Linesearch Newton-CG methods for convex optimization with noise

ANNALI DELL'UNIVERSITA' DI FERRARA
https://doi.org/10.1007/s11565-022-00435-4

Linesearch Newton-CG methods for convex optimization with noise

S. Bellavia[1] · **E. Fabrizi**[2] · **B. Morini**[1]

Received: 30 April 2022 / Accepted: 29 July 2022

Abstract

This paper studies the numerical solution of strictly convex unconstrained optimization problems by linesearch Newton-CG methods. We focus on methods employing inexact evaluations of the objective function and inexact and possibly random gradient and Hessian estimates. The derivative estimates are not required to satisfy suitable accuracy requirements at each iteration but with sufficiently high probability. Concerning the evaluation of the objective function we first assume that the noise in the objective function evaluations is bounded in absolute value. Then, we analyze the case where the error satisfies prescribed dynamic accuracy requirements. We provide for both cases a complexity analysis and derive expected iteration complexity bounds. We finally focus on the specific case of finite-sum minimization which is typical of machine learning applications.

Keywords Newton-CG · Evaluation complexity · Inexact function and derivatives · Probabilistic analysis · Finite-sum optimization · Subsampling

Mathematics Subject Classification 65K05 · 49M15 · 90C30

S. Bellavia, E. Fabrizi, B. Morini: Member of the INdAM Research Group GNCS.

✉ S. Bellavia
stefania.bellavia@unifi.it

E. Fabrizi
eugenio.fabrizi@unifi.it

B. Morini
benedetta.morini@unifi.it

[1] Department of Industrial Engineering, Università degli Studi di Firenze, Florence, Italy

[2] Department of Mathematics and Computer Science "Ulisse Dini", Università degli Studi di Firenze, Florence, Italy

Published online: 17 August 2022

1 Introduction

In this paper we consider globally convergent Inexact Newton methods for solving the strictly convex unconstrained optimization problem

$$\min_{x\in\mathbb{R}^n} f(x). \tag{1}$$

We focus on the Newton method where the linear systems are solved by the Conjugate Gradient (CG) method [25], usually denoted as Newton-CG method, and on the enhancement of its convergence properties by means of Armijo-type conditions.

The literature on globally convergent Newton-CG methods is well established as long as the gradient and the Hessian matrix are computed exactly or approximated sufficiently accurately in a deterministic way, see e.g., [15, 20, 31]. On the other hand, the research is currently very active for problems with inexact information on f and its derivatives and possibly such that the accuracy cannot be controlled in a deterministic way [1–11, 13, 14, 17, 19, 22, 23, 26, 32, 33, 35].

This work belongs to the recent stream of works and addresses the solution of (1) when the objective function f is computed with noise and gradient and Hessian estimates are random. Importantly, derivative estimates are not required to satisfy suitable accuracy requirements at each iteration but with sufficiently high probability. Concerning the evaluation of f we cover two cases: estimates of f subject to noise that is bounded in absolute value; estimates of f subject to a controllable error, i.e., computable with a prescribed dynamic accuracy. Such a class of problems has been considered in [4, 10] and our contribution consists in their solution with linesearch Newton-CG method; to our knowledge, this case has not been addressed in the literature. We provide two linesearch Newton-CG methods suitable for the class of problems specified above and provide bounds on the expected number of iterations required to reach a desired level of accuracy in the optimality gap.

The paper is organized as follows. In Sect. 2 we give preliminaries on Newton-CG and on the problems considered. In Sect. 3 we present and study a linesearch Newton-CG algorithm where function estimates are subject to a prefixed deterministic noise. In Sect. 4 we propose and study a linesearch Newton-CG algorithm where function estimates have controllable accuracy. In Sect. 5 we consider the specific case where f is a finite-sum which is typical of machine learning applications and compare our approach with the Inexact Newton methods specially designed for this class of problems given in [7, 12, 33].

In the rest of paper $\|\cdot\|$ denotes the 2-norm. Given symmetric matrices A and B, $A \preceq B$ means that $B - A$ is positive semidefinite.

2 Our setting

In this section we provide preliminaries on the solution of problem (1) and the assumptions made. Our methods belong to the class of the Inexact Newton methods [18] combined with a linesearch strategy for enhancing convergence properties. A key fea-

ture is that function, gradient and Hessian evaluations are approximated and the errors in such approximations are either deterministic or stochastic, as specified below.

The Inexact Newton methods considered here are iterative processes where, given the current iterate x_k, a random approximation g_k to $\nabla f(x_k)$ and a random approximation H_k to $\nabla^2 f(x_k)$, the trial step s_k satisfies

$$H_k s_k = -g_k + r_k, \quad \|r_k\| \leq \eta_k \|g_k\|, \tag{2}$$

for some $\eta_k \in (0, \bar{\eta}), 0 < \bar{\eta} < 1$, named forcing term.

With s_k and a trial steplength t_k at hand, some suitable sufficient decrease Armijo condition is tested on $x_k + t_k s_k$. Standard linesearch strategies are applied using the true function f. On the other hand, here we assume that the evaluation of f is subject to an error.

If H_k is positive definite we can solve inexactly the linear systems $H_k s = -g_k$ using the Conjugate Gradient (CG) method [25]. The resulting method is denoted as Newton-CG. If the initial guess for CG is the null vector, the following properties hold.

Lemma 2.1 *Suppose that H_k is symmetric positive definite and s_k is the vector in (2) obtained by applying the CG method with null initial guess to the linear system $H_k s = -g_k$. Let $0 < \lambda_1 \leq \lambda_n$ such that*

$$\lambda_1 I \preceq H_k \preceq \lambda_n I. \tag{3}$$

Then, there exist constants $\kappa_1, \kappa_2, \beta > 0$, such that:

$$\kappa_1 \|g_k\| \leq \|s_k\| \leq \kappa_2 \|g_k\|, \quad -g_k^T s_k \geq \beta \|s_k\|^2, \qquad \forall k > 0, \tag{4}$$

which satisfy

$$\frac{1-\bar{\eta}}{\lambda_n} \leq \kappa_1 \leq \kappa_2 \leq \frac{1}{\lambda_1}, \quad \beta \geq \lambda_1. \tag{5}$$

As a consequence

$$\beta \kappa_1 \leq \frac{-g_k^T s_k}{\|s_k\|^2} \frac{\|s_k\|}{\|g_k\|} = \frac{|g_k^T s_k|}{\|g_k\| \|s_k\|} \leq 1, \tag{6}$$

$$\lambda_n \kappa_1 \geq 1 - \bar{\eta}, \tag{7}$$

$$\kappa_1 \lambda_1 \leq \kappa_2 \lambda_1 \leq 1, \tag{8}$$

$$-g_k^T s_k \geq \beta \kappa_1^2 \|g_k\|^2. \tag{9}$$

Proof Lemma 7 in [21] guarantees that any step $\hat{s}$ returned by the CG method applied to $H_k s = g_k$ with null initial guess satisfies $\hat{s}^T H_k \hat{s} = -\hat{s}^T g_k$. Then, it holds

$$s_k^T H_k s_k = -s_k^T g_k, \tag{10}$$

and by (3) we have

$$-g_k^T s_k = s_k^T H_k s_k \geq \lambda_1 \|s_k\|^2 \tag{11}$$

which provides the lower bound on β in (5).
By (11) it also follows that

$$\|s_k\| \leq \lambda_1^{-1} \|g_k\|, \tag{12}$$

which provides the upper bound on κ_2 in (5).
Moreover, using (2)

$$\|g_k - r_k\| \geq \|g_k\| - \|r_k\| \geq (1 - \eta_k)\|g_k\| \geq (1 - \bar{\eta})\|g_k\|$$

while (2) and (3) give

$$\|g_k - r_k\| = \|H_k s_k\| \leq \|H_k\| \|s_k\| \leq \lambda_n \|s_k\|$$

and consequently

$$\|s_k\| \geq \frac{\|g_k - r_k\|}{\lambda_n} \geq \frac{1 - \bar{\eta}}{\lambda_n} \|g_k\|, \tag{13}$$

which provides the lower bound on κ_1 in (5).
Inequalities (6)–(9) are direct consequences of (4) and (5). □

2.1 Assumptions

We introduce the assumptions on the problem (1) and on the approximate evaluations of functions, gradients and Hessians.

Assumption 2.2 (smoothness and strong convexity of f) *The function f is twice continuously differentiable and there exist some $\lambda_n \geq \lambda_1 > 0$ such that the Hessian matrix $\nabla^2 f(x)$ satisfy*

$$\lambda_1 I \preceq \nabla^2 f(x) \preceq \lambda_n I, \quad \forall x \in \mathbb{R}^n. \tag{14}$$

As a consequence, f is strongly convex with constant λ_1, i.e.,

$$f(x) \geq f(y) + \nabla f(y)^T (x - y) + \frac{\lambda_1}{2} \|x - y\|^2 \text{ for all } x, y \in \mathbb{R}^n$$

and the gradient of f is Lipschitz-continuous with constant λ_n, i.e.,

$$\|\nabla f(x) - \nabla f(y)\| \leq \lambda_n \|x - y\| \text{ for all } x, y \in \mathbb{R}^n. \tag{15}$$

Further, letting x^* be the unique minimizer of the function f, for all $x \in \mathbb{R}^n$ we have

$$\lambda_1 \|x - x^*\| \leq \|\nabla f(x)\| \leq \lambda_n \|x - x^*\|, \tag{16}$$

and

$$\frac{\lambda_1}{2}\|x - x^*\|^2 \leq f(x) - f(x^*) \leq \frac{1}{2\lambda_1}\|\nabla f(x)\|^2, \tag{17}$$

see [29, Theorem 2.10].

As for approximated evaluations of the objective function, we consider two possible cases. The first one is such that the value $f(x)$ is approximated with a value $\tilde{f}(x)$ and the corresponding error is not controllable but its upper bound ε_f is known.

Assumption 2.3 (boundness of noise in f) *There exists a positive scalar ε_f such that*

$$|f(x) - \tilde{f}(x)| \leq \varepsilon_f, \quad \forall x \in \mathbb{R}^n. \tag{18}$$

The second case concerns a controllable error between $f(x)$ and $\tilde{f}(x)$ at the current iteration x_k and at the trial iteration $x_k + t_k s_k$.

Assumption 2.4 (controllable noise in f) *For all $k > 0$ and some given positive θ*

$$\begin{aligned} |f(x_k) - \tilde{f}(x_k)| &\leq -\theta t_k s_k^T g_k, \\ |f(x_k + t_k s_k) - \tilde{f}(x_k + t_k s_k)| &\leq -\theta t_k s_k^T g_k. \end{aligned} \tag{19}$$

The methods we are dealing with are globalized Inexact Newton methods employing random estimates g_k and H_k of the gradient and the Hessian and noisy values of the objective function. Then, they generate a stochastic process. We denote the random variables of our process as follows: the gradient estimator G_k, the hessian estimator $\mathcal{H}_k$, the step size parameter $\mathcal{T}_k$, the search direction $\mathcal{S}_k$, the iterate X_k. Their realizations are denoted as $g_k = G_k(\omega_k)$, $H_k = \mathcal{H}_k(\omega_k)$, $t_k = \mathcal{T}_k(\omega_k)$, $s_k = \mathcal{S}_k(\omega_k)$ and $x_k = X_k(\omega_k)$, respectively, with ω_k taken from a proper probability space. For brevity we will omit ω_k in the following. We let $\mathcal{E}_{k-1}$ denote all noise history up to iteration $k-1$ and we include $\mathcal{E}_{k-1}$ in the algorithmic history for completeness. We use $\mathcal{F}_{k-1} = \sigma(G_0, \ldots, G_{k-1}, \mathcal{H}_0, \ldots, \mathcal{H}_{k-1}, \mathcal{E}_{k-1})$ to denote the σ-algebra generated by $G_0, \ldots, G_{k-1}, \mathcal{H}_0, \ldots, \mathcal{H}_{k-1}$ and $\mathcal{E}_{k-1}$, up to the beginning of iteration k.

We assume that the random gradient estimators G_k are $(1 - \delta_g)$-probabilistically sufficiently accurate.

Assumption 2.5 (gradient estimate) *The estimator G_k is $(1 - \delta_g)$-probabilistically sufficient accurate in the sense that the indicator variable*

$$I_k = \mathbb{1}\{\|G_k - \nabla f(X_k)\| \leq t_k \eta_k \|G_k\|\} \tag{20}$$

satisfies the submartingale condition

$$P(I_k = 1 | \mathcal{F}_{k-1}) \geq 1 - \delta_g, \quad \delta_g \in (0, 1). \tag{21}$$

Iteration k is called *true* (iteration) if $I_k = 1$, *false* otherwise. Trivially, if kth iteration is true then the triangle inequality implies

$$\|\nabla f(x_k)\| \leq (1 + t_k \eta_k) \|g_k\|. \tag{22}$$

Finally, at each iteration and for any realization, the approximation H_k is supposed to be positive definite.

Assumption 2.6 *For all $k \geq 0$, H_k is symmetric positive definite and*

$$\lambda_1 I \preceq H_k \preceq \lambda_n I, \tag{23}$$

with λ_1, λ_n as in (14).

We remark that assuming f strictly convex, (14) and (23) hold, with suitable choices of the scalar λ_1, λ_n, as long as the sequence $\{H_k\}$ is symmetric positive definite and has eigenvalues uniformly bounded from below and above.

Some comments on Assumptions 2.3, 2.4, 2.5 and 2.6 are in order. They appear in a series of papers on unconstrained optimization where the evaluation of function, gradient and Hessian are inexact, either with a controllable or a random noise. Controlling the noise in a deterministic way, as in Assumptions 2.3 and 2.4 is a realistic request in applications such as those where the accuracy of f-values can be enforced by the magnitude of some discretization parameters or f is evaluated in variable precision or approximated by using smoothing operators [24, 27, 28, 30]. Probabilistically sufficient accurate gradients, as in Assumption 2.5, occur when the gradients are estimated by finite difference and some computation fails to complete, in derivative-free optimization or when the gradient are estimated by sample average approximation methods [1, 6, 16, 17]. Finally, Assumption 2.6 amounts to building a convex random model and is trivially enforced if f is the sum of strongly convex functions and the Hessian is estimated by sample average approximation methods. In literature, unconstrained optimization with inexact function and derivative evaluations covers many cases: exact function and gradient evaluations and possibly random Hessian [3, 13, 35], exact function and random gradient and Hessian [1, 2, 16], approximated function and random gradient and Hessian [4, 10], random function gradient and Hessian [5, 8, 11, 17, 32]. In the class of Inexact Newton method with random models we mention [7, 12, 14, 19, 26].

3 Bounded noise on *f*

In this section we present and analyze an Inexact Newton method with line-search where the function evaluation is noisy in the sense of Assumption 2.3.

At iteration k, given x_k and the steplength t_k, a non-monotone Armijo condition given in [10] is used. It employs the known upper bound ε_f introduced in Assumption 2.3 and has the form

$$\tilde{f}(x_k + t_k s_k) \leq \tilde{f}(x_k) + c\, t_k s_k^T g_k + 2\varepsilon_f, \tag{24}$$

$c \in (0, 1)$. If $x_k + t_k s_k$ satisfies (24) we say that the iteration is successful, we accept the step and increase the step-length t_k for the next iteration. Otherwise the step is rejected and the step-length t_k is reduced; at the next iterate g_k and H_k are supposed to be computed from scratch. Our procedure belongs to the framework given in Section 4.2 of [10] and it is sketched in Algorithm 3.1.

Algorithm 3.1: k-th iteration of Algorithm

Given x_k, $g_k \in \mathbb{R}^n$, $H_k \in \mathbb{R}^{n\times n}$ $c \in (0, 1)$, $\bar{\eta} \in (0, 1)$, $\tau \in (0, 1)$, $t_{\max} > 0$, $t_k \in (0, t_{\max}]$.

Step 1. Choose $\eta_k \in (0, \bar{\eta})$.
Step 2. Starting from the null vector, use CG to compute an approximate solution s_k of $H_k s = -g_k$ satisfying (2).
Step 3. If t_k satisfies condition (24) (successful iteration) then
$x_{k+1} = x_k + t_k s_k$, $t_{k+1} = \min\{t_{\max}, \tau^{-1} t_k\}$, $k = k + 1$.
else $x_{k+1} = x_k$, $t_{k+1} = \tau t_k$, $k = k + 1$.

Similarly to [10, 16], Algorithm 3.1 generates a stochastic process. Given x_k and t_k, the iterate x_{k+1} is fully determined by g_k, H_k and the noise in the function value estimation during iteration k.

Concerning the well definiteness of the linesearch strategy (24), we now prove that if the iteration is true and t_k is small enough, the linesearch condition is satisfied.

Lemma 3.1 *Suppose that Assumptions 2.2, 2.3, 2.5 and 2.6 hold. Suppose that iteration k is true and consider any realization of Algorithm 3.1. Then the iteration is successful whenever $t_k \le \bar{t} = \frac{2\beta\kappa_1(1-c)}{\kappa_1\lambda_n+2}$.*

Proof Let k be an arbitrary iteration. Inequalities (15) and (18) imply, using the standard arguments for functions with bounded Hessians,

$$
\begin{aligned}
\tilde{f}(x_k + t_k s_k) - \varepsilon_f &\le f(x_k + t_k s_k) \\
&= f(x_k) + \int_0^1 [\nabla f(x_k + \zeta t_k s_k)]^T (t_k s_k) d\zeta \\
&= f(x_k) + \int_0^1 t_k \left([\nabla f(x_k + \zeta t_k s_k)]^T s_k \pm \nabla f(x_k)^T s_k \right) d\zeta \\
&= f(x_k) + \int_0^1 t_k [\nabla f(x_k + \zeta t_k s_k) - \nabla f(x_k)]^T s_k d\zeta + t_k \nabla f(x_k)^T s_k \\
&\le f(x_k) + \int_0^1 t_k \|\nabla f(x_k + \zeta t_k s_k) - \nabla f(x_k)\| \|s_k\| d\zeta + t_k \nabla f(x_k)^T s_k \\
&\le f(x_k) + \frac{\lambda_n}{2} t_k^2 \|s_k\|^2 + t_k \nabla f(x_k)^T s_k \\
&\le \tilde{f}(x_k) + \varepsilon_f + \frac{\lambda_n}{2} t_k^2 \|s_k\|^2 + t_k \nabla f(x_k)^T s_k.
\end{aligned}
$$

Since iteration k is true by assumption then $\|\nabla f(x_k) - g_k\| \leq t_k \eta_k \|g_k\|$ holds, and using Lemma 2.1 we obtain

$$\begin{aligned}\tilde{f}(x_k + t_k s_k) &\leq \tilde{f}(x_k) + 2\varepsilon_f + \frac{\lambda_n}{2} t_k^2 \|s_k\|^2 + t_k \nabla f(x_k)^T s_k \pm t_k g_k^T s_k \\ &= \tilde{f}(x_k) + 2\varepsilon_f + \frac{\lambda_n}{2} t_k^2 \|s_k\|^2 + t_k [\nabla f(x_k) - g_k]^T s_k + t_k g_k^T s_k \\ &\leq \tilde{f}(x_k) + 2\varepsilon_f + \frac{\lambda_n}{2} t_k^2 \|s_k\|^2 + t_k^2 \frac{\eta_k}{\kappa_1} \|s_k\|^2 + t_k g_k^T s_k. \end{aligned} \tag{25}$$

Then, the linesearch condition (24) is clearly enforced whenever

$$\tilde{f}(x_k) + 2\varepsilon_f + \frac{\lambda_n}{2} t_k^2 \|s_k\|^2 + t_k^2 \frac{\eta_k}{\kappa_1} \|s_k\|^2 + t_k g_k^T s_k \leq \tilde{f}(x_k) + c t_k s_k^T g_k + 2\varepsilon_f$$

which gives

$$t_k \|s_k\|^2 \left(\frac{\lambda_n}{2} + \frac{\eta_k}{\kappa_1} \right) \leq -(1-c) g_k^T s_k.$$

Using (4) we have $-(1-c) g_k^T s_k \geq (1-c)\beta \|s_k\|^2$. Since $\eta_k < \bar{\eta} < 1$, if

$$t_k \|s_k\|^2 \left(\frac{\lambda_n}{2} + \frac{1}{\kappa_1} \right) \leq (1-c)\beta \|s_k\|^2,$$

then (24) holds and this yields the thesis. □

3.1 Complexity analysis of the stochastic process

In this section we carry out the convergence analysis of Algorithm 3.1. To this end we provide a bound on the expected number of iterations that the algorithm takes before it achieves a desired level of accuracy in the optimality gap $f(x_k) - f^*$ with $f^* = f(x^*)$ being the minimum value attained by f. Such a number of iteration is defined formally below.

Definition 3.2 Let x^* be the global minimizer of f and $f^* = f(x^*)$. Given some $\epsilon > 0$, N_ϵ is the number of iterations required until $f(x_k) - f^* \leq \epsilon$ occurs for the first time.

The number of iterations N_ϵ is a random variable and it can be defined as the hitting time for our stochastic process. Indeed it has the property $\sigma(\mathbb{1}\{N_\epsilon > k\}) \subset \mathcal{F}_{k-1}$.

Following the notation introduced in Sect. 2 we let $X_k, k \geq 0$, be the random variable with realization $x_k = X_k(\omega_k)$ and consider the following measure of progress towards optimality:

$$Z_k = \log \left(\frac{f(X_0) - f^*}{f(X_k) - f^*} \right). \tag{26}$$

Further, we let

$$Z_\epsilon = \log\left(\frac{f(X_0) - f^*}{\epsilon}\right) \tag{27}$$

be an upper bound for Z_k for any $k < N_\epsilon$. We denote with $z_k = Z_k(\omega_k)$ a realization of the random quantity Z_k.

A theoretical framework for analyzing a generic line search with noise has been developed in [10]. Under a suitable set of conditions, it provides the expected value for N_ϵ. We state a result from [10] and will exploit it for our algorithm.

Theorem 3.3 *Suppose that Assumptions 2.2, 2.3, 2.5, 2.6 hold. Let z_k a realization of Z_k in (26) and suppose that there exist a constant $\bar{t} > 0$, a nondecreasing function $h(t) : \mathbb{R}^+ \to \mathbb{R}$, which satisfies $h(t) > 0$ for any $t \in (0, t_{\max}]$, and a nondecreasing function $r(\varepsilon_f) : \mathbb{R} \to \mathbb{R}$, which satisfies $r(\varepsilon_f) \geq 0$ for any $\varepsilon_f \geq 0$, such that for any realization of Algorithm 3.1 the following holds for all $k < N_\epsilon$:*

(i) *If iteration k is true and successful, then $z_{k+1} \geq z_k + h(t_k) - r(\varepsilon_f)$.*
(ii) *If $t_k \leq \bar{t}$ and iteration k is true then iteration k is also successful, which implies $t_{k+1} = \tau^{-1} t_k$.*
(iii) *$z_{k+1} \geq z_k - r(\varepsilon_f)$ for all successful iterations k and $z_{k+1} \geq z_k$ for all unsuccessful iteration k.*
(iv) *The ratio $r(\varepsilon_f)/h(\bar{t})$ is bounded from above by some $\gamma \in (0, 1)$.*

Then under the condition that the probability δ_g in Assumption 2.5 is such that $\delta_g < \frac{1}{2} - \frac{\sqrt{\gamma}}{2}$, the stopping time N_ϵ is bounded in expectation as follows

$$\mathbb{E}[N_\epsilon] \leq \frac{2(1-\delta_g)}{(1-2\delta_g)^2 - \gamma}\left[\frac{2Z_\epsilon}{h(\bar{t})} + (1-\gamma)\log_\tau \frac{\bar{t}}{t_0}\right]. \tag{28}$$

Proof See [10, Assumption 3.3 and Theorem 3.13]. □

We show that our algorithm satisfies the assumptions in Theorem 3.3 if the magnitude of ϵ fulfills the following condition.

Assumption 3.4 *Let $c \in (0, 1)$ as in (24), β, κ_1 as in Lemma 2.1, λ_1 as in Assumption 2.2, $\bar{t}$ as in Lemma 3.1, $t_{\max}$ as in Algorithm 3.1. Assume that ϵ in Definition 3.2 is such that*

$$\epsilon > \frac{4\varepsilon_f}{(1-M)^{-\gamma} - 1} \tag{29}$$

where $M = \frac{c\beta\kappa_1^2\lambda_1\bar{t}}{(1+t_{\max})^2}$, for some $\gamma \in (0, 1)$ such that $(1-M)^{-\gamma} < 2$.

Note that $M \in (0, 1)$ due to the definition of $\bar{t}$ in Lemma 3.1 and the smaller M is, the larger is ϵ with respect to ε_f.

First, we provide a relation between z_k and z_{k+1} of the form specified in item (i), Theorem 3.3.

Lemma 3.5 *Suppose that Assumptions 2.2, 2.3, 2.5, 2.6 and 3.4 hold. Consider any realization of Algorithm 3.1. If the k-th iterate is true and successful, then*

$$z_{k+1} \geq z_k - \log\left(1 - c\beta\kappa_1^2\lambda_1 \frac{t_k}{(1+t_{\max})^2}\right) - \log\left(1 + \frac{4\varepsilon_f}{\epsilon}\right), \tag{30}$$

whenever $k < N_\epsilon$.

Proof By (17), $f(x_k) - f^* \leq \frac{1}{2\lambda_1}\|\nabla f(x_k)\|^2$. Using (22)

$$\|g_k\|^2 \geq \left(\frac{1}{1+t_k\eta_k}\right)^2 \|\nabla f(x_k)\|^2 \geq 2\lambda_1 \left(\frac{1}{1+t_k\eta_k}\right)^2 (f(x_k) - f^*). \tag{31}$$

Combining condition (18), (24) and Lemma 2.1 it holds

$$\begin{aligned} f(x_k) - f(x_{k+1}) + 2\varepsilon_f \geq \tilde{f}(x_k) - \tilde{f}(x_{k+1}) &\geq -ct_k g_k^T s_k - 2\varepsilon_f \\ &\geq ct_k\beta\kappa_1^2\|g_k\|^2 - 2\varepsilon_f, \end{aligned} \tag{32}$$

and thus, using (31),

$$f(x_k) - f(x_{k+1}) \pm f^* \geq 2ct_k\beta\kappa_1^2\lambda_1 \left(\frac{1}{1+t_k\eta_k}\right)^2 (f(x_k) - f^*) - 4\varepsilon_f.$$

Then it holds

$$f(x_{k+1}) - f^* \leq \left(1 - 2c\beta\kappa_1^2\lambda_1 \frac{t_k}{(1+t_k\eta_k)^2}\right)(f(x_k) - f^*) + 4\varepsilon_f.$$

We define $\Delta_k^f = f(x_k) - f^*$. Because of $f(x_k) - f^* > \epsilon$, we have

$$\begin{aligned} \Delta_{k+1}^f &\leq \left(1 - 2c\beta\kappa_1^2\lambda_1 \frac{t_k}{(1+t_k\eta_k)^2} + \frac{4\varepsilon_f}{\epsilon}\right)\Delta_k^f \\ &\leq \left(1 - c\beta\kappa_1^2\lambda_1 \frac{t_k}{(1+t_k\eta_k)^2} - \frac{4\varepsilon_f}{\epsilon} c\beta\kappa_1^2\lambda_1 \frac{t_k}{(1+t_k\eta_k)^2} + \frac{4\varepsilon_f}{\epsilon}\right)\Delta_k^f \\ &= \left(1 - c\beta\kappa_1^2\lambda_1 \frac{t_k}{(1+t_k\eta_k)^2}\right)\left(1 + \frac{4\varepsilon_f}{\epsilon}\right)\Delta_k^f \\ &\leq \left(1 - c\beta\kappa_1^2\lambda_1 \frac{t_k}{(1+t_{\max})^2}\right)\left(1 + \frac{4\varepsilon_f}{\epsilon}\right)\Delta_k^f \end{aligned}$$

where the second inequalities holds thanks to Assumption 3.4, because $4\varepsilon_f < \epsilon$ and the last one holds since $t_k \leq t_{\max}$ and $\eta_k < 1$.

Notice that since $\left(1+\frac{4\varepsilon_f}{\epsilon}\right) > 0$, $\Delta_k^f > 0$ and $\Delta_{k+1}^f \geq 0$, it holds $\left(1-c\beta\kappa_1^2\lambda_1\frac{t_k}{(1+t_{\max})^2}\right) \geq 0$. Now, taking the inverse and then the log of both sides, adding $\log \Delta_0^f$, we have

$$\log\left(\frac{\Delta_0^f}{\Delta_{k+1}^f}\right) \geq \log\left(\frac{\Delta_0^f}{\Delta_k^f}\right) - \log\left(1-c\beta\kappa_1^2\lambda_1\frac{t_k}{(1+t_{\max})^2}\right) - \log\left(1+\frac{4\varepsilon_f}{\epsilon}\right),$$

which completes the proof.

□

The next lemma analyzes item (ii) of Theorem 3.3

Lemma 3.6 *Suppose that Assumptions 2.2, 2.3, 2.5 and 2.6 hold. Consider any realization of Algorithm 3.1. For every iteration that is false and successful, we have*

$$z_{k+1} \geq z_k - \log\left(1+\frac{4\varepsilon_f}{\epsilon}\right).$$

Moreover $z_{k+1} = z_k$ for any unsuccessful iteration.

Proof For every false and successful iteration, using (18), (24) and (4) we have

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + ct_k s_k^\top g_k + 4\varepsilon_f \\ &\leq f(x_k) + 4\varepsilon_f, \end{aligned}$$

thus, because of $f(x_k) - f^* > \epsilon$,

$$\begin{aligned} f(x_{k+1}) - f^* &\leq f(x_k) - f^* + 4\varepsilon_f \\ &\leq \left(1+\frac{4\varepsilon_f}{\epsilon}\right)(f(x_k) - f^*). \end{aligned}$$

So it holds $\Delta_{k+1}^f \leq \left(1+\frac{4\varepsilon_f}{\epsilon}\right)\Delta_k^f$. Now taking the inverse and then the log of both sides, adding $\log \Delta_0^f$ we have

$$\log\left(\frac{\Delta_0^f}{\Delta_{k+1}^f}\right) \geq \log\left(\frac{\Delta_0^f}{\Delta_k^f}\right) - \log\left(1+\frac{4\varepsilon_f}{\epsilon}\right)$$

which completes the first part of the proof. Finally, for any unsuccessful iteration $z_{k+1} = z_k$ follows by Step 3 of Algorithm 3.1 that provides $x_{k+1} = x_k$ and hence $f(x_{k+1}) = f(x_k)$. □

We can now summarize our results. First, note that $\left(1 - c\beta\kappa_1^2\lambda_1 \frac{t_k}{(1+t_{\max})^2}\right) \geq 0$ for all $t_k \in [0, t_{\max}]$, due to (6) and (8). Second, let

$$h(t) = -\log\left(1 - c\beta\kappa_1^2\lambda_1 \frac{t}{(1+t_{\max})^2}\right) \quad \text{and} \quad r(\varepsilon_f) = \log\left(1 + \frac{4\varepsilon_f}{\epsilon}\right). \tag{33}$$

It is easy to see that $h(t)$ is monotone and non increasing if $t \in [0, t_{\max}]$.

Combining Lemmas 3.1, 3.5 and 3.6 , we have that for any realization of Algorithm 3.1 and $k < N_\epsilon$ with ϵ as in Assumption 3.4:

(i) (Lemma 3.5) If iteration k is true and successful, then $z_{k+1} \geq z_k + h(t_k) - r(\varepsilon_f)$.
(ii) (Lemma 3.1) If $t_k \leq \bar{t}$ and iteration k is true then iteration k is also successful, which implies $t_{k+1} = \tau^{-1} t_k$.
(iii) (Lemma 3.6) $z_{k+1} \geq z_k - r(\varepsilon_f)$ for all successful iterations k and $z_{k+1} = z_k$ for all unsuccessful iteration k.
(iv) (Assumption 3.4) The ratio $r(\varepsilon_f)/h(\bar{t})$ is bounded from above by some $\gamma \in (0, 1)$.

Hence, we can use Theorem 3.3 and get the following boun on $\mathbb{E}[N_\epsilon]$,

$$\mathbb{E}[N_\epsilon] \leq \frac{2(1-\delta_g)}{(1-2\delta_g)^2 - \gamma}\left[2\log_{1/(1-M)}\left(\frac{f(x_0) - f^*}{\epsilon}\right) + (1-\gamma)\log_\tau \frac{\bar{t}}{t_0}\right]$$

with M given in Assumption 3.4. This result is valid under Assumption 3.4, namely for sufficiently large values of ϵ. The fact that ϵ cannot be arbitrarily small is consistent with the presence of noise ε_f in f-evaluations. Trivially, if $\varepsilon_f = 0$ then the optimality gap $f(x_k) - f^*$ can be made arbitrarily small.

4 Decreasing noise on *f*

In this section we present an Inexact Newton algorithm suitable to the case where f-evaluations can be performed with adaptive accuracy. Letting $c \in (0, \frac{1}{2})$, we use the linesearch condition

$$\tilde{f}(x_k + t_k s_k) \leq \tilde{f}(x_k) + c t_k s_k^T g_k, \tag{34}$$

where $\tilde{f}(x_k)$ and $\tilde{f}(x_k + t_k s_k)$ satisfy Assumption 2.4 with $\theta < \frac{c}{2}$. In fact, (34) has the form of the classical Armijo condition but the true f is replaced by the approximation $\tilde{f}$.

The resulting algorithm is given below.

Algorithm 4.1: k**-th iteration of Algorithm**

Given x_k, $g_k \in \mathbb{R}^n$, $H_k \in \mathbb{R}^{n\times n}$, $c \in (0, \frac{1}{2})$, $\theta \in (0, \frac{c}{2})$, $\bar{\eta} \in (0, 1)$, $\tau \in (0, 1)$, $t_{\max} > 0$, $t_k \in (0, t_{\max}]$.

Step 1. Choose $\eta_k \in (0, \bar{\eta})$.
Step 2. Starting from the null vector, use CG to compute an approximate solution s_k of $H_k s = -g_k$ satisfying (2).
Step 3. Compute $\tilde{f}(x_k)$ and $\tilde{f}(x_k + t_k s_k)$ satisfying (19).
Step 4. If t_k satisfies condition (34) (successful step) then
$x_{k+1} = x_k + t_k s_k$, $t_{k+1} = \min\{t_{\max}, \tau^{-1} t_k\}$, $k = k + 1$.
else $x_{k+1} = x_k$, $t_{k+1} = \tau t_k$, $k = k + 1$.

The following Lemma shows that a successful iteration is guaranteed whenever it is true and t_k is sufficiently small.

Lemma 4.1 *Suppose that Assumptions 2.2, 2.4 with $\theta < \frac{c}{2}$, 2.5 and 2.6 hold. Suppose that iteration k is true and consider any realization of Algorithm 4.1. Then the iteration is successful whenever $t_k \le \bar{t} = \frac{2\kappa_1\beta}{\kappa_1\lambda_n+2}(1 - c - 2\theta)$.*

Proof Using the same arguments as in Lemma 3.1, using (19) and (34), rather than (18) and (24) we obtain

$$\tilde{f}(x_k + t_k s_k) \le \tilde{f}(x_k) - 2\theta t_k s_k^T g_k + \frac{\lambda_n}{2} t_k^2 \|s_k\|^2 + t_k^2 \frac{\eta_k}{\kappa_1} \|s_k\|^2 + t_k g_k^T s_k.$$

The linesearch condition (34) is clearly enforced whenever

$$\tilde{f}(x_k) - 2\theta t_k s_k^T g_k + \frac{\lambda_n}{2} t_k^2 \|s_k\|^2 + t_k^2 \frac{\eta_k}{\kappa_1} \|s_k\|^2 + t_k g_k^T s_k \le \tilde{f}(x_k) + c t_k s_k^T g_k$$

which gives

$$t_k \|s_k\|^2 \left(\frac{\lambda_n}{2} + \frac{\eta_k}{\kappa_1} \right) \le -(1 - c - 2\theta) g_k^T s_k.$$

Note that $1 - c - 2\theta > 0$ by $c \in (0, 1/2)$ and $\theta \in (0, \frac{c}{2})$. Using (4) we have $-(1 - c - 2\theta) g_k^T s_k \ge (1 - c - 2\theta)\beta \|s_k\|^2$. Then, since $\eta_k < \bar{\eta} < 1$, (34) holds if

$$t_k \|s_k\|^2 \left(\frac{\lambda_n}{2} + \frac{1}{\kappa_1} \right) \le (1 - c - 2\theta)\beta \|s_k\|^2,$$

and this yields the thesis. □

4.1 Complexity analysis of the stochastic process

The behaviour of the method is studied analyzing the hitting time N_ϵ in Definition 3.2. In particular, we first show the following two results on the realization z_k of the variable Z_k in (26).

Lemma 4.2 *Suppose that Assumptions 2.2, 2.4 with $\theta < \frac{c}{2}$, 2.5 and 2.6 hold. If the k-th iterate of Algorithm 4.1 is true and successful, for any realization of the Algorithm 4.1 we have*

$$z_{k+1} \geq z_k - \log\left(1 - 2\beta\kappa_1^2\lambda_1 (c - 2\theta)\, \frac{t_k}{(1 + t_{\max})^2}\right), \tag{35}$$

whenever $k < N_\epsilon$.

Proof Using the same arguments as in Lemma 3.5, using (19) and (34), rather than (18) and (24) we obtain

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\geq -(c - 2\theta) t_k g_k^T s_k \\ &\geq t_k \beta \kappa_1^2 (c - 2\theta) \|g_k\|^2, \end{aligned}$$

where the second inequality comes from (4). Thus

$$f(x_{k+1}) - f^* \leq f(x_k) - f^* - t_k \beta \kappa_1^2 (c - 2\theta) \|g_k\|^2$$

and using (31) we get

$$f(x_{k+1}) - f^* \leq \left(1 - 2\beta\kappa_1^2\lambda_1 (c - 2\theta)\, \frac{t_k}{(1 + t_{\max})^2}\right) (f(x_k) - f^*).$$

Now proceeding as in Lemma 3.5 we have the thesis. □

Lemma 4.3 *Suppose that Assumptions 2.2, 2.4 with $\theta < \frac{c}{2}$, 2.5 and 2.6 hold. For any realization of Algorithm 4.1 we have*

$$z_{k+1} > z_k,$$

if the iteration k is false and successful,

$$z_{k+1} = z_k,$$

if the iteration k is unsuccessful.

Proof For every false and successful iteration, using (19) and (34),we have

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + c t_k s_k^T g_k - 2\theta t_k g_k^T s_k \\ &= f(x_k) + (c - 2\theta) t_k s_k^T g_k < f(x_k), \end{aligned}$$

and in case of unsuccessful iteration Step 4 of the algorithm provides $x_{k+1} = x_k$. Then, due to the definition of Z_k in (26) the thesis follows.

□

Now we can state the main result on the expected value for the hitting time.

Theorem 4.4 *Suppose that Assumptions 2.2, 2.4 with $\theta < \frac{c}{2}$, 2.5 and 2.6 hold and let $\bar{t}$ given in Lemma 4.1. Then under the condition that the probability δ_g in Assumption 2.5 is such that $\delta_g < \frac{1}{2}$, the stopping time N_ϵ is bounded in expectation as follows*

$$\mathbb{E}[N_\epsilon] \leq \frac{2(1-\delta_g)}{(1-2\delta_g)^2}\left[2\log_{1/(1-M)}\left(\frac{f(x_0)-f^*}{\epsilon}\right)+\log_\tau \frac{\bar{t}}{t_0}\right]$$

with $M = \frac{2(c-2\theta)\beta\kappa_1^2\lambda_1\bar{t}}{(1+t_{\max})^2}$.

Proof Let

$$h(t) = -\log\left(1-2\beta\kappa_1^2\lambda_1\,(c-2\theta)\,\frac{t}{(1+t_{\max})^2}\right), \tag{36}$$

and note that $h(t)$ is non decreasing for $t \in [0, t_{\max}]$ and that $h(t) > 0$ for $t \in [0, t_{\max}]$. For any realization z_k of Z_k in (26) of Algorithm 4.1 the following hold for all $k < N_\epsilon$:

(i) If iteration k is true and successful, then $z_{k+1} \geq z_k + h(t_k)$ by Lemma 4.2.
(ii) If $t_k \leq \bar{t}$ and iteration k is true then iteration k is also successful, which implies $t_{k+1} = \tau^{-1}t_k$ by Lemma 4.1.
(iii) $z_{k+1} \geq z_k$ for all successful iterations k ($z_{k+1} = z_k$ for all unsuccessful iteration k), by Lemma 4.3.

Moreover, our stochastic process $\{\mathcal{T}_k, Z_k\}$ obeys the expressions below. By Lemma 4.1 and the definition of Algorithm 4.1 the update of the random variable $\mathcal{T}_k$ such that $t_k = \mathcal{T}_k(\omega_k)$ is

$$\mathcal{T}_{k+1} = \begin{cases} \tau^{-1}\mathcal{T}_k & \text{if } I_k = 1,\ \mathcal{T}_k \leq \bar{t} \ \text{(i.e., successful iteration)} \\ \tau^{-1}\mathcal{T}_k & \text{if the iteration is successful, } I_k = 0,\ \mathcal{T}_k \leq \bar{t} \\ \tau\,\mathcal{T}_k & \text{if the iteration is unsuccessful, } I_k = 0,\ \mathcal{T}_k \leq \bar{t} \\ \tau^{-1}\mathcal{T}_k & \text{if the iteration is successful, } \mathcal{T}_k > \bar{t} \\ \tau\,\mathcal{T}_k & \text{if the iteration is unsuccessful, } \mathcal{T}_k > \bar{t}, \end{cases}$$

where the event I_k is defined in 20. By Lemmas 4.1, 4.2 and 4.3 the random variable Z_k obeys the expression

$$Z_{k+1} \geq \begin{cases} Z_k + h(\mathcal{T}_k) & \text{if the iteration is successful and } I_k = 1 \\ Z_k & \text{if the iteration is successful and } I_k = 0 \\ Z_k & \text{if the iteration is unsuccessful} \end{cases}$$

Then Lemma 2.2–Lemma 2.7 and Theorem 2.1 in [16] hold which gives the thesis. □

4.2 Local convergence

We conclude our study analyzing the local behaviour of the Newton-CG method employing gradient estimates $(1-\delta_g)$-probabilistically sufficiently accurate, i.e. satisfying Assumption 2.5 and Hessian estimates satisfying the following assumption.

Assumption 4.5 *The Hessian of the objective function f is Lipschitz-continuous with constant $L_H > 0$,*

$$\|\nabla^2 f(x) - \nabla^2 f(y)\| \leq L_H \|x - y\|, \quad \forall x, y \in \mathbb{R}^n. \tag{37}$$

Given a constant $C > 0$, the Hessian estimator is $(1-\delta_H)$-probabilistically sufficiently accurate in the sense that the indicator variable

$$J_k = \mathbb{1}\{\|\mathcal{H}_k - \nabla^2 f(X_k)\| \leq C\eta_k\}$$

satisfies the submartingale condition

$$P(J_k = 1 | \mathcal{F}_{k-1}) \geq 1 - \delta_H, \quad \delta_H \in (0, 1). \tag{38}$$

We let $t_{\max} = 1$, so that the maximum step-size gives the full CG step s_k.

The following lemma shows that if the full CG step s_k is accepted then the error linearly decreases with a certain probability. Further, the same occurrence over ℓ successive iterations is analyzed.

Lemma 4.6 *Suppose that Assumptions 2.2, 2.4 with $\theta < \frac{c}{2}$, 2.5, 2.6 and 4.5 hold. Let $x_{\bar{k}}$ be a realization of Algorithm 4.1 with $t_{\bar{k}} = 1$. Assume that the iteration is successful and $\|x_{\bar{k}} - x^*\|$ and $\eta_{\bar{k}}$ are sufficiently small so that $\frac{1}{\lambda_1}\left[\frac{L_H}{2}\|x_{\bar{k}} - x^*\| + C\eta_{\bar{k}} + \frac{2\lambda_n \eta_{\bar{k}}}{1-\bar{\eta}}\right] < \tilde{C} < 1$. Then, at least with probability $p = (1-\delta_g)(1-\delta_H)$, it holds*

$$\|x_{\bar{k}+1} - x^*\| < \tilde{C}\|x_{\bar{k}} - x^*\|.$$

If $\{\eta_k\}$ is a non-increasing sequence and the iterations $\bar{k}, \ldots, \bar{k}+\ell-1$ are successful with $t_k = 1$ for $k = \bar{k}, \ldots, \bar{k}+\ell-1$, then it holds $\|x_{k+1} - x^*\| < \|x_k - x^*\|$ for $k = \bar{k}, \ldots, \bar{k}+\ell-1$, at least with probability p^l.

Proof

$$\begin{aligned}
\|x_{k+1} - x^*\| &= \|x_k + s_k - x^*\| \\
&= \|x_k - H_k^{-1} g_k + H_k^{-1} r_k - x^*\| \\
&= \|H_k^{-1}[\nabla^2 f(x_k)(x_k - x^*) - \nabla^2 f(x_k)(x_k - x^*) \\
&\quad + H_k(x_k - x^*) - g_k \pm \nabla f(x_k) + r_k]\| \\
&\leq \|H_k^{-1}\|(\|\nabla^2 f(x_k)(x_k - x^*) - \nabla f(x_k)\|
\end{aligned}$$

$$
\begin{aligned}
&+\|(\nabla^2 f(x_k) - H_k)(x_k - x^*)\| + \|g_k - \nabla f(x_k)\| + \|r_k\|) \\
&\leq \frac{1}{\lambda_1}\Big(\|\nabla^2 f(x_k)(x_k - x^*) - \nabla f(x_k)\| \\
&+\|(\nabla^2 f(x_k) - H_k)(x_k - x^*)\| + \|g_k - \nabla f(x_k)\| + \|r_k\|\Big).
\end{aligned}
$$

Thanks to (37) it holds

$$
\begin{aligned}
&\|\nabla^2 f(x_k)(x_k - x^*) - \nabla f(x_k)\| \\
&= \left\| \int_0^1 [\nabla^2 f(x_k) - \nabla^2 f(x^* + \zeta(x_k - x^*))](x_k - x^*) d\zeta \right\| \\
&\leq \int_0^1 \left\| \nabla^2 f(x_k) - \nabla^2 f(x_k - (1-\zeta)(x_k - x^*)) \right\| d\zeta \ \|x_k - x^*\| \\
&\leq \int_0^1 L_H (1-\zeta) d\zeta \ \|x_k - x^*\|^2 = \frac{L_H}{2}\|x_k - x^*\|^2.
\end{aligned}
$$

Let us assume that both the events I_k and J_k are true. Then, $\|g_k - \nabla f(x_k)\| \leq \eta_k \|g_k\|$,

$$
\begin{aligned}
\|(\nabla^2 f(x_k) - H_k)(x_k - x^*)\| &\leq \|\nabla^2 f(x_k) - H_k\| \|x_k - x^*\| \\
&\leq C\eta_k \|x_k - x^*\|,
\end{aligned}
$$

and by (2)

$$
\|g_k - \nabla f(x_k)\| + \|r_k\| \leq 2\eta_k \|g_k\|.
$$

Moreover,

$$
\|g_k\| \leq \|g_k - \nabla f(x_k)\| + \|\nabla f(x_k)\| \leq \eta_k \|g_k\| + \|\nabla f(x_k)\|
$$

i.e, $\|g_k\| \leq \frac{1}{1-\eta_k}\|\nabla f(x_k)\|$. Then combining with (16) we have

$$
\|g_k\| \leq \frac{1}{1-\eta_k}\|\nabla f(x_k)\| \leq \lambda_n \frac{1}{1-\eta_k}\|x_k - x^*\| \leq \lambda_n \frac{1}{1-\bar{\eta}}\|x_k - x^*\|.
$$

Therefore

$$
\|g_k - \nabla f(x_k)\| + \|r_k\| \leq \frac{2\eta_k \lambda_n}{1-\bar{\eta}}\|x_k - x^*\|.
$$

Then, since $P(I_k \cap J_k) \geq p$ it follows

$$
\|x_{k+1} - x^*\| \leq \frac{1}{\lambda_1}\left[\frac{L_H}{2}\|x_k - x^*\|^2 + C\eta_k \|x_k - x^*\| + \frac{2\lambda_n \eta_k}{1-\bar{\eta}}\|x_k - x^*\|\right]
$$

at least with probability p.

Therefore, since at iteration $\bar{k}$, $\frac{1}{\lambda_1}\left[\frac{L_H}{2}\|x_{\bar{k}} - x^*\| + C\eta_{\bar{k}} + \frac{2\lambda_n \eta_{\bar{k}}}{1-\bar{\eta}}\right] < \tilde{C} < 1$ by assumption, it follows $\|x_{\bar{k}+1} - x^*\| < \tilde{C}\|x_{\bar{k}} - x^*\|$.

At iteration $\bar{k}+1$, $t_{\bar{k}+1} = 1$ and the iteration is successful by hypothesis. Then, we can repeat the previous arguments and the thesis follows. □

5 Finite sum case

In this section we consider the finite-sum minimization problem that arises in machine learning and data analysis:

$$\min_{x\in\mathbb{R}^n} f(x) = \frac{1}{N}\sum_{i=1}^{N} f_i(x). \tag{39}$$

The objective function f is the mean of N component functions $f_i : \mathbb{R}^n \to \mathbb{R}$ and for large values of N, the exact evaluation of the function and derivatives might be computationally expensive. We suppose that each f_i is strongly convex.

Following [1, 2, 16] f is evaluated exactly while the approximations g_k and H_k to the gradient and the Hessian respectively satisfy accuracy requirements in probability.

The evaluations of g_k and H_k can be made using subsampling, that means picking randomly and uniformly chosen subsets of indexes $\mathcal{N}_{g,k}$ and $\mathcal{N}_{H,k}$ from $\mathcal{N} = \{1, \ldots, N\}$ and define

$$g_k = \frac{1}{|\mathcal{N}_{g,k}|}\sum_{i\in\mathcal{N}_{g,k}} \nabla f_i(x_k), \quad \text{and} \quad H_k = \frac{1}{|\mathcal{N}_{H,k}|}\sum_{i\in\mathcal{N}_{H,k}} \nabla^2 f_i(x_k). \tag{40}$$

If g_k and H_k are required to be probabilistically sufficiently accurate as in Definition 2.5 and in Assumption 4.5 respectively, the sample sizes $|\mathcal{N}_{g,k}|$ and $|\mathcal{N}_{H,k}|$ can be determined by using the operator-Bernstein inequality introduced in [34]. As shown in [6], g_k and H_k are $(1-\delta_g)$ and $(1-\delta_H)$ -probabilistically sufficiently accurate if

$$|\mathcal{N}_{g,k}| \geq \min\left\{N, \frac{4\kappa_{f,g}(x_k)}{\gamma_{g,k}}\left(\frac{\kappa_{f,g}(x_k)}{\gamma_{g,k}} + \frac{1}{3}\right)\log\left(\frac{n+1}{\delta_g}\right)\right\}, \tag{41}$$

$$|\mathcal{N}_{H,k}| \geq \min\left\{N, \frac{4\kappa_{f,H}(x_k)}{C\eta_k}\left(\frac{\kappa_{f,H}(x_k)}{C\eta_k} + \frac{1}{3}\right)\log\left(\frac{2n}{\delta_H}\right)\right\}, \tag{42}$$

where $\gamma_{g,k}$ is an approximation of the required gradient accuracy, namely $\gamma_{g,k} \approx t_k\eta_k\|G_k\|$ and under the assumption that, for any $x \in \mathbb{R}^n$, there exist non-negative upper bounds $\kappa_{f,g}$ and $\kappa_{f,H}$ such that

$$\max_{i\in\{1,\ldots,N\}} \|\nabla f_i(x)\| \leq \kappa_{f,g}(x),$$

$$\max_{i\in\{1,\ldots,N\}} \|\nabla^2 f_i(x)\| \leq \kappa_{f,H}(x).$$

A practical version of the procedure is shown in Algorithm 5.1. Gradient approximation requires a loop since the accuracy requirement is implicit; such a strategy is Step 2 of the following algorithm.

Algorithm 5.1: k-th iteration of Algorithm

Given $x_k, \in \mathbb{R}^n, \ c, \bar{\eta}, \tau, \kappa_\gamma \in (0, 1), \gamma_0 > 0, t_{\max} > 0, t_k \in (0, t_{\max}], C > 0.$

Step 1. Choose $\eta_k \in (0, \bar{\eta})$.
Step 2. Gradient approximation. Set $i = 0$ and initialize $\gamma_k^{(i)} = \gamma_0$.

2.1 compute g_k with (40) and (41) with $\gamma_{g,k} = \gamma_k^{(i)}$;
2.2 if $\gamma_k^{(i)} \leq t_k \eta_k \|g_k\|$, go to Step 3;
else, set $\gamma_k^{(i+1)} = \kappa_\gamma \gamma_k^{(i)}$, $i = i + 1$, go to Step 2.1.

Step 3. Compute H_k with (40) and (42).
Step 4. Starting from the null vector, use CG to compute an approximate solution s_k of $H_k s = -g_k$ satisfying (2).
Step 5. If t_k satisfies

$$f(x_k + t_k s_k) \leq f(x_k) + c t_k s_k^T g_k. \tag{43}$$

then $x_{k+1} = x_k + t_k s_k, \quad t_{k+1} = \min\{t_{\max}, \tau^{-1} t_k\}, \quad k = k + 1$ (successful iteration)
else $x_{k+1} = x_k, \quad t_{k+1} = \tau t_k, \quad k = k + 1$ (unsuccessful iteration).

Inexact Newton methods for the finite-sum minimization problems are investigated also in [7, 12, 33]. In [7] it is analyzed a linesearch Newton-CG method where the objective function and the gradient are approximated by subsampling with increasing samplesizes determined by a prefixed rule. Random estimates of the Hessian with adaptive accuracy requirements as in Assumption 4.5 are employed and local convergence results in the mean square are given. In [12] the local convergence of Inexact Newton method is studied assuming to use prefixed choice of the sample size used to estimate by subsampling both the gradient and the Hessian. The paper [33] studies the global as well as local convergence behavior of linesearch Inexact Newton algorithms, where the objective function is exact and the Hessian and/or gradient are sub-sampled. A high probability analysis of the local convergence of the method is given, whereas we prove complexity results is expectation with noise in the objective function. Moreover, the estimators g_k and H_k are supposed to be $(1 - \delta_g)$ and $(1 - \delta_H)$ -probabilistic sufficiently accurate as in our approach but with different accuracy requirements. Predetermined and increasing accuracy requirements are used in [33] rather than the adaptive accuracy requirements in Assumption 2.5 and in Assumption 4.5.

6 Conclusion

In this paper we presented three Inexact Newton-CG methods with linesearch suitable for strongly convex functions with deterministic noise. Two type of noise, bounded noise and controllable noise on the objective function were considered. Regarding gradients, random approximations were allowed and their accuracy was supposed to be sufficiently high with a certain probability. The Hessians were possiby approximated by means of positive definite matrices.

We presented algorithms for the above two cases of noise on the objective function and analyzed the iteration complexity of the stochastic processes generated. In particular, we established a bound on the expected number of iterations that the algorithms take until the optimality gap reaches a desired accuracy for the first time. Successively, we studied the local behavior of the algorithm with controllable noise on the objective function and random approximations of the Hessian sufficiently accurate with a certain probability. Finally, the discussion was specialized to the case where f is a finite-sum of strongly convex function.

Acknowledgements INdAM-GNCS partially supported the first and third authors under Progetti di Ricerca 2021.

Funding Open access funding provided by Universitá degli Studi di Firenze within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors have not conflict of interest to declare

References

1. Bandeira, A.S., Scheinberg, K., Vicente, L.N.: Convergence of trust-region methods based on probabilistic models. SIAM Journal on Optimization **24**(3), 1238–1264 (2014)
2. Bellavia, S., Gurioli, G.: Stochastic analysis of an adaptive cubic regularization method under inexact gradient evaluations and dynamic Hessian accuracy. Optimization **71**, 227–261 (2022)
3. Bellavia, S., Gurioli, G., Morini, B.: Adaptive cubic regularization methods with dynamic inexact Hessian information and applications to finite-sum minimization. IMA Journal of Numerical Analysis **41**(1), 764–799 (2021)
4. Bellavia, S., Gurioli, G., Morini, B., Toint, Ph.L.: Adaptive regularization for nonconvex optimization using inexact function values and randomly perturbed derivatives. Journal of Complexity **68**, 101591 (2022)
5. Bellavia, S., Gurioli, G., Morini, B., Toint, Ph.L.: Trust-region algorithms: probabilistic complexity and intrinsic noise with applications to subsampling techniques, arXiv preprint arXiv:2112.06176, (2021)

6. Bellavia, S., Gurioli, G., Morini, B., Toint, Ph.L.: Adaptive Regularization Algorithms with Inexact Evaluations for Nonconvex Optimization. SIAM Journal on Optimization **29**(4), 2281–2915 (2019)
7. Bellavia, S., Krejic, N., Krklec Jerinkic, N.: Subsampled Inexact Newton methods for minimizing large sums of convex functions. IMA Journal of Numerical Analysis **40**, 2309–2341 (2020)
8. Bellavia, S., Krejić, N., Morini, B.: Inexact restoration with subsampled trust-region methods for finite-sum minimization. Computational Optimization and Applications **76**, 701–736 (2020)
9. Berahas, A.S., Bollapragada, R., Nocedal, J.: An Investigation of Newton-Sketch and Subsampled Newton Methods. Optimization Methods and Software **35**, 661–680 (2020)
10. Berahas, A. S., Cao, L., Scheinberg, K.: Global convergence rate analysis of a generic line search algorithm with noise, SIAM Journal on Optimization, (2019)
11. Blanchet, J., Cartis, C., Menickelly, M., Scheinberg, K.: Convergence Rate Analysis of a Stochastic Trust Region Method via Submartingales. INFORMS Journal on Optimization **1**, 92–119 (2019)
12. Bollapragada, R., Byrd, R., Nocedal, J.: Exact and Inexact Subsampled Newton Methods for Optimization, IMA Journal Numerical Analysis, (2018)
13. Byrd, R.H., Chin, G.M., Nocedal, J., Wu, Y.: Sample size selection in optimization methods for machine learning. Mathematical Programming **134**(1), 127–155 (2012)
14. Byrd, R.H., Chin, G.M., Neveitt, W., Nocedal, J.: On the Use of Stochastic Hessian Information in Optimization Methods for Machine Learning. SIAM Journal on Optimization **21**(3), 977–995 (2011)
15. Carter, R.G.: On the global convergence of trust-region algorithms using inexact gradient information. SIAM Journal of Numerical Analysis **28**, 251–265 (1991)
16. Cartis, C., Scheinberg, K.: Global convergence rate analysis of unconstrained optimization methods based on probabilistic model. Mathematical Programming **169**(2), 337–375 (2017)
17. Chen, R., Menickelly, M., Scheinberg, K.: Stochastic optimization using a trust-region method and random models. Mathematical Programming **169**(2), 447–487 (2018)
18. Dembo, R.S., Eisenstat, S.C., Steinhaug, T.: Inexact Newton method. SIAM Journal on Numerical Analysis **19**(2), 400–409 (1982)
19. di Serafino, D., Krejic, N., Krklec Jerinkic, N., Viola, M.: LSOS: Line-search Second-Order Stochastic optimization methods for nonconvex finite sums, arXiv:2007.15966v2, (2021)
20. Eisenstat, S.C., Walker, H.F.: Choosing the Forcing Terms in an Inexact Newton Method. SIAM Journal on Scientific Computing **17**(1), 16–32 (1996)
21. Fountoulakis, K., Gondzio, J.: A second order method for strongly convex ℓ_1- regularization problems. Mathematical Programming **156**, 189–219 (2016)
22. Franchini, G., Ruggiero, V., Zanni, L.: Ritz-like values in steplength selections for stochastic gradient methods. Soft Computing **24**, 17573–17588 (2020)
23. Franchini, G., Porta, F., Ruggiero, V., Trombini, I.: A line search based proximal stochastic gradient algorithm with dynamical variance reduction, Optimization On Line, http://www.optimization-online.org/DB_HTML/2022/02/8810.html, (2022)
24. Gratton, S., Toint, Ph.L.: A note on solving nonlinear optimization problems in variable precision. Computational Optimization and Applications **76**, 917–933 (2020)
25. Hestenes, M.R., Stiefel, E.: Methods of conjugate gradients for solving linear systems. Journal of Research of the National Bureau of Standards **49**(6), 409–436 (1952)
26. Liu, Y., Roosta, F.: Convergence of Newton-MR under inexact hessian information. SIAM Journal on Optimization **31**(1), 59–90 (2021)
27. Maggiar, A., Wachter, A., Dolinskaya, I.S., Staum, J.: A derivative-free trust-region algorithm for the optimization of functions smoothed via gaussian convolution using adaptive multiple importance sampling. SIAM Journal on Optimization **28**, 1478–1507 (2018)
28. More, J.J., Wild, S.M.: Estimating computational noise. SIAM Journal on Scientific Computing **33**, 1292–1314 (2011)
29. Nesterov, Y.: Introductory Lectures on Convex Optimization: A Basic Course, vol. 87. Springer Science and Media (2013)
30. Nesterov, Y., Spokoiny, V.: Random gradient-free minimization of convex functions. Foundations of Computational Mathematics **17**, 527–566 (2017)
31. Nocedal, J., Wright, S.J.: Numerical Optimization. Springer Series in Operations Research, Springer (1999)
32. Paquette, C., Scheinberg, K.: A Stochastic Line Search Method with Expected Complexity Analysis. SIAM Journal of Optimization **30**, 349–376 (2020)

33. Roosta-Khorasani, F., Mahoney, M.W.: Sub-Sampled Newton Methods. Mathematical Programming **174**, 293–326 (2019)
34. Tropp, J.A.: An Introduction to Matrix Concentration Inequalities. Foundations and Trends in Machine Learning **8**(1–2), 1–230 (2015)
35. Xu, P., Roosta-Khorasani, F., Mahoney, M.W.: Newton-Type Methods for Non-Convex Optimization Under Inexact Hessian Information. Mathematical Programming **184**, 35–70 (2020)