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PRICING PERISHABLE PRODUCTS WITH COMPOUND POISSON DEMANDS

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We consider the dynamic pricing problem of perishable products in a system with a constant production rate. Potential demands arrive according to a compound Poisson process, and are price-sensitive. We carry out the sample path analysis of the inventory process and by using level-crossing method, we derive its stationary distribution given a pricing function. Based on the distribution, we express the average profit function. By a stochastic comparison approach, we characterize the pricing strategy given different customers willingness-to-pay functions. Finally, we provide an approximation algorithm to calculate the optimal pricing function.

Keywords: revenue management; dynamic pricing; perishable inventory; level-crossing

1. INTRODUCTION

We consider a continuous perishable inventory system where demands arrive according to a compound Poisson process. Inventories are replenished at a constant rate and have a fixed lifetime. Demands are satisfied by the oldest available units. A typical example for such perishable inventory systems is a blood bank where new blood units are generated through Red Cross at some average rate and can be kept only for a fixed time, if not used, before obsolescence. Some chemical products produced by a continuous processing plant can be regarded as another example. For such a system, Graves [12] considers a situation where demands arrive in a Poisson process with a constant rate. He modeled the inventory process in such a system as a virtual waiting time process of an $M/M/1$ queue. Here, we generalize Graves' model in two dimensions: First, we consider an inventory-dependent pricing mechanism and allow the demand to depend on the price, which, in turn, depends on the inventory level. The stationary distribution for the inventory process is derived. Second, we explore the properties of the optimal dynamic pricing function via stochastic comparison and provide an approximate algorithm to compute the optimal pricing function. Using dynamic pricing mechanism to maximize profit on perishable products is often observed in our daily life. For example, in supermarkets, the price of bananas often varies.

The performance evaluation of such perishable inventory systems and their extensions with various input (replenishment) and demand processes have been studied in the existing literature, including [14,20,21,23,24]. Recently, Bar-Lev, David, and Stadje [3] conducted the performance evaluation for a system with a second source of replenishment.

The literature on ordering policies for perishable inventory systems is extensive. Typical work on discrete-time review policy include [10,18,19]. Work on the continuous review policy incudes [16,17,25,26,30].

Different from the work reviewed above, our focus here is the issue of dynamic pricing for a perishable inventory system with compound Poisson demand, where the demand batch size has an exponential distribution. There exist plenty of literature on dynamic pricing. Pioneered by Gallego and van Ryzin [11], dynamic pricing of limited inventories has been carried out by many researchers. See the recent reviews by Bitran and Caldentey [5] and Elmaghraby and Keskinocak [9]. Recently, Zhang and Cooper [32] studied dynamic pricing for multiple products; Elmaghraby, Gulcu, and Keskinocak [8] and Yossi and Pazgal [31] studied dynamic pricing in the presence of strategic consumers. Aydin and Ziya [2] investigated the dynamic pricing of promotional products under upselling. In these work, inventories are limited, whereas in our system, they are perishable and continuously replenished.

We mainly apply stochastic comparison to explore the structure property of optimal pricing strategy. We note that Cooper and Gupta [7] applied such methodology in airline revenue management.

We first consider the sample path of inventory level process and, by using the level-crossing method, derive its stationary distribution. Based on that, we conduct a stochastic comparison on different pricing functions and prove that the optimal pricing function is monotone decreasing in the inventory level when the elasticity of customers willingness-to-pay function is larger than 1.

Cooper and Gupta [7] provided an example showing that a stochastically larger demand need not generate a larger revenue, even with optimal pricing strategies. Here, we show that when the distribution of customers' willingness to pay is inelastic in a certain sense (the elasticity is smaller than 1), charging the same price on the group with stochastically larger willingness-to-pay values must result in a larger average revenue. Therefore, we obtained a sufficient condition for the stochastically larger demand to be more beneficial.

We also show that given a linear pricing function, a more homogeneous customer group will result in a larger probability for perishability. Therefore, dynamic pricing tends to be more beneficial when customers are more heterogeneous.

We then show that the inventory process in this case can be approximated with a birth–death process and the optimal pricing function can be obtained by solving Bellman optimality equations on that birth–death process, from which we derive the decreasing monotonicity of the optimal dynamic pricing function. Finally, we consider a phase-type distribution of demand batch size and obtain the closed-form expression for the inventory level distribution.

The remainder of this article is organized as follows. In Section 2 we describe the system and derive the stationary distribution and the performance measures. In Section 3 we analyze the structural properties of the optimal pricing function. In Section 4 we provide an approximate algorithm for dynamic pricing and presents the numerical results to demonstrate the properties of the optimal policy. We extend our discussion to phase-type demand sizes and conclude the article in Section 5.

2. THE MODEL

2.1. Assumptions

We make the following assumptions about production and demand processes.

Potential demands arrive according to a compound Poisson process with rate λ and the size of each demand has an exponential distribution $G(\cdot)$ with rate μ.

Products are produced at a constant rate 1 and have a finite lifetime m; that is, an unconsumed product will perish and be discarded in m time units following its production. Unsatisfied demands are backlogged and will be satisfied first by new production outputs.

Customers are charged a price according to a pricing function $p(\cdot)$, which is a function of current inventory position $I(t)$. Customers value the product differently and their *willingness-to-pay* for the product has a distribution function $H(\cdot)$ across the whole group. Hence, given a price p, only those customers with value larger than p will purchase the product. Consequently, the probability that an arriving customer is willing to pay is $\bar{H}(p)$, where $\bar{H} = 1 - H$. The effective demand arrival rate given $I(t) = i$ is thus

$$\lambda_i = \lambda\bar{H}(p(i)).$$

Figure 1 shows the sample path of inventory position process $I(t)$. It increases at a constant arrival rate of 1 and it jumps down by the demand batch size when there is an effective demand arrival. This process has an upper bound at m (the oldest inventory has an age just under m).

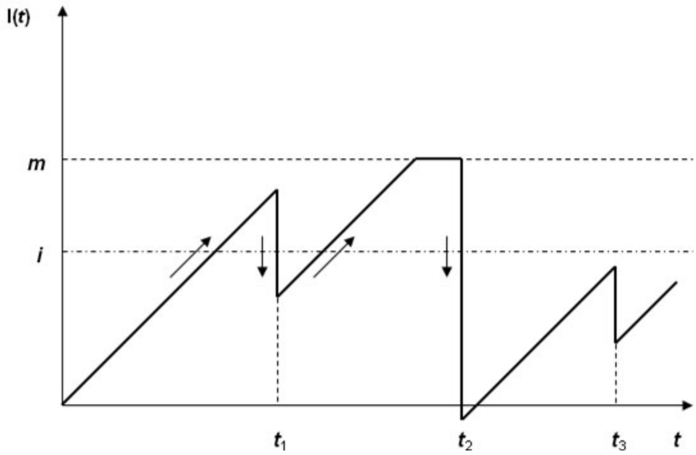

FIGURE 1. Sample path of $I(t)$.

2.2. Level Crossing and Stationary Distribution

Denote the stationary probability distribution function (p.d.f) of the inventory level i by $f(i), i \in (-\infty, m)$, and the probability mass at m by P_m. Using the level-crossing argument (e.g., Brill and Posner [6]), we can derive the integral equation for $f(i)$:

$$f(i) = \lambda_m P_m \bar{G}(m-i) + \int_i^m \lambda_w \bar{G}(w-i) f(w)\, dw, \tag{1}$$

where $f(w)$ equals the up-crossing rate of level w; the right-hand side is the down-crossing rate of level i with the first term being the down-crossing rate of level i starting from level m and the integration term representing the aggregate down-crossing rate starting from all levels in (i, m). Equation (1) follows from the fact that "up-crossing rate = down-crossing rate" (see, Brill and Posner [6]). The normalization condition is

$$P_m + \int_{-\infty}^m f(i)\, di = 1. \tag{2}$$

Because G is an exponential distribution with rate μ, (1) can be solved and the solution can be expressed as

$$f(i) = \lambda_m P_m e^{\int_i^m \lambda_w\, dw - \mu(m-i)}, \quad \forall i \in (-\infty, m) \tag{3}$$

in which P_m can be obtained from the normalization condition (2). Details of derivation can be found in Appendix A.

2.3. The Profit Function

After obtaining the stationary distribution for the inventory process I, the long-run average revenue function can then be expressed as $E[\lambda_I p(I)]/\mu$. Let the outdating cost per unit be c. Then the average outdating cost is cP_m.

Therefore, the average profit can be expressed as

$$\Pi = (\lambda/\mu)E[\bar{H}(p(I))p(I)] - cP_m.$$

In the above formula, $(\lambda/\mu)\bar{H}(p(I))p(I)$ is called revenue rate function. Ziya, Ayhan, and Foley [33] provided a discussion on the relationship of the concavity of this function and the property of the willingness-to-pay function. Our objective is to find an optimal pricing function $p(\cdot)$ that maximizes this average profit.

3. PROPERTIES OF THE OPTIMAL PRICING FUNCTION

Admittedly, obtaining the closed-form solution for the optimal pricing function is difficult since here the inventory state space is continuous and dynamic programming suffers from the curse of dimensionality. Here, we conduct stochastic comparisons of the average profits with different pricing functions to derive the structure properties of the optimal pricing function.

3.1. Monotonicity of Optimal Pricing Function

Consider two stationary inventory distributions I^1 and I^2, with distribution function F^1 and F^2, respectively. If $F^2(x) \geq F^1(x)$, for all x, I^1 is *stochastically larger* than I^2 (denoted $I^1 \succeq_{\mathrm{st}} I^2$).

We give a lemma on the first-order stochastic order of the inventory levels under two pricing functions and then provide results on the monotonicity of optimal pricing functions.

Consider two systems with pricing functions $p^1(\cdot)$ and $p^2(\cdot)$, respectively. We use superscripts 1 and 2 to indicate the performance measures in the corresponding two systems.

LEMMA 1: *If* $p^1(i) \geq p^2(i), \forall i \in (-\infty, m]$, $I^1 \succeq_{\mathrm{st}} I^2$ *and* $P^1_m \geq P^2_m$.

PROOF: Since $p^1(i) \geq p^2(i), \forall i \in (-\infty, m], \bar{H}(p^1(i)) \leq \bar{H}(p^2(i)), \forall i \in (-\infty, m]$ and, thus, $\lambda^1_i \leq \lambda^2_i, \forall i \in (-\infty, m]$.

From (3), we obtain

$$\frac{f^1(i)/P^1_m}{f^2(i)/P^2_m} = \frac{\lambda^1_m}{\lambda^2_m} e^{\int_i^m (\lambda^1_w - \lambda^2_w)\,dw}.$$

Additionally, from Appendix A, we know that $f(m) = \lambda_m P_m$. Hence, the above equation becomes

$$\frac{f^1(i)/f^1(m)}{f^2(i)/f^2(m)} = e^{\int_i^m (\lambda_w^1 - \lambda_w^2)\, dw}.$$

This ratio is increasing in i and reaches 1 at m. Clearly, $f^1(m) > f^2(m)$. Otherwise, if $f^1(m) \le f^2(m)$,

$$\frac{f^1(i)}{f^2(i)} < \frac{f^1(m)}{f^2(m)} \le 1, \quad \forall i < m;$$

and the normalization condition is violated.

Since the ratio $f^1(i)/f^2(i)$ is increasing in i and larger than 1 at m, it must cross 1 once from below; that is, there exists a number $k < m$ such that $f^1(i)/f^2(i) < 1$ when $i < k$; $=1$ when $i = k$ and >1 when $i > k$. (It is not possible that the ratio $f^1(i)/f^2(i)$ is always larger than 1, as the normalization condition will be violated.)

Therefore, f^1 singly crosses f^2 from below; that is, there exists a number $k < m$ such that $f^1(i) < f^2(i)$ when $i < k$, $=$ when $i = k$; and $>$ when $i > k$.

Since $p^1(m) \ge p^2(m)$, $\lambda_m^1 \le \lambda_m^2$. Therefore,

$$\frac{P_m^1}{P_m^2} = \frac{f^1(m)/\lambda_m^1}{f^2(m)/\lambda_m^2} > 1.$$

We can then conclude that $I^1 \succeq_{\mathrm{st}} I^2$, as the above inequalities on pdfs imply that $F^1(i) \le F^2(i)$, for all $i \le m$. ■

Intuitively, the lemma tells us that charging a higher price will result in more inventory in the system.

Before introducing the monotonicity conclusion of the optimal pricing function, we define the *generalized failure rate function* of H (denoted as $e(x)$)

$$e(x) = \frac{xh(x)}{\bar{H}(x)},$$

where h is the p.d.f. of H. See [15] for a detailed discussion of this function. It is also understood to be the elasticity of the willingness-to-pay function. It can be shown that the condition $e(x) \le (\ge) 1$ implies that the revenue rate function $\bar{H}(p)p$ is increasing (decreasing) in p.

We have the following proposition about the monotonicity of optimal pricing function.

PROPOSITION 2: *If $e(x) > 1$ for all $x \ge 0$, the optimal pricing function $p(\cdot)$ is (weakly) decreasing in the inventory level.*

PROOF: Suppose pricing function $p(\cdot)$ is optimal and strictly increasing in x over $[i_1, i_2]$. Since $e(x) > 1$, $\bar{H}(x)x$ is decreasing in x. Thus, $\bar{H}(p(i))p(i)$ is decreasing in

i on $[i_1, i_2]$. Consider another pricing function p^2, which is identical to p except that $p^2(i) = p(i_1), \forall i \in [i_1, i_2]$. Then $I \succeq_{\text{st}} I^2$ and $P_m \geq P_m^2$ by Lemma 1. We can derive that

$$E[(\lambda/\mu)\bar{H}(p(I))p(I)] - cP_m \leq E[(\lambda/\mu)\bar{H}(p(I^2))p(I^2)] - cP_m^2$$
$$< E[(\lambda/\mu)\bar{H}(p^2(I^2))p^2(I^2)] - cP_m^2.$$

where the first inequality follows from the property of $I \succeq_{\text{st}} I^2$; the second inequality follows from the decreasing property of the function $\bar{H}(x)x$. Thus p is not an optimal pricing function because the pricing function p^2 generates a higher profit than p. ■

Therefore, by a novel stochastic comparison approach, we proved the monotone decreasing property of the optimal pricing function when $e(x) > 1$. Unfortunately, we could not apply this approach to the case when $e(x) \leq 1$ as the first inequality can not be guaranteed in the proof of Proposition 2. We realize this limitation and hence provide an approximation algorithm for dynamic programming in Section 4, in which we show that the approximately optimal price is always decreasing in the inventory level.

3.2. A Necessary Condition

Here, we restrict our attention to a linear pricing function and obtain a necessary optimality condition. We first give the stochastic order between the inventory distributions under two pricing functions and then give the necessary optimality condition.

Consider two linear pricing functions p^1 and p^2.

LEMMA 3: *Suppose p^1 singly crosses p^2 from above; that is, there exists $\hat{i} > 0$ such that*

$$p^1(i) > p^2(i), \quad \forall i \leq \hat{i},$$

and the inequality is reversed for $i \geq \hat{i}$. Then $f^1(i)/f^2(i)$ is unimodal.

PROOF: From (3), we obtain that

$$\frac{f^1(i)/P_m^1}{f^2(i)/P_m^2} = \frac{\lambda_m^1}{\lambda_m^2} e^{\int_i^m (\lambda_w^1 - \lambda_w^2) dw}.$$

$\lambda_i^1 - \lambda_i^2$ is negative for $i \leq \hat{i}$ but positive for $i \geq \hat{i}$. So $\int_i^m (\lambda_w^1 - \lambda_w^2)\, dw$ is unimodal, and, therefore, so is $f^1(v)/f^2(v)$. ■

If p^1 singly crosses p^2 and the resulting two systems have the same expected inventory level (i.e., $E[I^1] = E[I^2]$), we call the crossing as *mean-preserving crossing*. We first provide some concepts on stochastic orders indicating the dispersion of the distribution function F. If $\int_v^m \bar{F}^1(i)\, di \leq \int_v^m \bar{F}^2(i)\, di$ for all $v \in (-\infty, m]$, I^1 is smaller than I^2 in the *increasing convex order* (denoted $I^1 \preceq_{\text{icx}} I^2$). If their expectations are equal, I^1 is smaller than I^2 in the *convex order* (denoted $I^1 \preceq_{cx} I^2$). This condition implies that I^1 has a smaller variance than I^2. Detailed discussions on these concepts can be found in [27].

We have the following lemma.

LEMMA 4: *If p^1 mean-preserving crosses p^2 from above, then $I^1 \preceq_{cx} I^2$ and $P_m^1 < P_m^2$.*

PROOF: From Lemma (3), we know that $f^1(i)/f^2(i)$ is unimodal; thus, f^1 crosses f^2 at least once and at most twice, first from below then from above. If crossing happens just once, we can derive that either $I^1 <_{st} I^2$ holds or $I^1 >_{st} I^2$ holds. In either way, $E[I^1] \neq E[I^2]$. Thus, the crossings must happen twice and, hence, $F^1(i)$ crosses $F^2(i)$ exactly once from below, which implies that $P_m^1 < P_m^2$. Since $E[I^1] = E[I^2]$, we conclude that $I^1 \preceq_{cx} I^2$. ■

We now give the necessary condition for a pricing function to be optimal.

PROPOSITION 5: *When $-xh'(x)/h(x) < 2$, $\bar{H}(p)p$ is strictly concave in p and, hence, strictly concave in i. Suppose the mode is $\hat{i}$. The optimal linear pricing function p must satisfy the condition that there is no other linear pricing function that can mean-preserving crosses p from above at $\hat{i}$.*

PROOF: Condition $-xh'(x)/h(x) < 2$ implies the strict concavity of the revenue rate function $\bar{H}(p)p$; see [33]. Since the price function is a linear function of inventory level, the revenue rate function $\bar{H}(p(i))p(i)$ is also strictly concave in inventory with the mode to be $\hat{i}$.

Now, suppose p^1 mean-preserving crosses p^2 from above at $\hat{i}$. Then

$$\bar{H}(p^1(i))p^1(i) > \bar{H}(p^2(i))p^2(i), \quad \forall\, i \neq \hat{i},$$

and the inequality becomes $=$ at point $\hat{i}$.

By Lemma 4, we can derive that

$$\begin{aligned} E[(\lambda/\mu)\bar{H}(p^1(I^1))p^1(I^1)] - cP_m^1 &\geq E[(\lambda/\mu)\bar{H}(p^1(I^2))p^1(I^2)] - cP_m^2 \\ &> E[(\lambda/\mu)\bar{H}(p^2(I^2))p^2(I^2)] - cP_m^2. \end{aligned}$$

The first $\geq$ follows from the concavity of $\bar{H}(p)p$ and $I^1 \preceq_{cx} I^2$.

Thus, the mean-preserving crossing increases the expected profit. ■

This proposition shows that the optimal linear pricing function must be the steepest, in the sense that no other steeper linear functions can mean-preserving cross it at $\hat{i}$. See Figure 2 for illustration.

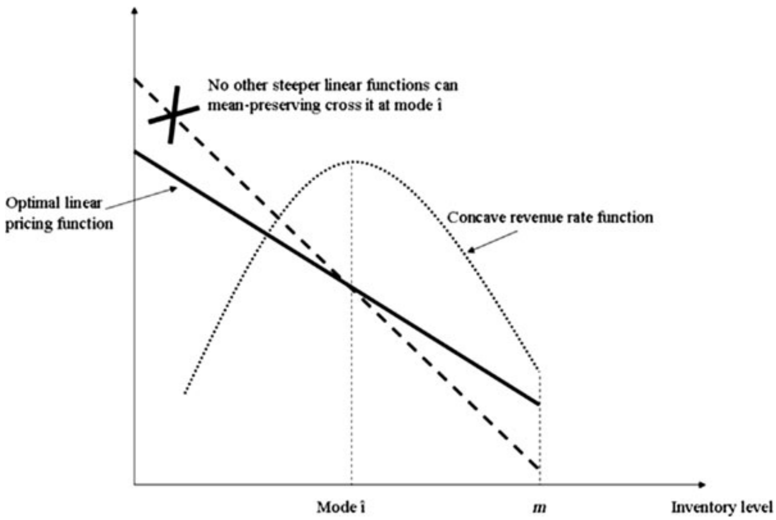

FIGURE 2. Mean-preserving crossing.

3.3. Sensitivity Analysis of H

In this subsection, we consider specific conditions on two willingness-to-pay distributions $H^k, k = 1, 2$. We consider two scenarios. In the first scenario, customers in one group have stochastically larger willingness-to-pay values than those in another group. In the second scenario, customers in one group are more concentrated on willingness-to-pay dimension than the other group. We are interested in the average profit performance with different groups of customers.

3.3.1. Impact of average willingness-to-pay value We first consider the situation that $H^1 \preceq_{\text{st}} H^2$. This means that system 2's customers have stochastically larger willingness-to-pay values than those in system 1. Cooper and Gupta [7] showed that a stochastically larger demand need not bring a larger profit for the firm. It is then interesting to know when it does. Here, we will provide a sufficient condition for a stochastically larger demand to be more beneficial for the firm.

We first give a conclusion on the stochastic comparison of the inventory levels with different groups of customers.

PROPOSITION 6: *If $H^1 \preceq_{\text{st}} H^2$, then $I^1 \succeq_{\text{st}} I^2$ and $P^1_m \geq P^2_m$.*

PROOF: The condition $H^1 \preceq_{\text{st}} H^2$ means $H^1(p) \geq H^2(p)$, for all p in $[0, \infty)$. Hence, $\lambda^1_w \leq \lambda^2_w$ for all $w \geq 0$.

From (3), we obtain that

$$\frac{f^1(i)/P_m^1}{f^2(i)/P_m^2} = e^{\int_i^m (\lambda_w^1 - \lambda_w^2)\, dw}.$$

This ratio is increasing in i. Considering the normalization condition and the above monotonicity, it can be shown that f^1 singly crosses f^2 from below and $P_m^1 \geq P_m^2$. Thus, we can derive $I^1 \succeq_{\mathrm{st}} I^2$. ■

Therefore, when system 2's customers have stochastically larger willingness-to-pay values than system 1's, system 2 has lower inventory than system 1 stochastically and the chance of perishability is smaller.

Next, we compare the average profit for the two systems.

PROPOSITION 7: *If $e^i(x) < 1$ for all $x \geq 0$ $(i = 1, 2)$, condition $H^1 \preceq_{\mathrm{st}} H^2$ implies that $\Pi^1 \leq \Pi^2$, given a decreasing pricing function.*

PROOF: By the definition of H^k, we have $\bar{H}^1(p)p \leq \bar{H}^2(p)p$. Additionally, the condition that $e(x) < 1$ for all $x \geq 0$ implies that $\bar{H}(p)p$ is increasing in p. Since p is a decreasing function, $\bar{H}(p(i))p(i)$ is decreasing in i. Hence,

$$\begin{aligned} E[(\lambda/\mu)\bar{H}^1(p(I^1))p(I^1)] - cP_m^1 &\leq [E(\lambda/\mu)\bar{H}^2(p(I^1))p(I^1)] - cP_m^1 \\ &\leq E[(\lambda/\mu)\bar{H}^2(p(I^2))p(I^2)] - cP_m^2, \end{aligned}$$

where the first inequality follows from the fact $\bar{H}^1(p)p \leq \bar{H}^2(p)p$; the second follows from the property of $I^1 \succeq_{\mathrm{st}} I^2$ obtained in Proposition 6. ■

Therefore, when customers' distribution on willingness to pay is inelastic in a certain sense (the elasticity is smaller than 1), charging the same price on the group with stochastically larger willingness-to-pay values must bring a larger average profit. Hence, we find a sufficient condition for a stochastically larger demand to be more beneficial for the firm.

3.3.2. Impact of dispersion of willingness-to-pay value. We now consider the condition that $H^1 \preceq_{\mathrm{icx}} H^2$. Intuitively, it means that system 1's customers are less heterogeneous than system 2's.

PROPOSITION 8: *If $H^1 \preceq_{\mathrm{icx}} H^2$, then $P_m^1 \geq P_m^2$, given a linear and decreasing pricing function.*

PROOF: $H^1 \preceq_{\text{icx}} H^2$ implies that $\int_l^n \bar{H}^1(x)\,dx \le \int_l^n \bar{H}^2(x)\,dx$ for all $l \le n$. Let $x = p(i)$. Then $dx = p(i)'\,di$. We can write the above as

$$\int_{p^{-1}(l)}^{p^{-1}(n)} \bar{H}^1(p(i))p(i)'\,di \le \int_{p^{-1}(l)}^{p^{-1}(n)} \bar{H}^2(p(i))p(i)'\,di.$$

Since $p(\cdot)$ is a decreasing and linear function, the above inequality can be rewritten as

$$\int_{p^{-1}(n)}^{p^{-1}(l)} \bar{H}^1(p(i))\,di \le \int_{p^{-1}(n)}^{p^{-1}(l)} \bar{H}^2(p(i))\,di.$$

Therefore, $\int_v^m \lambda_w^1\,dw \le \int_v^m \lambda_w^2\,dw$ for all $v \le m$. From the normalization condition, we can obtain $P_m^1 \ge P_m^2$. ■

Therefore, when customers are more homogeneous in their willingness-to-pay behavior, the inventory system will endure a higher perishability with a linear pricing function. We can go a little bit further to claim that when customers are more homogeneous in their willingness-to-pay behavior, a more aggressive pricing strategy should be applied to cut down the loss from perishing, for example, using 20% markdown rate instead of 5% when inventory level is high. Note that Proposition 8 only tells us the impact of customers' heterogeneity on the perishability, not the impact on the expected profit. The latter one, unfortunately, could not be obtained here with a stochastic comparison approach.

4. APPROXIMATE ALGORITHM FOR DYNAMIC PRICING

Here, we develop an approximate algorithm for dynamic pricing.

4.1. Converging Birth–Death Processes

We first show that we can construct from the inventory process a series of discrete-time birth–death processes, which are Markovian, that asymptotically converge to the original inventory process.

First we divide the state space of the inventory level into discrete pieces with equal interval length Δ. Define state $i_n = m - n\Delta$, $n = \{0, 1, 2, \ldots, \}$. The space for i_n is $\{m, m-\Delta, m-2\Delta, \ldots, \}$. Assume that the arrival rate in the interval $[m-(n+1)\Delta, m-n\Delta]$ equals the arrival rate at i_n (denoted as λ_n).

From the expression of $f(i)$, we obtain

$$\frac{f(i_{n+1})}{f(i_n)} = e^{\int_{i_{n+1}}^{i_n} \lambda_w\,dw - \mu\Delta}. \tag{4}$$

When Δ is sufficiently small, $\int_{i_{n+1}}^{i_n} \lambda_w\,dw \approx \lambda_{i_n}\Delta = \lambda\bar{H}(p(i_n))\Delta$. Therefore, (4) can be rewritten as

$$\frac{f(i_{n+1})}{f(i_n)} \approx e^{\lambda\bar{H}(p(i_n))\Delta - \mu\Delta}. \tag{5}$$

Define

$$\tilde{P}_n = \begin{cases} P_m, & n = 0 \\ f(i_{n-1})\Delta, & n \geq 1. \end{cases}$$

At level m, the level-crossing theory generates $P_m\lambda_m = f(m)$. We can rewrite it as

$$\tilde{P}_0\lambda_m\Delta = \tilde{P}_1. \quad \textbf{(6)}$$

Define

$$\tilde{\lambda}_n = \begin{cases} \lambda_m\Delta, & n = 0 \\ e^{\lambda\bar{H}(p(i_n))\Delta}, & n \geq 1 \end{cases} \quad \textbf{(7)}$$

and

$$\tilde{\mu}_n = \begin{cases} 1, & n = 1 \\ e^{\mu\Delta}, & n > 1. \end{cases} \quad \textbf{(8)}$$

Then from (4) and (6), we obtain

$$\tilde{P}_n\tilde{\lambda}_n = \tilde{P}_{n+1}\tilde{\mu}_{n+1}, \quad \forall n \geq 0.$$

The above equations are exactly the balance equations for a birth–death process with state-dependent arrival rates $\tilde{\lambda}_n$ and service rate $\tilde{\mu}_n$, and $\tilde{P}_n$ can be seen as the stable probability for state n. The system converges to the original system when Δ goes to zero. The average profit is then approximately

$$\Pi = \sum_{n=0}^{\infty} \tilde{P}_n\lambda_{i_n}p(i_n) - c\tilde{P}_0.$$

Therefore, the dynamic pricing problem is reduced to an occupancy-based pricing problem in a birth–death process that has Markovian property. The quality of approximation with this constructed birth–death process depends on how small Δ is.

4.2. Algorithm

To obtain this optimal solution, we can begin with standard optimality equation or Hamilton–Jacobi–Bellman equation, obtained by applying the uniformization method to a semi-Markov decision process with average-cost criterion (see, e.g., [4]).

We first provide the optimality equations for a truncated system with a large upper bound K on the queue length.

Suppose the arrival rate is ξ_m at the state m. As there is a one-to-one mapping between price and arrival rate as stated in (7), the price can be expressed as

$$p(\xi_m) = \begin{cases} \bar{H}^{-1}(\xi_m/(\lambda\Delta)), & m = 0 \\ \bar{H}^{-1}(\ln(\xi_m)/(\lambda\Delta)), & m \geq 1. \end{cases}$$

Define $R_m(\xi_m)$ as the average profit associated with state m, given the state-dependent arrival rate ξ_m. It can be expressed as follows:

$$R_m(\xi_m) = \begin{cases} \xi_m \bar{H}^{-1}(\xi_m/(\lambda\Delta))/\Delta - c, & m = 0 \\ \ln(\xi_m)\bar{H}^{-1}(\ln(\xi_m)/(\lambda\Delta))/\Delta, & m \geq 1. \end{cases}$$

The average profit maximization problem is to choose an arrival rate vector

$$\xi_m, m \in \{0, 1, \ldots, \}$$

that maximizes the average profit

$$\Pi = \sum_{m \in \{0,1,\ldots\}} R_m(\xi_m)\tilde{P}_m.$$

Using the uniformization technique, the average profit maximization problem can be transformed into solving the following optimality equations:

$$\begin{aligned} v_0 &= \max_{\xi} \left\{ \frac{R_0(\xi) - \gamma}{\lambda + \tilde{\mu}_1} + \frac{\xi}{\lambda + \tilde{\mu}_1} v_1 + \frac{\lambda + \tilde{\mu}_1 - \xi}{\lambda + \tilde{\mu}_1} v_0 \right\}, \\ v_m &= \max_{\xi} \left\{ \frac{R_m(\xi) - \gamma}{\lambda + \tilde{\mu}_m} + \frac{\xi}{\lambda + \tilde{\mu}_m} v_{m+1} + \frac{\tilde{\mu}_m}{\lambda + \tilde{\mu}_m} v_{m-1} \right. \\ &\quad \left. + \frac{\lambda + \tilde{\mu}_m - \xi - \tilde{\mu}_m}{\lambda + \tilde{\mu}_m} v_m \right\}, \quad \forall m \in \{1, \ldots, K-1\}, \\ v_K &= \max_{\xi} \left\{ \frac{R_K(\xi) - \gamma}{\lambda + \tilde{\mu}_K} + \frac{\tilde{\mu}_K}{\lambda + \tilde{\mu}_K} v_{K-1} + \frac{\lambda}{\lambda + \tilde{\mu}_K} v_K \right\}. \end{aligned}$$

In the above equations, γ is a guess of the maximal average profit and the vector of unknowns (v_m) is the *relative value function*. Take the first equation as example to explain the uniformization: Consider the rate for all events happening is $\lambda + \tilde{\mu}_1$. The chance for the next event to be the transition from state 0 to state 1 is $\xi/(\lambda + \tilde{\mu}_1)$; the chance for the system to stay at 0 is $1 - (\xi/(\lambda + \tilde{\mu}_1))$.

Define new notation y_m as

$$y_m = v_m - v_{m-1}, \quad m = 1, 2, \ldots, K.$$

Using the new notation, the optimality equations can be rewritten as

$$\gamma = \max_{\xi} \{R_0(\xi) + \xi y_1\}, \tag{9}$$

$$\gamma = \max_{\xi} \{R_m(\xi) + \xi y_{m+1} - \tilde{\mu}_m y_m\}, \quad \forall m \in \{1, 2, \ldots, K-1\}, \tag{10}$$

$$\gamma = \max_{\xi} \{R_K(\xi) - \tilde{\mu}_K y_K\}. \tag{11}$$

We can numerically solve these equations.

4.3. Structural Properties

PROPOSITION 9: *The optimal state-dependent arrival rate ξ_m^* is monotone decreasing in state m.*

PROOF: Following the approach as in [28], we can rewrite the optimality equation for state m ($m \in \{2, \ldots, K-1\}$) as

$$y_m = \max_{\xi} \left\{ \frac{R_m(\xi) - \gamma + \xi y_{m+1}}{\tilde{\mu}_m} \right\}. \tag{12}$$

If we can show that the function ξy_{m+1} is submodular in (ξ, m), then the optimal arrival rate ξ^* is decreasing in m (see [29]). The condition that ξy_{m+1} is submodular in (ξ, m) is again reduced to the proof of decreasing monotonicity of y_{m+1} in m, which can be proved by following the induction approach in the proof of Theorem 2 of [1]. ■

Therefore, the optimal price function must be increasing in state m in the approximation problem. This implies that it is decreasing in the inventory level in the original problem. This is consistent with Proposition 2.

Let us consider the following example: Consider H to be a gamma distribution with parameters (3, 1) and assume $\lambda = 1$, $\mu = 1$, $c = 2$, and $m = 3$. Our numerical result shows that the maximal average profit is 0.7141. Figure 3 shows the optimal state-dependent arrival rates and Figure 4 shows the optimal pricing function. We can see that the optimal pricing function is an inventory-level-dependent step function and so is the optimal arrival rate. Additionally, we observe that there is not much difference between the results with $\Delta = 0.01$ and $\Delta = 0.001$.

5. CONCLUDING REMARKS

In this article we consider a production–inventory system with inventory-level-dependent prices. The products are totally perishable after a fixed lifetime. By applying the level-crossing technique, we obtain the closed-form expression for the distribution of inventory and, hence, can express the expected profit for the company given a pricing function. We can then apply stochastic comparison technique to examine the structure properties of optimal pricing function.

The approach can be extended on several dimensions. First, we can consider the demand sizes having a phase-type distribution $G(\cdot)$ with representation $(\boldsymbol{\beta}, \mathbf{B})$ (see [22] for details on this distribution). It is well known that the phase-type distribution can approximate any other distribution. The complementary cumulative distribution function is

$$\bar{G}(i) = \boldsymbol{\beta}^T e^{\mathbf{B}i}\mathbf{1},$$

where **1** denotes a column vector of 1's.

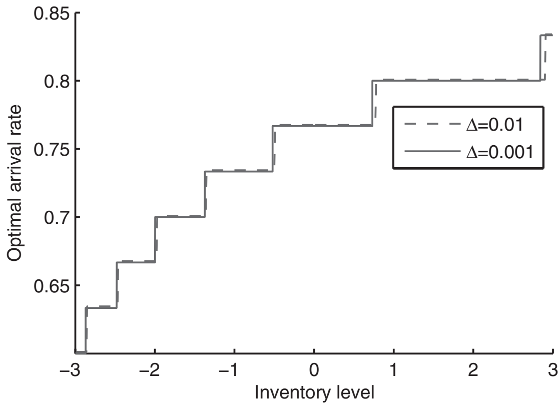

FIGURE 3. State-dependent arrival rates.

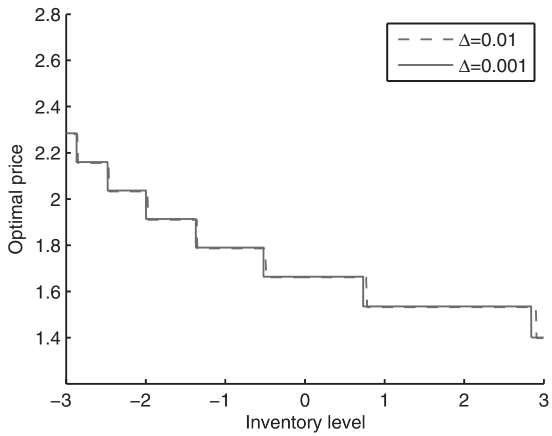

FIGURE 4. State-dependent prices.

It can be shown that the distribution for $f(i)$ can still be expressed explicitly, as in [13]. We omit the detail here and give the final result:

$$f(i) = \lambda_m P_m \boldsymbol{\beta}^T e^{(\int_i^m \lambda_w dw)\mathbf{1}\boldsymbol{\beta}^T + \mathbf{B}(m-i)}\mathbf{1}, \quad \forall i \in (-\infty, m), \tag{13}$$

where P_m can be obtained from the normalization condition (2). Dynamic pricing with this general inventory level distribution is worthy of further explorations.

Second, it is not hard to include inventory-related costs in the objective function. Denote the unit holding cost by h and unit penalty cost by s. The average holding cost can be expressed as

$$h\left(\int_0^m if(i)\,di + mP_m\right) = h\int_0^m \bar{F}(i)\,di.$$

The average penalty cost can be expressed as

$$s\left(-\int_{-\infty}^0 if(i)\,di\right) = s\int_{-\infty}^0 \bar{F}(i)\,di.$$

The approximate algorithm for the dynamic pricing still holds and one only needs to change the profit rate function into a more general form with the inventory-related costs embedded.

Third, we assume that the price affects the effective arrival rate only in the paper. In reality, it may also affect the demand size. This could be an interesting future research question.

Our model can also describe an production–inventory system using a base-stock policy where potential demands arrive in a Poisson process with inventory-level-dependent arrival rates and demand sizes have a phase-type distribution. The system stops producing when inventory hits a base-stock level. Conclusions in this article may be generalized to this production–inventory system.

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References

1. Ata, B. & Shneorson, S. (2006). Dynamic control of an $M/M/1$ service system with adjustable arrival and service rates. *Management Science* 52: 1778–1791.
2. Aydin, G. & Ziya, S. (2008). Pricing promotional products under upselling. *Manufacturing and Service Operations Management* 10: 360–376.
3. Bar-Lev, S., David, P., & Stadje, W. (2005). Control policies for inventory systems with perishable items: Outsourcing and urgency classes. *Probability in the Engineering and Informational Sciences* 19: 309–326.

4. Bertsekas, D. (1995). *Dynamic programming and optimal control*, Vol. 1. Belmont, MA: Athena Scientific.
5. Bitran, G. & Caldentey, R. (2003). An overview of pricing models for revenue management. *Manufacturing and Service Operations Management* 5: 203–229.
6. Brill, H. & Posner, M. (1977). Level crossings in point processes applied to queues: Single server case. *Operations Research* 25: 662–673.
7. Cooper, W. & Gupta, D. (2006). Stochastic comparisons in airline revenue management. *Manufacturing and Service Operations Management* 8: 221–234.
8. Elmaghraby, W., Gulcu, A., & Keskinocak, P. (2004). Optimal markdown mechanisms in the presence of rational customers with multi-unit demands. Working Paper, Georgia Insitute of Technology, School of Industrial and Systems Engineering.
9. Elmaghraby, W. & Keskinocak, P. (2003). Dynamic pricing in the presence of inventory considerations: Research overview, current practices and future directions. *Management Science* 49: 1287–1309.
10. Fries, B. (1975). Optimal ordering policy for a perishable commodity with fixed lifetime. *Operations Research* 23: 46–61.
11. Gallego, G. & van Ryzin, G. (1994). Optimal dynamic pricing of inventories with stochastic demand over finite horizions. *Management Science* 40: 999–1020.
12. Graves, S. (1982). The application of queueing theory to continuous perishable inventory systems. *Management Science* 28: 400–406.
13. Guo, P. & Zipkin, P. (2009). The effects of information on a queue with balking and phase-type service times. *Naval Research Logistics* 55: 406–411.
14. Kaspi, H. & Perry, D. (1983). Inventory system on perishable commodities. *Advances in Applied Probability* 15: 674–685.
15. Lariviere, M. (2006). A note on probability distributions with increasing generalized failure rates. *Operations Research* 54: 602–604.
16. Lian, Z. & Liu, L. (2001). An (s,S) continuous review perishable inventory system: Models and heuristics. *IIE Transactions* 33: 809–822.
17. Liu, L. & Lian, Z. (1999). An (s,S) continuous review model for perishable inventory. *Operations Research* 47: 150–158.
18. Nahmias, S. (1975). Optimal ordering policies for perishable inventory—II. *Operations Research* 23: 735–749.
19. Nahmias, S. (1982). Approximation techniques for several stochastic inventory models, *Computers and Operations Research* 30: 141–158.
20. Nahmias, S., Perry, D., & Stafje, W. (2004). Acturial valuation of perishable inventory systems. *Probability in the Engineering and Information Sciences* 18: 219–232.
21. Nahmias, S., Perry, D., & Stafje, W. (2004). Perishable inventory systems with variable input and demand rates. *Mathematical Methods of Operations Research* 60: 155–162.
22. Neuts, M. (1981). *Matrix-geometric solutions in stochastic models: An algorthmic approach*. New York: Dover.
23. Perry, D. (1999). Analysis of a sampling control scheme for a perishable inventory system. *Operations Research* 47: 966–973.
24. Perry, D. & Stadje, W. (2001). Disaters in Markovian inventory systems for perishable items. *Advances in Applied Probability* 33: 61–75.
25. Ravichandran, N. Stochastic analysis of a continuous review perishable inventory system with positive lead time and Poisson demand. (1995). *European Journal of Operational Research* 84: 444–457.
26. Schmidt, C. & Nahmias, S. (1985). Policies for perishable inventory. *Management Science* 31: 719–728.
27. Shaked, M. & Shanthikumar, J. (2006). *Stochastic orders*. New York: Springer-Verlag.
28. Stidham, S. & Weber, R. (1989). Monotonic and insensitive optimal policies for control of queues with undiscounted costs. *Operations Research* 87: 611–625.
29. Topkis, D. (1998). *Supermodularity and complementarity*. Princeton, NJ: Princeton University Press.
30. Weiss, H. (1980). Optimal ordering policies for continuous review perishable inventory models. *Operations Research* 28: 365–374.

31. Yossi, A. & Pazgal, A. (2008). Optimal pricing of seasonal products in the presence of forward-looking consumers. *Manufacturing and Service Operations Management*, 10: 339–359.
32. Zhang, D. & Cooper, W. (2005). Revenue management for parallel flights with customer choice behavior. *Operations Research* 53: 415–431.
33. Ziya, S., Ayhan, H. & Foley, R. (2004). Relationships among three assumptions in revenue management. *Operations Research* 52: 804–809.

APPENDIX A

Details of Derivation of Solution (3)

Equation (1) with an exponential G function can be expressed as

$$f(i) = \lambda_m P_m e^{-\mu(m-i)} + \int_i^m \lambda_w e^{-\mu(w-i)} f(w)\, dw.$$

Differentiating both sides of the above equation with respect to i yields the ordinary differential equation

$$f'(i) = \lambda_m \mu P_m e^{-\mu(m-i)} - \lambda_i f(i) + \mu \int_i^m \lambda_w e^{-\mu(w-i)} f(w)\, dw, \qquad i \le m$$

which can be rewritten as

$$f'(i) = \lambda_m \mu P_m e^{-(m-i)} - \lambda_i f(i) + \mu(f(i) - \lambda_m P_m e^{-\mu(m-i)}), \qquad i \le m$$

or

$$f'(i) = (\mu - \lambda_i) f(i), \qquad i \le m. \tag{A.1}$$

The solution can be expressed as

$$f(i) = A e^{\int_i^m \lambda_w\, dw - \mu(m-i)}, \quad i \le m.$$

Constant A is determined by letting $i \uparrow m$ in (1). That is,

$$A = f(m) = \lambda_m P_m.$$

Hence, the pdf of inventory level process I is

$$f(i) = \lambda_m P_m e^{\int_i^m \lambda_w dw - \mu(m-i)}, \quad \forall\, i \in (-\infty, m).$$