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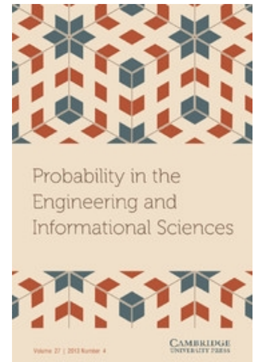

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Xiaoqiang Cai and Xian Zhou

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STOCHASTIC SCHEDULING WITH ASYMMETRIC EARLINESS AND TARDINESS PENALTIES UNDER RANDOM MACHINE BREAKDOWNS

Xiaoqiang Cai

Department of Systems Engineering & Engineering Management
The Chinese University of Hong Kong
Shatin, N.T., Hong Kong
E-mail: xqcai@se.cuhk.edu.hk

Xian Zhou

Department of Applied Mathematics
The Hong Kong Polytechnic University
Hung Hom, Kowloon, Hong Kong
E-mail: maxzhou@polyu.edu.hk

We study a stochastic scheduling problem of processing a set of jobs on a single machine. Each job has a random processing time P_i and a random due date D_i, which are independently and exponentially distributed. The machine is subject to stochastic breakdowns in either *preempt-resume* or *preempt-repeat* patterns, with the uptimes following an exponential distribution and the downtimes (repair times) following a general distribution. The problem is to determine an optimal sequence for the machine to process all jobs so as to minimize the expected total cost comprising asymmetric earliness and tardiness penalties, in the form of $\mathrm{E}[\sum \alpha_i \max\{0, D_i - C_i\} + \beta_i \max\{0, C_i - D_i\}]$. We find sufficient conditions for the optimal sequences to be V-shaped with respect to $\{\mathrm{E}(P_i)/\alpha_i\}$ and $\{\mathrm{E}(P_i)/\beta_i\}$, respectively, which cover previous results in the literature as special cases. We also find conditions under which optimal sequences can be derived analytically. An algorithm is provided that can compute the best V-shaped sequence.

1. INTRODUCTION

Scheduling to minimize both earliness and tardiness (E/T) penalties has emerged to be a main thrust of research in the scheduling field. Investigations of these

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problems have largely been motivated by the adoption of the just-in-time concept in the manufacturing industry, which aims to complete jobs exactly at their due dates, not earlier and not later. There are, however, many other applications that espouse the concept of earliness–tardiness minimization. One example is the harvest of agricultural products, which should be conducted around their ripe times (due dates). Another example is the production and delivery of fresh products such as meat or dairy products. Such a product should not be finished too early, in order to avoid its possible decay, and should not be completed too late, in order to meet the customer demand.

As indicated in Baker and Scudder [3] a basic model of E/T scheduling is to minimize

$$TC(\lambda) = \sum_{i=1}^{n} [\alpha_i \max\{0, D_i - C_i\} + \beta_i \max\{0, C_i - D_i\}], \tag{1.1}$$

where α_i and β_i are the unit costs due to earliness and tardiness of job i, respectively, D_i is the due date of job i, and C_i is its completion time under a given schedule λ. This model has been extensively studied in the deterministic scheduling area and widely used as a canonical model in the E/T scheduling literature. See, for example, Garey, Tarjan, and Wilfong [15], Hall and Posner [16], Kovalyov and Kubiak [18], and Sen, Sulek, and Dileepan [24].

In sharp contrast with the deterministic case, however, progress on the stochastic counterpart of the above canonical model is still limited. Forst [12] appears to be the first one to deal with the problem. Under the condition that all processing times are independent and identically distributed (i.i.d.), he identified some special situations in which optimal monotone sequences with respect to $\{\alpha_i\}$ or $\{\beta_i\}$ can be found. He concluded with the remark that "solving the general stochastic E/T problem ... appears to be a very mathematically formidable task." Soroush and Fredendall [25] considered the problem with symmetric E/T costs (i.e., $\alpha_i = \beta_i$ for all i) and Normally distributed processing times P_i. They obtained a deterministic equivalent, which becomes, however, a complicated, highly nonlinear function. Based on the central limit theorem, Cai and Zhou [7] addressed the situation where a job consists of a number elementary parts and thus the processing times of the jobs follow Normal distributions with variances proportional to means. They studied the problem with $\alpha_i = \alpha$ and $\beta_i = \beta$ for all i (but α might differ from β) and found a number of optimality properties and proposed a pseudopolynomial algorithm. Jia [17] considered the problem with $\alpha_i = \beta_i$ and $D_i = d$ for all i, where processing times P_i and the common due date d are exponentially distributed. He showed that the optimal sequence is Λ-shaped with respect to $\{w_i/\mathrm{E}(P_i)\}$. This result is extended to the case with machine breakdowns when the uptimes and downtimes are exponentially distributed. Qi, Yin, and Birge [23] studied a problem with linear E/T functions. Sufficient conditions to ensure an optimal SEPT (shortest expected processing time) sequence are developed. Some further results are also obtained when processing times follow exponential and normal distributions.

A related E/T scheduling model deals with quadratic penalty functions (i.e., to minimize $\sum[\alpha_i(\max\{0, D_i - C_i\})^2 + \beta_i(\max\{0, C_i - D_i\})^2]$). There has been considerable progress achieved on the stochastic variants of this quadratic E/T scheduling model; see, for example, Chakravarthy [10], Vani and Raghavachari [26], Mittenthal and Raghavachari [20], Cai and Tu [6], Luh, Chen, and Thakur [19], and Cai and Zhou [9]. To a certain extent, the quadratic model is relatively easy to handle in stochastic settings, because of the differentiability associated with the squared cost functions. The model with a piecewise linear objective function as in (1.1), on the other hand, does not enjoy such a convenience and becomes more complicated when stochastic variables such as random processing times and machine breakdowns are taken into account.

In this article we study the stochastic E/T scheduling problem under the following settings: (1) The objective is to minimize the expected total cost (1.1), where α_i and β_i might be different; (2) jobs have random processing times and due dates, which are independently and exponentially distributed; (3) the machine is subject to stochastic breakdowns, with the uptimes following an exponential distribution and the downtimes (repair times) following a general distribution. Note that in problems with regular objective functions,[1] Pinedo [21] and Pinedo and Rammouz [22] have considered jobs with exponential processing times. On the other hand, Cai and Zhou [8] have studied a problem to minimize the total E/T cost, where, for each job, the earliness cost is a general function, the tardiness cost is a fixed penalty, and the due date is a random number following an exponential distribution. As indicated by Cai and Zhou [8], an exponential due date models the arrival of an uncertain event, such as a transporter to deliver the finished jobs in a manufacturing system or a rainfall that is expected in a planting problem in agriculture. Frenk [13] has considered general settings on machine breakdowns, under the *preempt-resume* model (see below). He obtains a number of results on problems with regular objective functions.

We consider, in this article, both *preempt-resume* and *preempt-repeat* breakdown models (see, e.g., Birge, Frenk, Mittenthal, and Rinnoog Kan [4]). Under the preempt-resume model, it is assumed that if a machine breakdown occurs during the processing of a job, the work done on the job prior to the breakdown is not lost and the processing of the disrupted job can be continued from where it was interrupted. The preempt-repeat model, on the other hand, assumes that the work done prior to a breakdown is lost and the processing of the job will have to be redone again when the machine resumes operation. This is generally a much more difficult model (see Cai, Sun, and Zhou [5]), on which little has been found in the E/T scheduling literature. In our model considered in this article, however, the two breakdown models are actually equivalent (see Section 2.2).

The main contributions of our study in this article are as follows:

1. We derive several sufficient conditions under which the optimal sequence is V-shaped[2] with respect to $\{E(P_i)/\alpha_i\}$, or $\{E(P_i)/\beta_i\}$, or both. The V-shaped structure of optimal solutions is an important characteristic that exists in

some E/T scheduling problems (see, e.g., Eilon and Chowdhury [11] and Mittenthal and Raghavachari [20]), which makes it possible to develop efficient algorithms based on dynamic programming.
2. We derive a number of sufficient conditions under which optimal solutions can be constructed analytically.
3. We also present an algorithm that computes the best V-shaped sequence using dynamic programming.

The article is organized as follows. In Section 2 we describe the basic problem, the assumptions, and the breakdown models. The results on optimal V-shaped sequences are presented in Section 3. In Section 4 we identify several conditions under which optimal sequences can be constructed analytically. A dynamic programming algorithm to obtain the best V-shaped sequence is provided in Section 5. Finally, some concluding remarks are discussed in Section 6.

2. PROBLEM STATEMENT

Consider the problem where a set of n jobs are to be processed by a single machine. All jobs are ready for processing at time $t = 0$. The processing times P_i of job i, $i = 1,2,\ldots,n$, are random variables, which are independently and exponentially distributed with means $\mu_1,\ldots,\mu_n$, respectively. Each job i has a due date D_i, and $D_1,\ldots,D_n$ are independently and exponentially distributed with a common mean $1/\delta$, independent of $\{P_i\}$. Job preemption is not allowed; that is, if a job has been started, it has to be processed on the machine until it is completed.

The machine is subject to random breakdowns. The breakdown process of the machine is characterized by a sequence of finite-valued, positive random vectors $\{Y_k, Z_k\}_{k=1}^{\infty}$, where Y_k and Z_k represent the durations of the kth uptime and the kth downtime, respectively. If a machine breakdown occurs during the processing of a job, the work done on the job prior to the breakdown might follow either a *preempt-resume* model or a *preempt-repeat* model, depending on the nature and the processing requirements of the jobs. More details on the two breakdown models will be further elaborated in Sections 2.1 and 2.2. For the breakdown process of the machine, we make the following assumptions:

- The breakdown process is independent of the processing times $\{P_i\}$ and the due dates $\{D_i\}$.
- $\{Y_k, Z_k\}_{k=1}^{\infty}$ is a sequence of i.i.d. random vectors.
- The uptimes $\{Y_k\}$ of the machine are exponentially distributed with mean $1/\tau$, the downtimes $\{Z_k\}$ follow a general common distribution with mean ν, and the $\{Y_k\}$ are independent of $\{Z_k\}$.

The problem is to determine an optimal sequence to process all jobs, so as to minimize the expected total earliness and tardiness cost:

$$ETC(\lambda) = \mathrm{E}\left[\sum_{C_i<D_i} \alpha_i(D_i - C_i) + \sum_{C_i(\lambda)>D_i} \beta_i(C_i(\lambda) - D_i)\right]$$

$$= \sum_{i=1}^{n} \{\alpha_i \mathrm{E}[(D - C_i)I_{\{C_i<D\}}] + \beta_i \mathrm{E}[(C_i - D)I_{\{C_i>D\}}]\}, \quad \textbf{(2.1)}$$

where

- $\lambda = (i_1, i_2, \ldots, i_n)$ is a job sequence, which is a permutation of $\{1,2,\ldots,n\}$ that determines the order to process the n jobs, with $i_k = j$ if and only if job j is the kth to be processed;
- $C_i = C_i(\lambda)$ is the completion time of job i under sequence λ;
- α_i is the unit earliness cost to job i;
- β_i is the unit tardiness cost to job i;
- D is a representative of due dates $\{D_i\}$; in other words, an exponentially distributed random variable with mean $1/\delta$, independent of $\{P_i\}$;
- $\mathrm{E}[X]$ denotes the expectation of a random variable X;
- I_A is the indicator of an event A that takes the value 1 if A occurs and is zero otherwise.

We assume that the machine will start its first job at time 0, although this assumption is not necessary (see also remarks in Section 6).

The impact of machine breakdowns on job processing can follow the following two models.

2.1. The Preempt-Resume Model

Under this model, if a machine breakdown occurs during the processing of a job, the work done on the job prior to the breakdown is not lost and the processing of the disrupted job can be continued from the point where it was interrupted (cf. Birge et al. [4]).

Define a counting process $N(t) = \sup\{k \geq 0 : S_k \leq t\}$, where $S_k = Y_1 + Y_2 + \cdots + Y_k$ denotes the total uptime of the machine before its kth breakdown (with $S_0 = 0$). By the assumption that the $\{Y_k\}$ are exponentially distributed with mean $1/\tau$, $N(t)$ is a Poisson process with rate τ.

Let $\mathcal{B}_i(\lambda)$ denote the set of jobs sequenced no later than job i under λ. Then the completion time of job i under λ can be expressed as

$$C_i = R_i + \sum_{k=0}^{N(R_i-)} Z_k, \quad \textbf{(2.2)}$$

where

$$R_i = R_i(\lambda) = \sum_{k\in\mathcal{B}_i(\lambda)} P_k \quad \textbf{(2.3)}$$

represents the total uptime of the machine before job i is finished and $Z_0 \equiv 0$.

2.2. The Preempt-Repeat Model

In this model, if the machine breakdown occurs during the processing of a job, the work done on this job is lost and the processing will have to be done for this job all over again after the machine resumes its work (cf. Birge et al. [4]).

As in the previous literature (e.g., Frostig [14]), we assume that each time that a job is repeated, the processing time required is resampled independently according to its probability distribution. Then there is a sequence of independent processing times $\{P_{ik}\}_{k=0}^{\infty}$ to process job i, and each P_{ik} is a replicate of P_i, exponentially distributed with mean μ_i.

In general, the impact of preempt-repeat breakdowns can differ significantly from that of preempt-resume breakdowns. However, if the processing times are exponential, then due to the memoryless property of the exponential distribution, the remaining processing time at a breakdown has the same distribution as the original processing time, given that the job has not been completed. Thus, the two breakdown models become equivalent. As a result, we only need to consider the preempt-resume model for machine breakdowns, which will be assumed throughout the rest of the article.

3. V-SHAPED OPTIMAL SEQUENCES

We first derive an equivalent form of the objection function defined in (2.1), which is given in the following theorem.

THEOREM 1: *Under the preempt-resume model,*

$$ETC(\lambda) = \sum_{i=1}^{n} \beta_i (1 + \nu\tau) \sum_{k \in \mathcal{B}_i(\lambda)} \mu_k + \sum_{i=1}^{n} (\alpha_i + \beta_i) \frac{1}{\delta} \prod_{k \in \mathcal{B}_i(\lambda)} f_k - \frac{1}{\delta} \sum_{i=1}^{n} \beta_i, \quad \textbf{(3.1)}$$

where

$$f_k = \mathrm{E}[e^{-\eta P_k}] = \frac{1}{1 + \eta\mu_k}, \qquad k = 1, \ldots, n, \quad \textbf{(3.2)}$$

and

$$\eta = \delta + \tau \Pr(D \leq Z). \quad \textbf{(3.3)}$$

PROOF: By the independence between the processing times, the due dates, and the machine breakdowns, we have

$$\begin{aligned}
\mathrm{E}[(D-C_i)I_{\{C_i<D\}}|C_i=x] &= \mathrm{E}[(D-x)I_{\{x<D\}}] \\
&= \int_x^\infty (t-x)\delta e^{-\delta t}\,dt \\
&= \int_0^\infty y\delta e^{-\delta(y+x)}\,dy \\
&= e^{-\delta x}\frac{1}{\delta} \\
&= \frac{1}{\delta}\Pr(D>x).
\end{aligned}$$

Hence,

$$\mathrm{E}[(D-C_i)I_{\{C_i<D\}}] = \frac{1}{\delta}\mathrm{E}[e^{-\delta C_i}] = \frac{1}{\delta}\Pr(D>C_i). \tag{3.4}$$

Furthermore, as D is exponentially distributed, by (2.2) and (2.3) we have

$$\begin{aligned}
\Pr(D>C_i|R_i=x) &= \Pr\left(D>x+\sum_{k=0}^{N(x-)}Z_k\right) \\
&= \Pr(D>x)\Pr(D>Z_1+Z_2+\cdots+Z_{N(x-)}) \\
&= e^{-\delta x}\sum_{k=0}^{\infty}\Pr(D>Z_1+\cdots+Z_k)\Pr(N(x-)=k) \\
&= e^{-\delta x}\sum_{k=0}^{\infty}[\Pr(D>Z)]^k\frac{(\tau x)^k}{k!}e^{-\tau x} \\
&= e^{-\eta x}.
\end{aligned} \tag{3.5}$$

It follows that

$$\begin{aligned}
\Pr(D>C_i) &= \mathrm{E}[\Pr(D>C_i)|R_i] = \mathrm{E}[e^{-\eta R_i}] \\
&= \prod_{k\in\mathcal{B}_i(\lambda)}\mathrm{E}[e^{-\eta P_k}] \\
&= \prod_{k\in\mathcal{B}_i(\lambda)} f_k.
\end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.4), we obtain

$$\mathrm{E}[(D - C_i)I_{\{C_i<D\}}] = \frac{1}{\delta} \prod_{k\in\mathcal{B}_i(\lambda)} f_k. \tag{3.7}$$

Similarly, we can calculate

$$\begin{aligned}
\mathrm{E}[(C_i - D)I_{\{C_i>D\}} | C_i = x] &= \mathrm{E}[(x - D)I_{\{x>D\}}] \\
&= \int_0^x (x - t)\delta e^{-\delta t}\, dt \\
&= \int_0^x y\delta e^{-\delta(x-y)}\, dy \\
&= \delta e^{-\delta x} \int_0^x y\delta e^{\delta y}\, dy \\
&= \delta e^{-\delta x} \left\{ \frac{1}{\delta} x e^{\delta x} - \frac{1}{\delta^2}(e^{\delta x} - 1) \right\} \\
&= x - \frac{1}{\delta}(1 - e^{-\delta x}) \\
&= x - \frac{1}{\delta} + \frac{1}{\delta}\Pr(D > x).
\end{aligned}$$

Hence,

$$\mathrm{E}[(C_i - D)I_{\{C_i>D\}}] = \mathrm{E}[C_i] - \frac{1}{\delta} + \frac{1}{\delta}\Pr(D > C_i). \tag{3.8}$$

Since

$$\mathrm{E}[R_i] = \sum_{k\in\mathcal{B}_i(\lambda)} \mathrm{E}[P_k] = \sum_{k\in\mathcal{B}_i(\lambda)} \mu_k, \tag{3.9}$$

it follows from (2.2), (3.9), and the assumptions on P_k, $N(t)$, and Z_k that

$$\begin{aligned}
\mathrm{E}[C_i] &= \mathrm{E}\left[R_i + \sum_{k=0}^{N(R_i-)} Z_k \right] \\
&= \mathrm{E}[R_i] + \mathrm{E}[Z]\mathrm{E}[N(R_i-)] \\
&= \mathrm{E}[R_i] + \nu\mathrm{E}[\tau R_i] \\
&= (1 + \nu\tau)\mathrm{E}[R_i] \\
&= (1 + \nu\tau) \sum_{k\in\mathcal{B}_i(\lambda)} \mu_k.
\end{aligned} \tag{3.10}$$

Substituting (3.10) and (3.6) into (3.8), we get

$$\mathrm{E}[(C_i - D)I_{\{C_i > D\}}] = (1 + \nu\tau) \sum_{k \in \mathcal{B}_i(\lambda)} \mu_k + \frac{1}{\delta} \prod_{k \in \mathcal{B}_i(\lambda)} f_k - \frac{1}{\delta}, \tag{3.11}$$

which together with (3.7) gives (3.1). ■

In what follows we will suppress λ from $\mathcal{B}_i(\lambda)$ for ease of notation. Define

$$V_1(\lambda) = \sum_{i=1}^{n} \beta_i \sum_{k \in \mathcal{B}_i} \mu_k \quad \text{and} \quad V_2(\lambda) = \sum_{i=1}^{n} (\alpha_i + \beta_i) \prod_{k \in \mathcal{B}_i} f_k. \tag{3.12}$$

Then (3.1) can be rewritten as

$$ETC(\lambda) = (1 + \nu\tau)V_1(\lambda) + \frac{1}{\delta} V_2(\lambda) - \frac{1}{\delta} \sum_{i=1}^{n} \beta_i. \tag{3.13}$$

The next theorem is one of our main results, which shows that the optimal sequence is V-shaped with respect to $\{\mu_i/\beta_i\}$ under appropriate conditions.

THEOREM 2: *Define*

$$\gamma_{ij} = \left(\frac{\alpha_j}{\mu_j} - \frac{\alpha_i}{\mu_i}\right)\left(\frac{\beta_j}{\mu_j} - \frac{\beta_i}{\mu_i}\right)^{-1} \quad \text{if } \frac{\beta_j}{\mu_j} \neq \frac{\beta_i}{\mu_i}. \tag{3.14}$$

If $\{\gamma_{ij}\}$ *satisfy*

$$1 + \gamma_{jk} < \max\{(1 + \eta\mu_k)(1 + \gamma_{ij}), \delta(1 + \nu\tau)/\eta\} \tag{3.15}$$

for all distinct $i, j, k \in \{1, \ldots, n\}$ *such that* γ_{jk} *and* γ_{ij} *are defined, then an optimal sequence* λ^* *that minimizes* $ETC(\lambda)$ *is V-shaped with respect to* $\{\mu_i/\beta_i\}$.

PROOF: Given two jobs i and j, consider two sequences: $\lambda = \{\ldots, i, j, \ldots\}$ and $\lambda' = \{\ldots, j, i, \ldots\}$; that is, λ and λ' are identical except the order of jobs i and j is switched. Let $\mathcal{B}^* = \mathcal{B}_i(\lambda) - \{i\} = \mathcal{B}_i(\lambda') - \{j\}$ denote the set of jobs to be processed just before i and j under either λ or λ'. Then by (3.12),

$$\begin{aligned} V_1(\lambda) - V_1(\lambda') &= \beta_i\left(\sum_{k \in \mathcal{B}^*} \mu_k + \mu_i\right) + \beta_j\left(\sum_{k \in \mathcal{B}^*} \mu_k + \mu_i + \mu_j\right) \\ &\quad - \beta_j\left(\sum_{k \in \mathcal{B}^*} \mu_k + \mu_j\right) - \beta_i\left(\sum_{k \in \mathcal{B}^*} \mu_k + \mu_j + \mu_i\right) \\ &= -\beta_i\mu_j + \beta_j\mu_i \\ &= \mu_i\mu_j\left(\frac{\beta_j}{\mu_j} - \frac{\beta_i}{\mu_i}\right). \end{aligned} \tag{3.16}$$

Furthermore, by (3.2) we have

$$1 - f_i = \frac{\eta\mu_i}{1 + \eta\mu_i} = \eta\mu_i f_i. \tag{3.17}$$

Hence,

$$\begin{aligned} V_2(\lambda) - V_2(\lambda') &= (\alpha_i + \beta_i) f_i \prod_{k \in \mathcal{B}^*} f_k + (\alpha_j + \beta_j) f_i f_j \prod_{k \in \mathcal{B}^*} f_k \\ &\quad - (\alpha_j + \beta_j) f_j \prod_{k \in \mathcal{B}^*} f_k - (\alpha_i + \beta_i) f_i f_j \prod_{k \in \mathcal{B}^*} f_k \\ &= [(\alpha_i + \beta_i) f_i (1 - f_j) - (\alpha_j + \beta_j) f_j (1 - f_i)] \prod_{k \in \mathcal{B}^*} f_k \\ &= [(\alpha_i + \beta_i) f_i \eta\mu_j f_j - (\alpha_j + \beta_j) f_j \eta\mu_i f_i] \prod_{k \in \mathcal{B}^*} f_k \\ &= -\mu_i\mu_j \left(\frac{\alpha_j + \beta_j}{\mu_j} - \frac{\alpha_i + \beta_i}{\mu_i} \right) \eta f_i f_j \prod_{k \in \mathcal{B}^*} f_k. \end{aligned} \tag{3.18}$$

Combining (3.16) and (3.18) with (3.13), we obtain

$$\begin{aligned} &ETC(\lambda) - ETC(\lambda') \\ &\quad = \mu_i\mu_j \left(\frac{\beta_j}{\mu_j} - \frac{\beta_i}{\mu_i} \right)(1 + \nu\tau) - \frac{1}{\delta}\mu_i\mu_j \left(\frac{\alpha_j + \beta_j}{\mu_j} - \frac{\alpha_i + \beta_i}{\mu_i} \right) \eta f_i f_j \prod_{k \in \mathcal{B}^*} f_k \\ &\quad = \mu_i\mu_j \left(\frac{\beta_j}{\mu_j} - \frac{\beta_i}{\mu_i} \right) \left\{ 1 + \nu\tau - \frac{\eta}{\delta} f_i f_j \prod_{k \in \mathcal{B}^*} f_k \right\} \\ &\qquad - \mu_i\mu_j \left(\frac{\alpha_j}{\mu_j} - \frac{\alpha_i}{\mu_i} \right) \frac{\eta}{\delta} f_i f_j \prod_{k \in \mathcal{B}^*} f_k. \end{aligned} \tag{3.19}$$

By (3.14), (3.19) can be written as

$$ETC(\lambda) - ETC(\lambda') = \mu_i\mu_j \left(\frac{\beta_j}{\mu_j} - \frac{\beta_i}{\mu_i} \right) \left\{ 1 + \nu\tau - (1 + \gamma_{ij}) \frac{\eta}{\delta} f_i f_j \prod_{k \in \mathcal{B}^*} f_k \right\}. \tag{3.20}$$

Now let λ be an arbitrary sequence that is *not* V-shaped with respect to $\{\mu_i/\beta_i\}$. Without loss of generality, we can assume that $\lambda = \{1, 2, \ldots, n\}$. Then there are three consecutive jobs i, $i + 1$, and $i + 2$ under λ such that

$$\frac{\mu_i}{\beta_i} < \frac{\mu_{i+1}}{\beta_{i+1}} > \frac{\mu_{i+2}}{\beta_{i+2}}$$

or, equivalently,

$$\frac{\beta_i}{\mu_i} > \frac{\beta_{i+1}}{\mu_{i+1}} < \frac{\beta_{i+2}}{\mu_{i+2}}. \tag{3.21}$$

Let λ^1 denote the sequence that switches jobs i and $i+1$ in λ, and λ^2 the sequence that switches jobs $i+1$ and $i+2$ in λ so that

$$\lambda = \{\ldots, i, i+1, i+2, \ldots\},$$

$$\lambda^1 = \{\ldots, i+1, i, i+2, \ldots\},$$

and

$$\lambda^2 = \{\ldots, i, i+2, i+1, \ldots\}.$$

By (3.20),

$$ETC(\lambda) - ETC(\lambda^1) = \mu_i \mu_{i+1} \left(\frac{\beta_{i+1}}{\mu_{i+1}} - \frac{\beta_i}{\mu_i} \right) A_i, \tag{3.22}$$

where

$$A_i = 1 + \nu\tau - (1 + \gamma_{i,i+1}) \frac{\eta}{\delta} \prod_{k=1}^{i+1} f_k. \tag{3.23}$$

Similarly,

$$ETC(\lambda) - ETC(\lambda^2) = \mu_{i+1} \mu_{i+2} \left(\frac{\beta_{i+2}}{\mu_{i+2}} - \frac{\beta_{i+1}}{\mu_{i+1}} \right) A_{i+1}. \tag{3.24}$$

By condition (3.15), either

$$1 + \gamma_{i,i+1} > \frac{1 + \gamma_{i+1,i+2}}{1 + \eta\mu_{i+2}} = (1 + \gamma_{i+1,i+2}) f_{i+2}$$

or

$$1 + \gamma_{i+1,i+2} < \delta(1 + \nu\tau)/\eta. \tag{3.25}$$

If $A_i < 0$, then by (3.21) and (3.22), $ETC(\lambda) - ETC(\lambda^1) > 0$. When $A_i \geq 0$, by (3.23) and (3.25), either

$$\begin{aligned}
A_{i+1} &= 1 + \nu\tau - (1 + \gamma_{i+1,i+2})\,\frac{\eta}{\delta}\prod_{k=1}^{i+2} f_k \\
&= 1 + \nu\tau - (1 + \gamma_{i+1,i+2}) f_{i+2}\,\frac{\eta}{\delta}\prod_{k=1}^{i+1} f_k \\
&> 1 + \nu\tau - (1 + \gamma_{i,i+1})\,\frac{\eta}{\delta}\prod_{k=1}^{i+1} f_k \\
&= A_i \geq 0
\end{aligned}$$

or $A_{i+1} > 1 + \nu\tau - (1 + \gamma_{i+1,i+2})\eta/\delta > 0$. Therefore, $ETC(\lambda) - ETC(\lambda^2) > 0$ by (3.21) and (3.24). In either case, λ cannot be an optimal sequence. Thus, an optimal sequence must be V-shaped with respect to $\{\mu_i/\beta_i\}$. ■

If $\gamma_{ij} \geq 0$ for all i, j, then $\{\mu_i/\alpha_i\}$ have the same order as $\{\mu_i/\beta_i\}$. Hence, we have the following result on the V-shape property of the optimal sequence with respect to $\{\mu_i/\alpha_i\}$.

Corollary 1: *If $\gamma_{ij} \geq 0$ for all i and j and condition (3.15) holds for $\gamma_{ij} > 0$ and $\gamma_{jk} > 0$, then an optimal sequence λ^* that minimizes $ETC(\lambda)$ is V-shaped with respect to $\{\mu_i/\alpha_i\}$ (as well as to $\{\mu_i/\beta_i\}$).*

On the other hand, if $\gamma_{ij} \leq 0$ for all i and j, then $\{\mu_i/\alpha_i\}$ and $\{\mu_i/\beta_i\}$ have opposite orders. In such a case, an analytic solution is available. This result will be given in Theorem 3 in Section 4. Here we continue our analysis of V-shaped sequences.

Remark 1: When $\gamma_{ij} > 0$ for some i and j and $\gamma_{ij} < 0$ for other i and j, it is possible that an optimal sequence is V-shaped with respect to $\{\mu_i/\beta_i\}$, but not to $\{\mu_i/\alpha_i\}$. See the following example.

Example 1: Let $n = 3$, $\mu_1 = \mu_2 = \mu_3 = 1$, $(\beta_1, \beta_2, \beta_3) = (4,3,1)$, $(\alpha_1, \alpha_2, \alpha_3) = (2,1,2)$, and $\eta = 4$. Then it is not difficult to see that

$$\frac{1}{2} \leq 1 + \gamma_{ij} = 1 + \frac{\alpha_j - \alpha_i}{\beta_j - \beta_i} \leq 2, \quad \forall i, j,$$

so that

$$\frac{1 + \gamma_{jk}}{1 + \gamma_{ij}} \leq 4 < 5 = 1 + \eta\mu_k;$$

hence, (3.15) holds. Following Corollary 4 in Section 4, we will show that the optimal sequence is $\lambda^* = (1,2,3)$, which is in nondecreasing order of (hence V-shaped with respect to) $\{\mu_i/\beta_i\} = \{\frac{1}{4}, \frac{1}{3}, 1\}$, but not V-shaped with respect to $\{\mu_i/\alpha_i\} = \{\frac{1}{2}, 1, \frac{1}{2}\}$.

We now look at some special cases in which condition (3.15) holds.

Case I: $\{(\alpha_j/\mu_j) - (\alpha_i/\mu_i)\}$ and $\{(\beta_j/\mu_j) - (\beta_i/\mu_i)\}$ are proportional; that is,

$$\left(\frac{\alpha_j}{\mu_j} - \frac{\alpha_i}{\mu_i}\right) = K\left(\frac{\beta_j}{\mu_j} - \frac{\beta_i}{\mu_i}\right), \quad \forall i, j = 1, \ldots, n, \tag{3.26}$$

for some constant K. Then $\gamma_{ij} \equiv K$. When $K > 0$, $\{\alpha_i/\mu_i\}$ and $\{\beta_i/\mu_i\}$ have the same order and condition (3.15) holds obviously. If $K \leq 0$, then $\{\alpha_i/\mu_i\}$ and $\{\beta_i/\mu_i\}$ are in opposite orders. Thus, Theorems 1 and 2 lead to the following corollary.

COROLLARY 2: *If (3.26) holds for some constant K, then an optimal sequence λ^* that minimizes $ETC(\lambda)$ is V-shaped with respect to both $\{\mu_i/\beta_i\}$ and $\{\mu_i/\alpha_i\}$.*

Remark 2: If $\{\alpha_i\}$ and $\{\beta_i\}$ are proportional (i.e., $\alpha_i = c\beta_i$, $i = 1, \ldots, n$, for some constant c), then (3.26) holds with $K = c$. Hence, Corollary 2 covers this case. In particular, when $\alpha_i = \beta_i$, $i = 1, \ldots, n$, the objective function in (2.1) becomes

$$ETC(\lambda) = \sum_{i=1}^{n} \beta_i \mathrm{E}[|C_i - D|].$$

Hence, the main results of Jia [17] are a special case of our Corollary 2.

Case II: $\{(\alpha_j/\mu_j) - (\alpha_i/\mu_i)\}$ and $\{(\beta_j/\mu_j) - (\beta_i/\mu_i)\}$ are not equal, or proportional, but are close to each other in the sense that

$$\left(\frac{\alpha_j}{\mu_j} - \frac{\alpha_i}{\mu_i}\right) = (1 + \varepsilon_{ij})\left(\frac{\beta_j}{\mu_j} - \frac{\beta_i}{\mu_i}\right), \qquad i, j = 1, \ldots, n, \tag{3.27}$$

where $|\varepsilon_{ij}| \leq \varepsilon < 2$ for all i, j. Then $\gamma_{ij} = 1 + \varepsilon_{ij}$ and so

$$\frac{1 + \gamma_{jk}}{1 + \gamma_{ij}} = \frac{2 + \varepsilon_{jk}}{2 + \varepsilon_{ij}} \leq \frac{2 + \varepsilon}{2 - \varepsilon} = 1 + \frac{2\varepsilon}{2 - \varepsilon} < 1 + \eta\mu_k \quad \text{holds if } \varepsilon < \frac{2\eta\mu_k}{2 + \eta\mu_k}.$$

Furthermore, if $\varepsilon < 1$, then $\gamma_{ij} > 0$ for all i and j, so that $\{\mu_i/\beta_i\}$ and $\{\mu_i/\alpha_i\}$ have the same order. Thus, we obtain the following corollary.

COROLLARY 3: *If (3.27) holds with*

$$|\varepsilon_{ij}| \leq \varepsilon < \frac{2\eta\mu_{\min}}{2 + \eta\mu_{\min}}, \qquad \textit{where } \mu_{\min} = \min_{1 \leq i \leq n} \mu_i,$$

then an optimal sequence λ^* to minimize $ETC(\lambda)$ is V-shaped with respect to $\{\mu_i/\beta_i\}$. If, in addition, $\varepsilon < 1$, then λ^* is also V-shaped with respect to $\{\mu_i/\alpha_i\}$.

Remark 3: Although the above theorem and corollaries cover a fairly wide range of parameters such that an optimal sequence is V-shaped with respect to $\{\mu_i/\beta_i\}$ or $\{\mu_i/\alpha_i\}$ (or both), there are certainly situations in which the optimal sequence does not have a V-shape with respect to $\{\mu_i/\beta_i\}$ or $\{\mu_i/\alpha_i\}$. On the other hand, the conditions in each theorem or corollary are sufficient only, and there might exist a V-shaped optimum even though these conditions are not satisfied. See the following examples.

Example 2: Let $n = 3$, $\mu_1 = \mu_2 = \mu_3 = 1$, $(\beta_1,\beta_2,\beta_3) = (2,1,2)$, $(\alpha_1,\alpha_2,\alpha_3) = (3,1,9)$, $\delta = 1$, and $\tau = 0$. Then $f_i = \frac{1}{2}$, $i = 1,2,3$. By (3.1), the objective function in this case is given by

$$ETC(\lambda) = \sum_{i=1}^{3} \beta_i \sum_{k \in \mathcal{B}_i(\lambda)} \mu_k + \sum_{i=1}^{3} (\alpha_i + \beta_i) \prod_{k \in \mathcal{B}_i(\lambda)} f_k - \sum_{i=1}^{3} \beta_i.$$

Simple calculations show that

$$ETC(1,2,3) = 9.375, \qquad ETC(1,3.2) = 9.5, \qquad ETC(2,1,3) = 9.625,$$

$$ETC(2,3,1) = 10.375, \qquad ETC(3,1,2) = 11, \qquad ETC(1,2,3) = 11.625.$$

Thus, the optimal sequence is $\lambda^* = (1,2,3)$, which is not V-shaped with respect to $\{\mu_i/\beta_i\} = \{\frac{1}{2},1,\frac{1}{2}\}$ or to $\{\mu_i/\alpha_i\} = \{\frac{1}{3},1,\frac{1}{9}\}$.

Example 3: Let $n = 3$, $\mu_1 = \mu_2 = \mu_3 = 1$, $(\beta_1,\beta_2,\beta_3) = (6,4,2)$, $(\alpha_1,\alpha_2,\alpha_3) = (1,3,2)$, $\delta = 1$, and $\tau = 0$. Then $\gamma_{12} = -1$, $\gamma_{13} = -\frac{1}{4}$, $\gamma_{23} = \frac{1}{2}$, $1 + \eta\mu_i = 2$, $i = 1,2,3$, and $\delta(1 + \mu\tau)/\eta = 1$. Hence,

$$1 + \gamma_{23} = 1 + \frac{1}{2} > 1 = \max\{0,1\} = \max\left\{(1 + \eta\mu_3)(1 + \gamma_{12}), \frac{\delta}{\eta}(1 + \mu\tau)\right\}.$$

Therefore, the conditions of Theorem 2 do not hold. However, it is easy to verify that the conditions of Theorem 4 or Corollary 4 (see Section 4) hold, and so the optimal sequence is $\lambda^* = (1,2,3)$, which is not only nondecreasing in $\{\mu_i/\beta_i\} = \{\frac{1}{2},\frac{1}{4},\frac{1}{6}\}$ but also V-shaped with respect to $\{\mu_i/\alpha_i\} = \{1,\frac{1}{3},\frac{1}{2}\}$.

4. ANALYTICAL OPTIMAL SEQUENCES

We will reveal, in this section that there exist analytical optimal solutions for the problem if certain conditions are satisfied. The following theorem is one of our main results.

THEOREM 3: *If $\{\mu_i/\beta_i\}$ and $\{\mu_i/\alpha_i\}$ have opposite orders, that is,*

$$\frac{\alpha_j}{\mu_j} \le \frac{\alpha_i}{\mu_i} \Leftrightarrow \frac{\beta_j}{\mu_j} \ge \frac{\beta_i}{\mu_i} \quad \text{for all } i,j = 1,\ldots,n,$$

then a sequence in nondecreasing order of $\{\mu_i/\beta_i\}$ or in nonincreasing order of $\{\mu_i/\alpha_i\}$, is optimal to minimize $ETC(\lambda)$.

PROOF: Let $\beta_j/\mu_j \geq \beta_i/\mu_i$ so that $\alpha_j/\mu_j \leq \alpha_i/\mu_i$ by the condition of the theorem. Since

$$\begin{aligned}\frac{\eta}{\delta} f_i f_j \prod_{k \in \mathcal{B}^*} f_k < \frac{\eta}{\delta} \\ &= \frac{1}{\delta}(\delta + \tau \mathrm{E}[1 - e^{-\delta Z}]) \\ &\leq \frac{1}{\delta}(\delta + \tau \mathrm{E}[\delta Z]) \\ &= \frac{1}{\delta}(\delta + \tau\delta\nu) = 1 + \nu\tau,\end{aligned}$$

it follows from (3.19) that $ETC(\lambda) \geq ETC(\lambda')$. Thus, $ETC(\lambda) \geq ETC(\lambda')$ if and only if $\beta_j/\mu_j \geq \beta_i/\mu_i$ or $\mu_j/\beta_j \leq \mu_i/\beta_i$. The theorem then follows. ■

In addition to Theorem 3, there are a number of other situations in which an analytic solution is available to minimize $ETC(\lambda)$, as shown below.

Without loss of generality we assume $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ in Theorem 4.

THEOREM 4: *If*

$$\left|\frac{\alpha_j}{\mu_j} - \frac{\alpha_i}{\mu_i}\right| \leq B \left|\frac{\beta_j}{\mu_j} - \frac{\beta_i}{\mu_i}\right|,$$

where

$$B = \frac{\delta}{\eta}(1 + \nu\tau)(1 + \eta\mu_1)(1 + \eta\mu_2) - 1$$

for all $i, j = 1, \ldots, n$, then a sequence in nondecreasing order of $\{\mu_i/\beta_i\}$ is optimal to minimize $ETC(\lambda)$.

PROOF: Under the conditions of the theorem,

$$(1 + \gamma_{ij}) \frac{\eta}{\delta} f_i f_j \prod_{k \in \mathcal{B}^*} f_k \leq (1 + B) \frac{\eta}{\delta} f_i f_j = \frac{(1 + \nu\tau)(1 + \eta\mu_1)(1 + \eta\mu_2)}{(1 + \eta\mu_i)(1 + \eta\mu_j)} \leq 1 + \nu\tau.$$

Hence, by (3.20), $ETC(\lambda) \geq ETC(\lambda')$ if and only if

$$\frac{\beta_j}{\mu_j} \geq \frac{\beta_i}{\mu_i} \quad \text{or} \quad \frac{\mu_j}{\beta_j} \leq \frac{\mu_i}{\beta_i}.$$

This shows that an optimal sequence should schedule job j ahead of job i if $\mu_j/\beta_j \leq \mu_i/\beta_i$. The theorem follows. ■

Since $\eta \leq \delta(1 + \tau\nu)$, we have $B \geq (1 + \eta\mu_1)(1 + \eta\mu_2) - 1 > \eta(\mu_1 + \mu_2)$, which leads to the next corollary.

COROLLARY 4: *If*

$$\left| \frac{\alpha_j}{\mu_j} - \frac{\alpha_i}{\mu_i} \right| \leq \eta(\mu_1 + \mu_2) \left| \frac{\beta_j}{\mu_j} - \frac{\beta_i}{\mu_i} \right| \qquad \forall i, j = 1, \ldots, n,$$

then a sequence in nondecreasing order of $\{\mu_i/\beta_i\}$ *minimizes* $ETC(\lambda)$.

Example 1 (continued): Let $n = 3$, $\mu_1 = \mu_2 = \mu_3 = 1$, $(\beta_1, \beta_2, \beta_3) = (4,3,1)$, $(\alpha_1, \alpha_2, \alpha_3) = (2,1,2)$, and $\eta = 4$. Then

$$\left| \frac{\alpha_j}{\mu_j} - \frac{\alpha_i}{\mu_i} \right| = |\alpha_j - \alpha_i| \leq 1,$$

$$\left| \frac{\beta_j}{\mu_j} - \frac{\beta_i}{\mu_i} \right| = |\beta_j - \beta_i| \geq 1$$

$$\forall i, j = 1,2,3,$$

and $\eta(\mu_1 + \mu_2) = 4(1 + 1) = 8$. Hence, the condition of Corollary 4 holds and so $\lambda^* = (1,2,3)$ is the optimal sequence.

As a special case of Corollary 4, we have the following corollary.

COROLLARY 5: *If* $\{\alpha_i\}$ *are proportional to* $\{\mu_i\}$ *(i.e.,* $\alpha_i = c\mu_i$, $i = 1, \ldots, n$, *for some constant c), then a sequence in nondecreasing order of* $\{\mu_i/\beta_i\}$ *minimizes* $ETC(\lambda)$.

THEOREM 5: *If*

$$\left| \frac{\alpha_j}{\mu_j} - \frac{\alpha_i}{\mu_i} \right| \geq \tilde{B} \left| \frac{\beta_j}{\mu_j} - \frac{\beta_i}{\mu_i} \right|,$$

where

$$\tilde{B} = \frac{\delta}{\eta}(1 + \nu\tau) \prod_{k=1}^{n} (1 + \eta\mu_k) - 1$$

for all $i, j = 1, \ldots, n$, *then a sequence in nonincreasing order of* $\{\mu_i/\alpha_i\}$ *is optimal to minimize* $ETC(\lambda)$.

PROOF: By the definition of $\tilde{B}$,

$$(1 + \tilde{B}) \frac{\eta}{\delta} f_i f_j \prod_{k \in \mathcal{B}^*} f_k \geq (1 + \tilde{B}) \frac{\eta}{\delta} \prod_{k=1}^{n} f_k = (1 + \nu\tau) \prod_{k=1}^{n} (1 + \eta\mu_k) \prod_{k=1}^{n} f_k = 1 + \nu\tau. \tag{4.1}$$

Let

$$\frac{\mu_j}{\alpha_j} \leq \frac{\mu_i}{\alpha_i}$$

or, equivalently,

$$\frac{\alpha_j}{\mu_j} \geq \frac{\alpha_i}{\mu_i}.$$

Then by (3.19) and (4.1) together with the conditions of the theorem,

$$ETC(\lambda) - ETC(\lambda') \leq \frac{\mu_i \mu_j}{\tilde{B}} \left(\frac{\alpha_j}{\mu_j} - \frac{\alpha_i}{\mu_i} \right) \left\{ 1 + \nu\tau - (1 + \tilde{B}) \frac{\eta}{\delta} f_i f_j \prod_{k \in \mathcal{B}^*} f_k \right\} \leq 0.$$

Thus, an optimal sequence should place job i ahead of job j if $\mu_j/\alpha_j \leq \mu_i/\alpha_i$. This proves the theorem. ■

A special case of Theorem 5 is given next.

COROLLARY 6: *If $\{\beta_i\}$ are proportional to $\{\mu_i\}$, then a sequence in nonincreasing order of $\{\mu_i/\alpha_i\}$ minimizes $ETC(\lambda)$.*

5. AN ALGORITHM TO COMPUTE THE BEST V-SHAPED SEQUENCE

In this section we provide an algorithm that can search the best V-shaped sequence among all V-shaped sequences. Without loss of generality, we assume that the jobs have been numbered such that $\mu_1/\beta_1 \leq \mu_2/\beta_2 \leq \cdots \leq \mu_n/\beta_n$.

Consider a set of jobs $S_i = \{1, 2, \ldots, i\}$. In a V-shaped sequence, job i will be sequenced either the first or the last among all jobs in S_i. Assume that λ^* is the best V-shaped sequence and $\bar{S}_i$ is the set of jobs sequenced before all jobs in S_i under λ^*, and let

$$\Theta_i = \sum_{j \in \bar{S}_i} \mu_j \quad \text{and} \quad \Psi_i = \prod_{j \in \bar{S}_i} f_j. \tag{5.1}$$

Define $h_i(\Theta_i, \Psi_i)$ to be the contribution of all jobs in S_i to the cost function (3.1), given Θ_i and Ψ_i. Then it is easy to see that the costs arising from sequencing job i as the first and the last job among all jobs in the set S_i will be respectively

$$h_i^a(\Theta_i, \Psi_i) = h_{i-1}(\Theta_i + \mu_i, \Psi_i f_i) + \beta_i(1 + \nu\tau)(\Theta_i + \mu_i) + (\alpha_i + \beta_i) \frac{1}{\delta} \Psi_i f_i$$

and

$$h_i^b(\Theta_i, \Psi_i) = h_{i-1}(\Theta_i, \Psi_i) + \beta_i(1 + \nu\tau)\left(\Theta_i + \sum_{j\in S_i} \mu_j\right) + (\alpha_i + \beta_i)\frac{1}{\delta}\Psi_i \prod_{j\in S_i} f_j.$$

It follows from the principle of optimality of dynamic programming that, with $h_i^a(\Theta_i, \Psi_i)$ and $h_i^b(\Theta_i, \Psi_i)$ defined above, the best V-shaped sequence λ^* must sequence job i such that

$$h_i(\Theta_i, \Psi_i) = \min\{h_i^a(\Theta_i, \Psi_i), h_i^b(\Theta_i, \Psi_i)\} - \frac{1}{\delta}\sum_{j\in S_i} \beta_j. \tag{5.2}$$

We therefore have the following algorithm.

Algorithm 1

1. For $i = 1,2,\ldots,n$, compute $h_i(\Theta_i, \Psi_i)$ according to (5.2), for all possible values in the feasible sets of Θ_i and Ψ_i.
2. Let $H_n^* = \min_{\Theta_n,\Psi_n}\{h_n(\Theta_n, \Psi_n)\}$.
3. Construct, by a backward tracking process, the sequence λ^* that achieves H_n^*.

We have omitted the details of the backward tracking process to find λ^*. We have also omitted the definitions of the feasible sets for Θ_i and Ψ_i defined in (5.1). With certain assumptions (e.g., all μ_i are integers), one can identify finite feasible sets for Θ_i and Ψ_i, and the time complexity of Algorithm 1 might become pseudopolynomial.

6. CONCLUDING REMARKS

We have studied the stochastic scheduling problem with asymmetric linear E/T costs, a canonical model in the E/T scheduling literature. This model is regarded as rather difficult since the piecewise linear nature of its objective function complicates the analysis of optimal solutions. A significant advancement achieved in this article is the finding of a number of propositions regarding the optimal solutions under the settings of exponential processing times and exponential due dues. We have found that these propositions remain valid for fairly general machine breakdown processes, where the uptimes follow an exponential distribution and downtimes (repair times) can follow any general distribution, and the machine breakdowns can be of either preempt-resume or preempt-repeat patterns. Note that an uptime lasts until a machine breakdown occurs. Since the occurring time of a breakdown is usually quite uncertain, using an exponential distribution to model it is reasonable. The repair time, on the other hand, can depend on various factors, such as the severity of the machine breakdown, the amount of time needed to diagnose the cause of the failure, and the available expertise and manpower, and, therefore, a general distribution is obviously more desirable.

We have obtained sufficient conditions for the optimal sequences to be V-shaped or analytical. An algorithm has also been provided to show how the best V-shaped sequence can be computed using dynamic programming.

Although we have assumed that the machine will start the first job at time 0, all of our results obtained in this article can be generalized to allow a nonzero machine start time. Extensions to include other types of costs (such as a weighted flow time) are also possible.

The complexity of the problem in terms of NP-completeness remains an open question. We conjecture that it is NP-complete in the ordinary sense, under the sufficient conditions that we have derived here to ensure a V-shaped optimal sequence and with certain assumptions such as integer-valued μ_i. Another interesting future work is to look at the possibility of generalizing the results in this article to situations involving nonexponential processing times, with the objective of finding approximate solutions.

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Notes

1. An objective function is said to be regular if it is a nondecreasing function of the completion time of any job. An objective function involving an earliness cost is nonregular, since it might decrease as the completion time increases.

2. A sequence is V-shaped with respect to $\{a_i\}$, where a_i is an attribute of job i, $i = 1, 2, \ldots, n$, if it first arranges jobs in a nonincreasing order of $\{a_i\}$ and then in a nondecreasing order of $\{a_i\}$.

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