indeed, vanishing viscosity was advanced as a selection criterion for solutions with a mechanically feasible behavior at jumps, motivated by the observation that Energetic solutions jump 'too long and too early', cf. the characterization proved in [29] and the references therein. We refer to the pioneering [16], and the subsequent [26, 27], for the definition and properties of such solutions in the context of an 'abstract' rate-independent system. The vanishing-viscosity technique has also been adopted in various concrete applications, ranging from plasticity (cf., e.g., [7, 12, 14, 18, 32]), to damage, fracture, and fatigue (see for instance [1, 4, 9, 20, 21, 23]).

In this paper, we aim to gain further insight into the different ways of constructing *Balanced Viscosity* solutions to the model for damage and plasticity from [6], which were explored in [8] and [10]. We show that these notions of solutions essentially coincide if the hardening vanishes together with viscosities, while they retain different features if the hardening parameter is positive. In particular, it turns out that perfect plasticity coupled with damage may equivalently be approximated by means of processes where viscosity is confined to the flow rule for damage, or with viscosity also in the momentum equation and in the plastic flow rule.

1.1 The model

The rate-independent process we are going to address describes the evolution, in the time interval $(0, T)$, of the *displacement* $u : (0,T) \times \Omega \to \mathbb{R}^n$, of the plastic strain $p : (0,T) \times \Omega \to \mathbb{M}^{n\times n}_{\mathrm{D}}$, and of the damage variable $z : (0,T) \times \Omega \to [0,1]$ that describes the soundness of the material: for $z(t,x) = 1$ (respectively, $z(t,x) = 0$) the material is in the undamaged (fully damaged, resp.) state, at the time $t \in (0,T)$ and 'locally' around the point $x \in \Omega$. In fact, the related PDE system consists of

– The momentum balance

$$-\mathrm{div}\,\sigma = f \quad \text{in } \Omega \times (0,T)\,, \qquad \sigma \mathrm{n} = g \text{ on } \Gamma_{\mathrm{Neu}} \times (0,T), \tag{1.1a}$$

with f, g some external forces, n the outer unit normal vector to Ω, σ the *stress* tensor

$$\sigma = \mathbb{C}(z)e \quad \text{in } \Omega \times (0,T), \tag{1.1b}$$

$\mathbb{C}$ the elastic stress tensor, and $e : (0,T) \times \Omega \to \mathbb{M}^{n\times n}_{\mathrm{sym}}$ the elastic strain; together with the plastic strain p, the elastic strain e concurs to the kinematic admissibility condition for the *strain* $\mathrm{E}(u) = \frac{\nabla u + \nabla u^T}{2}$, i.e.,

$$\mathrm{E}(u) = e + p \quad \text{in } \Omega \times (0,T); \tag{1.1c}$$

– The flow rule for the damage variable z

$$\partial \mathrm{R}(\dot{z}) + A_{\mathrm{m}}(z) + W'(z) \ni -\tfrac{1}{2}\mathbb{C}'(z)e : e \quad \text{in } \Omega \times (0,T), \tag{1.1d}$$

where $\partial\mathrm{R} : \mathbb{R} \rightrightarrows \mathbb{R}$ denotes the convex analysis subdifferential of the density of dissipation potential

$$\mathrm{R} : \mathbb{R} \to [0,+\infty] \ \text{ defined by } \ \mathrm{R}(\eta) := \begin{cases} \kappa|\eta| & \text{if } \eta \le 0, \\ +\infty & \text{otherwise,} \end{cases}$$

encompassing the unidirectionality in the evolution of damage, A_{m} is the m-Laplacian operator, with $\mathrm{m} > \frac{n}{2}$, and W is a suitable nonlinear, possibly nonsmooth, function;

– The flow rule for the plastic tensor

$$\partial_{\dot{p}}\mathrm{H}(z,\dot{p}) \ni \sigma_{\mathrm{D}} \quad \text{in } \Omega \times (0,T), \tag{1.1e}$$

with σ_{D} the deviatoric part of the stress tensor σ and $\mathrm{H}(z,\cdot)$ the density of the plastic dissipation potential.

System (1.1a)–(1.1e) is complemented by the boundary conditions

$$u = w \text{ on } \Gamma_{\mathrm{Dir}} \times (0,T), \qquad \sigma\mathrm{n} = g \text{ on } \Gamma_{\mathrm{Neu}} \times (0,T), \qquad \partial_{\mathrm{n}} z = 0 \text{ on } \partial\Omega \times (0,T), \tag{1.1f}$$

where Γ_{Dir} is the Dirichlet part of the boundary $\partial\Omega$ and w a time-dependent Dirichlet loading, while Γ_{Neu} is the Neumann part of $\partial\Omega$ and g an assigned traction.

Alternative models for damage and plasticity have been analyzed in, e.g., [15, 30, 31], albeit from a different perspective. In fact, those papers address the rate-independent evolution of the damage and plastic processes coupled with a *rate-dependent* momentum balance, featuring viscosity and even inertial terms. Therefore, the resulting system has a *mixed* rate-dependent/independent character and is formulated in terms of a weak, energetic-type notion of solution. Instead, both in [8] and [10], (two distinct) viscous regularization procedures, described below, were advanced to construct Balanced Viscosity solutions to the fully rate-independent system (1.1).

1.2 Balanced viscosity solutions

In [8], the vanishing-viscosity approximation of system (1.1) was carried out by perturbing the damage flow rule by a viscous term, which led to the *viscously regularized* system

$$-\operatorname{div}\sigma = f \quad \text{with } \sigma = \mathbb{C}(z)e \quad \text{in } \Omega \times (0,T), \tag{1.2a}$$

$$\partial \mathrm{R}(\dot{z}) + \varepsilon \dot{z} + A_{\mathrm{m}}(z) + W'(z) \ni -\tfrac{1}{2}\mathbb{C}'(z)e : e \quad \text{in } \Omega \times (0,T), \tag{1.2b}$$

$$\partial_{\dot{p}} \mathrm{H}(z,\dot{p}) \ni \sigma_{\mathrm{D}} \quad \text{in } \Omega \times (0,T), \tag{1.2c}$$

supplemented by the boundary conditions

$$u = w \text{ on } \Gamma_{\mathrm{Dir}} \times (0,T), \quad \sigma \mathrm{n} = g \text{ on } \Gamma_{\mathrm{Neu}} \times (0,T), \quad \partial_{\mathrm{n}} z = 0 \text{ on } \partial\Omega \times (0,T). \tag{1.2d}$$

Passing to the limit in a reparameterized version of (1.2) led to a first construction of BV solutions to system (1.1). We shall illustrate the notion of *parameterized* Balanced Viscosity solution thus obtained in the forthcoming Sect. 3.3. In what follows, for quicker reference we will call the BV solutions from [8] BV_0 *solutions to system* (1.1), where the subscript 0 indicates that the solutions are obtained in the limit as $\varepsilon \downarrow 0$.

In [10], a different construction of BV solutions to system (1.1) was proposed, based on a viscous regularization of the momentum balance and of the plastic flow rule, *in addition* to that of the damage flow rule. This alternative approach was first proposed in [28] in a finite-dimensional context and extended to infinite-dimensional systems in the recent [25]. Both papers address the vanishing-viscosity analysis of an abstract evolutionary system, that can be thought of as prototypical of rate-dependent systems in solid mechanics, governing the evolution of an elastic variable u and of an internal variable z. In those papers, z has relaxation time ε, while u has a viscous damping with relaxation time ε^α, $\alpha > 0$. To emphasize the occurrence of these three time scales (the time scale $\varepsilon^0 = 1$ of the external loading, the relaxation time ε of z, and the - possibly different - relaxation time ε^α of u), the term 'multi-rate system' was used in [28]. Therein, as well as in [25], it was shown that, in the three cases $\alpha \in (0,1)$, $\alpha = 1$ and $\alpha > 1$, the vanishing-viscosity analysis as $\varepsilon \downarrow 0$ leads to different notions of Balanced-Viscosity solutions, including in particular different descriptions of time discontinuities. Indeed, the transition corresponding to a jump in time is characterized by a PDE system. In contrast, in the case of global minimization, the only condition to be satisfied at jumps is energy conservation.

Thus, along the lines of [28], in [10] the authors addressed the following, alternative, viscous regularization of system (1.1):

$$-\operatorname{div}(\varepsilon \nu \mathbb{D}\mathrm{E}(\dot{u}) + \sigma) = f \quad \text{in } \Omega \times (0,T), \tag{1.3a}$$

$$\partial \mathrm{R}(\dot{z}) + \varepsilon \dot{z} + A_{\mathrm{m}}(z) + W'(z) \ni -\tfrac{1}{2}\mathbb{C}'(z)e : e \quad \text{in } \Omega \times (0,T), \tag{1.3b}$$

$$\partial_{\dot{p}} \mathrm{H}(z,\dot{p}) + \varepsilon \nu \dot{p} + \mu p \ni \sigma_{\mathrm{D}} \quad \text{in } \Omega \times (0,T), \tag{1.3c}$$

supplemented by the boundary conditions

$$u = w \text{ on } \Gamma_{\mathrm{Dir}} \times (0,T), \quad (\varepsilon \nu \mathbb{D}\mathrm{E}(\dot{u}) + \sigma)\mathrm{n} = g \text{ on } \Gamma_{\mathrm{Neu}} \times (0,T), \quad \partial_{\mathrm{n}} z = 0 \text{ on } \partial\Omega \times (0,T), \tag{1.3d}$$

where $\mathbb{D}$ is a positive-definite fourth-order tensor. System (1.3) features a viscous regularization *both* in the damage flow rule *and* in the displacement equation and the plastic flow rule. Let us now illustrate the role of the various parameters appearing therein, namely the

- (Vanishing-)viscosity parameter $\varepsilon > 0$,
- (Vanishing-)hardening parameter $\mu > 0$,

– Additional parameter $\nu > 0$, which was required to fulfill $\nu \leq \mu$ in order to get suitable a priori estimates. We have referred to ν as a *rate* parameter, since it sets the mutual rate at which, on the one hand, the displacement and the plastic strain converge to equilibrium and rate-independent evolution, and, on the other hand, the damage parameter converges to rate-independent evolution. More precisely, if $\nu > 0$ stays fixed then u and p converge at the same rate as z, while their convergence occurs at a faster rate if $\nu \downarrow 0$. This is clear if one chooses, e.g., $\nu = \mu = \varepsilon$, so that the viscous terms $\mathrm{E}(\dot{u})$ and $\dot{p}$ in (1.3a) and (1.3c) are modulated by the coefficient ε^2, as opposed to the coefficient ε in the damage flow rule. Observe that, upon taking the vanishing-hardening limit $\mu \downarrow 0$, the constraint $\nu \leq \mu$ forces the joint vanishing-viscosity and vanishing-hardening limit to occur at a faster rate for u and p than for z.

We will refer to the vanishing-viscosity analyses in system (1.3) as *full*, as opposed to the *partial* vanishing-viscosity approximation provided by system (1.2), where only the damage flow rule is regularized.

Indeed, in [10] three *full* vanishing-viscosity analyses have been carried out for system (1.3), leading to three different notions of solution for system (1.1), possibly regularized by a hardening term. Let us briefly illustrate them.

(1) $\mathrm{BV}_0^{\mu,\nu}$ *solutions* The limit passage in a (reparameterized) version of (1.3) as $\varepsilon \downarrow 0$, while the positive parameters μ and ν stayed fixed, has led to BV solutions for a variant of the system (1.1), where the plastic flow rule was regularized by the hardening term μp, i.e., for the rate-independent system with hardening

$$-\operatorname{div}(\mathbb{C}(z)e) = f \quad \text{in } \Omega \times (0,T), \tag{1.4a}$$

$$\partial \mathrm{R}(\dot{z}) + A_{\mathrm{m}}(z) + W'(z) \ni -\tfrac{1}{2}\mathbb{C}'(z)e : e \quad \text{in } \Omega \times (0,T), \tag{1.4b}$$

$$\partial_{\dot{p}} \mathrm{H}(z,\dot{p}) + \mu p \ni \sigma_{\mathrm{D}} \quad \text{in } \Omega \times (0,T), \tag{1.4c}$$

coupled with the boundary conditions

$$\begin{aligned} &u = w \text{ on } \Gamma_{\mathrm{Dir}} \times (0,T), \quad \sigma \mathrm{n} = g \text{ on } \Gamma_{\mathrm{Neu}} \times (0,T), \\ &\partial_{\mathrm{n}} z = 0 \text{ on } \partial\Omega \times (0,T). \end{aligned} \tag{1.4d}$$

The BV solutions to system (1.4) thus constructed reflect their origin from a system viscously regularized in all of the three variables u, z, p. In fact, in the jump regime the system may switch to viscous behavior in the three variables u, z, and p. This means that the states of the system before and after a time discontinuity are connected by a trajectory (reparameterized in a certain time scale), whose evolution may be governed by viscosity in u, z, and p, similarly to (1.3). Since the convergence of u, z, and p to elastic equilibrium and rate-independent evolution has occurred at the *same rate* (as $\nu > 0$ stayed fixed), viscous behavior in u, z, and p may *equally* intervene in the jump regime. We shall refer to such solutions as $\mathrm{BV}_0^{\mu,\nu}$ *solutions to system* (1.4). The subscript 0 suggests that they have been obtained in the vanishing-viscosity limit $\varepsilon \downarrow 0$, while the occurrence of the parameter μ keeps track of the presence of hardening. Also, the parameter ν appears in the notation, since it still features in the limiting evolution as a coefficient of the viscous terms in the displacement equation and in the plastic flow rule, which may be active in the jump regime.

(2) $\mathrm{BV}_0^{\mu,0}$ *solutions* The limit passage in a (reparameterized) version of (1.3) as $\varepsilon \downarrow 0$ simultaneously with $\nu \downarrow 0$, while $\mu > 0$ stayed fixed, has again led to BV solutions for the rate-independent elasto-plastic damage system with hardening (1.4). These solutions still have the feature that, in the jump regime, the system may switch to viscous behavior in u, z, and p. However, BV solutions thus obtained reflect the fact that the convergence of u and p to elastic equilibrium and rate-independent evolution has occurred at a *faster rate* (as $\nu \downarrow 0$) than that for z. To emphasize this, such solutions were termed BV *solutions to the multi-rate system for damage with hardening*. We will refer to them as $\mathrm{BV}_0^{\mu,0}$ *solutions to system* (1.1). In this notation, the double occurrence of 0 relates to the fact that such solutions were obtained in the limit $\varepsilon, \nu \downarrow 0$, as opposed to the BV_0 solutions from [8] (arising in the limit of system (1.3) as $\varepsilon \downarrow 0$ and $\mu = \nu = 0$).

(3) $\mathrm{BV}_0^{0,0}$ *solutions* The limit passage in a (reparameterized) version of (1.3) as $\varepsilon,\ \mu,\ \nu \downarrow 0$ *jointly* led to BV *solutions to the multi-rate system for damage and perfect plasticity* (1.1), again reflecting the fact that the convergence of u and p to elastic equilibrium and rate-independent, *perfectly plastic* evolution happened at a rate faster than that for z. The vanishing-viscosity solutions arising from this joint limit will receive specific attention in this paper. In what follows, we will refer to them as $\mathrm{BV}_0^{0,0}$ *solutions to system* (1.1). Here, the triple occurrence of 0 relates to the fact that such solutions were obtained in the limit $\varepsilon,\ \mu,\ \nu \downarrow 0$ and thus immediately suggests the comparison with the BV_0 solutions from [8].

1.3 Our results

The aim of this paper is twofold:

(i) We propose to gain further insight into $\mathrm{BV}_0^{0,0}$ solutions to system (1.1) (cf. item #3 in the above list). More precisely, first of all we shall provide a *differential characterization* of such solutions, cf. Proposition 3.11 ahead. This will be compared with a corresponding characterization of BV_0 solutions proved in Proposition 3.5. Moreover, relying on Proposition 3.11, in Theorem 4.1 we will subsequently prove that, after an initial phase in which z is constant while u and p, evolving by viscosity, relax to elastic equilibrium and to rate-independent evolution, respectively, it turns out that u never leaves the equilibrium, and p the rate-independent regime. Afterward, the evolution of system (1.1) is captured by the notion of BV_0 solution as obtained in [8] by taking the vanishing-viscosity limit as $\varepsilon \downarrow 0$ of system (1.2). In other words, viscosity in u and p (may) intervene only in an initial phase in the reparameterized time scale, corresponding to a time discontinuity in the time scale of the loading. After this initial phase, the $\mathrm{BV}_0^{0,0}$ solutions to the perfectly plastic system for damage (1.1) arising by the *full* vanishing-viscosity approach of [10] comply with the same notion of solution of [8], where viscosity for u and p was neglected.

We point out that an analogous characterization can be proved for the $\mathrm{BV}_0^{\mu,0}$ solutions to the multi-rate system with *fixed* hardening parameter $\mu > 0$, obtained in the limit passage #2 of the above list; see Remark 5.2 ahead.

(ii) We aim to 'close the circle' in the analysis of the singular limits of system (1.3), by showing that, for two given sequences $(\mu_k)_k$, $(\nu_k)_k \subset (0,+\infty)$ with $0 < \nu_k \le \mu_k \downarrow 0$ as $k \to \infty$,

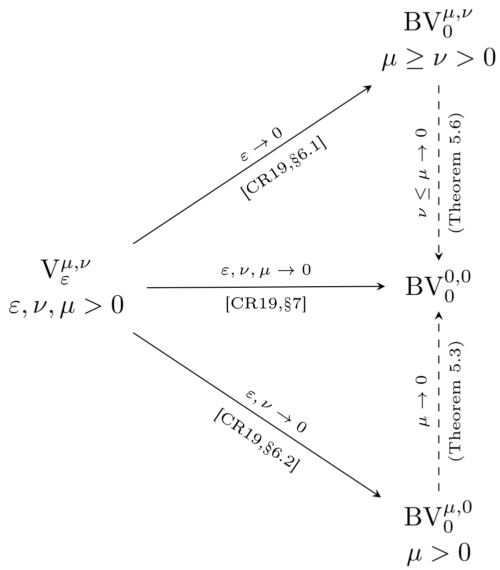

Fig. 1 The diagram displays the asymptotic relations between different notions of solution. The symbol $\mathrm{V}_\varepsilon^{\mu,\nu}$ indicates solutions to the viscous system (1.3). Solid lines represent convergences (along sequences) to the limiting solutions of type $\mathrm{BV}_0^{\mu,\nu}$, $\mathrm{BV}_0^{\mu,0}$, $\mathrm{BV}_0^{0,0}$, already proved in [10, Sections 6.1, 6.2, 7]. Dashed lines represent convergences proved in the present paper, the corresponding theorems being referred to in the diagram. Starting from $\mathrm{V}_\varepsilon^{\mu,\nu}$ one may either pass to the limit as $\varepsilon \downarrow 0$ and then as μ, $\nu \downarrow 0$; or pass to the limit as ε, $\nu \downarrow 0$ and then as $\mu \downarrow 0$. Since there is no uniqueness, it is not guaranteed that one gets the very same solution found in the joint limit ε, μ, $\nu \downarrow 0$. However, we prove that through the three different procedures one finds evolutions satisfying the same *notion* of solution. In this sense, we may say that the diagram commutes

(1) $\mathrm{BV}_0^{\mu_k,\nu_k}$ solutions to the *single*-rate system with hardening converge as $k \to \infty$ to a $\mathrm{BV}_0^{0,0}$ solution of the perfectly plastic damage system, which will be shown in Theorem 5.6 ahead;
(2) $\mathrm{BV}_0^{\mu_k,0}$ solutions to the *multi*-rate system with hardening converge as $k \to \infty$ to a $\mathrm{BV}_0^{0,0}$ solution, cf. Theorem 5.3.

In particular, we will prove that the diagram in Fig. 1 commutes.

We emphasize that these results establish asymptotic relations between BV solutions, already obtained as vanishing-viscosity limits. These convergence analyses show that $\mathrm{BV}_0^{0,0}$ solutions are robust enough to capture the asymptotic behavior of a wide class of BV solutions depending on different parameters. However, $\mathrm{BV}_0^{0,0}$ solutions reduce to BV_0 solutions after an initial phase in which u and p converge to elastic equilibrium and stability, respectively; in particular, if the initial conditions are at equilibrium, then the two notions of solution coincide. This feature may be traced back to the convex character of perfect plasticity and to the multi-rate character inherent to the system, since with $\nu \downarrow 0$ we have forced faster convergence to equilibrium in u and stability in p.

1.4 Plan of the paper

In Sect. 2, we detail the setup of the problem, list our assumptions and provide some preliminary results. In Sect. 3, we illustrate the notion of BV_0 solution to system (1.1) arising from the *partial* vanishing-viscosity approach of [8], and that of $\mathrm{BV}_0^{0,0}$ solution via the *full* vanishing-viscosity analysis in [10]. In Sect. 4, we establish Theorem 4.1, providing a complete characterization of $\mathrm{BV}_0^{0,0}$ solutions. Section 5 is devoted to the vanishing-hardening analysis of $\mathrm{BV}_0^{\mu,0}$ and $\mathrm{BV}_0^{\mu,\nu}$ solutions to the system with hardening. The proofs of Theorems 5.3 and 5.6 rely on some technical results collected in the "Appendix".

2 Setup and preliminaries

Throughout the paper, we will use the following

Notation 2.1 *(General notation and preliminaries)* Let X be a Banach space. By $\langle\cdot,\cdot\rangle_X$, we denote the duality between X^* and X or between $(X^n)^*$ and X^n (whenever X is a Hilbert space, $\langle\cdot,\cdot\rangle_X$ will be the inner product), while $\|\cdot\|_X$ stands for the norm in X or in X^n. The inner Euclidean product in $\mathbb{R}^n$, $n \geq 1$, is denoted by $\langle\cdot,\cdot\rangle$ and the Euclidean norm in $\mathbb{R}^n$ by $|\cdot|$. The symbol $B_r(0)$ stands for the open ball in $\mathbb{R}^n$ with radius r and center 0.

We write $\|\cdot\|_{L^p}$ for the L^p-norm on the space $L^p(O;\mathbb{R}^d)$, with O a measurable subset of $\mathbb{R}^n$ and $1 \leq p < +\infty$, and similarly $\|\cdot\|_{H^\mathrm{m}}$ for the norm of the Sobolev-Slobodeskij space $H^\mathrm{m}(O)$, for $0 \leq \mathrm{m} \in \mathbb{R}$. The symbol $\mathrm{M_b}(O;\mathbb{R}^d)$ stands for the space of $\mathbb{R}^d$-valued bounded Radon measures in O.

The space of symmetric $(n{\times}n)$-matrices is denoted by $\mathbb{M}^{n\times n}_{\mathrm{sym}}$, while the subspace of the deviatoric matrices with null trace is denoted by $\mathbb{M}^{n\times n}_{\mathrm{D}}$. One has $\mathbb{M}^{n\times n}_{\mathrm{sym}} = \mathbb{M}^{n\times n}_{\mathrm{D}} \oplus \mathbb{R}I$, where I is the identity matrix, i.e., any $\eta \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ can be decomposed as $\eta = \eta_\mathrm{D} + \frac{\mathrm{tr}(\eta)}{n}I$, where η_D is the orthogonal projection of η onto $\mathbb{M}^{n\times n}_{\mathrm{D}}$. The latter is called the *deviatoric part* of η. The symbol $\mathrm{Sym}(\mathbb{M}^{n\times n}_{\mathrm{D}};\mathbb{M}^{n\times n}_{\mathrm{D}})$ stands for the set of symmetric endomorphisms on $\mathbb{M}^{n\times n}_{\mathrm{D}}$.

Given a function $v : \Omega\times(0,T) \to \mathbb{R}$ differentiable, w.r.t. time a.e. on $\Omega\times(0,T)$, its (almost everywhere defined) partial time derivative is indicated by $\dot{v} : \Omega\times(0,T) \to \mathbb{R}$. A different notation will be employed when considering v as a (Bochner) function, from $(0,T)$ with values in a Lebesgue or Sobolev space X (with the Radon–Nikodým property): if $v \in \mathrm{AC}([0,T];X)$, then its (almost everywhere defined) time derivative is indicated by $v' : (0,T) \to X$.

The symbols c, c', C, C' will denote positive constants whose precise value may vary from line to line (or within the same line). We will sometimes employ the symbols I_i, $i = 0, 1, ...$, as place-holders for terms appearing in inequalities: also in this case, such symbols may appear in different proofs with different meaning.

2.1 Functions of bounded deformation

The state space for the displacement variable for the systems with hardening will be

$$H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n) := \{u \in H^1(\Omega;\mathbb{R}^n) \ : \ u = 0 \text{ on } \Gamma_{\mathrm{Dir}}\}$$

(recall that Γ_{Dir} is the Dirichlet part of $\partial\Omega$, cf. (2.Ω) ahead).

For the perfectly plastic damage system, displacements will belong to the space of *functions of bounded deformations*, defined by

$$\mathrm{BD}(\Omega) := \{u \in L^1(\Omega;\mathbb{R}^n) \ : \ \mathrm{E}(u) \in \mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})\},$$

with $\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ the space of $\mathbb{M}^{n\times n}_{\mathrm{sym}}$-valued bounded Radon measures on Ω.

We recall that $\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ can be identified with the dual of the space $\mathrm{C}^0_0(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ of continuous $\mathbb{M}^{n\times n}_{\mathrm{sym}}$-valued functions vanishing at the boundary of Ω. The space $\mathrm{BD}(\Omega)$ has a Banach structure if equipped with the norm

$$\|u\|_{\mathrm{BD}(\Omega)} := \|u\|_{L^1(\Omega;\mathbb{R}^n)} + \|\mathrm{E}(u)\|_{\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})}.$$

Indeed, $\mathrm{BD}(\Omega)$ is the dual of a normed space, cf. [33], and such duality provides a weak convergence on $\mathrm{BD}(\Omega)$: a sequence $(u_k)_k$ converges to u weakly in $\mathrm{BD}(\Omega)$ if $u_k \rightharpoonup u$ in $L^1(\Omega;\mathbb{R}^n)$ and $\mathrm{E}(u_k) \overset{*}{\rightharpoonup} \mathrm{E}(u)$ in $\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$. It holds $\mathrm{BD}(\Omega) \subset L^{n/(n-1)}(\Omega;\mathbb{R}^n)$. Up to subsequences, every bounded sequence in $\mathrm{BD}(\Omega)$ converges weakly*, weakly in $L^{n/(n-1)}(\Omega;\mathbb{R}^n)$, and strongly in $L^p(\Omega;\mathbb{R}^n)$ for any $1 \le p < \frac{n}{n-1}$. Finally, we recall that the trace $u|_{\partial\Omega}$ of a function $u \in \mathrm{BD}(\Omega)$ is well defined and is an element in $L^1(\partial\Omega;\mathbb{R}^n)$.

2.1.1 A divergence operator

First of all, we observe that any $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ such that $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^n)$ induces the distribution $[\sigma \mathrm{n}]$ defined by

$$\langle [\sigma\mathrm{n}], \psi\rangle_{\partial\Omega} := \langle \operatorname{div}\sigma, \psi\rangle_{L^2} + \langle \sigma, \mathrm{E}(\psi)\rangle_{L^2} \qquad \text{for every } \psi \in H^1(\Omega;\mathbb{R}^n). \tag{2.1}$$

By [22, Theorem 1.2] and [13, (2.24)] we have that $[\sigma\mathrm{n}] \in H^{-1/2}(\partial\Omega;\mathbb{R}^n)$; moreover, if $\sigma \in \mathrm{C}^0(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{sym}})$, then the distribution $[\sigma\mathrm{n}]$ fulfills $[\sigma\mathrm{n}] = \sigma\mathrm{n}$, where the right-hand side is the standard pointwise product of the matrix σ and the normal vector n in $\partial\Omega$.

For the treatment of the perfectly plastic system for damage, it will be crucial to work with the space

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) : \operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n),\ \sigma_{\mathrm{D}} \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\}. \tag{2.2}$$

Furthermore, our choice of external loadings (see (2.8a)) will ensure that the stress fields σ that we consider, at equilibrium, have the additional property that $[\sigma\mathrm{n}] \in L^\infty(\Omega;\mathbb{R}^n)$ and $\sigma \in \Sigma(\Omega)$ (cf. Lemma 3.1). Therefore, any of such fields induces a functional $-\mathrm{Div}\,\sigma \in \mathrm{BD}(\Omega)^*$ via

$$\langle -\mathrm{Div}\,\sigma, v\rangle_{\mathrm{BD}(\Omega)} := \langle -\operatorname{div}\sigma, v\rangle_{L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)} + \langle [\sigma\mathrm{n}], v\rangle_{L^1(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)} \qquad \text{for all } v \in \mathrm{BD}(\Omega). \tag{2.3}$$

With slight abuse of notation, we shall denote by $-\mathrm{Div}\,\sigma$ also the restriction of the above functional to $H^1(\Omega;\mathbb{R}^n)^*$.

2.1.2 The A_{m}-Laplacian

The damage flow rule features a gradient regularizing contribution in terms of the A_{m} -Laplacian operator, that is defined from the bilinear form $a_{\mathrm{m}} : H^{\mathrm{m}}(\Omega) \times H^{\mathrm{m}}(\Omega) \to \mathbb{R}$,

$$a_{\mathrm{m}}(z_1, z_2) := \int_\Omega \int_\Omega \frac{\big(\nabla z_1(x) - \nabla z_1(y)\big) \cdot \big(\nabla z_2(x) - \nabla z_2(y)\big)}{|x-y|^{n+2(\mathrm{m}-1)}} \, \mathrm{d}x \, \mathrm{d}y \text{ with } \mathrm{m} > \frac{n}{2}.$$

Then,

$$A_{\mathrm{m}} : H^{\mathrm{m}}(\Omega) \to H^{\mathrm{m}}(\Omega)^* \text{ is defined by}$$
$$\langle A_{\mathrm{m}}(z), w \rangle_{H^{\mathrm{m}}(\Omega)} := a_{\mathrm{m}}(z, w) \quad \text{for every } z, w \in H^{\mathrm{m}}(\Omega).$$

The inner product $\langle z_1, z_2 \rangle_{H^{\mathrm{m}}(\Omega)} := \int_\Omega z_1 z_2 \, \mathrm{d}x + a_{\mathrm{m}}(z_1, z_2)$ makes $H^{\mathrm{m}}(\Omega)$ a Hilbert space. Throughout the paper, we shall assume $\mathrm{m} > \frac{n}{2}$ and rely on the compact embedding $H^{\mathrm{m}}(\Omega) \Subset \mathrm{C}^0(\overline{\Omega})$.

2.2 Assumptions and preliminary results

This section and Sect. 2.3 collect all our assumptions on the constitutive functions of the model and on the problem data. We will omit to invoke them explicitly in the statement of the various results.

2.2.1 The reference configuration

In what follows, we will assume that $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, is a bounded Lipschitz domain satisfying the so-called *Kohn–Temam condition*:

$$\begin{aligned}
&\partial\Omega \text{ Lipschitz}, \quad \partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}} \cup \Sigma \quad \text{ with } \Gamma_{\mathrm{Dir}}, \Gamma_{\mathrm{Neu}}, \\
&\Sigma \text{ pairwise disjoint}, \Gamma_{\mathrm{Dir}} \text{ and } \Gamma_{\mathrm{Neu}} \text{ relatively open in } \partial\Omega, \\
&\text{and } \partial\Gamma_{\mathrm{Dir}} = \partial\Gamma_{\mathrm{Neu}} = \Sigma \text{ their relative boundary in } \partial\Omega, \\
&\text{with } \Sigma \text{ of class } \mathrm{C}^2 \text{ and } \mathcal{H}^{n-1}(\Sigma) = 0, \text{ and with } \partial\Omega \\
&\text{of class } \mathrm{C}^2 \text{ in a neighborhood of } \Sigma.
\end{aligned} \tag{2.Ω}$$

Notice that the additional C^2 regularity is only required close to the set Σ where the Dirichlet and the Neumann boundary meet. This is a technical assumption needed to interpret the stress-strain duality (2.9) as a measure.

2.2.2 The elasticity and viscosity tensors

We assume that the elastic tensor $\mathbb{C} : [0, +\infty) \to \mathrm{Lin}(\mathbb{M}^{n\times n}_{\mathrm{sym}}; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ fulfills the following conditions

$$\mathbb{C} \in \mathrm{C}^{1,1}([0, +\infty); \mathrm{Lin}(\mathbb{M}^{n\times n}_{\mathrm{sym}}; \mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{2.$\mathbb{C}_1$}$$

$$z \mapsto \mathbb{C}(z)\xi : \xi \text{ is nondecreasing for every } \xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}}, \tag{2.$\mathbb{C}_2$}$$

$$\exists\, \gamma_1, \gamma_2 > 0 \;\; \forall z \in [0,+\infty)\, \forall \xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}} : \quad \gamma_1|\xi|^2 \le \mathbb{C}(z)\xi : \xi \le \gamma_2|\xi|^2. \tag{2.$\mathbb{C}_3$}$$

For the viscosity tensor $\mathbb{D}$, we require that

$$\mathbb{D} \in \mathrm{C}^0(\overline{\Omega}; \mathrm{Sym}(\mathbb{M}^{n\times n}_{\mathrm{D}}; \mathbb{M}^{n\times n}_{\mathrm{D}})), \text{ and} \tag{2.$\mathbb{D}_1$}$$

$$\exists\, \delta_1, \delta_2 > 0\; \forall x \in \Omega \;\; \forall A \in \mathbb{M}^{d\times d}_{\mathrm{sym}} : \quad \delta_1|A|^2 \le \mathbb{D}(x)A : A \le \delta_2|A|^2. \tag{2.$\mathbb{D}_2$}$$

Thus, $\mathbb{D}$ induces an equivalent (by a Korn–Poincaré-type inequality) Hilbert norm on $H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)$, i.e.,

$$\begin{aligned} &\|u\|_{H^1,\mathbb{D}} := \left(\int_\Omega \mathbb{D}\mathrm{E}(u) : \mathrm{E}(u)\,\mathrm{d}x\right)^{1/2} \text{ and} \\ &\exists\, K_{\mathbb{D}} > 0\; \forall\, u \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n) : \; \|u\|_{H^1,\mathbb{D}} \le K_{\mathbb{D}}\|\mathrm{E}(u)\|_{L^2}. \end{aligned} \tag{2.4}$$

The related ‘dual norm’ is

$$\begin{aligned} &\|\eta\|_{(H^1,\mathbb{D})^*} := \left(\int_\Omega \mathbb{D}^{-1}\xi : \xi\right)^{1/2} \\ &\text{for all } \eta \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^* \text{ with } \eta = \mathrm{Div}\,\xi \text{ for some } \xi \in \widetilde{\Sigma}(\Omega). \end{aligned} \tag{2.5}$$

2.2.3 The potential energy for the damage variable

In addition to the regularizing, nonlocal gradient contribution featuring the bilinear form a_{m}, the z-dependent part of the mechanical energy functional shall feature a further term with density W satisfying

$$W \in \mathrm{C}^2((0,+\infty);\mathbb{R}^+) \cap \mathrm{C}^0([0,+\infty);\mathbb{R}^+ \cup \{+\infty\}), \tag{2.W1}$$

$$s^{2n}W(s) \to +\infty \text{ as } s \to 0^+, \tag{2.W2}$$

where $W \in \mathrm{C}^0([0,+\infty);\mathbb{R}^+ \cup \{+\infty\})$ means that $W(0) = \infty$ and $W(z) \to +\infty$ if $z \downarrow 0$, in accordance with (2.W_2).

Indeed, the energy contribution involving W forces z to be strictly positive; consequently, the material never reaches the most damaged state at any point.

2.2.4 The plastic and the damage dissipation densities

The plastic dissipation potential shall reflect the requirement that the admissible stresses belong to given constraint sets which, in turn, depend on the damage variable z. More precisely, as in [8] we ask that the constraint sets $(K(z))_{z\in[0,+\infty)}$ fulfill

$$K(z) \subset \mathbb{M}^{n\times n}_{\mathrm{D}} \text{ is closed and convex for all } z \in [0,+\infty), \tag{2.K_1}$$

$$\exists\, 0 < \bar r < \bar R \quad \forall\, 0 \le z_1 \le z_2 \ : \qquad B_{\bar r}(0) \subset K(z_1) \subset K(z_2) \subset B_{\bar R}(0), \tag{2.K_2}$$

$$\exists\, C_K > 0 \quad \forall\, z_1, z_2 \in [0,+\infty) \ : \qquad d_{\mathscr{H}}(K(z_1),K(z_2)) \le C_K|z_1 - z_2|, \tag{2.K_3}$$

with $d_{\mathscr{H}}$ the Hausdorff distance between two subsets of $\mathbb{M}^{n\times n}_{\mathrm{D}}$, defined by

$$d_{\mathscr{H}}(K_1,K_2) := \max\left(\sup_{x\in K_1} \mathrm{dist}(x,K_2),\ \sup_{x\in K_2} \mathrm{dist}(x,K_1)\right).$$

The associated support function $H : [0,+\infty)\times \mathbb{M}^{n\times n}_{\mathrm{D}} \to [0,+\infty)$, defined by

$$H(z,\pi) := \sup_{\sigma\in K(z)} \sigma : \pi \qquad \text{for all } (z,\pi) \in [0,+\infty)\times \mathbb{M}^{n\times n}_{\mathrm{D}}, \tag{2.6}$$

will act as density function for the plastic dissipation potential, cf. (2.14) later on.

We choose as damage dissipation density the function $\mathrm{R} : \mathbb{R} \to [0,+\infty]$ given by

$$\mathrm{R}(\zeta) := \begin{cases} -\kappa\zeta & \text{if } \zeta \le 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.7}$$

with $\kappa > 0$ a constant related to the toughness of the material.

2.2.5 Body and surface forces, Dirichlet loading, and initial data

We assume that the volume force f and the assigned traction g fulfill

$$f \in H^1(0,T;L^n(\Omega;\mathbb{R}^n)), \quad g \in H^1(0,T;L^\infty(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)). \tag{2.8a}$$

The induced total load is the function $F : [0,T] \to \mathrm{BD}(\Omega)^*$,

$$\langle F(t), v\rangle_{\mathrm{BD}(\Omega)} := \langle f(t), v\rangle_{L^{n/(n-1)}(\Omega;\mathbb{R}^n)} + \langle g(t), v\rangle_{L^1(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)} \quad \text{for all } v \in \mathrm{BD}(\Omega).$$

Furthermore, as customary for perfect plasticity, we shall impose a *uniform safe load condition*, namely that there exists

$$\varrho \in H^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})) \quad \text{with} \quad \varrho_{\mathrm{D}} \in H^1(0,T;L^\infty(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})) \tag{2.8b}$$

and there exists $\alpha > 0$ such that for every $t \in [0,T]$ (recall (2.1))

$$-\operatorname{div} \varrho(t) = f(t) \text{ a.e. on } \Omega, \qquad [\varrho(t)\mathrm{n}] = g(t) \text{ on } \Gamma_{\mathrm{Neu}}, \tag{2.8c}$$

$$\varrho_{\mathrm{D}}(t,x) + \xi \in K \qquad \text{for a.a. } x \in \Omega \text{ and for every } \xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}} \text{ s.t. } |\xi| \le \alpha. \tag{2.8d}$$

Observe that, combining (2.8a) with (2.8b)–(2.8d) yields $-\mathrm{Div}\, \varrho(t) = F(t)$ for all $t \in [0,T]$.

As for the time-dependent Dirichlet boundary condition w, we assume that

$$w \in H^1(0,T;H^1(\mathbb{R}^n;\mathbb{R}^n)). \tag{2.8e}$$

Finally, we shall consider initial data $q_0 = (u_0, z_0, p_0)$ with

$$u_0 \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n), \qquad z_0 \in H^{\mathrm{m}}(\Omega) \text{ with } W(z_0) \in L^1(\Omega) \text{ and } z_0 \le 1 \text{ in } \overline{\Omega}, \\ p_0 \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}). \tag{2.8f}$$

2.2.6 The stress-strain duality

For the treatment of the perfectly plastic damage system, it is essential to resort to a suitable notion of stress-strain duality that we borrow from [13, 22], also relying on [17] for the extension to Lipschitz boundaries satisfying (2.Ω). Following [13], we introduce the class $A(w)$ of *admissible displacements and strains* associated with a function $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, that is

$$A(w) := \{(u,e,p) \in \mathrm{BD}(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \times \mathrm{M_b}(\Omega \cup \Gamma_{\mathrm{Dir}};\, \mathbb{M}^{n\times n}_{\mathrm{D}}) : \\ \mathrm{E}(u) = e + p \text{ in } \Omega,\ p = (w-u)\odot \mathrm{n}\,\mathscr{H}^{n-1} \text{ on } \Gamma_{\mathrm{Dir}}\},$$

where n denotes the normal vector to $\partial\Omega$ and $\odot$ the symmetrized tensorial product. The *space of admissible plastic strains* is

$$\Pi(\Omega) := \{p \in \mathrm{M_b}(\Omega \cup \Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}}) : \\ \exists\,(u,w,e) \in \mathrm{BD}(\Omega) \times H^1(\mathbb{R}^n;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \text{ s.t. } (u,e,p) \in A(w)\}.$$

Given $\sigma \in \Sigma(\Omega)$ (cf. (2.2)), $p \in \Pi(\Omega)$, and u, e such that $(u,e,p) \in A(w)$, we define

$$\langle[\sigma_{\mathrm{D}} : p], \varphi\rangle := -\int_\Omega \varphi\sigma\cdot(e - \mathrm{E}(w))\,\mathrm{d}x - \int_\Omega \sigma\cdot[(u-w)\odot\nabla\varphi]\,\mathrm{d}x - \int_\Omega \varphi\,(\mathrm{div}\,\sigma)\cdot(u-w)\,\mathrm{d}x \tag{2.9}$$

for every $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n)$; in fact, this definition is independent of u and e. Under these assumptions, $\sigma \in L^r(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ for every $r < \infty$, and $[\sigma_{\mathrm{D}} : p]$ is a bounded Radon measure with $\|[\sigma_{\mathrm{D}} : p]\|_1 \le \|\sigma_{\mathrm{D}}\|_{L^\infty}\|p\|_1$ in $\mathbb{R}^n$. Restricting such measure to $\Omega \cup \Gamma_{\mathrm{Dir}}$, we set

$$\langle \sigma_{\mathrm{D}} \,|\, p\rangle := [\sigma_{\mathrm{D}} : p](\Omega \cup \Gamma_{\mathrm{Dir}}). \tag{2.10}$$

By (2.Ω) and (2.9), since $u \in \mathrm{BD}(\Omega) \subset L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, we get the following integration by parts formula, valid if the distribution $[\sigma\mathrm{n}]$ defined in (2.1) belongs to $\in L^\infty(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)$:

$$\langle \sigma_{\mathrm{D}} \,|\, p\rangle = -\langle \sigma, e - \mathrm{E}(w)\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})} + \langle -\mathrm{div}\,\sigma, u - w\rangle_{L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)} + \langle [\sigma\mathrm{n}], u - w\rangle_{L^1(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)}$$

for every $\sigma \in \Sigma(\Omega)$ and $(u,e,p) \in A(w)$. We refer to [17] for the properties mentioned above.

2.3 Energetics

A key ingredient for the construction of BV solutions to the rate-independent systems (1.1) (damage with perfect plasticity) and (1.4) (damage and plasticity with hardening) is the observation that their rate-dependent regularizations (1.2) and (1.3) have a *gradient-system structure*. Namely, they can be reformulated in terms of the generalized gradient flow

$$\partial\Psi(q'(t)) + \mathrm{D}\mathcal{E}(t, q(t)) \ni 0 \qquad \text{in } \mathbf{Q}^*, \text{ a.e. in } (0,T),$$

for suitable choices of

- The state space $\mathbf{Q}$ for the triple $q = (u, z, p)$;
- The driving energy functional $\mathcal{E} : (0, T) \times \mathbf{Q} \to \mathbb{R} \cup \{+\infty\}$;
- The dissipation potential $\Psi : \mathbf{Q} \to [0, +\infty]$, with convex analysis subdifferential $\partial\Psi : \mathbf{Q} \rightrightarrows \mathbf{Q}^*$,

as rigorously proved in [10]. Observe that also the rate-independent systems (1.1) and (1.4) have a gradient structure that is, however, only formal due to the fact that the functions u, z, and p may have jumps as functions of time. Nonetheless, for our analysis it is crucial to detail the energetics underlying both the rate-dependent and the rate-independent systems.

2.3.1 The state spaces

The state space for the rate-dependent/independent damage systems with hardening is

$$\mathbf{Q}_{\mathrm{H}} := H^1_{\mathrm{Dir}}(\Omega; \mathbb{R}^n) \times H^{\mathrm{m}}(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}}).$$

For the rate-independent damage system with perfect plasticity, the displacements are just functions of bounded deformation and the plastic strains are only bounded Radon measures on $\Omega \cup \Gamma_{\mathrm{Dir}}$, so that the associated state space is

$$\mathbf{Q}_{\mathrm{PP}} := \{ q = (u, z, p) \in \mathrm{BD}(\Omega) \times H^{\mathrm{m}}(\Omega) \times \mathrm{M_b}(\Omega \cup \Gamma_{\mathrm{Dir}}; \mathbb{M}^{n\times n}_{\mathrm{D}}) \ : \\ e := \mathrm{E}(u) - p \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}),\ u \odot \mathrm{n}\, \mathscr{H}^{n-1} + p = 0 \text{ on } \Gamma_{\mathrm{Dir}} \}. \tag{2.11}$$

Observe that in the definition of $\mathbf{Q}_{\mathrm{PP}}$ it is in fact required that $(u, e, p) \in A(0)$: indeed, the condition $u \odot \mathrm{n}\, \mathscr{H}^{n-1} + p = 0$ is a relaxation of the homogeneous Dirichlet condition $u = 0$ on Γ_{Dir}.

2.3.2 The energy functionals

The energy functional governing the rate-dependent and rate-independent systems *with hardening* (1.3) and (1.4), respectively, consists of

(1) a contribution featuring the elastic energy

$$\mathcal{Q}(z, e) = \int_\Omega \tfrac{1}{2}\mathbb{C}(z)e : e\,\mathrm{d}x;$$

(2) the potential energy for the damage variable and for the hardening term;
(3) the time-dependent volume and surface forces.

Namely, for $\mu > 0$ given, $\mathcal{E}_\mu : [0, T] \times \mathbf{Q}_{\mathrm{H}} \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathcal{E}_\mu(t, u, z, p) := \mathcal{Q}(z, \mathrm{E}(u{+}w(t)){-}p) + \int_\Omega \left(W(z){+}\frac{\mu}{2}|p|^2 \right) \mathrm{d}x + \frac{1}{2} a_{\mathrm{m}}(z, z) \\ - \langle F(t), u + w(t) \rangle_{H^1(\Omega; \mathbb{R}^n)}.$$

In what follows, we will use the short-hand notation

$$e(t) := \mathrm{E}(u+w(t))-p$$

for the elastic part of the strain tensor.

The energy functional for the rate-independent perfectly plastic damage system (1.1) is $\mathcal{E}_0 : [0,T] \times \mathbf{Q}_{\mathrm{PP}} \to \mathbb{R}\cup\{+\infty\}$ given by

$$\mathcal{E}_0(t,u,z,p) := \mathcal{Q}(z,e(t)) + \int_\Omega W(z)\,\mathrm{d}x + \frac{1}{2}a_{\mathrm{m}}(z,z) - \langle F(t), u+w(t)\rangle_{\mathrm{BD}(\Omega)}. \quad (2.12)$$

Observe that

$$\mathrm{dom}(\mathcal{E}_0) = [0,T]\times \mathrm{D} \qquad \text{with } \mathrm{D} = \{q \in \mathbf{Q}_{\mathrm{PP}} \ : \ W(z) \in L^1(\Omega)\}. \quad (2.13)$$

2.3.3 The dissipation potentials

The dissipation density R from (2.7) clearly induces an integral functional $\mathcal{R} : L^1(\Omega) \to [0,+\infty]$. However, since the damage flow rule will be posed in $H^{\mathrm{m}}(\Omega)^*$ (cf. (2.20) ahead), we will restrict $\mathcal{R}$ to the space $H^{\mathrm{m}}(\Omega)$, denoting the restriction by the same symbol. Hence, we will work with the functional

$$\mathcal{R} : H^{\mathrm{m}}(\Omega) \to [0,+\infty], \quad \mathcal{R}(\zeta) = \int_\Omega \mathrm{R}(\zeta(x))\,\mathrm{d}x \quad (2.14)$$

and with its convex analysis subdifferential $\partial\mathcal{R} : H^{\mathrm{m}}(\Omega) \rightrightarrows H^{\mathrm{m}}(\Omega)^*$. We will often use the following characterization of $\partial\mathcal{R}$, due to the 1-homogeneity of the potential $\mathcal{R}$:

$$\chi \in \partial\mathcal{R}(\zeta) \quad \Leftrightarrow \quad \begin{cases} \mathcal{R}(\tilde\zeta) \geq \langle \chi, \tilde\zeta\rangle_{H^{\mathrm{m}}} & \text{for all } \tilde\zeta \in H^{\mathrm{m}}(\Omega) \\ \mathcal{R}(\zeta) \leq \langle \chi, \zeta\rangle_{H^{\mathrm{m}}} \end{cases}. \quad (2.15)$$

For the systems with hardening (i.e., (1.3) and (1.4)), the *plastic dissipation potential* $\mathcal{H} : \mathrm{C}^0(\overline{\Omega};[0,+\infty)) \times L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \to \mathbb{R}$ is defined by

$$\mathcal{H}(z,\pi) := \int_\Omega H(z(x),\pi(x))\,\mathrm{d}x, \quad (2.16)$$

where H is given by (2.6) and π is a place-holder for the plastic rate $\dot p$. Its convex analysis subdifferential $\partial_\pi\mathcal{H} : \mathrm{C}^0(\overline{\Omega};[0,+\infty)) \times L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \rightrightarrows L^\infty(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, w.r.t. the second variable π, given by

$$\omega \in \partial_\pi\mathcal{H}(z,\pi) \quad \text{if and only if}$$
$$\mathcal{H}(z,\varrho) - \mathcal{H}(z,\pi) \geq \int_\Omega \omega(\varrho-\pi)\,\mathrm{d}x \quad \text{for all } \varrho \in L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}),$$

fulfills

$$\omega \in \partial_\pi\mathcal{H}(z,\pi) \quad \text{if and only if} \quad \omega(x) \in \partial_\pi H(z(x),\pi(x)) \quad \text{for a.a. } x \in \Omega. \quad (2.17)$$

A characterization analogous to (2.17) holds for the subdifferential $\partial_\pi\mathcal{H}(z,\cdot) : L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \rightrightarrows L^\infty(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$.

In order to handle the perfectly plastic system for damage, the plastic dissipation potential $\mathcal{H}(z,\cdot)$ has to be extended to the space $\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$. We define $\mathcal{H}_{\mathrm{PP}} : \mathrm{C}^0(\overline{\Omega};[0,1])\times \mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})\to\mathbb{R}$ by

$$\mathcal{H}_{\mathrm{PP}}(z,\pi) := \int_{\Omega\cup\Gamma_{\mathrm{Dir}}} H\left(z(x),\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\right)\mathrm{d}\mu(x),$$

where $\mu\in\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$ is a positive measure such that $\pi\ll\mu$ and $\frac{\mathrm{d}\pi}{\mathrm{d}\mu}$ is the Radon–Nikodým derivative of p with respect to μ; by one-homogeneity of $H(z(x),\cdot)$, the definition of $\mathcal{H}_{\mathrm{PP}}$ does not depend of μ. For the theory of convex functions of measures, we refer to [19]. By [2, Proposition 2.37], for every $z\in\mathrm{C}^0(\overline{\Omega};[0,1])$ the functional $p\mapsto\mathcal{H}_{\mathrm{PP}}(z,p)$ is convex and positively one-homogeneous. We recall that by Reshetnyak's lower semicontinuity theorem, if $(z_k)_k\subset\mathrm{C}^0(\overline{\Omega};[0,1])$ and $(\pi_k)_k\subset\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$ are such that $z_k\to z$ in $\mathrm{C}^0(\overline{\Omega})$ and $\pi_k\rightharpoonup\pi$ weakly in $\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$, then

$$\mathcal{H}_{\mathrm{PP}}(z,\pi)\le\liminf_{k\to+\infty}\mathcal{H}_{\mathrm{PP}}(z_k,\pi_k).$$

Finally, from [17, Proposition 3.9] it follows that for every $\sigma\in\mathcal{K}_z(\Omega)$

$$H\left(z,\frac{\mathrm{d}p}{\mathrm{d}|p|}\right)|p|\ge[\sigma_{\mathrm{D}} : p]\quad\text{as measures on }\Omega\cup\Gamma_{\mathrm{Dir}}. \tag{2.18}$$

In particular, we have

$$\mathcal{H}_{\mathrm{PP}}(z,p)\ge\sup_{\sigma\in\mathcal{K}_z(\Omega)}\langle\sigma_{\mathrm{D}}\,|\,p\rangle\qquad\text{for every }p\in\Pi(\Omega). \tag{2.19}$$

The (partially) viscously regularized system (1.2) also features the 2-homogeneous dissipation potential

$$\mathcal{R}_2(\zeta) := \int_\Omega \tfrac{1}{2}|\zeta(x)|^2\,\mathrm{d}x,$$

while the (fully) viscously regularized system (1.3) additionally involves the quadratic potentials

$$\mathcal{V}_{2,\nu}(v) := \int_\Omega \tfrac{\nu}{2}\mathbb{D}\mathrm{E}(v) : \mathrm{E}(v)\,\mathrm{d}x,\qquad \mathcal{H}_{2,\nu}(\pi)=\int_\Omega\tfrac{\nu}{2}|\pi|^2\,\mathrm{d}x.$$

2.3.4 The gradient structure for system (1.3)

It was proved in [10, Lemma 3.3] that for every $t\in[0,T]$ the functional $(u,z,p)=q\mapsto\mathcal{E}_\mu(t,q)$ is Fréchet differentiable on its domain $[0,T]\times\mathbf{D}$, with $\mathbf{D}=\{(u,z,p)\in\mathbf{Q}_{\mathrm{H}} : z>0 \text{ in }\overline{\Omega}\}$, and that for all $q\in\mathbf{Q}_{\mathrm{H}}$ the function $t\mapsto\mathcal{E}_\mu(t,q)$ belongs to $H^1(0,T)$. Relying on this, it was shown that system (1.3) reformulates as the generalized gradient system

$$\partial_{q'}\Psi_{\varepsilon,\nu}(q(t),q'(t))+\mathrm{D}_q\mathcal{E}_\mu(t,q(t))\ni 0\qquad\text{in }\mathbf{Q}_{\mathrm{H}}^*\quad\text{for a.a. }t\in(0,T), \tag{2.20}$$

involving the overall dissipation potential

$$\Psi_\nu(q,q') := \mathcal{V}_{2,\nu}(u') + \mathcal{R}(z') + \mathcal{R}_2(z') + \mathcal{H}(z,p') + \mathcal{H}_{2,\nu}(p')$$

and its rescaled version

$$\begin{aligned}\Psi_{\varepsilon,\nu}(q,q') := \frac{1}{\varepsilon}\Psi_\nu(q,\varepsilon q') = \int_\Omega \tfrac{\varepsilon\nu}{2}\mathbb{D}\mathrm{E}(u'):\mathrm{E}(u')\,\mathrm{d}x + \mathcal{R}(z') \\ + \int_\Omega \tfrac{\varepsilon}{2}|z'|^2\,\mathrm{d}x + \mathcal{H}(z,p') + \int_\Omega \tfrac{\varepsilon\nu}{2}|p'|^2\,\mathrm{d}x.\end{aligned}$$

3 The partial versus the full vanishing-viscosity approach

In this section, we aim to gain further insight into the notion of $\mathrm{BV}_0^{0,0}$ solution to system (1.1) arising from the *full* vanishing-viscosity approach of [10] and compare it with the BV_0 concept from [8]. In order to properly introduce both notions, it can be useful to recall the reparameterized energy-dissipation balance where one passes to the limit to obtain Balanced Viscosity solutions.

At this heuristic stage, we will treat the partial vanishing-viscosity approximation of [8] and the full approximation of [10] in a unified way, although at the level of the viscous approximation in [8] there is the significant difference that the plastic strain evolves rate-independently. Still, we will disregard this and instead focus on the similarities between the *rate-dependent* systems (1.3) (wherefrom the $\mathrm{BV}_0^{0,0}$ solutions of [10] stem), and (1.2) (wherefrom the BV_0 solutions of [8]). In fact, (1.2) can be formally obtained from [10] by setting $\nu = \mu = 0$. That is why, in what follows to fix the main ideas we will illustrate the limit passage in the energy-dissipation balance associated with system (1.3).

Throughout this section and the remainder of the paper, we will suppose that the assumptions of Sect. 2 are in force without explicitly invoking them.

3.1 The energy-dissipation balance for the viscous system

As observed in [10, Proposition 3.4], a curve $q = (u,z,p) \in H^1(0,T;\mathbf{Q}_\mathrm{H})$ is a solution to (1.3), namely to the generalized gradient system (2.20), if and only if it satisfies the energy-dissipation balance

$$\begin{aligned}&\int_0^t \Big(\Psi_{\varepsilon,\nu}(q(r),q'(r)) + \Psi^*_{\varepsilon,\nu}(q(r),-\mathrm{D}_q\mathcal{E}_\mu(r,q(r)))\Big)\,\mathrm{d}r + \mathcal{E}_\mu(t,q(t)) \\ &= \mathcal{E}_\mu(0,q(0)) + \int_0^t \partial_t\mathcal{E}_\mu(r,q(r))\,\mathrm{d}r\end{aligned} \tag{3.1}$$

for every $t \in [0,T]$. Let us now consider a family $(q^\mu_{\varepsilon,\nu})$ of solutions to (2.20). In [10], suitable a priori estimates, uniform w.r.t. the parameters $\varepsilon,\nu,\mu > 0$, for the length (measured in an appropriate norm) of the curves $q^\mu_{\varepsilon,\nu}$ were derived. Based on such estimates, it is possible to reparameterize such curves, obtaining *parameterized curves* defined on an 'artificial' time interval $[0,S]$

$$(\mathsf{t}^\mu_{\varepsilon,\nu},\mathsf{q}^\mu_{\varepsilon,\nu}) : [0,S]\to[0,T]\times\mathbf{Q}_{\mathrm{H}} \quad \text{with } \mathsf{t}^\mu_{\varepsilon,\nu} := (\mathsf{s}^\mu_{\varepsilon,\nu})^{-1}, \quad \mathsf{q}^\mu_{\varepsilon,\nu} := q^\mu_{\varepsilon,\nu}\circ\mathsf{t}^\mu_{\varepsilon,\nu}$$

and $\mathsf{s}^\mu_{\varepsilon,\nu}$ (suitable) arclength functions associated with the curves $(q^\mu_{\varepsilon,\nu})$. Now, in terms of the parameterized curves $(\mathsf{t}^\mu_{\varepsilon,\nu},\mathsf{q}^\mu_{\varepsilon,\nu})$, (3.1) translates into the *reparameterized* energy-dissipation balance

$$\begin{aligned} &\mathcal{E}(\mathsf{t}^\mu_{\varepsilon,\nu}(s),\mathsf{q}^\mu_{\varepsilon,\nu}(s)) + \int_0^s \mathcal{M}^{\mu,\nu}_\varepsilon(\mathsf{t}^\mu_{\varepsilon,\nu}(r),\mathsf{q}^\mu_{\varepsilon,\nu}(r),(\mathsf{t}^\mu_{\varepsilon,\nu})'(r),(\mathsf{q}^\mu_{\varepsilon,\nu})'(r))\,\mathrm{d}r \\ &\quad = \mathcal{E}(\mathsf{t}^\mu_{\varepsilon,\nu}(0),\mathsf{q}^\mu_{\varepsilon,\nu}(0)) + \int_0^s \partial_t\mathcal{E}(\mathsf{t}^\mu_{\varepsilon,\nu}(r),\mathsf{q}^\mu_{\varepsilon,\nu}(r))\,(\mathsf{t}^\mu_{\varepsilon,\nu})'(r)\,\mathrm{d}r \end{aligned} \tag{3.2}$$

featuring the functional $\mathcal{M}^{\mu,\nu}_\varepsilon : [0,T]\times\mathbf{Q}_{\mathrm{H}}\times(0,+\infty)\times\mathbf{Q}_{\mathrm{H}}\to[0,+\infty]$

$$\mathcal{M}^{\mu,\nu}_\varepsilon(t,q,t',q') := \mathcal{R}(z') + \mathcal{H}(z,p') + \mathcal{M}^{\mu,\nu}_{\varepsilon,\mathrm{red}}(t,q,t',q') \tag{3.3}$$

with the reduced functional

$$\mathcal{M}^{\mu,\nu}_{\varepsilon,\mathrm{red}} \text{ defined by } \mathcal{M}^{\mu,\nu}_{\varepsilon,\mathrm{red}}(t,q,t',q') := \frac{\varepsilon}{2t'}\mathcal{D}_\nu(q')^2 + \frac{t'}{2\varepsilon}(\mathcal{D}^{*,\mu}_\nu(t,q))^2,$$

and

$$\begin{aligned} &\mathcal{D}_\nu(q') := \sqrt{\nu\|u'(t)\|^2_{H^1,\mathbb{D}}+\|z'(t)\|^2_{L^2}+\nu\|p'(t)\|^2_{L^2}}, \\ &\mathcal{D}^{*,\mu}_\nu(t,q) := \sqrt{\frac{1}{\nu}\,\|-\mathrm{D}_u\mathcal{E}_\mu(t,q)\|^2_{(H^1,\mathbb{D})^*} + \widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_\mu(t,q),\partial\mathcal{R}(0))^2 + \frac{1}{\nu}\,d_{L^2}(-\mathrm{D}_p\mathcal{E}_\mu(t,q),\partial_\pi\mathcal{H}(z,0))^2}\,. \end{aligned} \tag{3.4}$$

In (3.4), $\|\cdot\|_{H^1,\mathbb{D}}$ and $\|\cdot\|_{(H^1,\mathbb{D})^*}$ are the norms introduced in (2.4) and (2.5), while the distance functional $\widetilde{d}_{L^2(\Omega)}(\cdot,\partial\mathcal{R}(0)) : H^{\mathrm{m}}(\Omega)^*\to[0,+\infty]$ is defined by

$$\widetilde{d}_{L^2(\Omega)}(\chi,\partial\mathcal{R}(0))^2 := \min_{\gamma\in\partial\mathcal{R}(0)} \mathfrak{f}_2(\chi-\gamma) \quad \text{with } \mathfrak{f}_2(\beta) := \begin{cases} \|\beta\|^2_{L^2(\Omega)} & \text{if } \beta\in L^2(\Omega), \\ +\infty & \text{if } \beta\in H^{\mathrm{m}}(\Omega)^*\setminus L^2(\Omega). \end{cases} \tag{3.5}$$

Clearly, the functional $\mathcal{M}^{\mu,\nu}_\varepsilon$ from (3.3) encompasses, in the energy-dissipation balance (3.2), the competition between viscous dissipation and tendency to relax toward equilibrium & rate-independent behavior. In fact, viscous dissipation is encoded in the term $\mathcal{D}_\nu(q')$, which is modulated by the viscosity parameter ε. In turn, the relaxation to rate-independent behavior is encompassed in the term $\mathcal{D}^{*,\mu}_\nu(t,q)$, modulated by $\frac{1}{\varepsilon}$.

Now, the concepts of BV_0 and $\mathrm{BV}_0^{0,0}$ solutions to system (1.1) are defined in terms of parameterized energy-dissipation balances akin to (3.2). These energy identities involve a (positive) *vanishing-viscosity potential* $\mathcal{M}$ that is defined on $[0,T]\times\mathbf{Q}_{\mathrm{PP}}\times[0,+\infty)\times\mathbf{Q}_{\mathrm{PP}}$ (recall that $\mathbf{Q}_{\mathrm{PP}}$ is the state space for the perfectly plastic damage system, cf. (2.11)). At least formally, $\mathcal{M}$ arises as Γ-limit

- Of the functionals $(\mathcal{M}^{0,0}_\varepsilon)_\varepsilon$ as $\varepsilon\downarrow 0$, in the case of the vanishing-viscosity analysis in [8] (recall that system (1.2) is formally a particular case of (1.3), with $\nu=\mu=0$);

– Of $(\mathcal{M}_{\varepsilon}^{\mu,\nu})_{\varepsilon,\nu,\mu}$ as $\varepsilon,\ \nu,\ \mu \downarrow 0$, in the case of the joint vanishing-viscosity and hardening analysis in [10].

However, in order to *rigorously* define the vanishing-viscosity contact potentials $\mathcal{M}$ relevant for the two concepts of BV solutions we will need to provide some technical preliminaries in the following section.

3.2 Preliminary definitions

The vanishing-viscosity potentials at the core of the definitions of BV_0 solutions in [8] and $\mathrm{BV}_0^{0,0}$ solutions in [10] have a structure akin to that of the functional $\mathcal{M}_{\varepsilon}^{\mu,\nu}$ from (3.3), but they are tailored to the driving energy $\mathcal{E}_0$ (2.12) for the perfectly plastic system. Recalling the definition of $\mathcal{M}_{\varepsilon}^{\mu,\nu}$, it is thus clear that, in order to define such vanishing-viscosity potentials, one needs suitable surrogates of the $(\mathbb{D}, H^1)^*$-norm of $\mathrm{D}_u\mathcal{E}_0(t,q)$, and of the L^2 -distance of $\mathrm{D}_p\mathcal{E}_0(t,q)$ from the stable set $\partial_\pi\mathcal{H}(z,0)$, cf. (3.4). Indeed, such quantities are no longer well defined for the functional $\mathcal{E}_0$ at every $(t,q) \in [0,T]\times \mathbf{Q}_{\mathrm{PP}}$; instead, note that the L^2-distance, in the sense of (3.5), of $\mathrm{D}_z\mathcal{E}_0(t,q)$ from the stable set $\partial\mathcal{R}(0)$ still makes sense. Following [10, Sect. 7], we will set at every $(t,q) \in \mathrm{dom}(\mathcal{E}_0) = [0,T]\times \mathbf{D}$ (cf. (2.13))

$$\mathcal{S}_u\mathcal{E}_0(t,q) := \sup_{\substack{\eta_u \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n) \\ \|\eta_u\|_{(H^1,\mathbb{D})} \le 1}} \langle -\mathrm{Div}\,\sigma(t) - F(t), \eta_u\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)}, \tag{3.6a}$$

$$\mathcal{W}_p\mathcal{E}_0(t,q) := \sup_{\substack{\eta_p \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \\ \|\eta_p\|_{L^2} \le 1}} \Big(\langle \sigma_{\mathrm{D}}(t), \eta_p\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} - \mathcal{H}(z,\eta_p)\Big). \tag{3.6b}$$

with $\sigma(t) = \mathbb{C}(z)e(t)$ and $e(t) = \mathrm{E}(u+w(t)) - p$. Observe that the expressions in (3.6a) and (3.6b) are well defined since $e(t)$ and, a fortiori, $\sigma(t)$ are elements in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ whenever $(u,z,p) \in \mathbf{Q}_{\mathrm{PP}}$. Formulae (3.6a) and (3.6b) have been inspired by the obvious fact that

$$\|\zeta\|_{(H^1,\mathbb{D})^*} = \sup_{\substack{\eta_u \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n) \\ \|\eta_u\|_{(H^1,\mathbb{D})} \le 1}} \langle \zeta, \eta_u\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)} \qquad \text{for every } \zeta \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^*,$$

and by the formula

$$d_{L^2}(\varsigma, \partial_\pi\mathcal{H}(z,0)) = \sup_{\substack{\eta \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \\ \|\eta\|_{L^2} \le 1}} \Big(\langle \varsigma, \eta\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} - \mathcal{H}(z,\eta)\Big)$$

for every $\varsigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, which was proved in [10, Lemma 3.6]. In particular, we note that for every $q \in \mathrm{D}$

$$\mathcal{W}_p\mathcal{E}_0(t,q) = d_{L^2}(\sigma_{\mathrm{D}}(t), \partial_\pi\mathcal{H}(z,0)), \tag{3.7a}$$

while

$$\mathcal{S}_u\mathcal{E}_0(t,q) = \| -\mathrm{Div}\,\sigma(t) - F(t)\|_{(H^1,\mathbb{D})^*} \qquad \text{if } u \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n). \tag{3.7b}$$

For later use, we also record here the following result.

Lemma 3.1 *For every* $(t,q) \in [0,T]\times\mathbf{D}$, *there holds*

$$\begin{aligned} &\mathcal{S}_u\mathcal{E}_0(t,q) = \mathcal{W}_p\mathcal{E}_0(t,q) = 0 \quad \Longleftrightarrow \\ &\quad \sigma(t)\in\mathcal{K}_z(\Omega)\,, \quad -\mathrm{div}\,\sigma(t) = f(t) \text{ a.e. in } \Omega\,, \quad [\sigma(t)\mathrm{n}] = g(t)\ \mathscr{H}^{n-1}\text{-a.e. on } \Gamma_{\mathrm{Neu}}, \end{aligned} \tag{3.8}$$

with $\sigma(t) = \mathbb{C}(z)e(t)$ *and* $e(t) = \mathrm{E}(e + w(t)) - p$.

Proof The implication ⇒ was proved in [10, Lemma 7.4]. A close perusal of the proof, also mimicking the convexity arguments from [13, Prop. 3.5], also yields the converse implication. □

We then set

$$\mathcal{D}^*(t,q) := \sqrt{(\mathcal{S}_u\mathcal{E}_0(t,q))^2 + (\mathcal{W}_p\mathcal{E}_0(t,q))^2}. \tag{3.9}$$

We are now in a position to define the vanishing-viscosity contact potentials involved in the definitions of BV solutions in [8] and [10]. We will use the notation $I_{\{0\}}$ for the indicator function of the singleton $\{0\}$, namely

$$I_{\{0\}}(\xi) = \begin{cases} 0 & \text{if } \xi = 0, \\ +\infty & \text{otherwise} \end{cases} \qquad \text{for all } \xi\in\mathbb{R}.$$

3.3 BV_0 solutions via the partial vanishing-viscosity approach

The vanishing-viscosity contact potential for the BV_0 solutions from [8] is the functional

$$\begin{aligned} &\mathcal{M}_0 : [0,T]\times\mathbf{Q}_{\mathrm{PP}}\times[0,+\infty)\times\mathbf{Q}_{\mathrm{PP}} \to [0,+\infty], \\ &\quad \mathcal{M}_0(t,q,t',q') = \mathcal{R}(z') + \mathcal{H}_{\mathrm{PP}}(z,p') + \mathcal{M}_{0,\mathrm{red}}(t,q,t',q'), \end{aligned} \tag{3.10a}$$

where for $q = (u,z,p)$ and $q' = (u',z',p')$ we have

$$\mathcal{M}_{0,\mathrm{red}}(t,q,t',q') = I_{\{0\}}(\mathcal{D}^*(t,q)) + \widetilde{\mathcal{M}}_{0,\mathrm{red}}(t,q,t',q') \tag{3.10b}$$

and

$$\begin{aligned} &\text{if } t' > 0, \quad \widetilde{\mathcal{M}}_{0,\mathrm{red}}(t,q,t',q') := I_{\{0\}}(\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t,q),\partial\mathcal{R}(0))), \\ &\text{if } t' = 0, \quad \widetilde{\mathcal{M}}_{0,\mathrm{red}}(t,q,t',q') := \begin{cases} \|z'\|_{L^2}\,\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t,q),\partial\mathcal{R}(0)) \\ \qquad \text{if } \widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t,q),\partial\mathcal{R}(0)) < +\infty, \\ +\infty \text{ otherwise.} \end{cases} \end{aligned} \tag{3.10c}$$

Observe that the definition of $\mathcal{M}_0(t,q,0,q')$ again reflects, in the jump regime, the tendencies of the system to evolve viscously and to relax toward equilibrium and rate-independent evolution. However, here viscous dissipation only affects the damage variable. Likewise, rate-independent behavior is solely encompassed by the relation $\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t,q),\partial\mathcal{R}(0))=0$.

Let us now detail the properties of the parameterized curves providing BV_0 solutions.

Definition 3.2 We call a parameterized curve $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) : [0,S]\to[0,T]\times\mathbf{Q}_{\mathrm{PP}}$ *admissible* if it satisfies

$$\mathsf{t} : [0,S]\to[0,T] \text{ is nondecreasing,} \tag{3.11a}$$

$$(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})\in \mathrm{AC}\big([0,S];[0,T]\times\mathrm{BD}(\Omega)\times H^{\mathrm{m}}(\Omega)\times\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\,\mathbb{M}^{n\times n}_{\mathrm{D}})\big), \tag{3.11b}$$

$$\mathsf{e}=\mathrm{E}(\mathsf{u}+w(\mathsf{t}))-\mathsf{p}\in \mathrm{AC}([0,S];L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{3.11c}$$

$$\begin{aligned}&\mathsf{t} \text{ is constant in every connected component of}\\ &A^\circ=\{s\in(0,S) : \widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\partial\mathcal{R}(0))>0\}.\end{aligned} \tag{3.11d}$$

We will denote by $\mathcal{A}(0,S;[0,T]\times\mathbf{Q}_{\mathrm{PP}})$ the class of admissible parameterized curves from $[0,S]$ to $[0,T]\times\mathbf{Q}_{\mathrm{PP}}$.

We are now in a position to give the definition of Balanced Viscosity solution in the sense of [8].

Definition 3.3 We call an admissible parameterized curve $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})\in\mathcal{A}(0,S;[0,T]\times\mathbf{Q}_{\mathrm{PP}})$ a Balanced Viscosity solution for the perfectly plastic damage system (1.1) in the sense of [8] (a $\mathrm{BV}_0^{0,0}$ *solution*, for short), if it satisfies the energy-dissipation balance

$$\begin{aligned}&\mathcal{E}_0(\mathsf{t}(s_2),\mathsf{q}(s_2))+\int_{s_1}^{s_2}\mathcal{M}_0(\mathsf{t}(r),\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r))\,\mathrm{d}r\\ &=\mathcal{E}_0(\mathsf{t}(s_1),\mathsf{q}(s_1))+\int_{s_1}^{s_2}\partial_t\mathcal{E}_0(\mathsf{t}(r),\mathsf{q}(r))\,\mathsf{t}'(r)\,\mathrm{d}r\end{aligned} \tag{3.12}$$

for every $0\le s_1\le s_2\le S$.

The existence of BV_0 solutions was proved in [8, Thm. 5.4] under the condition that the initial data $q_0=(u_0,z_0,p_0)$ for the perfectly plastic damage system (1.1) fulfill (2.8f) and the additional condition that

$$\begin{aligned}\mathrm{D}_q\mathcal{E}_0(0,q_0)=&\Big(-\mathrm{Div}\,\sigma_0-F(0),A_{\mathrm{m}}(z_0)+W'(z_0)+\tfrac{1}{2}\mathbb{C}'(z_0)e_0 : e_0,\mu p_0-(\sigma_0)_{\mathrm{D}}\Big)\\ &\in L^2(\Omega;\mathbb{R}^n\times\mathbb{R}\times\mathbb{M}^{n\times n}_{\mathrm{D}}).\end{aligned} \tag{3.13}$$

3.4 A differential characterization for BV_0 solutions

We now aim to provide a *differential characterization* for the notion of BV solution from Definition 3.2, in terms of a suitable system of subdifferential inclusions for the displacement variable (which in fact shall satisfy the elastic equilibrium equation), the damage variable, and the plastic strain. In order to properly formulate the flow rule governing the latter, we need the following result; the proof of one implication can be found [8], in turn based on arguments from [13].

Lemma 3.4 *Let an admissible parameterized curve* $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{e},\mathsf{p}) \in \mathcal{A}(0,S;[0,T]\times\mathbf{Q}_{\mathrm{PP}})$ *satisfy*

$$\mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s),\mathsf{p}'(s)) = \langle \sigma_{\mathrm{D}}(s) \,|\, \mathsf{p}'(s)\rangle \qquad \text{for a.a. } s \in (0,S) \tag{3.14}$$

with $\sigma(s) = \mathbb{C}(\mathsf{z}(s))\mathsf{e}(s)$, $\mathsf{e}(s) = e(\mathsf{t}(s))$, *and* $\langle\cdot\,|\,\cdot\rangle$ *the stress-strain duality from* (2.10).

Then, (t,q) *satisfies Hill's maximum work principle*

$$H\left(\mathsf{z}(s),\frac{\mathrm{d}\mathsf{p}'(s)}{\mathrm{d}|\mathsf{p}'(s)|}\right)|\mathsf{p}'(s)| = [\sigma_{\mathrm{D}}(s) : \mathsf{p}'(s)] \qquad \text{for a.a. } s \in (0,S), \tag{3.15}$$

where the above equality holds in the sense of measures on $\Omega\cup\Gamma_{\mathrm{Dir}}$, *with* $[\sigma_{\mathrm{D}}(s) : \mathsf{p}'(s)]$ *the distribution defined in* (2.9). *Furthermore, defining* $\mu(s) := \mathcal{L}^n + |\mathsf{p}'(s)|$ *for every* $s \in [0,S]$, *there exists* $\widehat{\sigma}_{\mathrm{D}}(s) \in L^\infty_{\mu(s)}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$ *such that for almost all* $s \in (0,S)$ *the following properties hold*:

$$\widehat{\sigma}_{\mathrm{D}}(s) = \sigma_{\mathrm{D}}(s) \quad \mathcal{L}^n\text{-a.e. on } \Omega, \tag{3.16a}$$

$$[\sigma_{\mathrm{D}}(s) : \mathsf{p}'(s)] = \left(\widehat{\sigma}_{\mathrm{D}}(s) : \frac{\mathrm{d}\mathsf{p}'(s)}{\mathrm{d}|\mathsf{p}'(s)|}\right)|\mathsf{p}'(s)| \quad \text{on } \Omega\cup\Gamma_{\mathrm{Dir}}, \tag{3.16b}$$

$$\partial_\pi H\left(\mathsf{z}(s),\frac{\mathrm{d}\mathsf{p}'(s)}{\mathrm{d}|\mathsf{p}'(s)|}\right) \ni \widehat{\sigma}_{\mathrm{D}}(s) \quad \text{for } |\mathsf{p}'(s)|\text{-a.e. } x \in \Omega\cup\Gamma_{\mathrm{Dir}}. \tag{3.16c}$$

Conversely, (3.16) *imply* (3.15) *which*, *in turn*, *gives* (3.14).

Proof We refer to [8, Prop. 6.5] for the proof of the fact that (3.14) implies (3.15) and (3.16). In turn, recalling (2.15), from (3.16c) we infer that

$$H\left(\mathsf{z}(s),\frac{\mathrm{d}\mathsf{p}'(s)}{\mathrm{d}|\mathsf{p}'(s)|}\right) = \left(\widehat{\sigma}_{\mathrm{D}}(s) : \frac{\mathrm{d}\mathsf{p}'(s)}{\mathrm{d}|\mathsf{p}'(s)|}\right) \qquad |\mathsf{p}'(s)|\text{-a.e. in } \Omega\cup\Gamma_{\mathrm{Dir}}.$$

Combining this with (3.16a) and (3.16b), we conclude (3.15), which yields (3.14) in view of the definition (2.16) of $\mathcal{H}_{\mathrm{PP}}$, and of the definition (2.10) of the stress-strain duality product. □

It is in the sense of (3.16) that we need to understand the (formally written) inclusion $\partial_\pi H(\mathsf{z},\mathsf{p}' \ni \sigma_{\mathrm{D}}$.

We are now in a position to prove the following differential characterization for the concept of BV solution from [8]. We mention in advance that, for notational simplicity, in (3.18b) we have simply written $\mathsf{z}'(s)$ in place of $J(\mathsf{z}'(s))$, with $J : L^2(\Omega) \to H^{\mathrm{m}}(\Omega)^*$ the Riesz operator.

Proposition 3.5 *An admissible parameterized curve* $(\mathsf{t}, \mathsf{q}) \in \mathcal{A}(0, S; [0, T]\times\mathbf{Q}_{\mathrm{PP}})$ *is a* BV_0 *solution to system* (1.1) *if and only if there exists a measurable function* $\lambda_{\mathsf{z}} : [0, S] \to [0, 1]$ *such that*

$$\mathsf{t}'(s)\lambda_{\mathsf{z}}(s) = 0 \quad \text{for a.a. } s \in (0, S), \tag{3.17}$$

and (t, q) *satisfies for a.a.* $s \in (0, S)$

$$\sigma(s) \in \mathcal{K}_{\mathsf{z}(s)}(\Omega), \quad -\mathrm{div}\,\sigma(s) = f(\mathsf{t}(s)) \text{ a.e. in } \Omega, \quad [\sigma(\mathsf{t}(s))\mathrm{n}] = g(\mathsf{t}(s))\ \mathscr{H}^{n-1}\text{-a.e. on } \Gamma_{\mathrm{Neu}}, \tag{3.18a}$$

$$\begin{aligned} &(1-\lambda_{\mathsf{z}}(s))\,\partial\mathcal{R}(\mathsf{z}'(s)) + \lambda_{\mathsf{z}}(s)\,\mathsf{z}'(s) \\ &\quad + (1-\lambda_{\mathsf{z}}(s))\left[A_{\mathrm{m}}\mathsf{z}(s) + W'(\mathsf{z}(s)) + \tfrac{1}{2}\mathbb{C}'(\mathsf{z}(s))\mathsf{e}(s) : \mathsf{e}(s)\right] \ni 0 \quad \text{in } H^{\mathrm{m}}(\Omega)^*, \end{aligned} \tag{3.18b}$$

$$\partial_\pi \mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s), \mathsf{p}'(s)) \ni \sigma_{\mathrm{D}}(s) \quad \text{in the sense of (3.16).} \tag{3.18c}$$

In fact, (3.18*a*) *holds at every* $s \in (0, S]$.

Remark 3.6 System (3.18) illustrates in a clear way how viscous behavior, *only w.r.t. the variable z*, may arise in the jump regime, namely when the system still evolves while $\mathsf{t}' = 0$ (i.e., the external time, recorded by the function t, is frozen). In that case, the parameter λ_{z} may be nonzero, thus activating the viscous contribution to the flow rule (3.18b).

Prior to carrying out the proof of Proposition 3.5, we record the following key chain-rule inequality, whose proof may be immediately inferred from that of [10, Lemma 7.6] (cf. also Lemma 3.13 ahead).

Lemma 3.7 *Along any admissible parameterized curve*

$$(\mathsf{t}, \mathsf{q}) \in \mathcal{A}(0, S; [0, T]\times\mathbf{Q}_{\mathrm{PP}}) \text{ s.t. } \mathcal{M}_0(\mathsf{t}, \mathsf{q}, \mathsf{t}', \mathsf{q}') < +\infty \text{ a.e. in } (0, S),$$

we have that $s \mapsto \mathcal{E}_0(\mathsf{t}(s), \mathsf{q}(s))$ *is absolutely continuous on* $[0, S]$ *and there holds*

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_0(\mathsf{t}(s), \mathsf{q}(s)) + \partial_t\mathcal{E}_0(\mathsf{t}(s), \mathsf{q}(s))\,\mathsf{t}'(s) \\ &= -\langle \mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s), \mathsf{q}(s)), \mathsf{z}'(s)\rangle_{H^{\mathrm{m}}} + \langle \sigma_{\mathrm{D}}(s) \mid \mathsf{p}'(s)\rangle + \langle \mathrm{Div}\,\sigma(s) + F(\mathsf{t}(s)), \mathsf{u}'(s)\rangle_{\mathrm{BD}} \\ &\le \mathcal{M}_0(\mathsf{t}(s), \mathsf{q}(s), \mathsf{t}'(s), \mathsf{q}'(s)) \qquad \text{for a.a. } s \in (0, S). \end{aligned} \tag{3.19}$$

As an immediate consequence of Lemma 3.7, we have the following characterization of the BV_0 solutions from Definition 3.2, which complements the other characterizations provided in [8, Prop. 5.3].

Corollary 3.8 *An admissible parameterized curve* $(\mathsf{t},\mathsf{q}) \in \mathcal{A}(0,S;[0,T]\times\mathbf{Q}_{\mathrm{PP}})$ *is a* BV_0 *solution to system* (1.1) *if and only if the function* $[0,S] \ni s \mapsto \mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))$ *is absolutely continuous and there holds*

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) + \partial_t\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))\,\mathsf{t}'(s)\\
&\quad = -\langle \mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{z}'(s)\rangle_{H^{\mathrm{m}}} + \langle \sigma_{\mathrm{D}}(s)\,|\,\mathsf{p}'(s)\rangle\\
&\qquad + \langle \mathrm{Div}\,\sigma(s)+F(\mathsf{t}(s)),\mathsf{u}'(s)\rangle_{\mathrm{BD}} = \mathcal{M}_0(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))
\end{aligned}
\tag{3.20}
$$

for almost all $s \in (0,S)$.

We are now in a position to carry out the

Proof of Proposition 3.5 By Corollary 3.8, BV_0 solutions in the sense of Def. 3.2 can be characterized in terms of (3.20), whence we deduce that $\mathcal{M}_0(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) < \infty$ for almost all $s \in (0,S)$. Therefore,

$$
\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) = 0 \qquad \text{for a.a. } s \in (0,S). \tag{3.21}
$$

In turn, (3.21) is equivalent to the validity of (3.18a). Observe that (3.18a) extends to every $s \in [0,S]$ by the continuity of the functions $s \mapsto \sigma(s)$, $s \mapsto f(\mathsf{t}(s))$, $s \mapsto g(\mathsf{t}(s))$ and by the continuity w.r.t. the Hausdorff distance of $s \mapsto \mathcal{K}_{\mathsf{z}(s)}(\Omega)$ thanks to $(2.K_3)$.

In view of (3.21) and recalling the definition (3.10) of $\mathcal{M}_0$, (3.20) can be then rewritten as

$$
\begin{aligned}
&-\langle \mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{z}'(s)\rangle_{H^{\mathrm{m}}} - \mathcal{R}(\mathsf{z}'(s)) - \widetilde{\mathcal{M}}_{0,\mathrm{red}}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))\\
&\quad = \mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s),\mathsf{p}'(s)) - \langle \sigma_{\mathrm{D}}(s)\,|\,\mathsf{p}'(s)\rangle
\end{aligned}
$$

for a.a. $s \in (0,S)$. Now, on the one hand the right-hand side of the above equality is positive thanks to (2.19). On the other hand, let $\omega \in \partial\mathcal{R}(0)$ satisfy $\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t,q),\partial\mathcal{R}(0)) = \|-\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) - \omega\|_{L^2}$. Then, we may estimate the left-hand side of the above equality via

$$
\begin{aligned}
&\langle -\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{z}'(s)\rangle_{H^{\mathrm{m}}} - \mathcal{R}(\mathsf{z}'(s)) - \widetilde{\mathcal{M}}_{0,\mathrm{red}}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))\\
&= \underbrace{\langle -\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) - \omega,\mathsf{z}'(s)\rangle_{H^{\mathrm{m}}} - \widetilde{\mathcal{M}}_{0,\mathrm{red}}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))}_{T_1}\\
&\quad + \underbrace{\langle \omega,\mathsf{z}'(s)\rangle_{H^{\mathrm{m}}} - \mathcal{R}(\mathsf{z}'(s))}_{T_2} \qquad \le 0,
\end{aligned}
$$

where we have used that $T_1 \le 0$ by the very definition of $\widetilde{\mathcal{M}}_{0,\mathrm{red}}$ (cf. (3.10c)), while $T_2 \le 0$ thanks to (2.15). Therefore, (3.20) is ultimately *equivalent* to

$$
\langle -\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) - \omega,\mathsf{z}'(s)\rangle_{H^{\mathrm{m}}} = \widetilde{\mathcal{M}}_{0,\mathrm{red}}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) \quad \text{for a.a. } s \in (0,S), \tag{3.22a}
$$

$$
\langle \omega,\mathsf{z}'(s)\rangle_{H^{\mathrm{m}}} = \mathcal{R}(\mathsf{z}'(s)) \quad \text{for a.a. } s \in (0,S), \tag{3.22b}
$$

$$\langle \sigma_{\mathrm{D}}(s) \,|\, \mathsf{p}'(s)\rangle = \mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s), \mathsf{p}'(s)) \quad \text{for a.a. } s \in (0, S) \tag{3.22c}$$

for every $\omega \in \partial\mathcal{R}(0)$ such that $\|-\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s), \mathsf{q}(s)) - \omega\|_{L^2} = \widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_0(t, q), \partial\mathcal{R}(0))$.

Now, the same argument as in, e.g., [21, Prop. 5.1] allows us to infer that (3.22a) and (3.22b) are *equivalent* to (3.18b), together with (3.17). In turn, by Lemma 3.4 it follows that (3.22c) is *equivalent* to (3.16). All in all, we have shown the equivalence between (3.20) and (3.17) &(3.18). This concludes the proof. □

3.5 BV solutions via the full vanishing-viscosity approach

The vanishing-viscosity contact potential for the BV solutions from [10] is the functional $\mathcal{M}_0^{0,0} : [0, T] \times \mathbf{Q}_{\mathrm{PP}} \times [0, +\infty) \times \mathbf{Q}_{\mathrm{PP}} \to [0, +\infty]$ defined via

$$\mathcal{M}_0^{0,0}(t, q, t', q') := \mathcal{R}(z') + \mathcal{H}_{\mathrm{PP}}(z, p') + \mathcal{M}_{0,\mathrm{red}}^{0,0}(t, q, t', q') \tag{3.23a}$$

where for $q = (u, z, p)$ and $q' = (u', z', p')$ we have

$$\text{if } t' > 0, \quad \mathcal{M}_{0,\mathrm{red}}^{0,0}(t, q, t', q') := \begin{cases} 0 & \text{if } \begin{cases} \mathcal{S}_u\mathcal{E}_0(t, q) = 0, \\ \widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t, q), \partial\mathcal{R}(0)) = 0, \text{ and} \\ \mathcal{W}_p\mathcal{E}_0(t, q) = 0, \end{cases} \\ +\infty & \text{otherwise,} \end{cases} \tag{3.23b}$$

$$\text{if } t' = 0, \quad \mathcal{M}_{0,\mathrm{red}}^{0,0}(t, q, t', q') := \begin{cases} \mathcal{D}(u', p')\,\mathcal{D}^*(t, q) & \text{if } z' = 0, \\ \|z'\|_{L^2}\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t, q), \partial\mathcal{R}(0)) & \\ & \text{if } \mathcal{D}^*(t, q) = 0 \\ & \text{and } \widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t, q), \partial\mathcal{R}(0)) < +\infty, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.23c}$$

with

$$\mathcal{D}(u', p') := \sqrt{\|u'\|^2_{H^1,\mathbb{D}} + \|p'\|^2_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}} \quad \text{and} \quad \mathcal{D}^* \text{ from (3.9).} \tag{3.23d}$$

As previously remarked, for every $(t, q) \in [0, T] \times \mathbf{Q}_{\mathrm{PP}}$ we have that $\mathcal{S}_u\mathcal{E}_0(t, q) < +\infty$ and $\mathcal{W}_p\mathcal{E}_0(t, q) < +\infty$, hence $\mathcal{D}^*(t, q) < +\infty$ and the product $\mathcal{D}(u', p')\,\mathcal{D}^*(t, q)$ is well defined as soon as $u' \in H^1(\Omega; \mathbb{R}^n)$ and $p' \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}})$; otherwise, we mean $\mathcal{D}(u', p')\,\mathcal{D}^*(t, q) = +\infty$.

The (reduced) vanishing-viscosity contact potentials $\mathcal{M}_{0,\mathrm{red}}$ from (3.10b) and $\mathcal{M}_{0,\mathrm{red}}^{0,0}$ differ from each other, both in their definition for $t' > 0$ and for $t' = 0$. For $t' > 0$, $\mathcal{M}_{0,\mathrm{red}}^{0,0}$ has to additionally enforce elastic equilibrium (i.e., $\mathcal{S}_u\mathcal{E}_0(t, q) = 0$) and the *stability* constraint that $\sigma_{\mathrm{D}} \in \partial_\pi\mathcal{H}(z, 0)$ (i.e., $\mathcal{W}_p\mathcal{E}_0(t, q) = 0$) since, for the fully rate-dependent viscous systems, these constraints are no longer fulfilled. Accordingly, viscous behavior in u and p may intervene in the jump regime of the rate-independent limit system. This is encoded in the new term $\mathcal{D}(u', p')\,\mathcal{D}^*(t, q)$ featuring in (3.23c), which appears in the energy-dissipation balance at jumps (i.e., for $t' = 0$), when $z' = 0$.

The following definition specifies the properties of the parameterized curves that are $\mathrm{BV}_0^{0,0}$ solutions and is to be compared with Definition 3.2 of admissible parameterized curves in the sense of [8].

Definition 3.9 We call a parameterized curve $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) : [0,S]\to[0,T]\times\mathbf{Q}_{\mathrm{PP}}$ *admissible in an enhanced sense* ('enhanced admissible' for short) if it satisfies (3.11a), (3.11b), (3.11c) and, in addition,

$$(\mathsf{u},\mathsf{p})\in \mathrm{AC}_{\mathrm{loc}}(B^\circ;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})),\ \text{where } B^\circ := \{s\in(0,S) : \mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))>0\}, \tag{3.24a}$$

$$\mathsf{t} \text{ is constant in every connected component of } B^\circ. \tag{3.24b}$$

We will denote by $\mathcal{EA}(0,S;[0,T]\times\mathbf{Q}_{\mathrm{PP}})$ the class of enhanced admissible parameterized curves from $[0,S]$ to $[0,T]\times\mathbf{Q}_{\mathrm{PP}}$.

Hence, enhanced admissible curves enjoy better spatial regularity, with $\mathsf{u}(\cdot)\in H^1(\Omega;\mathbb{R}^n)$ and $\mathsf{p}(\cdot)\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, in the set in which either $\mathcal{S}_u\mathcal{E}_0(\mathsf{t}(\cdot),\mathsf{q}(\cdot))>0$ or $\mathcal{W}_p\mathcal{E}_0(\mathsf{t}(\cdot),\mathsf{q}(\cdot))>0$. With that definition at hand, we are now in a position to give the definition of BV solution in the sense of [10].

Definition 3.10 We call an enhanced admissible parameterized curve $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) \in \mathcal{EA}(0,S;[0,T]\times\mathbf{Q}_{\mathrm{PP}})$ a Balanced Viscosity solution for the perfectly plastic damage system (1.1) in the sense of [10] (a $\mathrm{BV}_0^{0,0}$ *solution*, for short), if it satisfies the energy-dissipation balance

$$\begin{aligned}&\mathcal{E}_0(\mathsf{t}(s_2),\mathsf{q}(s_2))+\int_{s_1}^{s_2}\mathcal{M}_0^{0,0}(\mathsf{t}(r),\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r))\,\mathrm{d}r\\&=\mathcal{E}_0(\mathsf{t}(s_1),\mathsf{q}(s_1))+\int_{s_1}^{s_2}\partial_t\mathcal{E}_0(\mathsf{t}(r),\mathsf{q}(r))\,\mathsf{t}'(r)\,\mathrm{d}r\end{aligned}\tag{3.25}$$

for every $0\le s_1\le s_2\le S$.

The existence of $\mathrm{BV}_0^{0,0}$ solutions to system (1.1) was proved in [10, Thm. 7.9] for initial data $q_0=(u_0,z_0,p_0)$ complying with (2.8f) and (3.13).

3.6 A differential characterization for $\mathrm{BV}_0^{0,0}$ solutions

In this section, we provide a differential characterization for $\mathrm{BV}_0^{0,0}$ solutions. Preliminarily, we need to make precise in which sense we are going to understand the subdifferential inclusions governing the evolution of the reparameterized displacement and of the plastic variables. Indeed, by formally writing

$$-\lambda\mathrm{div}\,\mathbb{D}\mathrm{E}(\mathsf{u}')-(1-\lambda)\mathrm{Div}\,\sigma=(1-\lambda)F(\mathsf{t})\quad \text{a.e. in } (0,S),$$

with $\lambda : [0,S]\to[0,1]$ a measurable function (below we will have $\lambda=\lambda_{\mathsf{u},\mathsf{p}}$), we shall mean

$$-\mathrm{Div}\,\sigma(s)=F(\mathsf{t}(s))\qquad \text{in } \mathrm{BD}(\Omega)^*\quad \text{if } \lambda(s)=0, \tag{3.26a}$$

$$\begin{cases}\mathsf{u}'(s)\in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n),\\ -\lambda(s)\mathrm{div}\,\mathbb{D}\mathrm{E}(\mathsf{u}'(s))-(1-\lambda(s))\mathrm{Div}\,\sigma(s)=(1-\lambda(s))F(\mathsf{t}(s))\qquad \text{in } H^1(\Omega;\mathbb{R}^n)^*\end{cases}\quad \text{if } \lambda(s)>0, \tag{3.26b}$$

where in (3.26) $\mathrm{Div}\,\sigma(s)$ denotes the restriction of the functional from (2.3) to $H^1(\Omega;\mathbb{R}^n)$. In particular, let us emphasize that, when $\lambda > 0$ the displacement variable enjoys additional spatial regularity, and the quasistatic momentum balance (3.26b) allows for test functions in $H^1(\Omega;\mathbb{R}^n)$. Likewise, by writing

$$(1-\lambda)\partial_\pi \mathcal{H}_{\mathrm{PP}}(\mathsf{z},\mathsf{p}') + \lambda \mathsf{p}' \ni (1-\lambda)\sigma_{\mathrm{D}}$$

with $\lambda : [0,S] \to [0,1]$ a measurable function, we shall mean

$$\partial_\pi \mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s),\mathsf{p}'(s)) \ni \sigma_{\mathrm{D}}(s) \quad \text{in the sense of (3.16)} \quad \text{if } \lambda(s)=0, \tag{3.27a}$$

$$\begin{cases} \mathsf{p}'(s) \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}), \\ (1-\lambda(s))\partial_\pi \mathcal{H}(\mathsf{z}(s),\mathsf{p}'(s)) + \lambda(s)\mathsf{p}'(s) \ni (1-\lambda(s))\sigma_{\mathrm{D}}(s) \quad \text{a.e. in } \Omega \end{cases} \quad \text{if } \lambda(s) > 0. \tag{3.27b}$$

Namely, the plastic flow rule improves to a pointwise-in-space formulation in the set $\{\lambda > 0\}$, whereas in the set $\{\lambda = 0\}$ it only holds in the weak form (3.16).

We are now in a position to state our differential characterization of $\mathrm{BV}_0^{0,0}$ solutions. Observe that the definition of enhanced admissible curve is tailored to the subdifferential inclusions (3.29).

Proposition 3.11 *An enhanced admissible parameterized curve* $(\mathsf{t},\mathsf{q}) \in \mathcal{EA}(0,S;[0,T]\times\mathbf{Q}_{\mathrm{PP}})$ *is a* $\mathrm{BV}_0^{0,0}$ *solution to system* (1.1) *if and only if there exist two measurable functions* $\lambda_{\mathsf{u},\mathsf{p}}\ \lambda_{\mathsf{z}} : [0;S] \to [0;1]$ *such that*

$$\mathsf{t}'(s)\lambda_{\mathsf{u},\mathsf{p}}(s) = \mathsf{t}'(s)\lambda_{\mathsf{z}}(s) = 0 \quad \text{for a.a. } s \in (0,S), \tag{3.28a}$$

$$\lambda_{\mathsf{u},\mathsf{p}}(s)(1-\lambda_{\mathsf{z}}(s)) = 0 \quad \text{for a.a. } s \in (0,S), \tag{3.28b}$$

and the curves (t,q) *satisfy for a.a.* $s \in (0,S)$ *the system of subdifferential inclusions*

$$-\lambda_{\mathsf{u},\mathsf{p}}(s)\mathrm{div}\,\mathbb{D}\mathrm{E}(\mathsf{u}'(s)) - (1-\lambda_{\mathsf{u},\mathsf{p}}(s))\mathrm{Div}\,\sigma(s) = (1-\lambda_{\mathsf{u},\mathsf{p}}(s))F(\mathsf{t}(s)), \tag{3.29a}$$

$$\begin{aligned}&(1-\lambda_{\mathsf{z}}(s))\,\partial\mathcal{R}(\mathsf{z}'(s)) + \lambda_{\mathsf{z}}(s)\,\mathsf{z}'(s)\\ &\quad + (1-\lambda_{\mathsf{z}}(s))\left(A_{\mathrm{m}}\mathsf{z}(s)+W'(\mathsf{z}(s))+\tfrac{1}{2}\mathbb{C}'(\mathsf{z}(s))\mathsf{e}(s) : \mathsf{e}(s)\right) \ni 0 \quad \text{in } H^{\mathrm{m}}(\Omega)^*,\end{aligned} \tag{3.29b}$$

$$(1-\lambda_{\mathsf{u},\mathsf{p}}(s))\,\partial_\pi\mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s),\mathsf{p}'(s)) + \lambda_{\mathsf{u},\mathsf{p}}(s)\,\mathsf{p}'(s) \ni (1-\lambda_{\mathsf{u},\mathsf{p}}(s))\,\sigma_{\mathrm{D}}(s), \tag{3.29c}$$

where (3.29a) *and* (3.29c) *need to be interpreted as* (3.26) *and* (3.27), *respectively.*

Remark 3.12 In comparison with the differential characterization for BV_0 solutions provided by system (3.5), system (3.29) features *two* parameters, instead of one. Both λ_{z} and $\lambda_{\mathsf{u},\mathsf{p}}$ have the role of activating the viscous contributions to the damage flow rule, and to the displacement equation/plastic flow rule, respectively, in the jump regime (i.e., when $\mathsf{t}' = 0$). In fact, the viscous terms in (3.29a) and (3.29c) are modulated by the same parameter, which reflects the fact that viscous behavior intervenes for the variables u and p equally (or, in other terms, that u and p relax to elastic equilibrium and rate-independent evolution at the same rate, faster than z).

As in the case of Prop. 3.5, the proof of Prop. 3.11 will rely on a suitable chain-rule inequality, which we recall below.

Lemma 3.13 [10, *Lemma* 7.6] *Along any enhanced admissible parameterized curve*

$$(\mathsf{t},\mathsf{q}) \in \mathcal{EA}(0,S;[0,T]{\times}\mathbf{Q}_{\mathrm{PP}}) \text{ s.t. } \mathcal{M}_0^{0,0}(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}') < +\infty \quad \text{a.e. in } (0,S)$$

we have that

$$\begin{aligned}
&s \mapsto \mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) \text{ is absolutely continuous on } [0,S] \text{ and there holds for a.a. } s\in(0,S)\\
&\quad -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) + \partial_t\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))\,\mathsf{t}'(s)\\
&\quad = -\langle \mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{z}'(s)\rangle_{H^{\mathrm{m}}} + \langle \sigma_{\mathrm{D}}(s)\,|\,\mathsf{p}'(s)\rangle + \langle \mathrm{Div}\,\sigma(s){+}F(\mathsf{t}(s)),\mathsf{u}'(s)\rangle_{\mathrm{BD}}\\
&\quad \le \mathcal{M}_0^{0,0}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)).
\end{aligned} \tag{3.30}$$

For later use, we also record the following consequence of the chain-rule inequality, cf. [10, Proposition 7.7].

Corollary 3.14 *An enhanced admissible parameterized curve* $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) \in \mathcal{EA}(0,S;[0,T]{\times}\mathbf{Q}_{\mathrm{PP}})$ *is a* $\mathrm{BV}_0^{0,0}$ *solution if and only if it satisfies one of the following equivalent conditions*:

(1) *the energy-dissipation balance* (3.25) *holds as the inequality* $\le$;
(2) (t,q) *fulfills* (3.30) *as a chain of equalities*, *i.e.*,

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) + \partial_t\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))\,\mathsf{t}'(s)\\
&\quad = -\langle \mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{z}'(s)\rangle_{H^m} + \langle \sigma_{\mathrm{D}}(s)\,|\,\mathsf{p}'(s)\rangle + \langle \mathrm{Div}\,\sigma(s){+}F(\mathsf{t}(s)),\mathsf{u}'(s)\rangle_{\mathrm{BD}}\\
&\quad = \mathcal{M}_0^{0,0}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) \qquad \text{for a.a. } s\in(0,S).
\end{aligned} \tag{3.31}$$

We are now in a position to carry out the

Proof of Proposition 3.11 We exploit the characterization of $\mathrm{BV}_0^{0,0}$ solutions in terms of the chain of equalities (3.31). Now, we shall distinguish three cases:

Case 1 : $\mathsf{t}'(s) > 0$. Then, by the definition (3.23) of $\mathcal{M}_0^{0,0}$, from $\mathcal{M}_0^{0,0}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) < \infty$ we infer $\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\partial\mathcal{R}(0)) = 0$ and

$$\mathcal{S}_u\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) = \mathcal{W}_p\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) = 0.$$

By (3.8), the latter property is equivalent to (3.18a); hence, we find the validity of (3.29a) with $\lambda_{\mathsf{u},\mathsf{p}}(s) = 0$. All in all, identity (3.31) reduces to

$$\langle \sigma_{\mathrm{D}}(s)\,|\,\mathsf{p}'(s)\rangle - \mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s),\mathsf{p}'(s)) = \langle \mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{z}'(s)\rangle_{H^{\mathrm{m}}} + \mathcal{R}(\mathsf{z}'(s)). \tag{3.32}$$

Now, since $\sigma(s) \in \mathcal{K}_{\mathsf{z}(s)}$ for a.a. $s \in (0,S)$, by (2.18) the above left-hand-side is negative. On the other hand, from $\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\partial\mathcal{R}(0)) = 0$ we infer that

$\mathcal{R}(v) \geq \langle -\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)), v\rangle_{H^{\mathrm{m}}}$ for every $v \in H^{\mathrm{m}}(\Omega)$. Therefore, the above right-hand side is positive. Hence, both sides are equal to zero. From

$$\mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s),\dot{\mathsf{p}}'(s)) = \langle \sigma_{\mathrm{D}}(s) \,|\, \mathsf{p}'(s)\rangle, \tag{3.33}$$

we infer (recall Lemma 3.4) that (3.29c) holds with $\lambda_{\mathsf{u},\mathsf{p}}(s) = 0$. Likewise, from

$$\mathcal{R}(\mathsf{z}'(s)) = -\langle \mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)), \mathsf{z}'(s)\rangle_{H^{\mathrm{m}}}, \tag{3.34}$$

recalling (2.15) we deduce that $-\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s) \in \partial\mathcal{R}(\mathsf{z}'(s))$, i.e., the validity of (3.29b) with $\lambda_{\mathsf{z}}(s) = 0$.

Conversely, from (3.29b) with $\lambda_{\mathsf{z}}(s) = 0$ and (3.29c) with $\lambda_{\mathsf{u},\mathsf{p}}(s) = 1$ we deduce (3.33) and (3.34), respectively, hence (3.32) which, in this case, is equivalent to (3.31).

Case 2 : $\mathsf{t}'(s) = 0$ *and* $s \in B^\circ$ (with B° the set from (3.24a)). Then, from $\mathcal{M}_0^{0,0}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) < \infty$ we infer that

$$\mathsf{z}'(s) = 0, \tag{3.35}$$

and (3.31) reduces to

$$\begin{aligned}
&\langle \sigma_{\mathrm{D}}(s), \mathsf{p}'(s)\rangle_{L^2} + \langle \mathrm{Div}\,\sigma(s) + F(\mathsf{t}(s)), \mathsf{u}'(s)\rangle_{H^1} \\
&\quad = \mathcal{H}(\mathsf{z}(s),\mathsf{p}'(s)) + \mathcal{D}(\mathsf{u}'(s),\mathsf{p}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) \\
&\quad = \mathcal{H}(\mathsf{z}(s),\mathsf{p}'(s)) + \sqrt{\|\mathsf{u}'(s)\|^2_{H^1,\mathbb{D}} + \|\mathsf{p}'(s)\|^2_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}} \\
&\qquad \sqrt{\|\mathrm{Div}\,\sigma(s) + F(\mathsf{t}(s))\|^2_{(H^1,\mathbb{D})^*} + d_{L^2}(\sigma_{\mathrm{D}}(s), \partial_\pi\mathcal{H}(z,0))^2}\,.
\end{aligned} \tag{3.36}$$

For (3.36), we have used that, on the left-hand side, the duality pairing involving u' is between $H^1(\Omega;\mathbb{R}^n)$ and $H^1(\Omega;\mathbb{R}^n)^*$ since the admissible curve (t,q) enjoys the enhanced spatial regularity $\mathsf{u}' \in H^1(\Omega;\mathbb{R}^n)$ on the set B°. In turn, the right-hand side of (3.36) has been rewritten in view of (3.7). Likewise, $\mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s),\mathsf{p}'(s)) = \mathcal{H}(\mathsf{z}(s),\mathsf{p}'(s))$ and the stress-strain duality reduces to the scalar product in L^2 because $\mathsf{p}'(s) \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$. Let us now consider a measurable selection $s \mapsto \zeta(s) \in \partial_\pi\mathcal{H}(\mathsf{z}(s),0)$ such that

$$d_{L^2}(\sigma_{\mathrm{D}}(s), \partial_\pi\mathcal{H}(z,0)) = \|\sigma_{\mathrm{D}}(s) - \zeta(s)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}.$$

Then, (3.36) rewrites as

$$\begin{aligned}
&\langle \sigma_{\mathrm{D}}(s) - \zeta(s), \mathsf{p}'(s)\rangle_{L^2} + \langle \mathrm{Div}\,\sigma(s) + F(\mathsf{t}(s)), \mathsf{u}'(s)\rangle_{H^1} \\
&\quad - \sqrt{\|\mathsf{u}'(s)\|^2_{H^1,\mathbb{D}} + \|\mathsf{p}'(s)\|^2_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}}\sqrt{\|\mathrm{Div}\,\sigma(s) + F(\mathsf{t}(s))\|^2_{(H^1,\mathbb{D})^*} + \|\sigma_{\mathrm{D}}(s) - \zeta(s)\|^2_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}} \\
&= \mathcal{H}(\mathsf{z}(s),\mathsf{p}'(s)) - \langle \zeta(s), \mathsf{p}'(s)\rangle_{L^2}.
\end{aligned}$$

While the left-hand side is negative by Cauchy–Schwarz inequality, the right-hand side is positive since $\zeta(s) \in \partial_\pi\mathcal{H}(\mathsf{z}(s),0)$, cf. (2.15). All in all, we conclude that both sides are equal to zero. Now, combining the fact that $\zeta(s) \in \partial_\pi\mathcal{H}(\mathsf{z}(s),0)$ with the identity

$$\mathcal{H}(\mathsf{z}(s),\mathsf{p}'(s)) = \langle \zeta(s), \mathsf{p}'(s)\rangle_{L^2}$$

and again resorting to (2.15), we find that

$$\zeta(s) \in \partial_\pi \mathcal{H}(\mathsf{z}(s), \mathsf{p}'(s)). \tag{3.37}$$

From the equality

$$\sqrt{\|\mathsf{u}'(s)\|^2_{H^1,\mathbb{D}} + \|\mathsf{p}'(s)\|^2_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}}\sqrt{\|\mathrm{Div}\,\sigma(s) + F(\mathsf{t}(s))\|^2_{(H^1,\mathbb{D})^*} + \|\sigma_{\mathrm{D}}(s) - \zeta(s)\|^2_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}}$$
$$= \langle \sigma_{\mathrm{D}}(s) - \zeta(s), \mathsf{p}'(s)\rangle_{L^2} + \langle \mathrm{Div}\,\sigma(s) + F(\mathsf{t}(s)), \mathsf{u}'(s)\rangle_{H^1}$$

we infer that

$$-\tilde{\lambda}(s)\mathrm{div}\,\mathbb{D}\mathrm{E}(\mathsf{u}'(s)) = \mathrm{Div}\,\sigma(s) + F(\mathsf{t}(s)) \quad \text{in } H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^*, \tag{3.38a}$$

$$\tilde{\lambda}(s)\mathsf{p}'(s) = \sigma_{\mathrm{D}}(s) - \zeta(s) \quad \text{a.e. in } \Omega, \tag{3.38b}$$

$$\text{with } \tilde{\lambda}(s) = \frac{\sqrt{\|\mathrm{Div}\,\sigma(s) + F(\mathsf{t}(s))\|^2_{(H^1,\mathbb{D})^*} + \|\sigma_{\mathrm{D}}(s) - \zeta(s)\|^2_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}}}{\sqrt{\|\mathsf{u}'(s)\|^2_{H^1,\mathbb{D}} + \|\mathsf{p}'(s)\|^2_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}}}. \tag{3.38c}$$

Combining (3.35), (3.37), and (3.38), we deduce the validity of system (3.29) with

$$\lambda_{\mathsf{z}}(s) = 1 \quad \text{and} \quad \lambda_{\mathsf{u,p}}(s) = \frac{\tilde{\lambda}(s)}{1 + \tilde{\lambda}(s)} \in (0, 1).$$

Conversely, it can be easily checked that, if $\mathsf{t}'(s) = 0$ and $s \in B^\circ$, the validity of system (3.29) yields (3.36), hence (3.31).

Case 3 : $\mathsf{t}'(s) = 0$ *and* $s \notin B^\circ$. Hence, $\mathcal{D}^*(\mathsf{t}(s), \mathsf{q}(s)) = 0$, yielding

$$-\mathrm{Div}\,\sigma(s) = F(\mathsf{t}(s)),$$

i.e., (3.29a) with $\lambda_{\mathsf{u,p}}(s) = 0$. Then, (3.31) reduces to

$$\begin{aligned} &\langle -\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s), \mathsf{q}(s)), \mathsf{z}'(s)\rangle_{H^m} + \langle \sigma_{\mathrm{D}}(s) \,|\, \mathsf{p}'(s)\rangle \\ &\quad = \mathcal{R}(\mathsf{z}'(s)) + \|\mathsf{z}'(s)\|_{L^2}\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s), \mathsf{q}(s)), \partial\mathcal{R}(0)) + \mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s), \mathsf{p}'(s)) \end{aligned} \tag{3.39}$$

and, arguing in the same way as in the proof of Proposition 3.5 we conclude the validity of (3.29b) and (3.29c) with $\lambda_{\mathsf{u,p}}(s) = 0$ and some $\lambda_{\mathsf{z}}(s) \geq 0$.

Conversely, it can be proved that (3.29b) and (3.29c) with $\lambda_{\mathsf{u,p}}(s) = 0$ and some $\lambda_{\mathsf{z}}(s) \geq 0$ yield (3.39).

With this, we conclude the proof. □

4 A complete characterization of $\mathrm{BV}^{0,0}_0$ solutions

The main result of this section is the following theorem, whose proof adapts that of [28, Prop. 5.5].

Theorem 4.1 *Let* $(\mathsf{t}, \mathsf{q}) \in \mathcal{EA}(0, S; [0, T]\times\mathbf{Q}_{\mathrm{PP}})$ *be a* $\mathrm{BV}^{0,0}_0$ *solution to the perfectly plastic rate-independent system for damage* (1.1). *Suppose that* (t, q) *is non-degenerate, namely that*

$$\mathsf{t}'(s) + \|\mathsf{u}'(s)\|_{H^1(\Omega)} + \|\mathsf{z}'(s)\|_{H^{\mathrm{m}}(\Omega)} + \|\mathsf{p}'(s)\|_{L^2(\Omega)} > 0 \qquad \text{for a.a. } s \in (0,S). \tag{4.1}$$

Set

$$\mathfrak{S} := \{s \in [0,S] \ : \ \mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) = 0\}.$$

Then, $\mathfrak{S}$ *is either empty or it has the form* $[s_*, S]$ *for some* $s_* \in [0,S]$.

(a) *Assume* $s_* > 0$, *then for* $s \in [0,s_*) = [0,S] \setminus \mathfrak{S}$ *we have* $\mathsf{t}(s) \equiv \mathsf{t}(0)$ *and* $\mathsf{z}(s) \equiv \mathsf{z}(0)$, *whereas*

$$\mathcal{D}^*(\mathsf{t}(0),\mathsf{q}(0)) > 0 \tag{4.2a}$$

and (u,p) is a solution to the system

$$\begin{aligned} &-\lambda_{\mathsf{u,p}}(s)\mathrm{div}\,\mathbb{D}\mathrm{E}(\mathsf{u}'(s)) - (1-\lambda_{\mathsf{u,p}}(s))\mathrm{Div}\,\mathbb{C}(\mathsf{z}(0))\mathsf{e}(s) = (1-\lambda_{\mathsf{u,p}}(s))F(\mathsf{t}(s)), \\ &(1-\lambda_{\mathsf{u,p}}(s))\,\partial_\pi\mathcal{H}(\mathsf{z}(0),\mathsf{p}'(s)) + \lambda_{\mathsf{u,p}}(s)\,\mathsf{p}'(s) \ni (1-\lambda_{\mathsf{u,p}}(s))\,(\mathbb{C}(\mathsf{z}(0))\mathsf{e}(s))_{\mathrm{D}}. \end{aligned} \tag{4.2b}$$

(b) *Suppose that* $s_* < S$. *Then,* $\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) \equiv 0$ *for every* $s \in \mathfrak{S} = [s_*, S]$, *and the curve* (t,q) *is a* BV *solution to system* (1.1) *in the sense of Definition* 3.3.

Thus, Theorem 4.1 provides a complete characterization of (non-degenerate) $\mathrm{BV}_0^{0,0}$ solutions. It asserts that, if a $\mathrm{BV}_0^{0,0}$ solution (t,q) starts from an unstable datum q_0 with $\mathcal{D}^*(0,q_0) > 0$, then during an initial interval the damage variable z is frozen and the pair (u,p) evolves, possibly governed by viscosity in both variables. If it reaches, at some time s^*, the state in which elastic equilibrium $(\mathcal{S}_u\mathcal{E}_0(\mathsf{t}(s^*),\mathsf{q}(s^*)) = 0)$ and the plastic constraint $(\mathcal{W}_p\mathcal{E}_0(\mathsf{t}(s^*),\mathsf{q}(s^*)) = 0)$ are fulfilled, then it never leaves that state afterwards, and subsequently (t,q) behaves as a BV_0 solution.

Proof *Step 1:* As in the proof of [28, Prop. 5.5], we start by analyzing the behavior of a $\mathrm{BV}_0^{0,0}$ solution (t,q) on an interval $(s_1,s_2) \subset [0,S] \setminus \mathfrak{S}$. Since $\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) > 0$ for all $s \in (s_1,s_2)$, we read from (3.29a) and (3.29c) that $\lambda_{\mathsf{u,p}} > 0$ on (s_1,s_2). Thus, (3.28a) yields $\mathsf{t}' \equiv 0$ on (s_1,s_2), so that $\mathsf{t}(s) \equiv \mathsf{t}(s_1)$ for all $s \in [s_1,s_2]$. Furthermore, (3.28b) gives $\lambda_{\mathsf{z}} \equiv 1$ on (s_1,s_2). Combining this with (3.29b), we gather that $\mathsf{z}' \equiv 0$ on (s_1,s_2), so that $\mathsf{z}(s) \equiv \mathsf{z}(s_1)$ for all $s \in [s_1,s_2]$. From (4.1), we conclude that

$$\mathsf{u}'(s) \neq 0 \text{ or } \mathsf{p}'(s) \neq 0 \qquad \text{for a.a. } s \in (s_1,s_2), \tag{4.3a}$$

and, therefore, from (3.29a) and (3.29c) we infer that $\lambda_{\mathsf{u,p}}(s) < 1$ for almost all $s \in (s_1,s_2)$. Hence, the evolution of (t,q) in (s_1,s_2) is characterized by property (4.3a), joint with the previously found

$$\mathsf{t}(s) \equiv \mathsf{t}(s_1), \quad \mathsf{z}(s) \equiv \mathsf{z}(s_1) \qquad \text{for all } s \in [s_1,s_2], \tag{4.3b}$$

$$-\,\mathrm{div}\,\mathbb{D}\mathrm{E}(\mathsf{u}'(s)) - \widehat{\lambda}(s)\mathrm{Div}\,\sigma(s) = \widehat{\lambda}(s)F(\mathsf{t}(s_1)) \quad \text{for a.a. } s \in (s_1,s_2), \tag{4.3c}$$

$$\widehat{\lambda}(s)\zeta(s) + \mathsf{p}'(s) \ni \widehat{\lambda}(s)\sigma_{\mathrm{D}}(s) \qquad \text{for a.a. } s \in (s_1, s_2)\,,$$
$$\text{with } \widehat{\lambda} := \frac{1-\lambda_{\mathsf{u},\mathsf{p}}}{\lambda_{\mathsf{u},\mathsf{p}}} = \frac{\sqrt{\|\mathsf{u}'\|^2_{H^1,\mathbb{D}} + \|\mathsf{p}'\|^2_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}}}{\mathcal{D}^*(\mathsf{t},\mathsf{q})} \tag{4.3d}$$

(observe that $\widehat{\lambda} = \tilde{\lambda}^{-1}$ with $\tilde{\lambda}$ from, (3.38)). In turn, it can be easily checked that, for a *given* function $\widehat{\lambda} : (s_1, s_2) \to [0, +\infty)$, the Cauchy problem for system (4.3b)—(4.3d) does admit a unique solution.

Step 2: Since the function $[0, S] \ni s \mapsto \mathcal{D}^*(\mathsf{t}(s), \mathsf{q}(s))$ is lower semicontinuous by [10, Lemma 7.8], the set $\mathfrak{S}$ is closed; hence, its complement $[0, S] \setminus \mathfrak{S}$ is relatively open, and thus, it is the finite or countable union of disjoint intervals. Its connected components are of the form $(s_1, S]$, (s_2, s_3), $[0, s_4)$ or $[0, S]$ (if $\mathfrak{S} = \emptyset$). By the lower semicontinuity of $s \mapsto \mathcal{D}^*(\mathsf{t}(s), \mathsf{q}(s))$, it is immediate to check that $s_j \in \mathfrak{S}$ for all $j \in \{1, \dots, 4\}$.

Now, we aim to show that connected components of the type $(s_1, S]$ and (s_2, s_3) cannot occur. To this end, let us study the properties of the $\mathrm{BV}_0^{0,0}$ solution (t, q) on an interval of the type $(s_1, S]$ or (s_2, s_3), with $s_1,\ s_2 \in \mathfrak{S}$. Since $\mathcal{D}^*(\mathsf{t}(s_j), \mathsf{q}(s_j)) = 0$, we have that $\mathcal{S}_u\mathcal{E}_0(\mathsf{t}(s_j), \mathsf{q}(s_j)) = \mathcal{W}_p\mathcal{E}_0(\mathsf{t}(s_j), \mathsf{q}(s_j)) = 0$ for $j = 1, 2$. As shown in Step 1, the evolution on the intervals (s_1, S) and (s_2, s_3) is characterized by (4.3). Recall that system (4.3b)–(4.3d) admits a unique solution. Now, since $\mathcal{S}_u\mathcal{E}_0(\mathsf{t}(s_j), \mathsf{q}(s_j)) = \mathcal{W}_p\mathcal{E}_0(\mathsf{t}(s_j), \mathsf{q}(s_j)) = 0$ for $j = 1, 2$, it is immediate to check that the constant functions $\hat{\mathsf{u}}(s) \equiv \mathsf{u}(s_j)$ and $\hat{\mathsf{p}}(s) \equiv \mathsf{p}(s_j)$ provide the unique solution to (4.3b)–(4.3d). Thus, we conclude that $\mathrm{BV}_0^{0,0}$ solution (t, q) on an interval of the type $(s_1, S]$ or (s_2, s_3) must be constant, which is a contradiction to (4.3a).

Therefore, $[0, S] \setminus \mathfrak{S}$ does not possess connected components of the form $(s_1, S]$ or (s_2, s_3). Hence, either $\mathfrak{S} = \emptyset$, or $\mathfrak{S} = [s_*, S]$ for some $s_* > 0$. In the latter case, the calculations from Step 1 show that on $[0, S] \setminus \mathfrak{S} = [0, s_*)$ the evolution of (t, q) is characterized by (4.2).

Step 3: Suppose that $s_* < S$. Clearly, $\mathcal{D}^*(\mathsf{t}(s), \mathsf{q}(s)) \equiv 0$ for every $s \in \mathfrak{S} = [s_*, S]$. Hence, it satisfies system (3.29) with $\lambda_{\mathsf{u},\mathsf{p}}(s) \equiv 0$ for every $s \in [s_*, S]$, which coincides with system (3.18). This concludes the proof. □

5 Vanishing-hardening limit of BV solutions

In this section, we carry out the asymptotic analysis as the hardening parameter μ tends to 0 for the BV solutions to system (1.4), both in the *single-rate* case (i.e., for $\mathrm{BV}_0^{\mu,\nu}$ solutions, with $0 < \nu \le \mu$), and in the *multi-rate* case (i.e, for $\mathrm{BV}_0^{\mu,0}$ solutions). Indeed, we first address the latter case in Sect. 5.1 ahead, while the former will be sketched in Sect. 5.2. For both analyses, we will resort to some technical results collected in the "Appendix".

5.1 Vanishing-hardening analysis for *multi-rate* solutions

As recalled in the Introduction, BV solutions to the *multi-rate* system with hardening have been constructed in [10] (cf. Theorem 6.13 therein) by passing to the limit in the (reparameterized) version of (1.3) as the viscosity parameter ε tends to 0 simultaneously with the rate parameter $\nu \downarrow 0$, while the hardening parameter $\mu > 0$ stayed fixed. The solutions accordingly obtained, hereafter referred to as $\mathrm{BV}_0^{\mu,0}$ solutions, thus account for multiple rates in the system with hardening. In particular, like for $\mathrm{BV}_0^{0,0}$ solutions, the way in which viscous behavior in u, z, and p manifests itself in the jump regime reflects the fact that the convergence of u and p to elastic equilibrium and rate-independent evolution has occurred at a *faster rate* (as $\nu \downarrow 0$) than that for z, cf. Remark 5.2 ahead.

In order to recall the definition of $\mathrm{BV}_0^{\mu,0}$ solutions for fixed $\mu > 0$, we need to introduce the related vanishing-viscosity contact potential

$$\mathcal{M}_0^{\mu,0} : [0,T] \times \mathbf{Q}_\mathrm{H} \times [0,+\infty) \times \mathbf{Q}_\mathrm{H} \to [0,+\infty],$$
$$\mathcal{M}_0^{\mu,0}(t,q,t',q') := \mathcal{R}(z') + \mathcal{H}(z,p') + \mathcal{M}_{0,\mathrm{red}}^{\mu,0}(t,u,z,p,t',u',z',p')$$

where

$$\text{if } t' > 0, \qquad \mathcal{M}_{0,\mathrm{red}}^{\mu,0}(t,q,t',q') := \begin{cases} 0 & \text{if } \begin{cases} -\mathrm{D}_u\mathcal{E}_\mu(t,q) = 0, \\ -\mathrm{D}_z\mathcal{E}_\mu(t,q) \in \partial\mathcal{R}(0), \text{ and} \\ -\mathrm{D}_p\mathcal{E}_\mu(t,q) \in \partial_\pi\mathcal{H}(z,0), \end{cases} \\ +\infty & \text{otherwise,} \end{cases} \tag{5.1a}$$

$$\text{if } t' = 0, \qquad \mathcal{M}_{0,\mathrm{red}}^{\mu,0}(t,q,t',q') := \begin{cases} \mathcal{D}(u',p')\,\mathcal{D}^{*,\mu}(t,q) & \text{if } z' = 0, \\ \|z'\|_{L^2}\,\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_\mu(t,q),\partial\mathcal{R}(0)) & \\ & \text{if } \mathcal{D}^{*,\mu}(t,q) = 0 \\ & \text{and } \widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_\mu(t,q),\partial\mathcal{R}(0)) < +\infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.1b}$$

In (5.1), we have employed the notation

$$\begin{aligned} \mathcal{D}(u',p') &:= \sqrt{\|u'\|_{H^1,\mathbb{D}}^2 + \|p'\|_{L^2(\Omega;\mathbb{M}_\mathrm{D}^{n\times n})}^2}, \\ \mathcal{D}^{*,\mu}(t,q) &:= \sqrt{\|-\mathrm{D}_u\mathcal{E}_\mu(t,q)\|_{(H^1,\mathbb{D})^*}^2 + d_{L^2}(-\mathrm{D}_p\mathcal{E}_\mu(t,q),\partial_\pi\mathcal{H}(z,0))^2}. \end{aligned} \tag{5.1c}$$

We are now in a position to recall the notion of solution to system (1.4) from [10, Def. 6.10]. Observe that it involves (reparameterized) curves that are absolutely continuous on the *whole* interval $[0, S]$, with values in $\mathbf{Q}_\mathrm{H}$.

Definition 5.1 We call a parameterized curve $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) \in \mathrm{AC}([0,S];[0,T]\times\mathbf{Q}_\mathrm{H})$ a *(parameterized) Balanced Viscosity* solution to the *multi-rate* system with hardening (1.4) (a $\mathrm{BV}_0^{\mu,0}$ solution, for short), if $\mathsf{t} : [0,S] \to [0,T]$ is nondecreasing and (t,q) fulfills for all $0 \le s \le S$ the energy-dissipation balance

$$\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)) + \int_0^s \mathcal{M}_0^{\mu,0}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))\,\mathrm{d}\tau = \mathcal{E}_\mu(\mathsf{t}(0),\mathsf{q}(0)) + \int_0^s \partial_t\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau))\,\mathsf{t}'(\tau)\,\mathrm{d}\tau. \tag{5.2}$$

We say that (t,q) is *non-degenerate* if it fulfills (4.1).

Remark 5.2 In [10, Prop. 6.11], a characterization was provided for $\mathrm{BV}_0^{\mu,0}$ solutions in terms of a subdifferential system that features two positive parameters $\lambda_{\mathsf{u},\mathsf{p}}$ and λ_{z} activating viscous terms in the displacement equation & plastic flow rule, and in the damage flow rule, respectively. We refrain from recalling that system because it is completely analogous to (3.29) (with the same *switching conditions* (3.28)).

Accordingly, repeating the very same arguments as in the proof of Theorem 4.1, it is possible to provide an additional characterization of $\mathrm{BV}_0^{\mu,0}$ solutions completely analogous to that from the latter result. In particular, if a $\mathrm{BV}_0^{\mu,0}$ solution (t,q) originates from an unstable datum q_0 with $\mathcal{D}^{*,\mu}(0,q_0)>0$, then, during an initial interval the damage variable z is frozen and the pair (u,p) evolves, possibly in a viscous way. If (t,q) reaches, at some time s^*, the state in which elastic equilibrium $(-\mathrm{D}_u\mathcal{E}_\mu(\mathsf{t},\mathsf{q})=0)$ and stability $(-\mathrm{D}_p\mathcal{E}_\mu(\mathsf{t},\mathsf{q})\in\partial_\pi\mathcal{H}(z,0))$ are fulfilled, then it never leaves that state afterwards and subsequently behaves as a Balanced Viscosity solution with viscous behavior in the variable z, only (namely, the counterpart, for the system with hardening, of the BV_0 concept).

We mention in advance that the analogue of Theorem 4.1 does not hold, instead, for $\mathrm{BV}_0^{\mu,\nu}$ solutions.

We now consider a vanishing sequence $(\mu_k)_k$ and set $\mu=\mu_k$. By [10, Theorem 6.13], under the assumptions of Sects. 2 and (3.13) (cf. also Remark 5.4 below), for any fixed k there exists a parameterized Balanced Viscosity solution $(\mathsf{t}_k,\mathsf{q}_k)$ in the sense of the previous definition. Moreover, for the sequence $(\mathsf{t}_k,\mathsf{q}_k)_k$ we may assume the validity of the following a priori estimates:

$$
\begin{aligned}
&\exists\, C>0 \;\; \forall\, k\in\mathbb{N} \;\text{ for a.a. } s\in(0,S)\ :\\
&\mathsf{t}_k'(s)+\|\mathsf{u}_k'(s)\|_{W^{1,1}(\Omega)}+\|\mathsf{z}_k'(s)\|_{H^{\mathrm{m}}(\Omega)}+\|\mathsf{p}_k'(s)\|_{L^1(\Omega)}+\sqrt{\mu_k}\,\|\mathsf{p}_k'(s)\|_{L^2(\Omega)}\\
&\qquad+\|\mathsf{e}_k'(s)\|_{L^2(\Omega)}+\mathcal{D}(\mathsf{u}_k'(s),\mathsf{p}_k'(s))\,\mathcal{D}^{*,\mu_k}(\mathsf{t}_k(s),\mathsf{q}_k(s))\le C.
\end{aligned}
\tag{5.3}
$$

Indeed, the existence of a sequence $(\mathsf{t}_k,\mathsf{q}_k)_k$ enjoying the bounds (5.3) follows by time-discretization, cf. [10, Prop. 4.4], and by a reparameterization argument. Up to a further time reparameterization, we may also assume that the solutions are non-degenerate, cf. (4.1) and [10, Remark 6.9].

Theorem 5.3 *Let* $(\mu_k)_k$ *be a vanishing sequence and* $(\mathsf{t}_k,\mathsf{q}_k)_k$ *be a sequence of* $\mathrm{BV}_0^{\mu_k,0}$ *solutions to system* (1.4), *such that estimate* (5.3) *holds.*

Then, there exist a (not relabeled) subsequence and a curve $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})\in\mathcal{EA}(0,S;[0,T]\times\mathbf{Q}_{\mathrm{PP}})$ *such that*

(1) *for all* $s\in[0,S]$, *the following convergences hold as* $k\to+\infty$

$$
\begin{aligned}
&\mathsf{t}_k(s)\to\mathsf{t}(s),\quad \mathsf{u}_k(s)\overset{*}{\rightharpoonup}\mathsf{u}(s)\text{ in }\mathrm{BD}(\Omega),\quad \mathsf{z}_k(s)\rightharpoonup\mathsf{z}(s)\text{ in }H^{\mathrm{m}}(\Omega),\\
&\mathsf{e}_k(s)\rightharpoonup\mathsf{e}(s)\text{ in }L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}),\quad \mathsf{p}_k(s)\overset{*}{\rightharpoonup}\mathsf{p}(s)\text{ in }\mathrm{M}_{\mathrm{b}}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}});
\end{aligned}
\tag{5.4}
$$

(2) *there exists* $\overline{C}>0$ *such that for a.e.* $s\in(0,S)$ *there holds*

$$
\begin{aligned}
&\mathsf{t}'(s)+\|\mathsf{u}'(s)\|_{\mathrm{BD}(\Omega)}+\|\mathsf{z}'(s)\|_{H^{\mathrm{m}}(\Omega)}+\|\mathsf{p}'(s)\|_{\mathrm{M}_{\mathrm{b}}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})}\\
&+\|\mathsf{e}'(s)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})}+\mathcal{D}(\mathsf{u}'(s),\mathsf{p}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s))\le\overline{C};
\end{aligned}
\tag{5.5}
$$

(3) (t,q) *is a Balanced Viscosity solution for the perfectly plastic damage system* (1.1) *in the sense of Definition* 3.10.

Remark 5.4 The validity of Theorem 5.3 extends to sequences $(\mathsf{t}_k,\mathsf{q}_k)_k$ originating from initial data $q_0^k=(u_0^k,z_0^k,p_0^k)_k$ fulfilling

$$\|u_0^k\|_{H^1_{\mathrm{Dir}}(\Omega)}+\|z_0^k\|_{H^{\mathrm{m}}(\Omega)}+\|W(z_0^k)\|_{L^1(\Omega)}+\|p_0^k\|_{L^2(\Omega)}+\|\mathrm{D}_q\mathcal{E}_0(0,q_0^k)\|_{L^2(\Omega)}\le C$$

with C independent of k.

In particular, the last condition yields $\mu_k p_0^k\to 0$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, as needed for (5.10) below.

Proof The proof is divided in two steps.

Step 1: *Compactness.* For later use, we observe that, due to estimate (5.3),

$$\exists\, R>0\ \ \forall\, k\in\mathbb{N}\ \ \forall\, s\in[0,S]\ :\quad \|\mathsf{u}_k(s)\|_{\mathrm{BD}(\Omega)}+\|\mathsf{z}_k(s)\|_{H^{\mathrm{m}}(\Omega)}+\|\mathsf{p}_k(s)\|_{\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}})}\le R. \tag{5.6}$$

By the assumptions on initial data and external loading and by (5.3), we have $\sup_{s\in[0,S]}\mathcal{E}_{\mu_k}(\mathsf{t}_k(s),\mathsf{q}_k(s))\le C$ for a constant independent of k. In particular, this implies that, as $k\to\infty$,

$$\mu_k\,\mathsf{p}_k(s)\to 0 \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \text{ for every } s\in[0,S]. \tag{5.7}$$

In view of (5.3), we find a (not relabeled) subsequence and a Lipschitz curve $(\mathsf{t},\mathsf{q})\in W^{1,\infty}([0,S];[0,T]\times\mathbf{Q}_{\mathrm{PP}})$ such that the following convergences hold as $k\to\infty$

$$\mathsf{t}_k\overset{*}{\rightharpoonup}\mathsf{t}\quad \text{in } W^{1,\infty}(0,S;[0,T]),\quad \mathsf{u}_k\overset{*}{\rightharpoonup}\mathsf{u}\quad \text{in } W^{1,\infty}(0,S;\mathrm{BD}(\Omega)), \tag{5.8a}$$

$$\mathsf{z}_k\overset{*}{\rightharpoonup}\mathsf{z}\quad \text{in } W^{1,\infty}(0,S;H^{\mathrm{m}}(\Omega)), \tag{5.8b}$$

$$\mathsf{e}_k\overset{*}{\rightharpoonup}\mathsf{e}\quad \text{in } W^{1,\infty}(0,S;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})),\quad \mathsf{p}_k\overset{*}{\rightharpoonup}\mathsf{p}\quad \text{in } W^{1,\infty}(0,S;\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})), \tag{5.8c}$$

where $\mathsf{e}=\mathrm{E}(\mathsf{u}+w(\mathsf{t}))-\mathsf{p}$. Furthermore, an argument based on the Ascoli-Arzelà theorem (cf. [3, Prop. 3.3.1]) also yields

$$\mathsf{u}_k\to\mathsf{u}\quad \text{in } \mathrm{C}^0([0,S];\ \mathrm{BD}(\Omega)_{\mathrm{w}^*}), \tag{5.9a}$$

$$\mathsf{e}_k\to\mathsf{e}\quad \text{in } \mathrm{C}^0([0,S];L^2(\Omega;\ \mathbb{M}^{n\times n}_{\mathrm{sym}})_{\mathrm{w}}), \tag{5.9b}$$

$$\mathsf{z}_k\to\mathsf{z}\quad \text{in } \mathrm{C}^0([0,S];\ H^{\mathrm{m}}(\Omega)_{\mathrm{w}}), \tag{5.9c}$$

$$\mathsf{p}_k\to\mathsf{p}\quad \text{in } \mathrm{C}^0([0,S];\ \mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})_{\mathrm{w}^*}). \tag{5.9d}$$

Indeed, convergences (5.9a) and (5.9d) are to be intended in the spaces $\mathrm{C}^0([0,S];(\mathrm{BD}(\Omega),d_{\mathrm{BD,weak}^*}))$ and in $\mathrm{C}^0([0,S];(\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\ \mathbb{M}^{n\times n}_{\mathrm{D}}),d_{\mathrm{M_b,weak}^*}))$, where $d_{\mathrm{BD,weak}^*}$ and $d_{\mathrm{M_b,weak}^*}$ metrize the weak topologies of $\mathrm{BD}(\Omega)$ and $\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$, respectively, on the balls of radius R that contain $(\mathsf{u}_k)_k$ and $(\mathsf{p}_k)_k$, resp. (cf. (5.6); here, we use that $\mathrm{BD}(\Omega)$ is the dual of a separable space). The second and third convergences have an analogous meaning. Hence, (5.4) follows.

With the very same arguments as in the proof of [10, Prop. 7.9], based on the estimate

$$\exists\, C>0\ \forall\, k\in\mathbb{N} \text{ for a.a. } s\in(0,S)\ :\ \mathcal{D}(\mathsf{u}_k'(s),\mathsf{p}_k'(s))\,\mathcal{D}^{*,\mu_k}(\mathsf{t}_k(s),\mathsf{q}_k(s))\leq C,$$

we also prove the enhanced regularity $(\mathsf{t},\mathsf{q})\in\mathcal{EA}(0,S;[0,T]\times\mathbf{Q}_{\mathrm{PP}})$.

Step 2: *energy-dissipation upper estimate*. By Corollary 3.14, it is sufficient to show that the pair (t,q) complies with the energy-dissipation inequality

$$\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))+\int_0^s\mathcal{M}_0^{0,0}(\mathsf{t}(r),\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r))\,\mathrm{d}r\leq\mathcal{E}_0(\mathsf{t}(0),\mathsf{q}(0))+\int_0^s\partial_t\mathcal{E}_0(\mathsf{t}(\tau),\mathsf{q}(\tau))\,\mathsf{t}'(\tau)\,\mathrm{d}\tau$$

for every $s\in[0,S]$. We start from (5.2) for the solution $(\mathsf{t}_k,\mathsf{q}_k)$ with $\mu=\mu_k$. It is straightforward to see that

$$\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))\leq\liminf_{k\to+\infty}\mathcal{E}_{\mu_k}(\mathsf{t}(s),\mathsf{q}(s)),\qquad\mathcal{E}_0(\mathsf{t}(0),\mathsf{q}(0))=\lim_{k\to+\infty}\mathcal{E}_{\mu_k}(\mathsf{t}(0),\mathsf{q}(0)),\tag{5.10}$$

and

$$\int_0^s\partial_t\mathcal{E}_0(\mathsf{t}(\tau),\mathsf{q}(\tau))\,\mathsf{t}'(\tau)\,\mathrm{d}\tau=\lim_{k\to+\infty}\int_0^s\partial_t\mathcal{E}_{\mu_k}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))\,\mathsf{t}_k'(\tau)\,\mathrm{d}\tau.$$

It remains to show that

$$\int_0^s\mathcal{M}_0^{0,0}(\mathsf{t}(r),\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r))\,\mathrm{d}r\leq\liminf_{k\to+\infty}\int_0^s\mathcal{M}_0^{\mu_k,0}(\mathsf{t}_k(r),\mathsf{q}_k(r),\mathsf{t}_k'(r),\mathsf{q}_k'(r))\,\mathrm{d}r.\tag{5.11}$$

In fact, it will be sufficient to obtain the above estimate only for the reduced functionals $\mathcal{M}_{0,\mathrm{red}}^{0,0}$ and $\mathcal{M}_{0,\mathrm{red}}^{\mu_k,0}$. In view of (3.23), we distinguish two cases. Let $A:=\{s\in[0,S]\ :\ \mathsf{t}'(s)>0\}$.

Case $\mathsf{t}'>0$. We prove that the function $s\mapsto\mathcal{M}_{0,\mathrm{red}}^{0,0}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))$ is finite for a.a. $s\in A$. We apply Lemma A.1 ahead with the choices $\mathsf{f}_k:=\mathsf{t}_k$, $\mathsf{f}:=\mathsf{t}$. Thus, we conclude that for a.a. $s\in A$ there is a subsequence $(k_j)_j$ and, for every j, there is $s_{k_j}\in(0,S)$ such that $|s_{k_j}-s|<\frac{1}{j}$ and $\mathsf{t}_{k_j}'(s_{k_j})>0$. In particular, $\mathcal{M}_0^{\mu_k,0}(\mathsf{t}_{k_j}(s_{k_j}),\mathsf{q}_{k_j}(s_{k_j}),\mathsf{t}_{k_j}'(s_{k_j}),\mathsf{q}_{k_j}'(s_{k_j}))=0$. By (3.23), (5.7), convergences (5.9), and Lemma A.2 ahead, we obtain

$$\mathcal{S}_u\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))=0,\qquad-\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))\in\partial\mathcal{R}(0),\qquad\mathcal{W}_p\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))=0\quad\text{for a.a. } s\in A,$$

which is equivalent to state that $\mathcal{M}_{0,\mathrm{red}}^{0,0}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))=0$. Hence, we obviously have the *pointwise* estimate

$$\mathcal{M}_{0,\mathrm{red}}^{0,0}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))\leq\liminf_{k\to+\infty}\mathcal{M}_{0,\mathrm{red}}^{\mu_k,0}(\mathsf{t}_k(s),\mathsf{q}_k(s),\mathsf{t}_k'(s),\mathsf{q}_k'(s))\qquad\text{for a.a. } s\in A.\tag{5.12}$$

Case $\mathsf{t}'=0$. By virtue of Lemma A.2, we are in a position to apply Lemma A.3 below in the context of the space $\boldsymbol{Q}:=\mathbf{Q}_{\mathrm{PP}}$, with $\boldsymbol{S}$ the ball of radius R from (5.6), to the functionals $\mathscr{M}_k:=\mathcal{M}_{0,\mathrm{red}}^{\mu_k,0}$ and $\mathscr{M}_0:=\mathcal{M}_{0,\mathrm{red}}^{0,0}$, with $\mathcal{M}_{0,\mathrm{red}}^{\mu_k,0}$ extended to $\mathbb{R}\times(\mathbf{Q}_{\mathrm{PP}}\backslash\mathbf{Q}_{\mathrm{H}})\times\mathbb{R}\times(\mathbf{Q}_{\mathrm{PP}}\backslash\mathbf{Q}_{\mathrm{H}})$ by setting $\mathcal{M}^{\mu_k,0}_{0,\mathrm{red}}(t,q,t',q')=+\infty$ when q or $q'\in\mathbf{Q}_{\mathrm{PP}}\backslash\mathbf{Q}_{\mathrm{H}}$. We may then observe that the Γ-lim inf estimate (A.2) in Lemma A.3 follows from Lemma A.2. Thus, we conclude

$$\int_{(0,s)\backslash A}\mathcal{M}^{0,0}_{0,\mathrm{red}}(\mathsf{t}(r),\mathsf{q}(r),0,\mathsf{q}'(r))\,\mathrm{d}r\le\liminf_{k\to+\infty}\int_{(0,s)\backslash A}\mathcal{M}^{\mu_k,0}_{0,\mathrm{red}}(\mathsf{t}_k(r),\mathsf{q}_k(r),\mathsf{t}_k'(r),\mathsf{q}_k'(r))\,\mathrm{d}r.$$

All in all, (5.11) follows.

This finishes the proof. □

5.2 Vanishing-hardening analysis for *single-rate* solutions

BV solutions to the *single-rate* system with hardening have been obtained in [10, Section 6.1] by performing the asymptotic analysis of the reparameterized energy-dissipation balance (3.2) as the viscosity parameter ε tends to 0, while keeping the hardening and the rate parameters μ and ν fixed. That is why we refer to them as $\mathrm{BV}^{\mu,\nu}_0$ solutions to the system with hardening. Their definition involves the corresponding vanishing-viscosity contact potential $\mathcal{M}^{\mu,\nu}_0:[0,T]\times\mathbf{Q}_{\mathrm{H}}\times[0,+\infty)\times\mathbf{Q}_{\mathrm{H}}\to[0,+\infty]$ given by

$$\mathcal{M}^{\mu,\nu}_0(t,q,t',q'):=\mathcal{R}(z')+\mathcal{H}(z,p')+\mathcal{M}^{\mu,\nu}_{0,\mathrm{red}}(t,q,t',q'),\quad\text{where}$$

$$\text{if } t'>0,\quad \mathcal{M}^{\mu,\nu}_{0,\mathrm{red}}(t,q,t',q'):=\begin{cases}0 & \text{if }\begin{cases}-\mathrm{D}_u\mathcal{E}_\mu(t,q)=0,\\ -\mathrm{D}_z\mathcal{E}_\mu(t,q)\in\partial\mathcal{R}(0),\ \text{and}\\ -\mathrm{D}_p\mathcal{E}_\mu(t,q)\in\partial_\pi\mathcal{H}(z,0),\end{cases}\\ +\infty & \text{otherwise,}\end{cases}\tag{5.13a}$$

$$\text{if } t'=0,\quad \mathcal{M}^{\mu,\nu}_{0,\mathrm{red}}(t,q,0,q'):=\mathcal{D}_\nu(q')\,\mathcal{D}^{*,\mu}_\nu(t,q).\tag{5.13b}$$

For better readability, we also recall that

$$\mathcal{D}_\nu(q'):=\sqrt{\nu\|u'(t)\|^2_{H^1,\mathbb{D}}+\|z'(t)\|^2_{L^2}+\nu\|p'(t)\|^2_{L^2}},$$
$$\mathcal{D}^{*,\mu}_\nu(t,q):=\sqrt{\frac{1}{\nu}\|-\mathrm{D}_u\mathcal{E}_\mu(t,q)\|^2_{(H^1,\mathbb{D})^*}+\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_\mu(t,q),\partial\mathcal{R}(0))^2+\frac{1}{\nu}d_{L^2}(-\mathrm{D}_p\mathcal{E}_\mu(t,q),\partial_\pi\mathcal{H}(z,0))^2}.$$

It is worthwhile to remark that the *reduced* functional $\mathcal{M}^{\mu,\nu}_{0,\mathrm{red}}$, at $t'=0$, *simultaneously* encompasses viscosity for the three variables u, z, and p. Instead, its counterpart $\mathcal{M}^{\mu,0}_{0,\mathrm{red}}$ for $\mathrm{BV}^{\mu,0}_0$ solutions features, in the jump regime $t'=0$, a viscous contribution in the variables (u,p) when $z'=0$, and viscosity in z when $\mathcal{D}^{*,\mu}(t,q)=0$, i.e., when u is at elastic equilibrium and p is locally stable.

We are now in a position to recall the notion of $\mathrm{BV}^{\mu,\nu}_0$ solution, cf. [10, Definition 6.2].

Definition 5.5 We call a parameterized curve $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})\in\mathrm{AC}([0,S];[0,T]\times\mathbf{Q}_{\mathrm{H}})$ a *(parameterized) Balanced Viscosity* solution to the *single-rate* system with hardening (1.4) (a $\mathrm{BV}^{\mu,\nu}_0$ solution, for short), if $\mathsf{t}:[0,S]\to[0,T]$ is nondecreasing and (t,q) fulfills for all $0\le s\le S$ the energy-dissipation balance

$$\begin{aligned} &\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)) + \int_0^s \mathcal{M}_0^{\mu,\nu}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))\,\mathrm{d}\tau \\ &\quad = \mathcal{E}_\mu(\mathsf{t}(0),\mathsf{q}(0)) + \int_0^s \partial_t\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau))\,\mathsf{t}'(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{5.14}$$

We say that (t,q) is *non-degenerate* if it fulfills (4.1).

Let us now address the asymptotic analysis of the above solutions for a vanishing sequence $(\mu_k)_k$. As mentioned in the Introduction, in the construction of $\mathrm{BV}_0^{\mu,\nu}$ solutions the rate parameter is always supposed smaller than the hardening parameter, which forces us to also consider a sequence $(\nu_k)_k$ such that $\nu_k \le \mu_k$ for all $k \in \mathbb{N}$, so that $\nu_k \to 0$ as well. In fact, the technical condition $\nu_k \le \mu_k$ comes into play in the proof of [10, Prop. 4.4]. The latter result and [10, Theorem 6.8] ensure the existence of $\mathrm{BV}_0^{\mu_k,\nu_k}$ solutions $(\mathsf{t}_k,\mathsf{q}_k)_k$ enjoying the following a priori estimates

$$\begin{aligned} &\exists\, C>0 \ \ \forall\, k\in\mathbb{N} \ \text{ for a.a.}\, s\in(0,S) \ : \\ &\mathsf{t}_k'(s)+\|\mathsf{u}_k'(s)\|_{W^{1,1}(\Omega)}+\|\mathsf{z}_k'(s)\|_{H^{\mathrm{m}}(\Omega)}+\|\mathsf{p}_k'(s)\|_{L^1(\Omega)}+\sqrt{\mu_k}\,\|\mathsf{p}_k'(s)\|_{L^2(\Omega)} \\ &\qquad +\|\mathsf{e}_k'(s)\|_{L^2(\Omega)}+\mathcal{D}_{\nu_k}(\mathsf{u}_k'(s),\mathsf{p}_k'(s))\,\mathcal{D}_{\nu_k}^{*,\mu_k}(\mathsf{t}_k(s),\mathsf{q}_k(s)) \le C \end{aligned} \tag{5.15}$$

(and, up to a reparametrization, the non-degeneracy condition).

In Theorem 5.6 below, we are going to show that, as the hardening and rate parameters μ_k and ν_k vanish, (up to a subsequence) $\mathrm{BV}_0^{\mu_k,\nu_k}$ solutions converge to a $\mathrm{BV}_0^{0,0}$ solution of system (1.1).

Theorem 5.6 *Let* $(\mu_k)_k$, $(\nu_k)_k$ *be two vanishing sequences, and let* $(\mathsf{t}_k,\mathsf{q}_k)_k$ *be a sequence of* $\mathrm{BV}_0^{\mu_k,\nu_k}$ *solutions to system* (1.4) *such that estimate* (5.15) *holds.*

Then, there exist a (not relabeled) subsequence and a curve $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) \in \mathcal{EA}(0,S;[0,T]\times\mathbf{Q}_{\mathrm{PP}})$ *such that items* (1), (2), (3) *of the statement of Theorem 5.3 hold.*

Proof The argument is split in the same steps as the proof of Theorem 5.3.

Step 1: *Compactness.* With minor changes, from estimate (5.15) we derive estimate (5.6) and convergences (5.8) and (5.9), whence convergences (5.4), for the sequence of $\mathrm{BV}_0^{\mu_k,\nu_k}$ solutions. Analogously, for the limiting curve (t,q) estimate (5.5) holds.

Step 2*a*: *energy-dissipation upper estimate when* $\mathsf{t}' > 0$. The analogue of (5.12) at all $s \in A := \{s\in[0,S] \ : \ \mathsf{t}'(s) > 0\}$ can be obtained in the same way as in the proof of Theorem 5.3, taking into account that

$$\mathcal{M}_{0,\mathrm{red}}^{\mu_k,0}(t,q,t',q') = \mathcal{M}_{0,\mathrm{red}}^{\mu_k,\nu_k}(t,q,t',q') \qquad \text{whenever } t'>0.$$

Step 2*b*: *energy-dissipation upper estimate when* $\mathsf{t}' = 0$. We will now show that

$$\int_{(0,s)\setminus A} \mathcal{M}_{0,\mathrm{red}}^{0,0}(\mathsf{t}(r),\mathsf{q}(r),0,\mathsf{q}'(r))\,\mathrm{d}r \le \liminf_{k\to+\infty}\int_{(0,s)\setminus A} \mathcal{M}_{0,\mathrm{red}}^{\mu_k,\nu_k}(\mathsf{t}_k(r),\mathsf{q}_k(r),\mathsf{t}_k'(r),\mathsf{q}_k'(r))\,\mathrm{d}r. \tag{5.16}$$

As in the proof of Thm. 5.3, we will apply Lemma A.3 below in the context of the space $\boldsymbol{Q} := \mathbf{Q}_{\mathrm{PP}}$, with $\boldsymbol{S}$ the ball of radius R from (5.6), to the functionals $\mathscr{M}_k := \mathcal{M}_{0,\mathrm{red}}^{\mu_k,\nu_k}$ (extended to $\mathbb{R}\times(\mathbf{Q}_{\mathrm{PP}}\setminus\mathbf{Q}_{\mathrm{H}})\times\mathbb{R}\times(\mathbf{Q}_{\mathrm{PP}}\setminus\mathbf{Q}_{\mathrm{H}})$ as described for $\mathcal{M}_{0,\mathrm{red}}^{\mu_k,0}$ in the proof of

Thm 5.3), and $\mathscr{M}_0 := \mathcal{M}^{0,0}_{0,\mathrm{red}}$. With this aim, we only need to check that the Γ-liminf estimate in (A.2) holds in our context. Clearly, it is sufficient to check that for any sequence $(t_k, q_k, t'_k, q'_k)_k$ with $(t_k, q_k, t'_k, q'_k) \overset{*}{\rightharpoonup} (t, q, 0, q')$ in $\mathbb{R} \times \boldsymbol{S} \times \mathbb{R} \times \mathbf{Q}_{\mathrm{PP}}$ as $k \to \infty$ there holds

$$\mathcal{M}^{0,0}_{0,\mathrm{red}}(t, q, 0, q') \leq \liminf_{k\to+\infty} \mathcal{M}^{\mu_k,\nu_k}_{0,\mathrm{red}}(t_k, q_k, t'_k, q'_k). \tag{5.17}$$

The above estimate easily follows from Lemma A.2 in the case in which $(t_k)_k$ admits a strictly positive subsequence. Instead, if there exists $\bar{k} \in \mathbb{N}$ such that $t'_k \equiv 0$ for $k \geq \bar{k}$, then $\mathcal{M}^{\mu_k,\nu_k}_{0,\mathrm{red}}(t_k, q_k, t'_k, q'_k) = \mathcal{D}_{\nu_k}(q'_k)\, \mathcal{D}^{*,\mu_k}_{\nu_k}(t_k, q_k)$ for all $k \geq \bar{k}$ and we may argue in the following way. When $z' = 0$ and $\mathcal{D}^*(t, q) = 0$, we use that

$$\mathcal{M}^{\mu_k,\nu_k}_{0,\mathrm{red}}(t_k, q_k, t'_k, q'_k) \geq \begin{cases} \mathcal{D}(u'_k, p'_k)\, \mathcal{D}^{*,\mu_k}(t_k, q_k), \\ \|z'_k\|_{L^2}\, \tilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_{\mu_k}(t_k, q_k), \partial\mathcal{R}(0)), \end{cases}$$

which follows by neglecting some terms in the expression for $\mathcal{M}^{\mu_k,\nu_k}_{0,\mathrm{red}}$. Then, as in Thm. 5.3 we use Lemma A.2 to pass to the limit in the two terms on the right-hand side of the above inequality in the cases $z' = 0$ and $\mathcal{D}^{*,\mu}(t, q) = 0$, respectively. In the remaining case $\|z'\|_{L^2}\mathcal{D}^*(t, q) > 0$, it holds $\lim_{k\to\infty} \mathcal{M}^{\mu_k,\nu_k}_{0,\mathrm{red}}(t_k, q_k, t'_k, q'_k) = +\infty$: indeed, by Lemma A.2 and since $z'_k \rightharpoonup z'$ in $L^2(\Omega)$, we have that $\|z'_k\|_{L^2}\mathcal{D}^{*,\mu_k}(t_k, q_k) > c$ for a suitable $c > 0$. Since

$$\mathcal{M}^{\mu_k,\nu_k}_{0,\mathrm{red}}(t_k, q_k, t'_k, q'_k) \geq \frac{1}{\sqrt{\nu_k}}\|z'_k\|_{L^2}\mathcal{D}^{*,\mu_k}(t_k, q_k),$$

we again conclude estimate (5.17) as $(\mu_k)_k$ and $(\nu_k)_k$ vanish.

Thus, by Lemma A.3, we have proven (5.16). This finishes the proof. □

Appendix A: Some technical results

We collect the results employed in the proofs of Theorems 5.3 and 5.6.

Lemma A.1 *Let* $\mathsf{f}, \mathsf{f}_k : [0, S] \to [0, T]$ *be nondecreasing functions such that* $\mathsf{f}_k \to \mathsf{f}$ *uniformly. Let* $A := \{s \in [0, S] : \mathsf{f}'(s) > 0\}$. *Then for a.a.* $s \in A$, *there is a subsequence* k_j *and, for every* j, *there is* $s_{k_j} \in (0, S)$ *such that* $|s_{k_j} - s| < \frac{1}{j}$ *and* $\mathsf{f}'_{k_j}(s_{k_j}) > 0$.

Proof Let

$$\Xi := \{s \in A \ :\ \exists\, k(s)\ \forall\, k > k(s)\ \exists\, \delta_k > 0\ :\ \mathsf{f}'_k(\sigma) = 0 \text{ for a.a. } \sigma \in (s - \delta_k, s + \delta_k)\}.$$

We shall prove that Ξ is at most countable, which implies the statement of the lemma. Let $s^1, s^2 \in \Xi$. Then, one has $\mathsf{f}(s^1) \neq \mathsf{f}(s^2)$; indeed, since f is nondecreasing, $\mathsf{f}(s^1) = \mathsf{f}(s^2)$ would imply that f is constant in (s^1, s^2), which is in contrast with the assumption $s^1, s^2 \in A$ (and thus $\mathsf{f}'(s^1), \mathsf{f}'(s^2) > 0$). Let now $y^i_k := \mathsf{f}_k(s^i)$ for $i = 1, 2$. Since $y^i_k \to \mathsf{f}(s^i)$ for $i = 1, 2$, for k sufficiently large one has $|y^1_k - y^2_k| > \frac{1}{2}|\mathsf{f}(s^1) - \mathsf{f}(s^2)| > 0$. Let $\phi_k \in \mathrm{BV}(0, T)$ denote the inverse function of f_k. It turns out that y^1_k, y^2_k are both jump points of ϕ_k for every $k > \max\{k(s^1), k(s^2)\}$. Since the jump points of a BV function are countable, it follows that Ξ is countable, too. □

Lemma A.2 [10, *Lemma* 7.8] *Let* $t_k \to t$ *in* $[0, T]$, $\mu_k \to 0$, $(q_k)_k = (u_k, z_k, p_k)_k \subset \mathbf{Q}_{\mathrm{PP}}$ *such that the following convergences hold as* $k \to +\infty$: $q_k \overset{*}{\rightharpoonup} q = (u, z, p)$ *in* $\mathbf{Q}_{\mathrm{PP}}$, $e(t_k) = \mathrm{E}(u_k + w(t_k)) - p_k \to e(t) = \mathrm{E}(u + w(t)) - p$ *in* $L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ *and* $\mu_k p_k \to 0$ *in* $L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}})$. *Then,*

$$\mathcal{S}_u\mathcal{E}_0(t, q) \le \liminf_{k\to+\infty} \|\mathrm{D}_u\mathcal{E}_{\mu_k}(t_k, q_k)\|_{(H^1,\mathbb{D})^*}, \tag{A.1a}$$

$$\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t, q), \partial\mathcal{R}(0)) \le \liminf_{k\to+\infty} \widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_{\mu_k}(t_k, q_k), \partial\mathcal{R}(0)), \tag{A.1b}$$

$$\mathcal{W}_p\mathcal{E}_0(t, q) \le \liminf_{k\to+\infty} d_{L^2}(-\mathrm{D}_p\mathcal{E}_{\mu_k}(t_k, q_k), \partial_\pi\mathcal{H}(z_k, 0)). \tag{A.1c}$$

We borrow our final auxiliary result from [25]. The proof, therein developed in the case of a sequence (t_k, q_k) with values in $\mathbb{R} \times \boldsymbol{Q}$ with $\boldsymbol{Q}$ a reflexive space, can be straightforwardly adapted to the case of the dual of a separable space.

Lemma A.3 [25, *Prop*. 5.2] *Let* $\boldsymbol{Q}$ *be the dual of a separable Banach space, let* $\boldsymbol{S}$ *be a weakly closed subset of* $\boldsymbol{Q}$, *and let* $(\mathscr{M}_k)_k$, $\mathscr{M}_0 : \mathbb{R} \times \boldsymbol{S} \times \mathbb{R} \times \boldsymbol{Q} \to [0, \infty]$ *be measurable and weakly lower semicontinuous functionals fulfilling the* Γ-$\liminf$ *estimate*

$$\begin{aligned}&\left((t_k, q_k, t'_k, q'_k) \overset{*}{\rightharpoonup} (t, q, t', q') \text{ in } \mathbb{R} \times \boldsymbol{S} \times \mathbb{R} \times \boldsymbol{Q} \text{ as } k \to \infty\right)\\ &\quad\Longrightarrow \mathscr{M}_0(t, q, t', q') \le \liminf_{k\to\infty} \mathscr{M}_k(t_k, q_k, t'_k, q'_k).\end{aligned} \tag{A.2}$$

Suppose that, the functionals $\mathscr{M}_0(t, q, \cdot, \cdot)$ *and* $\mathscr{M}_k(t, q, \cdot, \cdot)$ *are convex for every* $k \in \mathbb{N}$ *and* $(t, q) \in \mathbb{R} \times \boldsymbol{S}$. *Let* $(\mathsf{t}_k, \mathsf{q}_k)$, $(\mathsf{t}, \mathsf{q}) \subset \mathrm{AC}([a, b]; \mathbb{R} \times \boldsymbol{S})$ *fulfill*

$$\mathsf{t}_k(s) \to \mathsf{t}(s), \quad \mathsf{q}_k(s) \overset{*}{\rightharpoonup} \mathsf{q}(s) \text{ for all } s \in [a, b], \qquad (\mathsf{t}'_k, \mathsf{q}'_k) \rightharpoonup (\mathsf{t}', \mathsf{q}') \text{ in } L^1(a, b; \mathbb{R} \times \boldsymbol{Q}).$$

Then,

$$\liminf_{k\to\infty} \int_a^b \mathscr{M}_k(\mathsf{t}_k(s), \mathsf{q}_k(s), \mathsf{t}'_k(s), \mathsf{q}'_k(s))\,\mathrm{d}s \ge \int_a^b \mathscr{M}_0(\mathsf{t}(s), \mathsf{q}(s), \mathsf{t}'(s), \mathsf{q}'(s))\,\mathrm{d}s.$$

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