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MULTISCALE ASYMPTOTIC METHOD FOR MAXWELL'S EQUATIONS IN COMPOSITE MATERIALS*

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Abstract. In this paper we discuss the multiscale analysis of Maxwell's equations in composite materials with a periodic microstructure. The new contributions in this paper are the determination of higher-order correctors and the explicit convergence rate for the approximate solutions (see Theorem 2.3). Consequently, we present the multiscale finite element method and derive the convergence result (see Theorem 4.1). The numerical results demonstrate that higher-order correctors are essential for solving Maxwell's equations in composite materials.

Key words. Maxwell's equations, homogenization, multiscale asymptotic expansion, composite materials, edge element

AMS subject classifications. 65F10, 78M05

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1. Introduction.

The classical macroscopic electromagnetic field is described by four vector functions of position $\mathbf{x} \in R^3$ and time $t \in R$ denoted by $\mathcal{E}$, $\mathcal{D}$, $\mathcal{H}$, and $\mathcal{B}$. The fundamental field vectors $\mathcal{E}$ and $\mathcal{H}$ are called the electric and magnetic field intensities, respectively. The vector functions $\mathcal{D}$ and $\mathcal{B}$ are called the electric displacement and magnetic induction, respectively. The following are Maxwell's equations:

$$
\begin{aligned}
\frac{\partial \mathcal{B}}{\partial t} + \nabla \times \mathcal{E} &= 0,\\
\nabla \cdot \mathcal{D} &= \rho,\\
\frac{\partial \mathcal{D}}{\partial t} - \nabla \times \mathcal{H} &= -\mathcal{J},\\
\nabla \cdot \mathcal{B} &= 0.
\end{aligned} \tag{1.1}
$$

Either by using the Fourier transform in time, or because we wish to analyze electromagnetic propagation at a single frequency, the time-dependent problem (1.1) can be reduced to the time-harmonic Maxwell's system. If the radiation has a temporal frequency $\omega > 0$, then the electromagnetic field is said to be time-harmonic, provided

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that

$$
\begin{aligned}
\mathcal{E}(\mathbf{x},t) &= \mathcal{R}(\exp(-i\omega t)\hat{\mathbf{E}}(\mathbf{x})),\\
\mathcal{D}(\mathbf{x},t) &= \mathcal{R}(\exp(-i\omega t)\hat{\mathbf{D}}(\mathbf{x})),\\
\mathcal{H}(\mathbf{x},t) &= \mathcal{R}(\exp(-i\omega t)\hat{\mathbf{H}}(\mathbf{x})),\\
\mathcal{B}(\mathbf{x},t) &= \mathcal{R}(\exp(-i\omega t)\hat{\mathbf{B}}(\mathbf{x})),\\
\mathcal{J}(\mathbf{x},t) &= \mathcal{R}(\exp(-i\omega t)\hat{\mathbf{J}}(\mathbf{x})),\\
\rho(\mathbf{x},t) &= \mathcal{R}(\exp(-i\omega t)\hat{\rho}(\mathbf{x})),
\end{aligned}
\tag{1.2}
$$

where $i=\sqrt{-1}$ and $\mathcal{R}$ denotes the real part of the expression in parentheses.

Substituting (1.2) into (1.1) leads to the time-harmonic Maxwell's equations:

$$
\begin{aligned}
-i\omega\hat{\mathbf{B}} + \nabla\times\hat{\mathbf{E}} &= 0,\\
\nabla\cdot\hat{\mathbf{D}} &= \hat{\rho},\\
-i\omega\hat{\mathbf{D}} - \nabla\times\hat{\mathbf{H}} &= -\hat{\mathbf{J}},\\
\nabla\cdot\hat{\mathbf{B}} &= 0.
\end{aligned}
\tag{1.3}
$$

Constitutive equations for linear media are

$$
\hat{\mathbf{D}} = \epsilon\hat{\mathbf{E}}, \quad \hat{\mathbf{B}} = \mu\hat{\mathbf{H}}, \tag{1.4}
$$

where $\epsilon = (\epsilon_{ij})$, $\mu = (\mu_{ij})$ are 3×3 positive-definite matrix functions of position, respectively.

By using (1.3) and (1.4), it is easy to get

$$
\nabla\times(\mu^{-1}\nabla\times\hat{\mathbf{E}}) - \omega^2\epsilon\hat{\mathbf{E}} = i\omega\hat{\mathbf{J}}, \quad \hat{\mathbf{H}} = \frac{1}{i\omega}\mu^{-1}\nabla\times\hat{\mathbf{E}} \tag{1.5}
$$

and

$$
\nabla\times(\epsilon^{-1}\nabla\times\hat{\mathbf{H}}) - \omega^2\mu\hat{\mathbf{H}} = \nabla\times(\epsilon^{-1}\hat{\mathbf{J}}), \quad \hat{\mathbf{E}} = \frac{1}{i\omega}\Big(\epsilon^{-1}\hat{\mathbf{J}} - \epsilon^{-1}\nabla\times\hat{\mathbf{H}}\Big), \tag{1.6}
$$

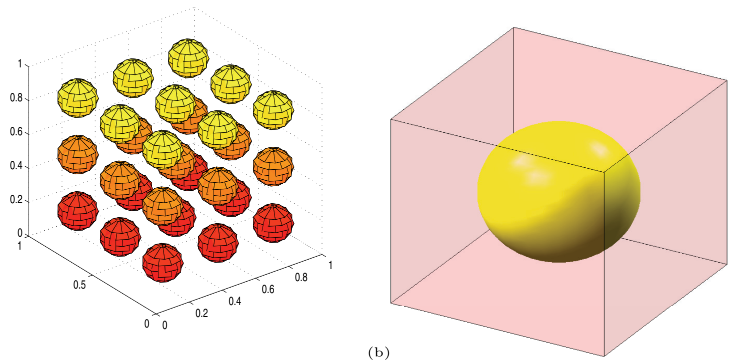
where ϵ^{-1}, μ^{-1} denote the inverse matrices of ϵ, μ, respectively.

REMARK 1.1. *In mathematics, the choice of eliminating* $\hat{\mathbf{H}}$*, rather than* $\hat{\mathbf{E}}$*, is arbitrary. But, in physics, we first solve for* $\hat{\mathbf{H}}$ *and then determine* $\hat{\mathbf{E}}$*; see* [25, 22, 39, 38].

In this paper we study the electromagnetic properties of composite materials with a periodic microstructure. When the wavelength is much larger than the typical scale of the microstructure, we can change (1.1) into the time-harmonic electric field equations (1.3). From (1.5), we consider the following equations with rapidly oscillating coefficients given by

$$
\begin{cases}
\mathbf{curl}(A^\varepsilon \mathbf{curl}\,\mathbf{u}^\varepsilon) - \omega^2\mathbf{u}^\varepsilon = \mathbf{f}(\mathbf{x}), & \mathbf{x}\in\Omega,\\
\nabla\cdot\mathbf{u}^\varepsilon = 0, & \mathbf{x}\in\Omega,\\
\mathbf{u}^\varepsilon\times\mathbf{n} = 0, & \mathbf{x}\in\partial\Omega,
\end{cases}
\tag{1.7}
$$

where $\Omega\subset R^3$ is a bounded Lipschitz polygonal convex domain or a smooth domain with a microstructure as shown in Figure 1.1(a). Here $\varepsilon>0$ denotes the relative size of a periodic microstructure of composite materials, i.e., $0<\varepsilon=\frac{l}{L}<1$, where l, L are, respectively, the sizes of a periodic cell and the entire domain. If we assume that $L=1$, without loss of generality, then the reference periodic cell Q is defined as



(a) (b)

FIG. 1.1. (a) *A whole domain* Ω. (b) *Reference cell* Q.

$Q = \{\xi = (\xi_1, \xi_2, \xi_3) : 0 < \xi_i < 1,\ i = 1, 2, 3\}$ as shown in Figure 1.1(b). $\mathbf{curl} = \nabla\times$, $\nabla\cdot\ = \text{div}$, and the matrix-valued permeability $A^\varepsilon(\mathbf{x}) = (a^\varepsilon_{ij}(\mathbf{x})) = (a_{ij}(\frac{\mathbf{x}}{\varepsilon}))$. We denote by ω the angular frequency, which is a real number, while $\mathbf{n} = (n_1, n_2, n_3)$ is the outward unit normal to $\partial\Omega$.

We make the following assumptions on the coefficients:

(A_1) Let $\xi = \varepsilon^{-1}\mathbf{x}$ and assume that $a_{ij}(\frac{\mathbf{x}}{\varepsilon})$ are 1-periodic functions.

(A_2) The matrix $A^\varepsilon(\mathbf{x}) = (a_{ij}(\frac{\mathbf{x}}{\varepsilon})) = (a_{ij}(\xi))$ satisfies

$$\mathcal{R}\Big(\eta^* A^\varepsilon(\mathbf{x})\eta\Big) \geq \gamma_0|\eta|^2 \quad \forall \eta \in C^n, \quad \text{a.e.} \quad \mathbf{x} \in \Omega, \tag{1.8}$$

where η^* is the Hermitian of a vector η, $|\eta|^2 = \eta_j\eta_j$, γ_0 is a constant independent of ε, and $\mathcal{R}(u)$ denotes the real part of u.

(A_3) $a_{ij}(\xi) = a_{ji}(\xi)$.

(A_4) $a_{ij} \in L^\infty(\Omega)$, $\mathbf{f} \in L^2(\Omega)^3$.

LEMMA 1.1. *If conditions* (A_2)–(A_4) *are satisfied, then there is a unique solution of problem* (1.7).

The proof of Lemma 1.1 can be found in [45, 6, 17].

Costabel, Dauge, and Nicaise [13] investigated the regularity of time-harmonic Maxwell's equations in heterogeneous media, where the permeability μ and the permittivity ϵ are piecewise constant. The important difference with the homogeneous case is that the regularity for the interface problem can be much lower, even with regular data. Thus, in the homogeneous case, one has at least $H^{1/2}$ regularity for Lipschitz domain and H^1 regularity for convex domains. However, for the composite structures, the authors found only 0 as a limit for the regularity. Hence for any $s > 0$ there are examples where the solution is not in H^s. If there are only two materials, the lower limit of regularity is 1/4 for arbitrary polyhedra and 1/2 for convex domains. Therefore, both the theoretical analysis and the numerical computations for solving Maxwell's equations in heterogeneous media are extremely difficult.

Problem (1.7) has a wide range of applications in electric, communication, and materials science and so forth (see, e.g., [19, 22, 23, 25, 26, 38, 39] and the references therein). It involves materials with a large number of heterogeneities (inclusions or holes). In such cases, the direct accurate numerical computation of the solution becomes difficult because it would require a very fine mesh and a prohibitive amount of computation time. The homogenization method is a way to give the overall behavior by incorporating the fluctuations due to the heterogeneities.

Bensoussan, Lions, and Papanicolaou [6] studied the homogenization method of the Maxwell's type equations with rapidly oscillating coefficients early on and derived convergence results. Homogenization for the nonstationary Maxwell's system was considered in the book by Sanchez-Palencia [40]. Further results for the nonstationary Maxwell's equations for a nonconducting medium can be found in the book [24]. Wellander [47] used the two-scale convergence method (see, e.g., [35, 1]) to obtain convergence results for the homogenization method for the time-dependent Maxwell's equations with linear constitutive relations in a heterogeneous medium and further expanded them in the case of nonlinear conductivity, which might be nonperiodic. Bossavit, Griso, and Miara [7] investigated the behavior of the electromagnetic field of a medium presenting periodic microstructures made of bianisotropic material and obtained convergence results for the homogenization method based upon the periodic unfolding method. In particular, they proved that the limit law differed from the initial one regarding the convolution term accounting for the memory effects. Banks et al. [3] used the periodic unfolding method, which was introduced in [12] in the abstract framework of stationary elliptic equations, to derive homogenization results of the nonstationary Maxwell's system with bianisotropy, chiral symmetry, thermal, and memory effects. Some numerical results for calculating the effective parameters for a Debye dielectric medium in the cases of circular and square microstructures in two dimensions were advanced. Barbatis and Stratis [4] studied the periodic homogenization of Maxwell's equations for dissipative bianisotropic media in the time domain, both in R^3 and in a bounded domain with perfect conductor boundary condition. When the wavelength was much larger than the typical scale of the microstructure in a material, Sjöberg et al. [18, 43, 44, 27] applied a Floquet–Bloch decomposition of Maxwell's equations to homogenization and gave a comparison of two numerical results between the classical method of homogenization and Floquet–Bloch homogenization for Maxwell's equations.

The engineering literature is dominated by the simple mixture formulae, which are derived using physical arguments. For an excellent overview and history of the mixture formulae, see [42]. In [3], a comparison was made between the effective parameters obtained by the exact classical homogenization method and those computed by traditional mixture formulae, such as the Maxwell Garnett formulae or Bruggeman formulae.

It is well known that homogenization describes the asymptotic behavior of the solution as $\varepsilon \to 0$. Numerous numerical results have shown that numerical accuracy of the homogenization method may not be satisfactory if ε is not sufficiently small (see, e.g., [8, 9]). This is the motivation for multiscale asymptotic methods and associated numerical algorithms. In [47] and [48] the authors obtained, respectively, the convergence results of the multiscale correctors in the norm of the space $L^2(\Omega \times (0, T))$ for the time-dependent Maxwell's equations in the linear and nonlinear cases. Wellander and Kristensson [49] analyzed the homogenization of the Maxwell's equations for a bounded object with penetrable boundary conditions. In addition to the interior homogenization problem, there is an exterior scattering problem that couples via the boundary conditions to the interior problem. They solved this problem by introducing the Calderón operators and the multiscale correctors. To our knowledge, this is the first theoretical result of the strong convergence in the norm of the space $\mathbf{H}(\mathbf{curl}; \Omega)$ for the time-harmonic Maxwell's equations. However, as mentioned in [49], there was still an open question of how irregular a function could be and be an admissible test function. The general convergence rate they obtained did not contain explicit terms.

The object of this paper is to present the multiscale asymptotic method and the associated numerical algorithm for the time-harmonic Maxwell's equations (1.7). We are able to obtain explicit convergence rates in the norm of the space $\mathbf{H}(\mathbf{curl};\Omega)$ under some proper assumptions and develop the multiscale finite element method for solving the problem (1.7); see Theorems 2.3 and 4.1. The main difficulty to overcome in this paper is how to treat the multiscale asymptotic solution near the boundary $\partial\Omega$.

The remainder of this paper is organized as follows. In section 2, we first present the formal multiscale asymptotic expansions of the solution of problem (1.7) and then derive the main convergence theorem of this paper (see Theorem 2.3). The key steps are the proofs of Propositions 2.1 and 2.2. Section 3 is devoted to the finite element computations for solving related problems, where the adaptive edge element is used to solve the cell problems. In order to improve the numerical accuracy, the postprocessing technique is introduced to solve the homogenized Maxwell's equations. We derive the proofs of the convergence results. In section 4, we present the multiscale finite element method for solving problem (1.7) and obtain the final error estimates. The numerical examples are given in section 5 to validate the theoretical results of this paper. For the sake of clearness the proofs of Propositions 2.1 and 2.2 will be given in Appendix A. To validate the theoretical results of Propositions 2.1 and 2.2, the numerical results will be presented in Appendix B.

Throughout the paper the Einstein summation convention on repeated indices is adopted. By C we shall denote a positive constant independent of ε without distinction.

2. Multiscale asymptotic expansions and the main convergence theorem.

2.1. The basic function spaces. We define the **curl** of a distribution $\mathbf{u} = (u_1, u_2, u_3)$ of $D'(\Omega)^3$ by

$$\mathbf{curl}\ \mathbf{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right),$$

and $\nabla \cdot \mathbf{u} = \mathrm{div}(\mathbf{u})$. Suppose that

$$\mathbf{H}(\mathbf{curl};\Omega) = \{\mathbf{u} \in L^2(\Omega)^3;\ \mathbf{curl}\ \mathbf{u} \in L^2(\Omega)^3\}$$

with the norm

$$\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl};\Omega)} = \{\|\mathbf{u}\|_{0,\Omega}^2 + \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega}^2\}^{1/2}$$

and

$$\mathbf{H}_0(\mathbf{curl};\Omega) = \text{closure of } (C_0^\infty(\Omega))^3 \text{ in } \mathbf{H}(\mathbf{curl};\Omega),$$

or

$$\mathbf{H}_0(\mathbf{curl};\Omega) = \{\mathbf{u} \in \mathbf{H}(\mathbf{curl};\Omega);\ \mathbf{u} \times \mathbf{n}|_{\partial\Omega} = 0\}.$$

2.2. Multiscale asymptotic expansions. We first present the formal multiscale asymptotic expansions of the solution for problem (1.7). Let $\xi = \varepsilon^{-1}\mathbf{x}$, and set formally

$$\mathbf{u}^\varepsilon(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \varepsilon\Theta_1(\xi)\mathbf{curl}\,\mathbf{u}^0(\mathbf{x}) + \varepsilon^2\Theta_2(\xi)\mathbf{curl}^2\,\mathbf{u}^0(\mathbf{x}) + \cdots, \tag{2.1}$$

where $\mathbf{curl}^l = \mathbf{curl}^{l-1}(\mathbf{curl})$, $l \geq 1$, $\mathbf{curl}^0 = I$, I is an identity operator.

Formally substituting (2.1) into (1.7), taking into account that $\mathbf{curl} \to \mathbf{curl}_{\mathbf{x}} + \varepsilon^{-1}\mathbf{curl}_\xi$, and equating the coefficients of like powers of ε, we define

$$\left\{\begin{array}{l} \mathbf{curl}_\xi(A(\xi)\mathbf{curl}_\xi\Theta_1^p(\xi)) = -\mathbf{curl}_\xi(A(\xi)\mathbf{e}_p), \quad \xi \in Q, \\ \nabla_\xi \cdot \Theta_1^p(\xi) = 0, \quad \xi \in Q, \\ \Theta_1^p(\xi) \times \nu = 0, \quad \xi \in \partial Q, \quad p = 1,2,3, \end{array}\right. \tag{2.2}$$

where $Q = (0,1)^3$, $\Theta_1^p(\xi)$, $p = 1,2,3$, are the vector-valued functions, and the matrix-valued function $\Theta_1(\xi) = (\Theta_1^1(\xi), \Theta_1^2(\xi), \Theta_1^3(\xi))$. $\nu = (\nu_1, \nu_2, \nu_3)$ is the outward unit normal to ∂Q, $\mathbf{e}_1 = \{1,0,0\}^T$, $\mathbf{e}_2 = \{0,1,0\}^T$, $\mathbf{e}_3 = \{0,0,1\}^T$.

REMARK 2.1. *The definition of* $\Theta_1(\xi) = (\Theta_1^1(\xi), \Theta_1^2(\xi), \Theta_1^3(\xi))$ *in* (2.2) *is similar to* (11.42) *of* [6, p. 145]. *However, the essential difference is that we take a perfect conductor boundary condition instead of the periodic boundary condition of* [6]. *Similarly to* (11.42) *of* [6, p. 145], *under assumptions* (A$_1$)–(A$_4$), *it can be shown that problem* (2.2) *has a unique weak solution.*

In order to define second-order cell functions, we first set

$$\left\{\begin{array}{l} \mathbf{curl}_\xi(A(\xi)\mathbf{curl}_\xi\tilde{\Theta}_2^p(\xi)) = -\mathbf{curl}_\xi(A(\xi)\Theta_1^p(\xi)) \\ \quad -A(\xi)\mathbf{curl}_\xi\Theta_1^p(\xi) - A(\xi)\mathbf{e}_p + \hat{A}\mathbf{e}_p, \quad \xi \in Q, \\ \tilde{\Theta}_2^p(\xi) \times \nu = 0, \quad \xi \in \partial Q, \quad p = 1,2,3, \end{array}\right. \tag{2.3}$$

where the matrix-valued function $\tilde{\Theta}_2(\xi) = (\tilde{\Theta}_2^1(\xi), \tilde{\Theta}_2^2(\xi), \tilde{\Theta}_2^3(\xi))$, and the homogenized coefficients matrix $\hat{A}$ will be given below.

Let $\tilde{G}(\xi) = -A(\xi)\mathbf{curl}_\xi\,\Theta_1^p(\xi) - A(\xi)\mathbf{e}_p + \hat{A}\mathbf{e}_p$. Observing (2.3), if $\nabla_\xi \cdot \tilde{G}(\xi) \neq 0$, then there is no solution for this equation. To overcome this difficulty, we introduce a scalar function $\zeta_2^p(\xi)$, $p = 1,2,3$, as follows:

$$\left\{\begin{array}{l} -\Delta_\xi\zeta_2^p(\xi) = \nabla_\xi \cdot \tilde{G}(\xi), \quad \xi \in Q, \\ \zeta_2^p(\xi) = 0, \quad \xi \in \partial Q, \end{array}\right. \tag{2.4}$$

which satisfies

$$\zeta_2^p \in H^2(Q) \cap H_0^1(Q).$$

It is easy to verify that

$$\nabla_\xi \cdot (\tilde{G}(\xi) + \nabla_\xi\zeta_2^p(\xi)) = 0. \tag{2.5}$$

We thus define

$$\left\{\begin{array}{l} \mathbf{curl}_\xi(A(\xi)\mathbf{curl}_\xi\Theta_2^p(\xi)) = -\mathbf{curl}_\xi(A(\xi)\Theta_1^p(\xi)) \\ \quad -A(\xi)\mathbf{curl}_\xi\Theta_1^p(\xi) - A(\xi)\mathbf{e}_p + \hat{A}\mathbf{e}_p + \nabla_\xi\zeta_2^p(\xi), \quad \xi \in Q, \\ \nabla_\xi \cdot \Theta_2^p(\xi) = 0, \quad \xi \in Q, \\ \Theta_2^p(\xi) \times \nu = 0, \quad \xi \in \partial Q, \quad p = 1,2,3, \end{array}\right. \tag{2.6}$$

where the matrix-valued function $\Theta_2(\xi) = (\Theta_2^1(\xi), \Theta_2^2(\xi), \Theta_2^3(\xi))$.

REMARK 2.2. *Using an idea similar to that reported for* (11.42) *in* [6], *under assumptions* (A$_1$)–(A$_4$), *the existence and uniqueness of the weak solutions of the cell problems* (2.6) *can be proved. The definitions of the second-order cell functions* $\Theta_2^p(\xi)$, $p = 1,2,3$, *are given in this paper. Numerical simulation results presented in* *section 5 clearly show that the second-order corrector terms are crucial in obtaining accurate computed solutions.*

As usual, we can define the solution $\mathbf{u}^0(\mathbf{x})$ of the homogenized Maxwell's equations as

$$\left\{\begin{array}{l}\mathbf{curl}(\hat{A}\,\mathbf{curl}\,\mathbf{u}^0(\mathbf{x}))-\omega^2\mathbf{u}^0(\mathbf{x})=\mathbf{f}(\mathbf{x}),\quad \mathbf{x}\in\Omega,\\ \nabla\cdot\mathbf{u}^0(\mathbf{x})=0,\quad \mathbf{x}\in\Omega,\\ \mathbf{u}^0(\mathbf{x})\times\mathbf{n}=0,\quad \mathbf{x}\in\partial\Omega,\end{array}\right. \tag{2.7}$$

where $\mathbf{n}=(n_1,n_2,n_3)$ is the outward unit normal to $\partial\Omega$.

To compute the homogenized coefficients matrix $\hat{A}$, there are two types of methods; see [6, pp. 142–145].

(i) The first method is to solve for $\chi^j(\xi)$

$$\left\{\begin{array}{l}\nabla_\xi\cdot(A^{-1}(\xi)\nabla_\xi\chi^j(\xi))=-\nabla_\xi\cdot(A^{-1}(\xi)\mathbf{e_j}),\\ \mathbf{e}_1=\{1,0,0\}^T,\ \mathbf{e}_2=\{0,1,0\}^T,\ \mathbf{e}_3=\{0,0,1\}^T,\end{array}\right. \tag{2.8}$$

where $A^{-1}(\xi)$ denotes the inverse matrix of $A(\xi)$, $\chi^j(\xi)$, $j=1,2,3$, are the scalar functions, and the vector-valued function $\chi(\xi)=(\chi^1(\xi),\chi^2(\xi),\chi^3(\xi))$. We thus define

$$\hat{A}=\Big(\mathcal{M}(A^{-1}(I_3+\nabla_\xi\chi(\xi)))\Big)^{-1}, \tag{2.9}$$

where $\mathcal{M}v=\int_Q v(\xi)d\xi$.

(ii) The second method is to set

$$\hat{A}=\mathcal{M}\Big(A(\xi)+A(\xi)\,\mathbf{curl}_\xi\,\Theta_1(\xi)\Big), \tag{2.10}$$

where the matrix-valued function $\Theta_1(\xi)=(\Theta_1^1(\xi),\Theta_1^2(\xi),\Theta_1^3(\xi))$ is as given in (2.2), and $\mathcal{M}$ has been defined in (2.9).

Remark 2.3. *Theoretically the above two methods for calculating the homogenized coefficients are equivalent. But there exist differences between the two numerical results. In section* 5*, some numerical tests and comparisons are given for two methods.*

Remark 2.4. *By verifying that the homogenized coefficients matrix $\hat{A}$ satisfies conditions* (A_2)–(A_3) *(see* [6]*), we then show that there exists a unique weak solution to* (2.7)*. Since the homogenized problem* (2.7) *is a Maxwell's system with constant coefficients, it is a much simpler case on which to carry out the theoretical analysis and numerical computations than for the original problem* (1.7)*.*

We thus define the multiscale asymptotic expansions for problem (1.7):

$$\begin{array}{l}\mathbf{u}_1^\varepsilon(\mathbf{x})=\mathbf{u}^0(\mathbf{x})+\varepsilon\Theta_1(\xi)\mathbf{curl}\,\mathbf{u}^0(\mathbf{x}),\\ \mathbf{u}_2^\varepsilon(\mathbf{x})=\mathbf{u}^0(\mathbf{x})+\varepsilon\Theta_1(\xi)\mathbf{curl}\,\mathbf{u}^0(\mathbf{x})+\varepsilon^2\Theta_2(\xi)\mathbf{curl}^2\,\mathbf{u}^0(\mathbf{x}),\end{array} \tag{2.11}$$

where $\mathbf{u}^0(\mathbf{x})$ is the solution of the homogenized Maxwell's equations (2.7), and the matrix-valued functions $\Theta_1(\xi)$, $\Theta_2(\xi)$ are defined as in (2.2) and (2.6), respectively.

It should be stated that the expansions (2.11) are purely formal. Next we will derive the convergence results for the multiscale asymptotic method. To this end, let us make the following assumptions for the coefficients matrix $A^\varepsilon(\mathbf{x})=A(\frac{\mathbf{x}}{\varepsilon})=A(\xi)=(a_{ij}(\xi))$, $\xi=\varepsilon^{-1}\mathbf{x}$.

(H_1) $A(\xi)$ is a diagonal matrix, i.e. $A(\xi)=\mathrm{diag}(a_{11}(\xi),a_{22}(\xi),a_{33}(\xi))$.

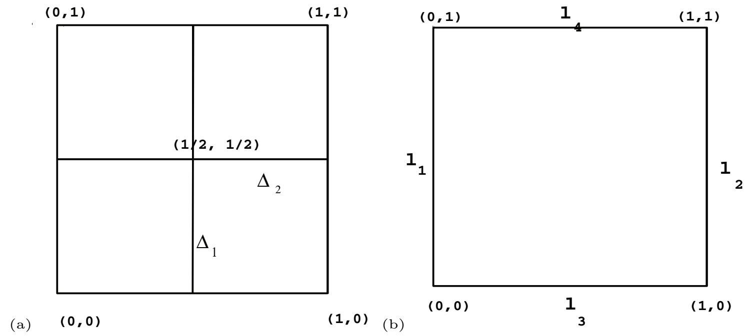


FIG. 2.1. (a) *The symmetry of* Q. (b) *The sides of* Q.

(H_2) $a_{kk}(\xi)$, $k = 1, 2, 3$, are symmetric with respect to the middleplane Δ_k of $Q = (0, 1)^3$, where Δ_k, $k = 1, 2$, are illustrated in Figure 2.1(a) in the two dimensional case.

REMARK 2.5. *The conditions* (H_1) *and* (H_2) *imply that composite materials satisfy geometric symmetric properties in a periodic microstructure.*

PROPOSITION 2.1. *Let* $\Theta_1^p(\xi)$*,* $p = 1, 2, 3$*, be the solutions of cell problems* (2.2)*. Under assumptions* (A_1)–(A_4)*,* (H_1)–(H_2)*, we can prove that*

$$[A(\xi)\mathbf{curl}_\xi \Theta_1^p(\xi) \times \nu]|_{\partial Q} = 0, \tag{2.12}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ *is the outward unit normal on the boundary* ∂Q *of the reference cell* $Q = (0, 1)^3$*;* $[v]|_{\partial Q}$ *denotes the jump of a function* v *on* ∂Q*.*

PROPOSITION 2.2. *Let* $\Theta_2^p(\xi)$*,* $p = 1, 2, 3$*, be the solutions of cell problems* (2.6)*. Under assumptions* (A_1)–(A_4)*,* (H_1)–(H_2)*, we have*

$$[A(\xi)\mathbf{curl}_\xi \Theta_2^p(\xi) \times \nu]|_{\partial Q} = 0. \tag{2.13}$$

The proofs of Propositions 2.1 and 2.2 will be given in Appendix A.

REMARK 2.6. *Propositions* 2.1 *and* 2.2 *are essential for the convergence analysis of the proposed method in this paper, since they allow us to obtain explicit convergence rates in the norm of the space* $\mathbf{H}(\mathbf{curl}; \Omega)$ *under some proper assumptions.*

We derive the main convergence theorem of this paper as follows.

THEOREM 2.3. *Suppose that* $\Omega \subset R^3$ *is the union of entire periodic cells, i.e.,* $\overline{\Omega} = \bigcup_{z \in T_\varepsilon} \varepsilon(z + \overline{Q})$*, where* $T_\varepsilon = \{z \in Z^3, \ \varepsilon(z + \overline{Q}) \subset \overline{\Omega}\}$*. Let* $\mathbf{u}^\varepsilon$ *be the solution of* (1.7)*, and let* $\mathbf{u}_s^\varepsilon$ *be the multiscale asymptotic solutions as shown in* (2.11)*. Under assumptions* (A_1)–(A_4) *and* (H_1)–(H_2)*, if* $\mathbf{f} \in \mathbf{H}^s(\Omega)$*,* $\mathbf{u}^0 \in \mathbf{H}^{s+2}(\Omega)$*,* $s = 1, 2$*, we obtain the following error estimate:*

$$\|\mathbf{u}^\varepsilon - \mathbf{u}_s^\varepsilon\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\varepsilon, \quad s = 1, 2, \tag{2.14}$$

where C *is a constant independent of* ε*.*

Proof. We first prove Theorem 2.3 in the case $s = 1$. Set

$$\mathbf{u}_1^\varepsilon(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \varepsilon \Theta_1(\xi)\mathbf{curl}_\mathbf{x}\, \mathbf{u}^0(\mathbf{x}). \tag{2.15}$$

By taking into account $\mathbf{curl} \to \mathbf{curl}_\mathbf{x} + \varepsilon^{-1}\mathbf{curl}_\xi$, we obtain the following equalities which hold in the sense of distributions:

$$\begin{aligned}\mathbf{curl}\Big(A\left(\frac{\mathbf{x}}{\varepsilon}\right)\mathbf{curl}\,\mathbf{u}_1^\varepsilon(\mathbf{x})\Big) &= \mathbf{curl}_\mathbf{x}(A(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}))\\ &+ \varepsilon\mathbf{curl}_\mathbf{x}(A(\xi)\mathbf{curl}_\mathbf{x}(\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x})))\\ &+ \mathbf{curl}_\mathbf{x}(A(\xi)\mathbf{curl}_\xi\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}))\\ &+ \varepsilon^{-1}\mathbf{curl}_\xi\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x})\\ &+ \mathbf{curl}_\xi(A(\xi)\mathbf{curl}_\mathbf{x}(\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x})))\\ &+ \varepsilon^{-1}\mathbf{curl}_\xi(A(\xi)\mathbf{curl}_\xi\Theta_1(\xi))\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}).\end{aligned} \tag{2.16}$$

Given $\overline{\Omega} = \bigcup_{z\in T_\varepsilon}\varepsilon(z+\overline{Q})$, let $E_z = \varepsilon(z+\overline{Q})$, and assume that ∂E_z denotes the boundary of E_z. When we apply Green's formulae to (2.16) on $\bigcup_{z\in T_\varepsilon}\partial E_z$, using Proposition 2.1, we get

$$\sum_{z\in T_\varepsilon}\int_{\partial E_z}\Big[\Big(A(\frac{\mathbf{x}}{\varepsilon})\mathbf{curl}\,\mathbf{u}_1^\varepsilon\Big)\times\nu\Big]\mathbf{v}d\sigma = 0 \quad \forall\mathbf{v}\in(H^1(\Omega))^3,$$

where $\nu = (\nu_1,\nu_2,\nu_3)$ denotes the outward unit normal to the boundary $\sum_{z\in T_\varepsilon}\partial E_z$. Therefore, (2.16) is valid in the sense of distributions.

Combining (2.2) and (2.7), it follows that

$$\mathbf{curl}\Big(A\left(\frac{\mathbf{x}}{\varepsilon}\right)\mathbf{curl}\,\mathbf{u}_1^\varepsilon(\mathbf{x})\Big) = \mathbf{curl}_\mathbf{x}\Big(\hat{A}\,\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x})\Big) + \mathcal{F}_0(\mathbf{x},\xi,\varepsilon) + \varepsilon\tilde{\mathcal{F}}_1(\mathbf{x},\xi,\varepsilon), \tag{2.17}$$

where

$$\begin{aligned}\mathcal{F}_0 = &\,\mathbf{curl}_\mathbf{x}(A(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}))\\ &+\mathbf{curl}_\mathbf{x}(A(\xi)\mathbf{curl}_\xi\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}))\\ &-\mathbf{curl}_\mathbf{x}(\hat{A}\,\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}))\\ &+\mathbf{curl}_\xi(A(\xi)\mathbf{curl}_\mathbf{x}(\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}))),\end{aligned}$$

$$\tilde{\mathcal{F}}_1 = \mathbf{curl}_\mathbf{x}(A(\xi)\mathbf{curl}_\mathbf{x}(\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}))),$$

and the homogenized coefficients matrix $\hat{A}$ is given as in (2.10) or (2.9).

From (1.7), (2.7), and (2.17), we have

$$\begin{aligned}&\mathbf{curl}\Big(A\left(\frac{\mathbf{x}}{\varepsilon}\right)\mathbf{curl}\,(\mathbf{u}^\varepsilon(\mathbf{x})-\mathbf{u}_1^\varepsilon(\mathbf{x}))\Big) - \omega^2(\mathbf{u}^\varepsilon(\mathbf{x})-\mathbf{u}_1^\varepsilon(\mathbf{x}))\\ &\quad= \mathcal{F}_0(\mathbf{x},\xi,\varepsilon) + \varepsilon\mathcal{F}_1(\mathbf{x},\xi,\varepsilon),\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}\mathcal{F}_1 = &\,\mathbf{curl}_\mathbf{x}(A(\xi)\mathbf{curl}_\mathbf{x}(\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x})))\\ &-\omega^2\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\,\mathbf{u}^0(\mathbf{x}).\end{aligned}$$

We recall (2.18) and assume that

$$B(\xi) = \Big(\hat{A} - A(\xi) - A(\xi)\mathbf{curl}_\xi\Theta_1(\xi)\Big),$$

$\mathbf{v}^0(\mathbf{x}) = \mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x})$, and consequently

$$\begin{aligned}\mathcal{F}_0(\mathbf{x},\xi,\varepsilon) = &\,\mathbf{curl}_\mathbf{x}\Big[B(\xi)\mathbf{v}^0(\mathbf{x})\Big]\\ &+\mathbf{curl}_\xi\Big(A(\xi)\mathbf{curl}_\mathbf{x}(\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}))\Big).\end{aligned}$$

Thanks to (A_2) and (A_4), from (2.2), it is obvious that

$$A(\xi) \in (L^\infty(Q))^{3\times 3}, \quad A(\xi)\mathbf{curl}_\xi \Theta_1(\xi) \in (L^\infty(Q))^{3\times 3}.$$

Given $\mathbf{curl}_\mathbf{x} B(\xi) = 0$, $\mathbf{u}^0 \in (H^3(\Omega))^3$, we check that $\mathcal{F}_0(\mathbf{x}, \xi, \varepsilon)$ is bounded and measurable in $(\xi, \mathbf{x})$, 1-periodic in ξ, and Lipschitz continuous with respect to $\mathbf{x}$ uniformly in ξ. From (2.10), we have $\int_Q \mathcal{F}_0(\mathbf{x}, \xi, \varepsilon) d\xi = 0$. It follows from Lemma 1.6 of [37, p. 8] that

$$\left| \int_\Omega \mathcal{F}_0 \mathbf{v} d\mathbf{x} \right| \leq C\varepsilon \|\mathbf{v}\|_{(H^1(\Omega))^3} \quad \forall \mathbf{v} \in (H^1(\Omega))^3, \tag{2.19}$$

where C is a constant independent of ε. Furthermore, using Corollary 3.6 of [21, p. 55], we obtain

$$\left| \int_\Omega \mathcal{F}_0 \mathbf{v} d\mathbf{x} \right| \leq C\varepsilon \|\mathbf{v}\|_{(H^1(\Omega))^3} \leq C\varepsilon \Big\{ \|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\nabla \cdot \mathbf{v}\|_{L^2(\Omega)} \Big\}. \tag{2.20}$$

Under the assumptions of Theorem 2.3, it is easy to verify that

$$\left| \int_\Omega \mathcal{F}_1 \mathbf{v} d\mathbf{x} \right| \leq C \|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega), \tag{2.21}$$

where C is a constant independent of ε.

Using (2.2) and (2.7), we get

$$\begin{aligned} \nabla \cdot \mathbf{u}_1^\varepsilon &= \nabla_\mathbf{x} \cdot \mathbf{u}^0(\mathbf{x}) + \nabla_\xi \cdot \Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}) \\ &+ \varepsilon \nabla_\mathbf{x} \cdot \Big(\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}) \Big) = \varepsilon \nabla_\mathbf{x} \cdot \Big(\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}) \Big), \end{aligned}$$

and consequently

$$\nabla \cdot (\mathbf{u}^\varepsilon - \mathbf{u}_1^\varepsilon) = \varepsilon \mathcal{R}(\mathbf{x}, \xi, \varepsilon), \tag{2.22}$$

where $\mathcal{R}(\mathbf{x}, \xi, \varepsilon) = \nabla_\mathbf{x} \cdot (\Theta_1(\xi)\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}))$, $\|\mathcal{R}\|_{L^2(\Omega)} \leq C$, where C is a constant independent of ε.

Let $\eta^\varepsilon(\mathbf{x})$ be the solution of

$$\begin{cases} \nabla \cdot (\nabla \eta^\varepsilon(\mathbf{x})) \equiv \Delta \eta^\varepsilon(\mathbf{x}) = \varepsilon \mathcal{R}(\mathbf{x}, \xi, \varepsilon), & \mathbf{x} \in \Omega, \\ \eta^\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega. \end{cases} \tag{2.23}$$

Since $\Omega \subset R^3$ is a bounded convex domain, we can show that

$$\|\eta^\varepsilon(\mathbf{x})\|_{H^2(\Omega)} \leq C\varepsilon, \tag{2.24}$$

where C is a constant independent of ε.

Setting $\tilde{\mathbf{u}}_1^\varepsilon(\mathbf{x}) = \mathbf{u}_1^\varepsilon(\mathbf{x}) - \nabla \eta^\varepsilon(\mathbf{x})$, we find

$$\nabla \cdot (\mathbf{u}^\varepsilon(\mathbf{x}) - \tilde{\mathbf{u}}_1^\varepsilon(\mathbf{x})) = 0. \tag{2.25}$$

If we assume that Ω is the union of entire cells, then we obtain by using the homogeneous Dirichlet boundary conditions of cell functions $\Theta_1^p(\xi)$, $p = 1, 2, 3$, that

$$(\mathbf{u}^\varepsilon(\mathbf{x}) - \mathbf{u}_1^\varepsilon(\mathbf{x})) \times \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega. \tag{2.26}$$

Furthermore, if $\eta^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$, then we (cf. [6, p. 144]) get

$$\nabla \eta^\varepsilon(\mathbf{x}) \times \mathbf{n} = 0, \tag{2.27}$$

and consequently,

$$(\mathbf{u}^\varepsilon(\mathbf{x}) - \tilde{\mathbf{u}}_1^\varepsilon(\mathbf{x})) \times \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega. \tag{2.28}$$

We thus obtain

$$\begin{cases} \mathbf{curl}\Big(A\left(\dfrac{\mathbf{x}}{\varepsilon}\right)\mathbf{curl}\,(\mathbf{u}^\varepsilon - \tilde{\mathbf{u}}_1^\varepsilon)\Big) - \omega^2(\mathbf{u}^\varepsilon - \tilde{\mathbf{u}}_1^\varepsilon) = \tilde{\mathcal{F}}_0(\mathbf{x}, \xi, \varepsilon) \\ +\varepsilon \mathcal{F}_1(\mathbf{x}, \xi, \varepsilon), \quad \mathbf{x} \in \Omega, \\ \nabla \cdot (\mathbf{u}^\varepsilon - \tilde{\mathbf{u}}_1^\varepsilon) = 0, \quad \mathbf{x} \in \Omega, \\ (\mathbf{u}^\varepsilon - \tilde{\mathbf{u}}_1^\varepsilon) \times \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega, \end{cases} \tag{2.29}$$

where $\tilde{\mathcal{F}}_0(\mathbf{x}, \xi, \varepsilon) = \mathcal{F}_0(\mathbf{x}, \xi, \varepsilon) + \omega^2 \nabla \eta^\varepsilon$ and $\mathcal{F}_0(\mathbf{x}, \xi, \varepsilon)$, $\mathcal{F}_1(\mathbf{x}, \xi, \varepsilon)$ are given above.

Setting $\mathbf{v} = \mathbf{u}^\varepsilon - \tilde{\mathbf{u}}_1^\varepsilon$ and using (2.19), (2.20), and (2.29), we derive

$$\left| \int_\Omega \tilde{\mathcal{F}}_0 \cdot \mathbf{v} d\mathbf{x} \right| \le C\varepsilon \Big\{ \|v\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\nabla \cdot v\|_{L^2(\Omega)} \Big\} \le C\varepsilon \|v\|_{\mathbf{H}(\mathbf{curl};\Omega)}, \tag{2.30}$$

where C is a constant independent of ε.

We now define the bilinear form as follows:

$$a_\varepsilon(\mathbf{u}, \mathbf{v}) = \int_\Omega \left[A\left(\frac{\mathbf{x}}{\varepsilon}\right) \mathbf{curl}\,\mathbf{u}(\mathbf{x}) \cdot \mathbf{curl}\,\mathbf{v}(\mathbf{x}) - \omega^2 \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \right] d\mathbf{x} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega), \tag{2.31}$$

where the scalar product $\mathbf{u} \cdot \mathbf{v} = u_i v_i$.

Due to the presence of the term $-\omega^2 \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})$ on the left side of (2.31), $a_\varepsilon(\mathbf{u}, \mathbf{v})$ is not a coercive bilinear form. To overcome this difficulty, we use the trick of [29, p. 89]. To this end, we define the modified bilinear form given by

$$a_\varepsilon^+(\mathbf{u}, \mathbf{v}) = \int_\Omega \left[A\left(\frac{\mathbf{x}}{\varepsilon}\right) \mathbf{curl}\,\mathbf{u}(\mathbf{x}) \cdot \mathbf{curl}\,\mathbf{v}(\mathbf{x}) + \omega^2 \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \right] d\mathbf{x}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega). \tag{2.32}$$

Using Lemma 4.10 of [29, p. 90], we can prove that the bilinear form $a_\varepsilon^+(\mathbf{u}, \mathbf{v})$ is coercive, i.e.,

$$|a_\varepsilon^+(\mathbf{u}, \mathbf{u})| \ge \gamma_0 \|\mathbf{u}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \quad \forall \mathbf{u} \in \mathbf{H}(\mathbf{curl}; \Omega), \tag{2.33}$$

where γ_0 is a constant independent of $\mathbf{u}$.

We consider the problem associated with the bilinear form $a_\varepsilon^+(\mathbf{u}, \mathbf{v})$ as follows:

$$\begin{cases} \mathbf{curl}\left(A\left(\dfrac{\mathbf{x}}{\varepsilon}\right)\mathbf{curl}\,\mathbf{w}^\varepsilon\right) + \omega^2 \mathbf{w}^\varepsilon = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ \nabla \cdot \mathbf{w}^\varepsilon = 0, \quad \mathbf{x} \in \Omega, \\ \mathbf{w}^\varepsilon \times \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega. \end{cases} \tag{2.34}$$

Similarly to $\tilde{\mathbf{u}}_1^\varepsilon$, we can define the first-order multiscale solution $\tilde{\mathbf{w}}_1^\varepsilon$. Furthermore, we have

$$\begin{cases} \mathbf{curl}\Big(A\left(\dfrac{\mathbf{x}}{\varepsilon}\right)\mathbf{curl}\,(\mathbf{w}^\varepsilon(\mathbf{x}) - \tilde{\mathbf{w}}_1^\varepsilon(\mathbf{x}))\Big) + \omega^2(\mathbf{w}^\varepsilon(\mathbf{x}) - \tilde{\mathbf{w}}_1^\varepsilon(\mathbf{x})) \\ \quad = \tilde{\mathcal{G}}_0(\mathbf{x}, \xi, \varepsilon) + \varepsilon \mathcal{G}_1(\mathbf{x}, \xi, \varepsilon), \quad \mathbf{x} \in \Omega, \\ \nabla \cdot (\mathbf{w}^\varepsilon(\mathbf{x}) - \tilde{\mathbf{w}}_1^\varepsilon(\mathbf{x})) = 0, \quad \mathbf{x} \in \Omega, \\ (\mathbf{w}^\varepsilon(\mathbf{x}) - \tilde{\mathbf{w}}_1^\varepsilon(\mathbf{x})) \times \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega, \end{cases} \tag{2.35}$$

where

$$\left|\int_\Omega \tilde{\mathcal{G}}_0 \mathbf{v} d\mathbf{x}\right| \le C\varepsilon \|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl};\Omega), \tag{2.36}$$

and

$$\left|\int_\Omega \mathcal{G}_1 \mathbf{v} d\mathbf{x}\right| \le C \|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl};\Omega), \tag{2.37}$$

where C is a constant independent of ε.

Combining (2.33), (2.35), (2.36), and (2.37), it follows that

$$\begin{aligned} \gamma_0 \|\mathbf{w}^\varepsilon - \tilde{\mathbf{w}}_1^\varepsilon\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} &\le |a_\varepsilon^+(\mathbf{w}^\varepsilon - \tilde{\mathbf{w}}_1^\varepsilon, \mathbf{w}^\varepsilon - \tilde{\mathbf{w}}_1^\varepsilon)| \\ &\le \left\{ \left|\int_\Omega \tilde{\mathcal{G}}_0(\mathbf{w}^\varepsilon - \tilde{\mathbf{w}}_1^\varepsilon) d\mathbf{x}\right| + \varepsilon \left|\int_\Omega \mathcal{G}_1(\mathbf{w}^\varepsilon - \tilde{\mathbf{w}}_1^\varepsilon) d\mathbf{x}\right| \right\} \\ &\le C\varepsilon \|\mathbf{w}^\varepsilon - \tilde{\mathbf{w}}_1^\varepsilon\|_{\mathbf{H}(\mathbf{curl};\Omega)}, \end{aligned} \tag{2.38}$$

and consequently

$$\gamma_0 \|\mathbf{w}^\varepsilon - \tilde{\mathbf{w}}_1^\varepsilon\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\varepsilon, \tag{2.39}$$

where C is a constant independent of ε.

Similarly to (4.18) of [29, p. 91], we get

$$(I + K)(\mathbf{u}^\varepsilon - \tilde{\mathbf{u}}_1^\varepsilon) = (\mathbf{w}^\varepsilon - \tilde{\mathbf{w}}_1^\varepsilon), \tag{2.40}$$

where I is an identity operator from $(L^2(\Omega))^3 \to (L^2(\Omega))^3$, and the operator K is a bounded and compact map from $(L^2(\Omega))^3 \to (L^2(\Omega))^3$ shown as in (4.15) of [29, p. 90]. Furthermore, we have the estimates

$$\|\mathbf{u}^\varepsilon - \tilde{\mathbf{u}}_1^\varepsilon\|_{(L^2(\Omega))^3} \le C \|\mathbf{w}^\varepsilon - \tilde{\mathbf{w}}_1^\varepsilon\|_{(L^2(\Omega))^3} \tag{2.41}$$

and

$$\|\mathbf{u}^\varepsilon - \tilde{\mathbf{u}}_1^\varepsilon\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\Big\{ \|\mathbf{w}^\varepsilon - \tilde{\mathbf{w}}_1^\varepsilon\|_{(L^2(\Omega))^3} + \|\mathbf{u}^\varepsilon - \tilde{\mathbf{u}}_1^\varepsilon\|_{(L^2(\Omega))^3} \Big\}. \tag{2.42}$$

Combining (2.39), (2.41), and (2.42), we find

$$\|\mathbf{u}^\varepsilon - \tilde{\mathbf{u}}_1^\varepsilon\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\varepsilon.$$

Since $\tilde{\mathbf{u}}_1^\varepsilon(\mathbf{x}) = \mathbf{u}_1^\varepsilon(\mathbf{x}) - \nabla \eta^\varepsilon(\mathbf{x})$, using (2.23) and (2.24), we derive

$$\|\mathbf{u}^\varepsilon - \mathbf{u}_1^\varepsilon\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\varepsilon,$$

where C is a constant independent of ε.

We next prove Theorem 2.3 for $s = 2$. Set

$$\mathbf{u}_2^\varepsilon(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \varepsilon \Theta_1(\xi) \mathbf{curl}_\mathbf{x}\, \mathbf{u}^0(\mathbf{x}) + \varepsilon^2 \Theta_2(\xi) \mathbf{curl}_\mathbf{x}{}^2\, \mathbf{u}^0(\mathbf{x}). \tag{2.43}$$

Similar to the analysis above, we derive the following equalities which hold in the sense of distributions:

$$\begin{aligned} \mathbf{curl}\Big(A\left(\frac{\mathbf{x}}{\varepsilon}\right) \mathbf{curl}\, \mathbf{u}_2^\varepsilon(\mathbf{x})\Big) &= \mathbf{curl}_\mathbf{x}(A(\xi)\mathbf{curl}_\mathbf{x} u^0(\mathbf{x})) \\ &+ \varepsilon \mathbf{curl}_\mathbf{x}(A(\xi)\mathbf{curl}_\mathbf{x}(\Theta_1(\xi)\mathbf{curl}_\mathbf{x} u^0(\mathbf{x}))) \\ &+ \varepsilon^2 \mathbf{curl}_\mathbf{x}(A(\xi)\mathbf{curl}_\mathbf{x}\,[\Theta_2(\xi)\mathbf{curl}_\mathbf{x}(\mathbf{curl}_\mathbf{x} u^0)]) \\ &+ \mathbf{curl}_\mathbf{x}(A(\xi)\mathbf{curl}_\xi \Theta_1(\xi)\mathbf{curl}_\mathbf{x} u^0(\mathbf{x})) \\ &+ \varepsilon \mathbf{curl}_\mathbf{x}(A(\xi)\mathbf{curl}_\xi \Theta_2(\xi)\mathbf{curl}_\mathbf{x}(\mathbf{curl}_\mathbf{x} u^0)) \\ &+ \mathbf{curl}_\xi(A(\xi)\mathbf{curl}_\mathbf{x}(\Theta_1(\xi)\mathbf{curl}_\mathbf{x} u^0(\mathbf{x}))) \\ &+ \varepsilon \mathbf{curl}_\xi(A(\xi)\mathbf{curl}_\mathbf{x}[\Theta_2(\xi)\mathbf{curl}_\mathbf{x}(\mathbf{curl}_\mathbf{x} u^0)]) \\ &+ \mathbf{curl}_\xi(A(\xi)\mathbf{curl}_\xi \Theta_2(\xi))\mathbf{curl}_\mathbf{x}(\mathbf{curl}_\mathbf{x} u^0(\mathbf{x})). \end{aligned} \tag{2.44}$$

Here we have used the conclusions of Propositions 2.1 and 2.2.

By a arguments similar to the case for $s = 1$, Theorem 2.3 follows. $\square$

REMARK 2.7. *The error estimates are obtained provided that the solution* $\mathbf{u}^0(\mathbf{x})$ *of the homogenized Maxwell's equations* (2.7) *with constant coefficients is smooth enough. Rigorous regularity analysis for the solutions of three dimensional time-harmonic Maxwell's equations is very challenging. We refer the reader to articles reported in* [13, 14, 30].

REMARK 2.8. *We recall that, in Theorem* 2.3, $\mathbf{u}_1^\varepsilon(\mathbf{x})$, $\mathbf{u}_2^\varepsilon(\mathbf{x})$ *are the first-order and the second-order multiscale asymptotic expansions, respectively. We obtain theoretically the same convergence rate with the order of* ε*. However, the numerical results presented in section* 5 *clearly show that the second-order corrector terms are crucial.*

REMARK 2.9. *We assume that the whole domain* Ω *is the union of entire cells in Theorem* 2.3*. So far we cannot determine whether Theorem* 2.3 *is valid or not in a general domain.*

3. Edge finite element computations.

In this section we will discuss the finite element computations of the solutions for the related problems on the basis of the multiscale asymptotic expansions (2.11). We will use the edge finite elements originated from Nédélec (cf. [33]), which have a wide range of applications in solving the Maxwell's equations.

3.1. Finite element computations of cell functions in unit cell Q.

We first present the adaptive edge finite elements for solving the cell problems (2.2) and (2.6) and derive the convergence results.

The variational problem of (2.2) is to find $\Theta_1^p(\xi) \in \mathbf{H}_0(\mathbf{curl}; Q)$ such that

$$b(\Theta_1^p, \mathbf{v}) = -\int_Q A(\xi)\mathbf{e}_p \cdot \mathbf{curl}_\xi \mathbf{v}(\xi) d\xi \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; Q), \quad p = 1, 2, 3, \tag{3.1}$$

where the bilinear form

$$b(\mathbf{u}, \mathbf{v}) = \int_Q \mathbf{curl}_\xi \mathbf{u}(\xi) \cdot \mathbf{curl}_\xi \mathbf{v}(\xi) d\xi \tag{3.2}$$

and the reference cell $Q = (0, 1)^3$.

Let $\mathcal{F}^{h_0} = \{K\}$ be a regular family of tetrahedrons of the reference cell Q, $h_0 = \max_K\{h_K\}$. We define the finite element subspace of $\mathbf{H}_0(\mathbf{curl}; Q)$ consisting of linear edge elements by

$$W_{h_0} = \{\mathbf{v}_{h_0} \in \mathbf{H}(\mathbf{curl}; Q) : \quad \mathbf{v}_{h_0}|_K \in R_1,\ \mathbf{v}_{h_0} \times \nu = 0 \text{ on } \partial Q\}, \tag{3.3}$$

where ν is the outward unit normal to the boundary ∂Q. For the linear edge elements, Nédélec (cf. [33]) defined

$$R_1 = \{\mathbf{v}(\mathbf{x}) = \mathbf{a} + \mathbf{b} \times \mathbf{x}, \quad \mathbf{a}, \mathbf{b} \in R^3\},$$

where the six constants (the components of $\mathbf{a}$ and $\mathbf{b}$) are determined from the moments $\int_{\partial K} \mathbf{v} \cdot \tau ds$ on the six edges of an element K, and τ is the unit tangent vector of K. Direct computation shows that the basis functions with unit integral on the edge joining vertices i and j are given by

$$\psi_{i,j} = \lambda_i \nabla\lambda_j - \lambda_j \nabla\lambda_i,$$

where λ_i is the barycentric coordinate function corresponding to node $\mathbf{a}_i$. We refer the interested reader to Monk's book [29, Chap. V, p. 139]; see also [33].

The discrete variational problem of (3.1) is to find $\Theta^p_{1,h_0} \in W_{h_0}$ such that

$$b(\Theta^p_{1,h_0}, \mathbf{v}_{h_0}) = -\int_Q A(\xi)\mathbf{e}_p \cdot \mathbf{curl}_\xi \mathbf{v}_{h_0}(\xi) d\xi \quad \forall \mathbf{v}_{h_0} \in W_{h_0}, \tag{3.4}$$

where $p = 1, 2, 3$.

We have the following propositions.

PROPOSITION 3.1. *Let $\Theta^p_1(\xi)$, $p = 1, 2, 3$, be the weak solutions of problems* (2.2)*, and let $\Theta^p_{1,h_0}(\xi)$, $p = 1, 2, 3$, be the corresponding finite element solutions in W_{h_0}. If the elements $a_{ij}(\xi) \in C^0(Q)$ of the coefficients matrix $A(\xi)$, $\Theta^p_1 \in \mathbf{H}^\sigma(Q)$, $\nabla \times \Theta^p_1 \in \mathbf{H}^\sigma(Q)$ for some σ with $\frac{1}{2} < \sigma \leq 1$, then it holds that*

$$\|\Theta^p_1 - \Theta^p_{1,h_0}\|_{\mathbf{H}(\mathbf{curl};Q)} \leq Ch_0^\sigma \Big(\|\Theta^p_1\|_{\mathbf{H}^\sigma(Q)} + \|\nabla \times \Theta^p_1\|_{\mathbf{H}^\sigma(Q)}\Big), \quad 0 < \sigma \leq 1, \tag{3.5}$$

where C is a constant independent of the mesh parameter h_0.

Following the reasoning of the proof of Theorem 7.1 of [29, p. 169], we can complete the proof of Proposition 3.1.

REMARK 3.1. *It is well known that the solution of the time-harmonic Maxwell's equations could have much stronger singularities than the corresponding Dirichlet or Neumann singular functions of the Laplace operator when the computational domain is nonconvex or the structure of composite materials. In these situations, the regularity of the solution is only in H^σ with $0 < \sigma < \frac{1}{2}$; see* [10]*, and also see* [13]*. For the structures of composite materials (i.e., the elements $a_{ij}(\xi)$ of the coefficients matrix $A(\xi)$ of problem* (2.2) *are discontinuous), it might be difficult to obtain similar convergence results to Proposition* 3.1*. But, following the reasoning of the proof of Theorem* 7.25 *of* [29, p. 187]*, we can derive the pointwise convergence results via collective compactness.*

In order to compute $\zeta^p_2(\xi)$ and $\Theta^p_2(\xi)$, from (2.4) and (2.6), we need to numerically solve problem (2.4). To this end, let $\mathcal{F}^{h_0} = \{K\}$ be a regular family of tetrahedrons of the reference cell Q, $h_0 = \max_K\{h_K\}$. We define a linear finite element space given by

$$S^0_{h_0}(Q) = \{v_{h_0} \in C^0(\overline{Q}) : \quad v_{h_0}|_K \in P_1,\, v_{h_0}|_{\partial Q} = 0\} \subset H^1_0(Q), \tag{3.6}$$

where P_1 is the set of all piecewise linear polynomials.

The discrete variational problem of (2.4) is to find $\zeta^p_{2,h_0} \in S^0_{h_0}(Q)$ such that

$$\int_Q \nabla\zeta^p_{2,h_0} \cdot \nabla v_{h_0} d\xi = \int_Q \tilde{G}(\xi) \cdot \nabla v_{h_0} d\xi, \tag{3.7}$$

where $p = 1, 2, 3$.

As usual, we have the following error estimates.

PROPOSITION 3.2. *If $\tilde{G} \in (L^2(Q))^3$, then we have*

$$\|\zeta^p_2 - \zeta^p_{2,h_0}\|_{1,Q} \leq Ch_0, \quad p = 1, 2, 3, \tag{3.8}$$

where C is a constant independent of h_0, ε.

A weak solution of the cell problem (2.6) is to find $\Theta_2^p(\xi) \in \mathbf{H}_0(\mathbf{curl}; Q)$ such that

$$\begin{aligned} b(\Theta_2^p, \mathbf{v}) &= -\int_Q A(\xi)\Theta_1^p(\xi) \cdot \mathbf{curl}_\xi \mathbf{v} d\xi + \int_Q \tilde{G}(\xi) \cdot \mathbf{v} d\xi \\ &- \int_Q \nabla_\xi \zeta_2^p(\xi) \cdot \mathbf{v} d\xi \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; Q), \quad p = 1, 2, 3, \end{aligned} \tag{3.9}$$

where the bilinear form $b(\mathbf{u}, \mathbf{v})$ is defined as in (3.2), $\Theta_1^p(\xi)$, $p = 1, 2, 3$, are given as in (3.1), and $\tilde{G}(\xi)$ is defined as in (2.4).

The discrete variational problem of (3.9) is to find $\Theta_{2,h_0}^p \in W_{h_0}$ such that

$$\begin{aligned} b(\Theta_{2,h_0}^p, \mathbf{v}_{h_0}) &= -\int_Q A(\xi)\Theta_{1,h_0}^p(\xi) \cdot \mathbf{curl}_\xi \mathbf{v}_{h_0} d\xi + \int_Q \tilde{G}_{h_0}(\xi) \cdot \mathbf{v}_{h_0} d\xi \\ &- \int_Q \nabla_\xi \zeta_{2,h_0}^p(\xi) \cdot \mathbf{v}_{h_0} d\xi \quad \forall \mathbf{v}_{h_0} \in W_{h_0}, \quad p = 1, 2, 3, \end{aligned} \tag{3.10}$$

where

$$\tilde{G}_{h_0}(\xi) = -A(\xi)\mathbf{curl}_\xi \Theta_{1,h_0}^p(\xi) - A(\xi)\mathbf{e}_p + \hat{A}^{h_0}\mathbf{e}_p$$

and

$$\hat{A}^{h_0} = \int_Q [A(\xi) + A(\xi)\, \mathbf{curl}_\xi\, \Theta_{1,h_0}(\xi)] d\xi.$$

PROPOSITION 3.3. *We can verify that the constant coefficients matrix $\hat{A}^{h_0}$ is a Hermitian matrix. Furthermore, if h_0 is sufficiently small , then it holds that*

$$\mathcal{R}\Big(\eta^* \hat{A}^{h_0} \eta\Big) \geq \bar{\mu}_0 |\eta|^2 \quad \forall \eta \in C^n, \quad a.e. \quad \mathbf{x} \in \Omega, \tag{3.11}$$

where η^* is the Hermitian of a vector η, $|\eta|^2 = \eta_j \eta_j$, $\bar{\mu}_0$ is a constant independent of h_0, and $\mathcal{R}(u)$ denotes the real part of u.

Proof. We first prove that $\hat{A}^{h_0}$ is a Hermitian matrix. To this end, we recall (2.8) and let the scalar function $\chi_{h_0}^j(\xi)$ be the finite element approximate solution of $\chi^j(\xi)$ in $S_{h_0}^0(Q)$. Let us introduce the notation $\chi_{h_0}(\xi) = (\chi_{h_0}^1(\xi), \chi_{h_0}^2(\xi), \chi_{h_0}^3(\xi))$, $A^{-1}(\xi) = (b_{ij}(\xi))$, $\hat{A}^{-1} = (\hat{b}_{ij})$, $(\hat{A}^{h_0})^{-1} = (\hat{b}_{ij}^{h_0})$. It is easy to check that

$$\hat{b}_{ij}^{h_0} = \int_Q \left[b_{ij}(\xi) + b_{ik}(\xi) \frac{\partial \chi_{h_0}^k}{\partial \xi_k} \right] d\xi. \tag{3.12}$$

From (2.8), we can get

$$\hat{b}_{ij}^{h_0} = \int_Q \frac{\partial}{\partial \xi_l} (\chi_{h_0}^i(\xi) + \xi_i) b_{lm}(\xi) \frac{\partial}{\partial \xi_m} (\chi_{h_0}^j(\xi) + \xi_j) d\xi. \tag{3.13}$$

We know $A^{-1}(\xi) = (b_{lm}(\xi))$ is a Hermitian matrix because $A(\xi) = (a_{ij}(\xi))$ is a Hermitian matrix. Equation (3.13) gives $\hat{b}_{ij}^{h_0} = \hat{b}_{ji}^{h_0}$. Thus we verify that $\hat{A}^{h_0}$ is a Hermitian matrix.

We can check that

$$\hat{b}_{ij}^{h_0} - \hat{b}_{ij} = \int_Q \frac{\partial}{\partial \xi_l} (\chi_{h_0}^i(\xi) - \chi^i(\xi)) b_{lm}(\xi) \frac{\partial}{\partial \xi_m} (\chi_{h_0}^j(\xi) - \chi^j(\xi)) d\xi. \tag{3.14}$$

Using the fact that $\|\chi^i_{h_0}(\xi) - \chi^i(\xi)\|_{1,Q} \le Ch_0$, we get

$$|\hat{b}^{h_0}_{ij} - \hat{b}_{ij}| \le Ch_0^2, \tag{3.15}$$

and consequently

$$\|(\hat{A}^{h_0})^{-1} - \hat{A}^{-1}\|_F \le Ch_0^2, \tag{3.16}$$

where $\|B\|_F$ denotes the Frobenius norm of the matrix B, and C is a constant independent of h_0.

Under the assumptions of (A_4), if $\Theta_1^p \in \mathbf{H}_0(\mathbf{curl};\Omega)$, using (2.10), then we have

$$\|\hat{A}\|_F \le C. \tag{3.17}$$

Furthermore, by using Proposition 3.1, for a sufficiently small $h_0 > 0$, we can prove that

$$\|\hat{A}^{h_0}\|_F \le \|\hat{A}\|_F + \|\hat{A}^{h_0} - \hat{A}\|_F \le C(1 + h_0^\sigma) \le C, \quad \frac{1}{2} < \sigma \le 1, \tag{3.18}$$

where C is a constant independent of h_0.

It is obvious that

$$\hat{A}^{h_0} - \hat{A} = -\hat{A}((\hat{A}^{h_0})^{-1} - \hat{A}^{-1})\hat{A}^{h_0}. \tag{3.19}$$

From (3.16)–(3.19), we get

$$\|\hat{A}^{h_0} - \hat{A}\|_F \le Ch_0^2, \tag{3.20}$$

where C is a constant independent of h_0.

Inequality (3.11) follows by choosing a sufficiently small $h_0 > 0$. The proof of Proposition 3.3 is complete. ∎

PROPOSITION 3.4. *Let $\Theta_2^p(\xi)$, $p = 1, 2, 3$, be the weak solutions of problems* (2.6)*, and let $\Theta^p_{2,h_0}(\xi)$, $p = 1, 2, 3$, be the corresponding finite element solutions defined as in* (3.10) *in W_{h_0}. If the elements $a_{ij}(\xi) \in C^0(Q)$ for the coefficients matrix $A(\xi)$, $\Theta_2^p \in \mathbf{H}^\sigma(Q)$, $\nabla \times \Theta_2^p \in \mathbf{H}^\sigma(Q)$ for some σ with $\frac{1}{2} < \sigma \le 1$, then it holds that*

$$\|\Theta_2^p - \Theta^p_{2,h_0}\|_{\mathbf{H}(\mathbf{curl};Q)} \le Ch_0^\sigma, \quad 0 < \sigma \le 1, \tag{3.21}$$

where C is a constant independent of the mesh parameter h_0.

Using Propositions 3.1 and 3.2 and (3.20), we can complete the proof of Proposition 3.4.

REMARK 3.2. *Similarly to Remark* 3.1*, when the elements $a_{ij}(\xi)$ of the coefficients matrix $A(\xi)$ of problems* (2.6) *are discontinuous, it is difficult to obtain the convergence results of Proposition* 3.4 *with explicit convergence rates. But, using Remark* 3.1*,* (3.8)*, and* (3.20)*, and following the reasoning of the proof of Theorem* 7.25 *of* [29, p. 187]*, we can derive the pointwise convergence results about Θ^p_{2,h_0}.*

REMARK 3.3. *It is well known that the regularity for the interface problem can be much lower, even with regular data (see, e.g.,* [13]*). In this case the H^1-conforming discretization cannot be used directly to solve the cell problems* (2.2) *and* (2.6)*. Next we use an adaptive multilevel method presented in* [10]*. For convenience, here we give a posteriori error estimates for solving cell problems* (2.2)*. In solving* (2.4) *and* (2.6)*, we take the same mesh as* (2.2)*. We first introduce the notation: Let $\mathcal{T}_k$ be a* *sequence of tetrahedrons of the reference cell Q and $\mathcal{F}_k$ be the set of faces not lying on ∂Q, $k \geq 0$. The finite element space U_k over $\mathcal{T}_k$ is defined by*

$$\begin{aligned} U_k = \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}; Q) : \quad & \mathbf{v} \times \nu|_{\partial Q} = 0 \quad \textit{and} \\ & \mathbf{v}|_T = \mathbf{a}_T + \mathbf{b}_T \times \mathbf{x} \quad \textit{with} \quad \mathbf{a}_T, \mathbf{b}_T \in R^3 \quad \forall T \in \mathcal{T}_k\}. \end{aligned}$$

Degrees of freedom on every $T \in \mathcal{T}_k$ are $\int_{E_i} \mathbf{v} \cdot d\mathbf{l}$, $i = 1, \ldots, 6$, where $E_1, \ldots, E_6$ are six edges of T. For any $T \in \mathcal{T}_k$ and $F \in \mathcal{F}_k$, we denote the diameters of T and F by h_T and h_F, respectively.

Let $\Theta_{1,k}^p$, $p = 1, 2, 3$, denote the finite element approximate solutions of Θ_1^p, $p = 1, 2, 3$, in the finite element space U_k, respectively. To derive a posteriori error estimates, we introduce the Scott–Zhang operator $\mathcal{I}_k : H_0^1(Q) \to V_k$ [41] and the Beck–Hiptmair–Hoppe–Wohlmuth operator II_k: $\mathbf{H}^1(Q) \cap \mathbf{H_0}(\mathbf{curl}; Q) \to \mathbf{U}_k$ [5], where V_k is the piecewise linear $\mathbf{H_0}(\mathbf{curl}; Q)$-conforming finite element space over $\mathcal{T}_k$ defined by

$$V_k = \{w \in H_0^1(Q) : \quad w|_T = a_T + \mathbf{b}_T \cdot \mathbf{x} \quad \text{ with } a_T \in R^1 \text{ and } \mathbf{b}_T \in R^3 \ \ \forall T \in \mathcal{T}_k\}.$$

Following the reasoning of Theorems 3.3 and 3.4 of [10], we can give a posteriori error estimates for Θ_1^p, $p = 1, 2, 3$, given by

$$\|\Theta_1^p(\xi) - \Theta_{1,k}^p(\xi)\|^2_{\mathbf{H}_0(\mathbf{curl}; Q)} \leq C\Big(\sum_{T \in \mathcal{T}_k} \eta_T^2 + \sum_{F \in \mathcal{F}_k} \eta_F^2\Big), \tag{3.22}$$

where

$$\begin{aligned} \eta_T^2 &= h_T^2 \|\mathbf{curl}\, A(\xi)\mathbf{e}_p + \mathbf{curl}(A(\xi)\mathbf{curl}\, \Theta_{1,k}^p(\xi))\|^2_{0,T} \\ \eta_F^2 &= h_F \|[(A(\xi)\mathbf{e}_p + A(\xi)\mathbf{curl}\, \Theta_{1,k}^p(\xi)) \times \nu]|_F\|^2_{0,F}. \end{aligned}$$

3.2. Finite element computation for the homogenized Maxwell's equations. We now discuss the finite element computation for the homogenized Maxwell's equations. We introduce superconvergence results for the finite element method and present the postprocessing technique.

We recall (2.7) and (3.10). In practice, we have to solve the modified homogenized Maxwell's equations as follows:

$$\begin{cases} \mathbf{curl}(\hat{A}^{h_0}\, \mathbf{curl}\, \tilde{\mathbf{u}}^0(\mathbf{x})) - \omega^2 \tilde{\mathbf{u}}^0(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ \nabla \cdot \tilde{\mathbf{u}}^0(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \\ \tilde{\mathbf{u}}^0(\mathbf{x}) \times \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega, \end{cases} \tag{3.23}$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is the outward unit normal to $\partial\Omega$, and $\hat{A}^{h_0}$ is as given in (3.10).

The variational problem of (3.23) is to find $\tilde{\mathbf{u}}^0 \in \mathbf{H}_0(\mathbf{curl}; \Omega)$ such that

$$A(\tilde{\mathbf{u}}^0, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} d\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega), \tag{3.24}$$

where the bilinear form $A(\tilde{\mathbf{u}}^0, \mathbf{v})$ is defined by

$$A(\mathbf{u}, \mathbf{v}) = \int_\Omega (\hat{A}^{h_0} \mathbf{curl}\, \mathbf{u} \cdot \mathbf{curl} \mathbf{v} - \omega^2 \mathbf{u} \cdot \mathbf{v}) d\mathbf{x}. \tag{3.25}$$

Let $\tau_h = \{e\}$ be a regular family of tetrahedrons of the whole domain Ω, $h = \max_e\{h_e\}$. We define the finite element subspace of $\mathbf{H}_0(\mathbf{curl}; \Omega)$ consisting of degree-k edge elements by

$$X_h = \{\mathbf{u}_h \in \mathbf{H}(\mathbf{curl}; \Omega) : \quad \mathbf{u}_h|_e \in R_k,\, \mathbf{u}_h \times \mathbf{n} = 0 \text{ on } \partial\Omega\}, \tag{3.26}$$

where $\mathbf{n}$ is the outward unit normal to the boundary $\partial\Omega$ and R_k is defined as in (5.32) of [29, p. 128].

The problem of approximating $\tilde{\mathbf{u}}^0$ by finite elements then reduces to using the finite element space X_h defined in (3.26) and computing $\tilde{\mathbf{u}}_h^0 \in X_h$ such that

$$A(\tilde{\mathbf{u}}_h^0, \mathbf{v}_h) = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h d\mathbf{x} \quad \forall \mathbf{v}_h \in X_h. \tag{3.27}$$

LEMMA 3.5 (see [29, p. 169]). *Let Ω be a simply connected Lipschitz domain with connected boundary $\partial\Omega$. Let τ_h be a regular mesh and suppose X_h is given as in* (3.26)*. Then if $\tilde{\mathbf{u}}^0$ satisfies* (3.24) *and $\tilde{\mathbf{u}}_h^0 \in X_h$ satisfies* (3.27)*, there is a constant C independent of h, $\tilde{\mathbf{u}}^0$, and $\tilde{\mathbf{u}}_h^0$ and a h_0 independent of $\tilde{\mathbf{u}}^0$ and $\tilde{\mathbf{u}}_h^0$ such that, for $0 < h < h_0$,*

$$\|\tilde{\mathbf{u}}^0 - \tilde{\mathbf{u}}_h^0\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le \frac{1}{1 - Ch^{1/2+\delta}} \inf_{\mathbf{v}_h \in X_h} \|\tilde{\mathbf{u}}^0 - \mathbf{v}_h\|_{\mathbf{H}(\mathbf{curl};\Omega)}, \quad \delta > 0. \tag{3.28}$$

Furthermore, if $\tilde{\mathbf{u}}^0 \in \mathbf{H}^\gamma(\Omega)$, $\nabla \times \tilde{\mathbf{u}}^0 \in \mathbf{H}^\gamma(\Omega)$ for some γ with $\frac{1}{2} < \gamma \le k$, then we have

$$\|\tilde{\mathbf{u}}^0 - \tilde{\mathbf{u}}_h^0\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le Ch^\gamma, \quad \frac{1}{2} < \gamma \le k, \tag{3.29}$$

where C is a constant independent of h.

PROPOSITION 3.6. *Let $\mathbf{u}^0(\mathbf{x})$ and $\tilde{\mathbf{u}}^0(\mathbf{x})$ be the solutions of problems* (2.7) *and* (3.23)*, respectively. Under the assumptions of Lemma* 3.5*, we can prove that*

$$\|\mathbf{u}^0 - \tilde{\mathbf{u}}^0\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le Ch_0^2. \tag{3.30}$$

Furthermore, if $\tilde{\mathbf{u}}^0 \in \mathbf{H}^k(\Omega)$, $\nabla \times \tilde{\mathbf{u}}^0 \in \mathbf{H}^k(\Omega)$ for $k \ge 1$, then we have

$$\|\mathbf{u}^0 - \tilde{\mathbf{u}}_h^0\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\Big\{h^k + h_0^2\Big\}, \tag{3.31}$$

where C is a constant independent of h_0, h.

Proof. Subtracting (2.7) from (3.23), we get

$$\begin{cases} \mathbf{curl}(\hat{A}\,\mathbf{curl}\,(\mathbf{u^0} - \tilde{\mathbf{u}}^\mathbf{0})) - \omega^2(\mathbf{u^0} - \tilde{\mathbf{u}}^0) \\ \quad = \mathbf{curl}((\hat{A}^{h_0} - \hat{A})\,\mathbf{curl}\,\tilde{\mathbf{u}}^\mathbf{0}), \quad \mathbf{x} \in \Omega, \\ \nabla \cdot (\mathbf{u^0} - \tilde{\mathbf{u}}^0) = 0, \quad \mathbf{x} \in \Omega, \\ (\mathbf{u^0} - \tilde{\mathbf{u}}^0) \times \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega. \end{cases} \tag{3.32}$$

Following the reasoning of the proof of Theorem 4.17 of [29, p. 95], and using Proposition 3.3, we derive

$$\|\tilde{\mathbf{u}}^0\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\|\mathbf{f}\|_{0,\Omega}, \tag{3.33}$$

where C is a constant independent of h_0.

By means of the trick of [29, p. 89], we obtain

$$\|\mathbf{u}^0 - \tilde{\mathbf{u}}^0\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\|\hat{A}^{h_0} - \hat{A}\|_F \|\tilde{\mathbf{u}}^0\|_{\mathbf{H}(\mathbf{curl};\Omega)}. \tag{3.34}$$

Combining (3.20), (3.33), and (3.34) yields

$$\|\mathbf{u}^0 - \tilde{\mathbf{u}}^0\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le Ch_0^2, \tag{3.35}$$

where C is a constant independent of h_0.

Using the triangle inequality and Lemma 3.5, we have

$$\begin{aligned}\|\mathbf{u}^0-\tilde{\mathbf{u}}_h^0\|_{\mathbf{H}(\mathbf{curl};\Omega)} &\le \|\mathbf{u}^0-\tilde{\mathbf{u}}^0\|_{\mathbf{H}(\mathbf{curl};\Omega)}+\|\tilde{\mathbf{u}}^0-\tilde{\mathbf{u}}_h^0\|_{\mathbf{H}(\mathbf{curl};\Omega)}\\ &\le C\Big\{h^k+h_0^2\Big\}.\end{aligned}\tag{3.36}$$

Therefore, the proof of Proposition 3.6 is complete. □

To implement the postprocessing technique for calculating higher-order derivatives $\mathbf{curl}\,\tilde{\mathbf{u}}^0(\mathbf{x})$, $\mathbf{curl}\,(\mathbf{curl}\,\tilde{\mathbf{u}}^0(\mathbf{x}))$, we now introduce the superconvergence results of [28]. Here we assume that Ω is the union of entire cells. Let τ_h be a cubic mesh on Ω with the largest size h.

LEMMA 3.7 (see [28]). *Let $\tilde{\mathbf{u}}^0$ be the solution of* (3.23) *and $\tilde{\mathbf{u}}_h^0$ be the finite element solution of $\tilde{\mathbf{u}}^0$ in the finite element space consisting of degree-k edge elements. If $\tilde{\mathbf{u}}^0\in\mathbf{H}^{k+2}(\Omega)$, then one can prove that*

$$\|\tilde{\mathbf{u}}^0-\Pi_{2h}\tilde{\mathbf{u}}_h^0\|_{0,\Omega}\le Ch^{k+1},\tag{3.37}$$

$$\|\nabla\times(\tilde{\mathbf{u}}^0-\Pi_{3h}\tilde{\mathbf{u}}_h^0)\|_{0,\Omega}\le Ch^{k+1},\tag{3.38}$$

where Π_{2h} and Π_{3h} are the interpolation postprocessing operators as defined in [28], *and C is a constant independent of h_0, h.*

PROPOSITION 3.8. *Let $\mathbf{u}^0(\mathbf{x})$, $\tilde{\mathbf{u}}^0(\mathbf{x})$ be the solutions of problems* (2.7) *and* (3.23), *respectively, and let $\tilde{\mathbf{u}}_h^0$ be the finite element solution of $\tilde{\mathbf{u}}^0$ in the finite element space consisting of degree-k edge elements. Under the assumptions of Proposition* 3.6 *and Lemma* 3.7, *we have*

$$\|\mathbf{u}^0-\Pi_{2h}\tilde{\mathbf{u}}_h^0\|_{0,\Omega}\le C\Big\{h^{k+1}+h_0^2\Big\},\tag{3.39}$$

$$\|\nabla\times(\mathbf{u}^0-\Pi_{3h}\tilde{\mathbf{u}}_h^0)\|_{0,\Omega}\le C\Big\{h^{k+1}+h_0^2\Big\},\tag{3.40}$$

where C is a constant independent of h_0, h.

Proposition 3.8 is the straightforward consequence of Proposition 3.6 and Lemma 3.7.

4. Multiscale finite element method and the error estimates.

We recall (2.11) and summarize the above theoretical results as follows: the multiscale finite element method for solving the time-harmonic Maxwell's equations in composite materials consists of the following parts.

Part I. Compute the matrix-valued cell functions $\Theta_1(\xi)$, $\Theta_2(\xi)$ on a reference cell $Q=(0,1)^3$.

Part II. Solve numerically the modified homogenized Maxwell's equations (3.23) over the whole domain Ω in a coarse mesh.

Part III. Calculate numerically higher-order derivatives $\mathbf{curl}\,\tilde{\mathbf{u}}^0(\mathbf{x})$, $\mathbf{curl}\,(\mathbf{curl}\,\tilde{\mathbf{u}}^0(\mathbf{x}))$ by means of the finite difference method.

We now define the first-order $\mathbf{curl}$ difference quotients given by

$$\delta\tilde{\mathbf{u}}_h^0=\frac{1}{\tau(N_p)}\sum_{e\in\sigma(N_p)}\mathbf{curl}\,\tilde{\mathbf{u}}_h^0|_e(N_p),\tag{4.1}$$

where $\sigma(N_p)$ is the set of elements with node N_p, $\tau(N_p)$ is the number of elements of $\sigma(N_p)$, $\tilde{\mathbf{u}}_h^0$ is the finite element solution of $\tilde{\mathbf{u}}^0$, and $\mathbf{curl}\,\tilde{\mathbf{u}}_h^0|_e(N_p)$ is the value of a function $\mathbf{curl}\,\tilde{\mathbf{u}}_h^0$ at node N_p associated with element e.

Analogously, we define the second-order **curl** difference quotients as

$$\delta^2\tilde{\mathbf{u}}_h^0 = \frac{1}{\tau(N_p)} \sum_{e\in\sigma(N_p)} \sum_{j=1}^{d} \delta\mathbf{u}_h^0(P_j)\,\mathbf{curl}\,\psi_j|_e(N_p), \tag{4.2}$$

where d is the number of nodes on e, P_j, $j = 1, \dots, d$, are the nodes of e, and ψ_j, $j = 1, \dots, d$, are Lagrange's shape functions.

In summary, the multiscale finite element method for solving the Maxwell's equations (1.7) in composite materials with a periodic structure is defined by

$$\mathbf{U}_{1,h}^{\varepsilon,h_0}(\mathbf{x}) = \tilde{\mathbf{u}}_h^0(\mathbf{x}) + \varepsilon\Theta_{1,h_0}(\xi)\delta\tilde{\mathbf{u}}_h^0(\mathbf{x}), \tag{4.3}$$

and

$$\mathbf{U}_{2,h}^{\varepsilon,h_0}(\mathbf{x}) = \tilde{\mathbf{u}}_h^0(\mathbf{x}) + \varepsilon\Theta_{1,h_0}(\xi)\delta\tilde{\mathbf{u}}_h^0(\mathbf{x}) + \varepsilon^2\Theta_{2,h_0}(\xi)\delta^2\tilde{\mathbf{u}}_h^0(\mathbf{x}), \tag{4.4}$$

where the matrix-valued functions $\Theta_{1,h_0}(\xi) = (\Theta_{1,h_0}^1(\xi), \Theta_{1,h_0}^2(\xi), \Theta_{1,h_0}^3(\xi))$, $\Theta_{2,h_0}(\xi) = (\Theta_{2,h_0}^1(\xi), \Theta_{2,h_0}^2(\xi), \Theta_{2,h_0}^3(\xi))$, and $\Theta_{1,h_0}^p(\xi)$, $\Theta_{2,h_0}^p(\xi)$, $p = 1, 2, 3$, are defined as in (3.4) and (3.10), respectively.

To improve the numerical accuracy of the multiscale finite element method, we present a postprocessing technique as follows:

$$P\mathbf{U}_{s,h}^{\varepsilon,h_0}(\mathbf{x}) = S_h\tilde{\mathbf{u}}_h^0(\mathbf{x}) + \sum_{l=1}^{s} \varepsilon^l\Theta_{l,h_0}(\xi)\delta^l S_h\tilde{\mathbf{u}}_h^0(\mathbf{x}), \quad s = 1, 2, \tag{4.5}$$

where $S_h = \Pi_{2h}$ or $S_h = \Pi_{3h}$; see (3.39) and (3.40).

Next we give the final error estimates of the multiscale finite element method for solving the Maxwell's equations in composite materials.

THEOREM 4.1. *Suppose that* $\Omega \subset R^3$ *is the union of entire periodic cells, i.e.,* $\overline{\Omega} = \bigcup_{z\in T_\varepsilon} \varepsilon(z + \overline{Q})$, *where* $T_\varepsilon = \{z \in Z^3,\ \varepsilon(z + \overline{Q}) \subset \overline{\Omega}\}$. *Let* $\mathbf{u}^\varepsilon$ *be the solution of* (1.7), *and let* $\mathbf{U}_{s,h}^{\varepsilon,h_0}$, $P\mathbf{U}_{s,h}^{\varepsilon,h_0}$ *be defined as in* (4.4) *and* (4.5), *respectively. Under the assumptions of Theorem* 2.3 *and Propositions* 3.1 *and* 3.6, *if* $\mathbf{f} \in \mathbf{H}^s(\Omega)$, $\mathbf{u}^0 \in \mathbf{H}^{s+2}(\Omega)$, *then we have*

$$\|\mathbf{u}^\varepsilon - \mathbf{U}_{s,h}^{\varepsilon,h_0}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\Big\{\varepsilon + h_0^\sigma + h^{min(2,k)}\Big\}, \quad s = 1, 2, \tag{4.6}$$

where $\frac{1}{2} < \sigma \le 1$, $k \ge 1$ *is the degree of the edge element space* X_h *for solving the modified homogenized Maxwell's equations,* h_0, h *are the mesh parameters of* Q *and* Ω, *respectively, and* C *is a constant independent of* ε, h_0, *and* h.

Furthermore, we obtain

$$\|\mathbf{u}^\varepsilon - P\mathbf{U}_{s,h}^{\varepsilon,h_0}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\Big\{\varepsilon + h_0^\sigma + h^{min(2,k+1)}\Big\}, \quad s = 1, 2. \tag{4.7}$$

Proof. We prove Theorem 4.1 only in the case $s = 2$. The remainder can be completed similarly.

From (4.4), we have

$$
\begin{aligned}
\mathbf{u}^\varepsilon(\mathbf{x}) - \mathbf{U}_{2,h}^{\varepsilon,h_0}(\mathbf{x}) &= \mathbf{u}^\varepsilon(\mathbf{x}) - \mathbf{u}_2^\varepsilon(\mathbf{x}) + \mathbf{u}_2^\varepsilon(\mathbf{x}) - \mathbf{U}_{s,h}^{\varepsilon,h_0}(\mathbf{x}) \\
&= \mathbf{u}^\varepsilon(\mathbf{x}) - \mathbf{u}_2^\varepsilon(\mathbf{x}) + \mathbf{u}^0(\mathbf{x}) - \tilde{\mathbf{u}}_h^0(\mathbf{x}) \\
&+ \varepsilon(\Theta_1(\xi) - \Theta_{1,h_0}(\xi))\mathbf{curl}_\mathbf{x}\mathbf{u}^0(\mathbf{x}) \\
&+ \varepsilon\Theta_{1,h_0}(\xi)[\mathbf{curl}_\mathbf{x}(\mathbf{u}^0(\mathbf{x}) - \tilde{\mathbf{u}}^0(\mathbf{x})) \\
&+ (\mathbf{curl}_\mathbf{x}\,\tilde{\mathbf{u}}^0(\mathbf{x}) - \delta\tilde{\mathbf{u}}_h^0(\mathbf{x}))] \\
&+ \varepsilon^2(\Theta_2(\xi) - \Theta_{2,h_0}(\xi))\mathbf{curl}_\mathbf{x}{}^2\mathbf{u}^0(\mathbf{x}) \\
&+ \varepsilon^2\Theta_{2,h_0}(\xi)[\mathbf{curl}_\mathbf{x}{}^2(\mathbf{u}^0(\mathbf{x}) - \tilde{\mathbf{u}}^0(\mathbf{x})) \\
&+ (\mathbf{curl}_\mathbf{x}{}^2\tilde{\mathbf{u}}^0(\mathbf{x}) - \delta^2\tilde{\mathbf{u}}_h^0(\mathbf{x}))].
\end{aligned} \tag{4.8}
$$

By using Theorem 2.3 and Propositions 3.6 and 3.1, we can get (4.6). Similarly, by means of Theorem 2.3 and Propositions 3.8 and 3.1, we can prove (4.7).

Therefore, the proof of Theorem 4.1 is complete. ∎

5. Numerical tests. To validate the developed multiscale algorithm and to confirm the theoretical analysis reported in this paper, we present numerical simulations for the following case studies. We consider the time-harmonic Maxwell's equations with rapidly oscillating coefficients given by

$$
\begin{cases}
\mathbf{curl}\left(A\left(\dfrac{\mathbf{x}}{\varepsilon}\right)\mathbf{curl}\,\mathbf{u}^\varepsilon\right) - \omega^2\mathbf{u}^\varepsilon = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}\in\Omega,\\
\nabla\cdot\mathbf{u}^\varepsilon = 0, \quad \mathbf{x}\in\Omega,\\
\mathbf{u}^\varepsilon\times\mathbf{n} = 0, \quad \mathbf{x}\in\partial\Omega.
\end{cases} \tag{5.1}
$$

EXAMPLE 5.1. *In* (5.1), *assume that* $\Omega = (0,1)^3$ *is a periodic structure, where* $\varepsilon = \frac{1}{3}$, $\omega^2 = 1$. *The elements of the coefficients matrix* $A(\frac{\mathbf{x}}{\varepsilon}) = (a_{ij}(\frac{\mathbf{x}}{\varepsilon}))$ *are continuous and rapidly oscillating periodic functions, as in the following.*

Case 5.1.1. Let $\mathbf{x} = (x_1, x_2, x_3)$;

$$
a_{ij}\left(\frac{\mathbf{x}}{\varepsilon}\right) = \frac{20}{\left(2+1.5\sin\left(2\pi\left(\frac{x_1}{\varepsilon}\right)+0.75\right)\right)\left(2+1.5\sin\left(2\pi\left(\frac{x_2}{\varepsilon}\right)+0.75\right)\right)\left(2+1.5\sin\left(2\pi\left(\frac{x_3}{\varepsilon}\right)+0.75\right)\right)}\delta_{ij};
$$

$\mathbf{f} = (30, 30, 30)^T$, δ_{ij} is the Kronecker symbol.

EXAMPLE 5.2. *In* (5.1), *let* $\varepsilon = \frac{1}{3}$, $\omega^2 = 1$, *and assume that the entire domain* $\Omega = (0,1)^3$ *and the reference cell* Q *are shown in Figures* 1.1(a) *and* 1.1(b), *respectively. Let* a_{ij1} *denote the value of* a_{ij} *in the inside ellipsoid of the reference cell* Q, *and let* a_{ij0} *denote the value of* a_{ij} *in the other part of* Q. *Let* δ_{ij} *be a Kronecker symbol, and let* $\mathbf{a}^T$ *denote the transpose of* $\mathbf{a}$. *The equation of an ellipsoid is given by*

$$
\frac{(\xi_1-0.5)^2}{0.16} + \frac{(\xi_2-0.5)^2}{0.16} + \frac{(\xi_3-0.5)^2}{0.16} = 1.
$$

The following three cases are investigated.

Case 5.2.1. $a_{ij0} = 10\delta_{ij}$, $a_{ij1} = \delta_{ij}$; $\mathbf{f} = (100, 100, 100)^T$.

Case 5.2.2. $a_{ij0} = 100\delta_{ij}$, $a_{ij1} = \delta_{ij}$; $\mathbf{f} = (300, 300, 300)^T$.

Case 5.2.3. $a_{ij0} = 1000\delta_{ij}$, $a_{ij1} = \delta_{ij}$; $\mathbf{f} = (100, 100, 100)^T$.

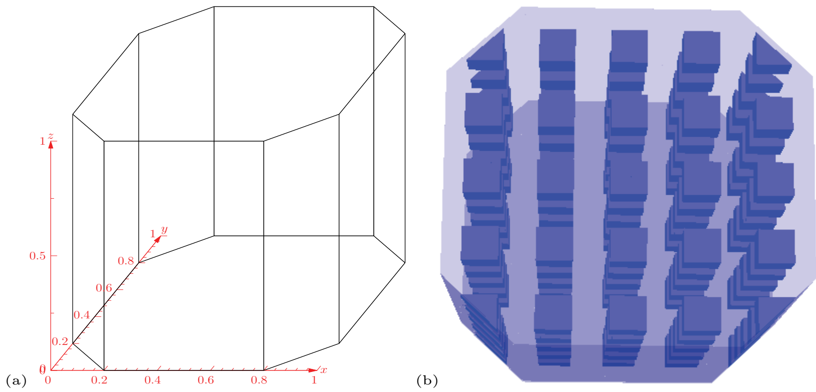


FIG. 5.1. (a) *A whole domain* Ω. (b) *The top view.*

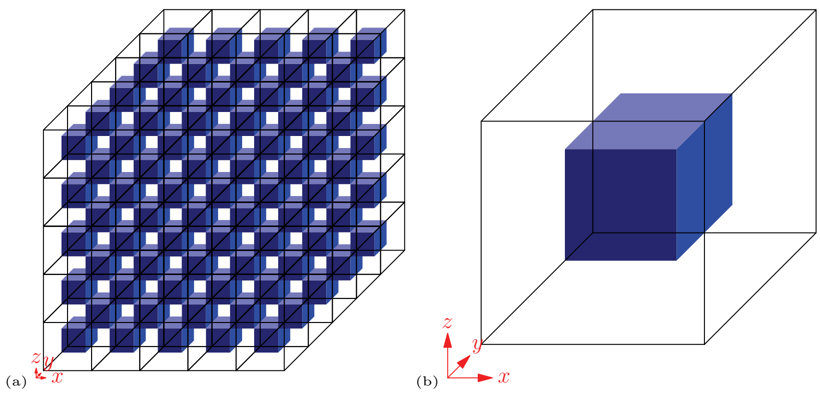


FIG. 5.2. (a) *A periodic structure.* (b) *The reference cell* $Q = (0, 1)^3$.

EXAMPLE 5.3. *In this case study, we consider the same equations as given in Example* 5.2. *But we assume that the entire domain* Ω *is a general convex domain as shown in Figure* 5.1(a) *and* (b) *and the reference cell* Q *is illustrated in Figure* 5.2(b). *Let* $\varepsilon = \frac{1}{5}$, $\omega^2 = 1$. *Let* a_{ij1} *denote the value of* a_{ij} *in the inside cube of the reference cell* Q, *and let* a_{ij0} *denote the value of* a_{ij} *in the other part of* Q.

Case 5.3.1. $a_{ij0} = 100\delta_{ij}$, $a_{ij1} = \delta_{ij}$; $\mathbf{f} = (300, 300, 300)^T$.

Case 5.3.2. $a_{ij0} = 1000\delta_{ij}$, $a_{ij1} = \delta_{ij}$; $\mathbf{f} = (300, 300, 300)^T$.

We will give the numerical results of the solution for problem (5.1) in Cases 5.1.1, 5.2.1–5.2.3, and 5.3.1–5.3.2. In order to show the numerical accuracy of the method presented in this paper, we have to know the exact solution of problem (5.1). However, it is very difficult to find it. We replace it with the finite element solution in a fine mesh. We employ the linear tetrahedral edge elements to solve the original problem (5.1). In real applications, this step is not necessary. We apply linear tetrahedral edge elements to solve cell problems (2.2) and (2.6), and the modified homogenized Maxwell's equations (3.23), respectively. The numbers of elements and degrees of freedom in Examples 5.1 and 5.2 and in Example 5.3 are listed in Tables 5.1 and 5.2, respectively.

It should be noted that $\mathbf{u}^\varepsilon(\mathbf{x})$ denotes the finite element solution in a fine mesh, and $\mathbf{u}^0(\mathbf{x})$ is the finite element solution for the modified homogenized Maxwell's equa-

Table 5.1
Computational cost in Examples 5.1 *and* 5.2.

	Original equations	Cell problem	Homogenized equations
Elements	253987	13882	48000
dof	298963	16876	59660

Table 5.2
Computational cost in Example 5.3.

	Original equations	Cell problem	Homogenized equations
Elements	146839	12000	30548
dof	178910	14930	38766

Table 5.3
Computational errors: Method I.

	$\frac{\|\mathbf{e}_0\|_{(0)}}{\|\mathbf{u}_h^0\|_{(0)}}$	$\frac{\|\mathbf{e}_1\|_{(0)}}{\|\mathbf{U}_{1,h}^{\varepsilon,h_0}\|_{(0)}}$	$\frac{\|\mathbf{e}_2\|_{(0)}}{\|\mathbf{U}_{2,h}^{\varepsilon,h_0}\|_{(0)}}$	$\frac{\|\mathbf{e}_0\|_{(1)}}{\|\mathbf{u}_h^0\|_{(1)}}$	$\frac{\|\mathbf{e}_1\|_{(1)}}{\|\mathbf{U}_{1,h}^{\varepsilon,h_0}\|_{(1)}}$	$\frac{\|\mathbf{e}_2\|_{(1)}}{\|\mathbf{U}_{2,h}^{\varepsilon,h_0}\|_{(1)}}$
Case 5.1.1	0.2274	0.1810	0.1752	0.8103	0.6645	0.6650
Case 5.2.1	0.1436	0.1073	0.0935	0.8616	0.7577	0.7041
Case 5.2.2	0.9352	0.9287	0.1835	3.6309	3.2331	0.8391
Case 5.2.3	9.52967	9.51272	0.2497	33.9376	30.7439	0.7951
Case 5.3.1	0.2559	0.2472	0.1681	1.6305	1.4607	0.8183
Case 5.3.2	2.4285	2.4208	0.4531	13.5377	12.5054	0.9058

Table 5.4
Computational errors: Method II.

	$\frac{\|\mathbf{e}_0\|_{(0)}}{\|\mathbf{u}_h^0\|_{(0)}}$	$\frac{\|\mathbf{e}_1\|_{(0)}}{\|\mathbf{U}_{1,h}^{\varepsilon,h_0}\|_{(0)}}$	$\frac{\|\mathbf{e}_2\|_{(0)}}{\|\mathbf{U}_{2,h}^{\varepsilon,h_0}\|_{(0)}}$	$\frac{\|\mathbf{e}_0\|_{(1)}}{\|\mathbf{u}_h^0\|_{(1)}}$	$\frac{\|\mathbf{e}_1\|_{(1)}}{\|\mathbf{U}_{1,h}^{\varepsilon,h_0}\|_{(1)}}$	$\frac{\|\mathbf{e}_2\|_{(1)}}{\|\mathbf{U}_{2,h}^{\varepsilon,h_0}\|_{(1)}}$
Case 5.1.1	0.3341	0.2858	0.2495	1.0108	0.7711	0.7188
Case 5.2.1	0.1443	0.1027	0.0590	0.8931	0.7732	0.7079
Case 5.2.2	1.151	1.143	0.177	4.2069	3.7566	0.7867
Case 5.2.3	11.700	11.679	0.2573	41.3304	37.491	0.8210
Case 5.3.1	0.2944	0.2826	0.1270	1.8397	1.6465	0.8098
Case 5.3.2	2.9309	2.9217	0.4477	15.998	14.784	0.8705

tions (3.23) in a coarse mesh. Here we employ two kinds of methods to calculate the homogenized coefficients, respectively. In Method I and Method II, we apply a divergence operator and a **curl** operator to compute the homogenized coefficients matrix $\hat{A}$, respectively; see (2.7) and (2.8). Let $\mathbf{U}_{1,h}^{\varepsilon,h_0}$, $\mathbf{U}_{2,h}^{\varepsilon,h_0}$ be, respectively, the first-order and the second-order multiscale finite element solutions based on (4.3) and (4.4). Set $\mathbf{e}_0 = \mathbf{u}^\varepsilon - \mathbf{u}^0$, $\mathbf{e}_1 = \mathbf{u}^\varepsilon - \mathbf{U}_{1,h}^{\varepsilon,h_0}$, $\mathbf{e}_2 = \mathbf{u}^\varepsilon - \mathbf{U}_{2,h}^{\varepsilon,h_0}$. For simplicity, let $\|\mathbf{u}\|_{(0)} = \|\mathbf{u}\|_{(L^2(\Omega))^3}$, $\|\mathbf{u}\|_{(1)} = \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl};\Omega)}$.

The numerical results are illustrated in Tables 5.3 and 5.4, respectively.

Figure 5.3: (a)–(h) clearly show the numerical results of related solutions at the intersection $x_3 = 0.5$ in Case 5.2.2.

Figure 5.4: (a)–(h) clearly show the numerical results of related solutions at the intersection $x_3 = 0.5$ in Case 5.3.1.

Remark 5.1. *From the numerical simulations for the above case studies, we note that when the difference between various materials is large, the homogenization method and the first-order multiscale method fail to provide satisfactory results. The second-order multiscale approach, however, clearly is the best among the three methods, and it results in accurate numerical solutions for the time-harmonic Maxwell's equations*

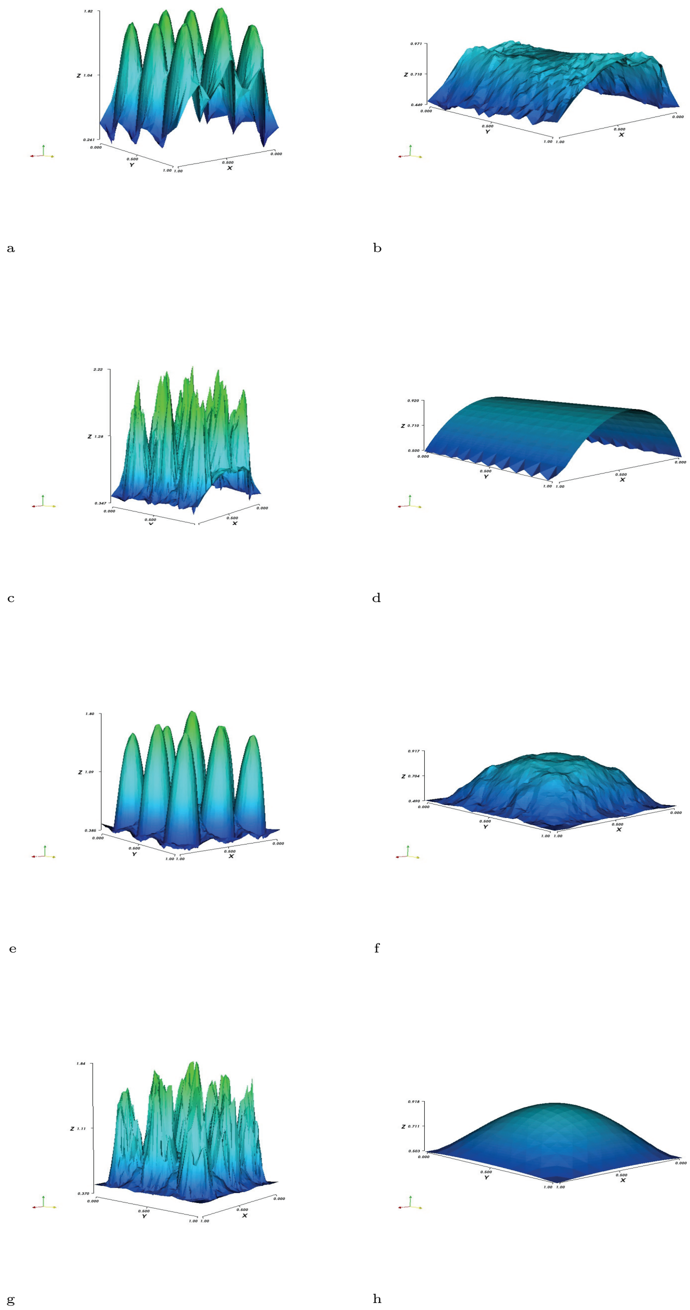


FIG. 5.3. *In Case* 5.2.2, (a) *the second component of the solution* $\mathbf{u}^{\varepsilon}$ *in a fine mesh;* (b) *the second component of the first-order multiscale finite element solution* $\mathbf{U}_{1,h}^{\varepsilon,h_0}$*;* (c) *the second component of the second-order multiscale finite element solution* $\mathbf{U}_{2,h}^{\varepsilon,h_0}$*;* (d) *the second component of the homogenized solution* $\mathbf{u}^0$ *in a coarse mesh;* (e) *the third component of the solution* $\mathbf{u}^{\varepsilon}$ *in a fine mesh;* (f) *the third component of the first-order multiscale finite element solution* $\mathbf{U}_{1,h}^{\varepsilon,h_0}$*;* (g) *the third component of the second-order multiscale finite element solution* $\mathbf{U}_{2,h}^{\varepsilon,h_0}$*;* (h) *the third component of the homogenized solution* $\mathbf{u}^0$ *in a coarse mesh.*

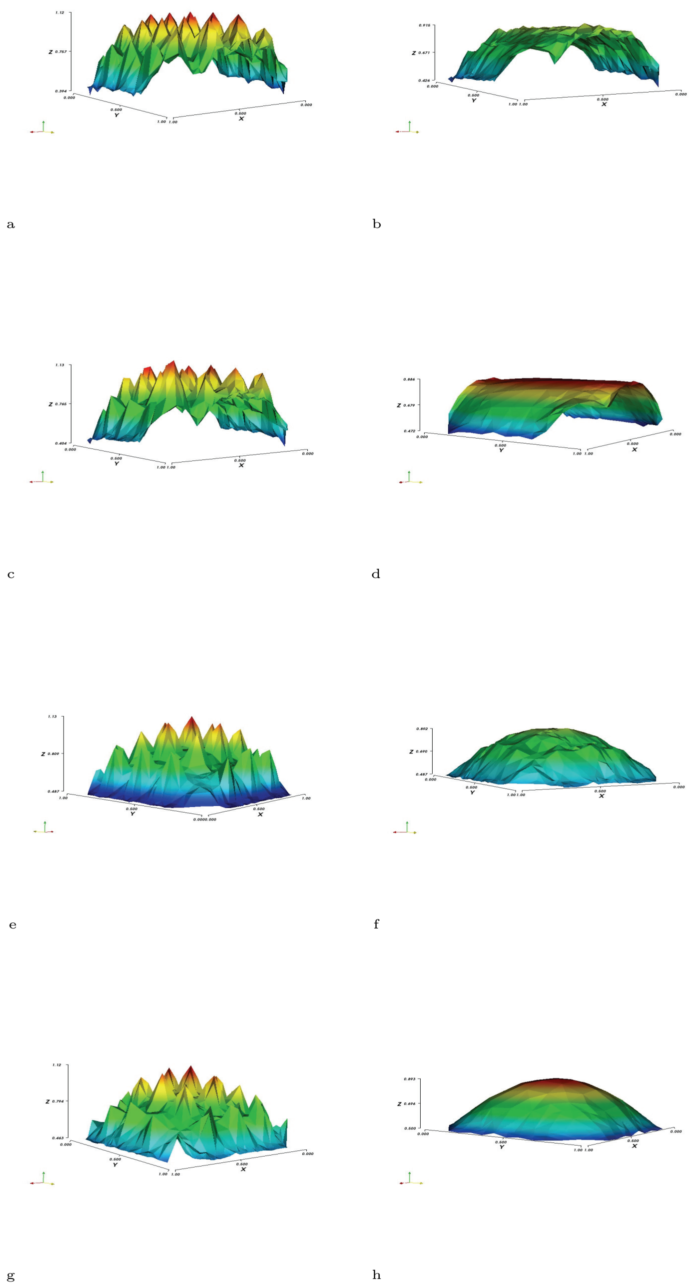


FIG. 5.4. *In Case* 5.3.1, (a) *the second component of the solution* $\mathbf{u}^{\varepsilon}$ *in a fine mesh;* (b) *the second component of the first-order multiscale finite element solution* $\mathbf{U}_{1,h}^{\varepsilon,h_0}$*;* (c) *the second component of the second-order multiscale finite element solution* $\mathbf{U}_{2,h}^{\varepsilon,h_0}$*;* (d) *the second component of the homogenized solution* $\mathbf{u}^0$ *in a coarse mesh;* (e) *the third component of the solution* $\mathbf{u}^{\varepsilon}$ *in a fine mesh;* (f) *the third component of the first-order multiscale finite element solution* $\mathbf{U}_{1,h}^{\varepsilon,h_0}$*;* (g) *the third component of the second-order multiscale finite element solution* $\mathbf{U}_{2,h}^{\varepsilon,h_0}$*;* (h) *the third component of the homogenized solution* $\mathbf{u}^0$ *in a coarse mesh.*

in composite materials. Moreover, the simulation results validate the convergence analysis.

Remark 5.2. *We recall Remark* 2.9 *and cannot determine whether Theorem* 2.3 *is valid or not in a general domain. But the numerical results presented in Example* 5.3 *show that the multiscale method developed in this paper is also effective for a general domain with a periodic microstructure in some cases. This seems to be an unsolved problem.*

Remark 5.3. *When solving Maxwell's equations in a perforated domain, we usually fill the holes with an almost degenerated phase, which is known as the hole-filling method. In practice, engineers often use this method to predict the effective properties of perforated materials. From a physical point of view, when the material properties of the weak phase go to zero, this limit procedure is clear. However, a rigorous mathematical justification has not been reported in the literature. In* [46]*, the authors presented a justification for this limiting process for the elastostatic equations. Now, in order to deal with a perforated domain, we cover the holes with an almost degenerated phase using the method reported* [46]*, and the algorithm developed in this paper is applied to solve the Maxwell's equations in a nonperforated domain.*

Remark 5.4. *From the numerical viewpoint, the multiscale method presented in this work is effective for the subdivided periodic domains (i.e., local periodic structures).*

Conclusions. This paper discussed the multiscale analysis of Maxwell's equations in composite materials with a periodic microstructure. The new contributions obtained in this paper were the determination of the explicit convergence rate for the approximate solutions and higher-order correctors. Consequently, numerical approximation techniques and some numerical examples were presented.

Appendix A. The proofs of Propositions 2.1 and 2.2. In order to prove Propositions 2.1 and 2.2, we first assume that the elements $a_{kk}(\xi)$, $k = 1, 2, 3$, of a matrix $A(\xi)$ are sufficiently smooth functions, i.e., $a_{kk} \in C^\infty(Q)$, $k = 1, 2, 3$, and then extend these results into general cases: $a_{kk} \in L^\infty(Q)$.

Suppose that $a_{kk} \in C^\infty(Q)$, $k = 1, 2, 3$. Here we prove Proposition 2.1 only for $\Theta_1^1(\xi)$. The other cases can be completed similarly.

We recall cell problem (2.2) and have

$$\text{(A.1)} \qquad \begin{cases} \mathbf{curl}_\xi(A(\xi)\mathbf{curl}_\xi\Theta_1^1(\xi)) = -\mathbf{curl}_\xi(A(\xi)\mathbf{e}_1), \quad \xi \in Q, \\ \nabla_\xi \cdot \Theta_1^1(\xi) = 0, \quad \xi \in Q, \\ \Theta_1^1(\xi) \times \nu = 0, \quad \xi \in \partial Q, \quad \mathbf{e}_1 = (1, 0, 0), \end{cases}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is the outward unit normal to ∂Q.

We set $\Theta_1^1(\xi) = (\psi_1(\xi), \psi_2(\xi), \psi_3(\xi))$. From (A.1), under the assumptions of (H_1)–(H_2), we can obtain the following lemma.

Lemma A.1. *Let the reference cell* $Q = \{\xi : 0 < \xi_k < 1,\ k = 1, 2, 3\}$*. Then we can directly check that*

$$\text{(A.2)} \qquad \begin{array}{ll} \psi_1(\xi_1, \xi_2, \xi_3) = -\psi_1(1 - \xi_1, \xi_2, \xi_3), & \psi_2(\xi_1, \xi_2, \xi_3) = \psi_2(1 - \xi_1, \xi_2, \xi_3), \\ \psi_3(\xi_1, \xi_2, \xi_3) = \psi_3(1 - \xi_1, \xi_2, \xi_3), & \psi_1(\xi_1, \xi_2, \xi_3) = -\psi_1(\xi_1, 1 - \xi_2, \xi_3), \\ \psi_2(\xi_1, \xi_2, \xi_3) = \psi_2(\xi_1, 1 - \xi_2, \xi_3), & \psi_3(\xi_1, \xi_2, \xi_3) = -\psi_3(\xi_1, 1 - \xi_2, \xi_3), \\ \psi_1(\xi_1, \xi_2, \xi_3) = -\psi_1(\xi_1, \xi_2, 1 - \xi_3), & \psi_2(\xi_1, \xi_2, \xi_3) = -\psi_2(\xi_1, \xi_2, 1 - \xi_3), \\ \psi_3(\xi_1, \xi_2, \xi_3) = \psi_3(\xi_1, \xi_2, 1 - \xi_3). & \end{array}$$

To begin, we introduce the notation. Set $\partial Q = \bigcup_{j=1}^{6} S_j$, where

$$\begin{array}{ll} S_1 = \{\xi : \quad \xi_1 = 1,\, 0 \le \xi_2, \xi_3 \le 1\}, & S_2 = \{\xi : \quad \xi_1 = 0,\, 0 \le \xi_2, \xi_3 \le 1\}, \\ S_3 = \{\xi : \quad \xi_2 = 1,\, 0 \le \xi_1, \xi_3 \le 1\}, & S_4 = \{\xi : \quad \xi_2 = 0,\, 0 \le \xi_1, \xi_3 \le 1\}, \\ S_5 = \{\xi : \quad \xi_3 = 1,\, 0 \le \xi_1, \xi_2 \le 1\}, & S_6 = \{\xi : \quad \xi_3 = 0,\, 0 \le \xi_1, \xi_2 \le 1\}. \end{array}$$

The outward unit normal vectors on ∂Q are the following:

$$\begin{array}{ll} S_1 : \nu = (1,0,0), & S_2 : \nu = (-1,0,0), \\ S_3 : \nu = (0,1,0), & S_4 : \nu = (0,-1,0), \\ S_5 : \nu = (0,0,1), & S_6 : \nu = (0,0,-1). \end{array}$$

We set $A(\xi)\mathbf{curl}_\xi \Theta_1^1(\xi) = (P, H, R)$, where

$$P = a_{11}(\xi)\Big(\frac{\partial \psi_3}{\partial \xi_2} - \frac{\partial \psi_2}{\partial \xi_3}\Big),\ H = a_{22}(\xi)\Big(\frac{\partial \psi_1}{\partial \xi_3} - \frac{\partial \psi_3}{\partial \xi_1}\Big),\ R = a_{33}(\xi)\Big(\frac{\partial \psi_2}{\partial \xi_1} - \frac{\partial \psi_1}{\partial \xi_2}\Big). \tag{A.3}$$

It is obvious that

$$\begin{array}{ll} A(\xi)\mathbf{curl}_\xi \Theta_1^1(\xi) \times \nu|_{S_1} = (0, R, -H), & A(\xi)\mathbf{curl}_\xi \Theta_1^1(\xi) \times \nu|_{S_2} = (0, -R, H), \\ A(\xi)\mathbf{curl}_\xi \Theta_1^1(\xi) \times \nu|_{S_3} = (-R, 0, P), & A(\xi)\mathbf{curl}_\xi \Theta_1^1(\xi) \times \nu|_{S_4} = (R, 0, -P), \\ A(\xi)\mathbf{curl}_\xi \Theta_1^1(\xi) \times \nu|_{S_5} = (H, -P, 0), & A(\xi)\mathbf{curl}_\xi \Theta_1^1(\xi) \times \nu|_{S_6} = (-H, P, 0). \end{array} \tag{A.4}$$

From (A.3) and (A.4), using Lemma A.1, we can obtain directly

$$[P]|_{\partial Q} = 0, \quad [H]|_{S_5} = 0, \quad [R]|_{S_3} = 0, \tag{A.5}$$

where $[v]|_S$ denotes the jump of a function v on S.

In (A.1), if we set $\Lambda_1(\xi) = A(\xi)[e_1 + \mathbf{curl}_\xi \Theta_1^1(\xi)]$, then we get

$$\mathbf{curl}_\xi\, \Lambda_1(\xi) = 0, \tag{A.6}$$

and consequently

$$\begin{cases} \dfrac{\partial R}{\partial \xi_2} - \dfrac{\partial H}{\partial \xi_3} = 0, \\ \dfrac{\partial \overline{P}}{\partial \xi_3} - \dfrac{\partial R}{\partial \xi_1} = 0, \\ \dfrac{\partial H}{\partial \xi_1} - \dfrac{\partial \overline{P}}{\partial \xi_2} = 0, \quad \overline{P} = (a_{11} + P). \end{cases} \tag{A.7}$$

LEMMA A.2. *Under assumptions* (A_1)–(A_3), (A_1)–(A_2), *we can prove that the corresponding Fourier series of the vector-valued function* $\Lambda_1(\xi)$ *is absolutely uniform convergent on* $\overline{Q}$, $k = 1, 2, 3$.

Following the reasoning of the proof of Lemma A.3 of [9], the proof of Lemma A.2 can be completed.

Let $m \cdot \xi = m_j \xi_j$, where $m = (m_1, m_2, m_3) \in Z^3$, $\xi = (\xi_1, \xi_2, \xi_3)$. By the variational forms of the first equation of (A.7), we take, respectively, test functions

$$\begin{aligned} &v_1(\xi) = (e^{i2\pi m_1 \xi_1} - 1)(e^{i2\pi m_2 \xi_2} - 1)(e^{i2\pi m_3 \xi_3} - 1), \quad m_1 \neq 0,\, m_2 \neq 0,\, m_3 \neq 0, \\ &v_2(\xi) = (e^{i2\pi m_2 \xi_2} - 1)(e^{i2\pi m_3 \xi_3} - 1), \quad m_2 \neq 0,\, m_3 \neq 0, \\ &v_3(\xi) = (e^{i2\pi m_2 \xi_2} - 1), \quad m_2 \neq 0, \\ &v_4(\xi) = (e^{i2\pi m_3 \xi_3} - 1), \quad m_3 \neq 0, \\ &v_5(\xi) = 1 \end{aligned}$$

and get five equations. Similarly, from the variational forms of the second equation of (A.7), we take, respectively, test functions

$$\begin{aligned} &w_1(\xi) = (e^{i2\pi m_1\xi_1} - 1)(e^{i2\pi m_2\xi_2} - 1)(e^{i2\pi m_3\xi_3} - 1), \quad m_1 \neq 0,\, m_2 \neq 0,\, m_3 \neq 0,\\ &w_2(\xi) = (e^{i2\pi m_1\xi_1} - 1)(e^{i2\pi m_3\xi_3} - 1), \quad m_1 \neq 0,\, m_3 \neq 0,\\ &w_3(\xi) = (e^{i2\pi m_1\xi_1} - 1), \quad m_1 \neq 0,\\ &w_4(\xi) = (e^{i2\pi m_3\xi_3} - 1), \quad m_3 \neq 0,\\ &w_5(\xi) = 1 \end{aligned}$$

and also obtain another five equations. To give the above ten equations, here we have used Lemma A.2.

Using (A.5) and combining the above ten equations, we derive

$$\text{(A.8)} \qquad \begin{cases} \int_{S_1} [R] e^{i2\pi(m_2\xi_2 + m_3\xi_3)} d\sigma_\xi = 0, \quad m_2 \neq 0,\, m_3 \neq 0,\\ \int_{S_1} [R] d\sigma_\xi = 0,\\ \int_{S_1} [R] e^{i2\pi m_2\xi_2} d\sigma_\xi = 0, \quad m_2 \neq 0,\\ \int_{S_1} [R] e^{i2\pi m_3\xi_3} d\sigma_\xi = 0, \quad m_3 \neq 0. \end{cases}$$

Because of the completeness of the functions family $\{e^{i2\pi(m_2\xi_2+m_3\xi_3)}\}|_{\{(m_2,m_3)\in Z^2\}}$, we infer that

$$\text{(A.9)} \qquad [R]|_{S_1} = 0.$$

From $\frac{\partial H}{\partial \xi_1} - \frac{\partial \overline{P}}{\partial \xi_2} = 0$, we use $[\overline{P}]|_{\partial Q} = 0$ and prove similarly that

$$\text{(A.10)} \qquad [H]|_{S_1} = 0.$$

Combining (A.4), (A.5), (A.9), and (A.10) yields

$$\text{(A.11)} \qquad \begin{aligned} &[A(\xi)\mathbf{curl}_\xi \Theta_1^1(\xi) \times \nu]|_{S_1} = 0, \quad [A(\xi)\mathbf{curl}_\xi \Theta_1^1(\xi) \times \nu]|_{S_3} = 0,\\ &[A(\xi)\mathbf{curl}_\xi \Theta_1^1(\xi) \times \nu]|_{S_5} = 0. \end{aligned}$$

Let us turn to the proof of Proposition 2.2. We prove Proposition 2.2 only for $p = 1$. The remainder can be proved similarly. We recall (2.6) and have

$$\text{(A.12)} \qquad \begin{cases} \mathbf{curl}_\xi(A(\xi)\mathbf{curl}_\xi \Theta_2^1(\xi)) = -\mathbf{curl}_\xi(A(\xi)\Theta_1^1(\xi))\\ \quad -A(\xi)\mathbf{curl}_\xi\Theta_1^1(\xi) - A(\xi)\mathbf{e}_1 + \hat{A}\mathbf{e}_1 + \nabla_\xi \zeta_2^1(\xi), \quad \xi \in Q,\\ \nabla_\xi \cdot \Theta_2^1(\xi) = 0, \quad \xi \in Q,\\ \Theta_2^1(\xi) \times \nu = 0, \quad \xi \in \partial Q, \end{cases}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is the outward unit normal to ∂Q. We set $\Theta_2^1(\xi) = (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi))$. Under the assumptions of (H_1)–(H_2), we have the following lemma.

Lemma A.3. *We can directly verify that*

$$\text{(A.13)} \qquad \begin{aligned} &\phi_1(\xi_1,\xi_2,\xi_3) = \phi_1(1-\xi_1,\xi_2,\xi_3), \qquad \phi_2(\xi_1,\xi_2,\xi_3) = -\phi_2(1-\xi_1,\xi_2,\xi_3),\\ &\phi_3(\xi_1,\xi_2,\xi_3) = -\phi_3(1-\xi_1,\xi_2,\xi_3), \quad \phi_1(\xi_1,\xi_2,\xi_3) = \phi_1(\xi_1,1-\xi_2,\xi_3),\\ &\phi_2(\xi_1,\xi_2,\xi_3) = -\phi_2(\xi_1,1-\xi_2,\xi_3), \quad \phi_3(\xi_1,\xi_2,\xi_3) = \phi_3(\xi_1,1-\xi_2,\xi_3),\\ &\phi_1(\xi_1,\xi_2,\xi_3) = \phi_1(\xi_1,\xi_2,1-\xi_3), \qquad \phi_2(\xi_1,\xi_2,\xi_3) = \phi_2(\xi_1,\xi_2,1-\xi_3),\\ &\phi_3(\xi_1,\xi_2,\xi_3) = -\phi_3(\xi_1,\xi_2,1-\xi_3). \end{aligned}$$

We set $A(\xi)\mathbf{curl}_\xi\Theta_2^1(\xi) = (\hat{P}, \hat{H}, \hat{R})$. From (A.12), we get

(A.14)
$$\begin{cases} \dfrac{\partial \hat{R}}{\partial \xi_2} - \dfrac{\partial \hat{H}}{\partial \xi_3} = -P_1 - P - a_{11} + \hat{a}_{11} + \dfrac{\partial \zeta_2^1(\xi)}{\partial \xi_1}, \\ \dfrac{\partial \hat{P}}{\partial \xi_3} - \dfrac{\partial \hat{R}}{\partial \xi_1} = -H_1 - H + \dfrac{\partial \zeta_2^1(\xi)}{\partial \xi_2}, \\ \dfrac{\partial \hat{H}}{\partial \xi_1} - \dfrac{\partial \hat{P}}{\partial \xi_2} = -R_1 - R + \dfrac{\partial \zeta_2^1(\xi)}{\partial \xi_3}, \end{cases}$$

where the scalar function $\zeta_2^1(\xi)$ is defined as in (2.4), (P, H, R) are given as in (A.3), the vector-valued function (P_1, H_1, R_1) is

(A.15)
$$\begin{aligned} P_1 &= \frac{\partial}{\partial \xi_2}(a_{33}\psi_3) - \frac{\partial}{\partial \xi_3}(a_{22}\psi_2), \\ H_1 &= \frac{\partial}{\partial \xi_3}(a_{11}\psi_1) - \frac{\partial}{\partial \xi_1}(a_{33}\psi_3), \\ R_1 &= \frac{\partial}{\partial \xi_1}(a_{22}\psi_2) - \frac{\partial}{\partial \xi_2}(a_{11}\psi_1), \end{aligned}$$

and the vector-valued function $\Theta_1^1(\xi) = (\psi_1, \psi_2, \psi_3)$ is given as in (A.2).

Similarly to (A.4), we get

(A.16)
$$\begin{aligned} &A(\xi)\mathbf{curl}_\xi\Theta_2^1(\xi) \times \nu|_{S_1} = (0, \hat{R}, -\hat{H}), \quad A(\xi)\mathbf{curl}_\xi\Theta_2^1(\xi) \times \nu|_{S_2} = (0, -\hat{R}, \hat{H}), \\ &A(\xi)\mathbf{curl}_\xi\Theta_2^1(\xi) \times \nu|_{S_3} = (-\hat{R}, 0, \hat{P}), \quad A(\xi)\mathbf{curl}_\xi\Theta_2^1(\xi) \times \nu|_{S_4} = (\hat{R}, 0, -\hat{P}), \\ &A(\xi)\mathbf{curl}_\xi\Theta_2^1(\xi) \times \nu|_{S_5} = (\hat{H}, -\hat{P}, 0), \quad A(\xi)\mathbf{curl}_\xi\Theta_2^1(\xi) \times \nu|_{S_6} = (-\hat{H}, \hat{P}, 0). \end{aligned}$$

From (A.13), we can check that

(A.17)
$$[\hat{H}]|_{S_1} = 0, \quad [\hat{R}]|_{S_1} = 0.$$

Following the reasoning of proof of Theorem A.4 of [9], from the second equation of (A.14), we take, respectively, the test functions

$$\begin{aligned} &u_1(\xi) = (e^{i2\pi m_1\xi_1} - 1)(e^{i2\pi m_2\xi_2} - 1)(e^{i2\pi m_3\xi_3} - 1), \quad m_1 \neq 0,\, m_2 \neq 0,\, m_3 \neq 0, \\ &u_2(\xi) = (e^{i2\pi m_1\xi_1} - 1)(e^{i2\pi m_2\xi_2} - 1), \quad m_1 \neq 0,\, m_2 \neq 0, \\ &u_3(\xi) = (e^{i2\pi m_1\xi_1} - 1)(e^{i2\pi m_3\xi_3} - 1), \quad m_1 \neq 0,\, m_3 \neq 0, \\ &u_4(\xi) = (e^{i2\pi m_2\xi_2} - 1)(e^{i2\pi m_3\xi_3} - 1), \quad m_2 \neq 0,\, m_3 \neq 0, \\ &u_5(\xi) = (e^{i2\pi m_1\xi_1} - 1), \quad m_1 \neq 0, \\ &u_6(\xi) = (e^{i2\pi m_2\xi_2} - 1), \quad m_2 \neq 0, \\ &u_7(\xi) = (e^{i2\pi m_3\xi_3} - 1), \quad m_3 \neq 0, \\ &u_8(\xi) = 1 \end{aligned}$$

and obtain

(A.18)
$$\begin{cases} \int_{S_5} [\hat{P}] e^{i2\pi(m_1\xi_1 + m_2\xi_2)} d\sigma_\xi = 0, \quad m_1 \neq 0,\, m_2 \neq 0, \\ \int_{S_5} [\hat{P}] d\sigma_\xi = 0, \\ \int_{S_5} [\hat{P}] e^{i2\pi m_1\xi_1} d\sigma_\xi = 0, \quad m_1 \neq 0, \\ \int_{S_5} [\hat{P}] e^{i2\pi m_2\xi_2} d\sigma_\xi = 0, \quad m_2 \neq 0. \end{cases}$$

It follows from the completeness of the function family $\{e^{i2\pi(m_1\xi_1+m_2\xi_2)}\}|_{\{(m_1,m_2)\in Z^2\}}$ that

$$[\hat{P}]|_{S_5} = 0. \tag{A.19}$$

From the third equation of (A.14), we can similarly prove

$$[\hat{P}]|_{S_3} = 0. \tag{A.20}$$

In the same way, from the first equation of (A.14), we see that

$$[\hat{R}]|_{S_3} = 0, \quad [\hat{H}]|_{S_5} = 0. \tag{A.21}$$

Combining (A.16), (A.17), (A.19), (A.20), and (A.21), it follows that

$$\begin{array}{l} [A(\xi)\mathbf{curl}_\xi\Theta_2^1(\xi)\times\nu]|_{S_1} = 0, \quad [A(\xi)\mathbf{curl}_\xi\Theta_2^1(\xi)\times\nu]|_{S_3} = 0, \\ [A(\xi)\mathbf{curl}_\xi\Theta_2^1(\xi)\times\nu]|_{S_5} = 0. \end{array} \tag{A.22}$$

Next we prove Propositions 2.1 and 2.2 in general cases: $a_{kk} \in L^\infty(Q)$, $k = 1, 2, 3$. It is well known that smooth functions are not dense in $L^\infty(Q)$. To this end, we introduce q such that

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{2}. \tag{A.23}$$

We can find a sequence of smooth functions $a_{kk}^{(\beta)}(\xi) \in C^\infty(Q)$ such that (see [6, p. 104])

$$\|a_{kk}^{(\beta)} - a_{kk}\|_{0,q,Q} \to 0 \quad \text{as} \quad \beta \to \infty. \tag{A.24}$$

Let $\Theta_1^{1,(\beta)}(\xi)$ be the weak solution of the following problem:

$$\left\{ \begin{array}{l} \mathbf{curl}_\xi(A^{(\beta)}(\xi)\mathbf{curl}_\xi\Theta_1^{1,(\beta)}(\xi)) = -\mathbf{curl}_\xi(A^{(\beta)}(\xi)\mathbf{e}_1), \quad \xi \in Q, \\ \nabla_\xi \cdot \Theta_1^{1,(\beta)}(\xi) = 0, \quad \xi \in Q, \\ \Theta_1^{1,(\beta)}(\xi) \times \nu = 0, \quad \xi \in \partial Q, \quad \mathbf{e}_1 = (1, 0, 0), \end{array} \right. \tag{A.25}$$

where $A^{(\beta)}(\xi) = \mathrm{diag}(a_{11}^{(\beta)}(\xi), a_{22}^{(\beta)}(\xi), a_{33}^{(\beta)}(\xi))$.

From (A.1) and (A.25), for $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; Q)$, we can obtain

$$\begin{aligned} &\left| \int_Q A(\xi)\mathbf{curl}_\xi(\Theta_1^1 - \Theta_1^{1,(\beta)}) \cdot \mathbf{curl}_\xi \mathbf{v} d\xi \right| \\ &\leq C \left\{ \left(\sum_{k=1}^3 \|a_{kk} - a_{kk}^{(\beta)}\|_{0,q,Q} \right) \|\mathbf{curl}_\xi\Theta_1^{1,(\beta)}\|_{0,r,Q} \|\mathbf{curl}_\xi \mathbf{v}\|_{0,Q} \right. \\ &\quad \left. + \left(\sum_{k=1}^3 \|a_{kk} - a_{kk}^{(\beta)}\|_{0,Q} \right) \|\mathbf{curl}_\xi \mathbf{v}\|_{0,Q} \right\}, \end{aligned} \tag{A.26}$$

where C is a constant independent of β, ε.

Since $a_{kk}^{(\beta)}(\xi) \in C^\infty(Q)$, from (A.25), we can infer that $\|\mathbf{curl}_\xi\Theta_1^{1,(\beta)}\|_{0,r,Q} \leq C$ (see [14]).

If we let $\mathbf{v} = (\Theta_1^1 - \Theta_1^{1,(\beta)})$, using (A.24), we have

$$\|\mathbf{curl}_\xi(\Theta_1^1 - \Theta_1^{1,(\beta)})\|_{0,Q} \to 0 \quad \text{as} \quad \beta \to \infty. \tag{A.27}$$

Recalling (A.1) and (A.25), and using the regularity results that the coefficients are sufficiently smooth (see (A.11)), for any $\mathbf{v} \in (H^1_{per}(Q))^3$, we derive

$$\begin{aligned}\int_{\partial Q}(A(\xi)\mathbf{curl}_\xi \Theta_1^1 \times \nu)\cdot \mathbf{v}d\sigma = &-\int_Q A(\xi)\mathbf{curl}_\xi(\Theta_1^1 - \Theta_1^{1,(\beta)})\cdot \mathbf{curl}_\xi\,\mathbf{v}d\xi \\ &-\int_Q (A(\xi)-A^{(\beta)}(\xi))\mathbf{curl}_\xi\Theta_1^{1,(\beta)}\cdot \mathbf{curl}_\xi\,\mathbf{v}d\xi \\ &-\int_Q (A(\xi)-A^{(\beta)}(\xi))e_p\cdot \mathbf{curl}_\xi\,\mathbf{v}d\xi,\end{aligned} \tag{A.28}$$

and consequently

$$\begin{aligned}\left|\int_{\partial Q}(A(\xi)\mathbf{curl}_\xi \Theta_1^1 \times \nu)\cdot \mathbf{v}d\sigma\right| \le C\Bigg\{&\|\mathbf{curl}_\xi(\Theta_1^1 - \Theta_1^{1,(\beta)})\|_{0,Q}\|\mathbf{curl}_\xi\,\mathbf{v}\|_{0,Q} \\ &\left(\sum_{k=1}^{3}\|a_{kk}-a_{kk}^{(\beta)}\|_{0,q,Q}\right)\|\mathbf{curl}_\xi\Theta_1^{1,(\beta)}\|_{0,r,Q}\|\mathbf{curl}_\xi\,\mathbf{v}\|_{0,Q}\Bigg\},\end{aligned} \tag{A.29}$$

where $(H^1_{per}(Q))^3 = \{\mathbf{v} \in (H^1(Q))^3, \mathbf{v}$ takes equal values on opposite faces of $Q\}$.

Thanks to $\|\mathbf{curl}_\xi\Theta_1^{1,(\beta)}\|_{0,r,Q} \le C$, using (A.24) and (A.27), we get

$$\int_{\partial Q}(A(\xi)\mathbf{curl}_\xi \Theta_1^1 \times \nu)\cdot \mathbf{v}d\sigma = 0 \quad \forall \mathbf{v} \in (H^1_{per}(Q))^3. \tag{A.30}$$

Furthermore, we obtain

$$[A(\xi)\mathbf{curl}_\xi \Theta_1^1 \times \nu]|_{\partial Q} = 0,$$

where $[\mathbf{v}]$ denotes the jump of a vector-valued function $\mathbf{v}$ on the boundary ∂Q. Therefore, the proof of Proposition 2.1 is complete. Similarly, we can complete the proof of Proposition 2.2.

Appendix B. The numerical tests for Propositions 2.1 and 2.2. To validate the theoretical results of Propositions 2.1 and 2.2, we do a simulation for continuous coefficients that approximate the piecewise constant coefficients and compare the convergence. For Case 5.2.2, we define a continuous coefficient that approximates the piecewise constant coefficient $a_{kk}(\xi)$, i.e., $a_{kk}^{(\beta)}(\xi) = \frac{99}{(1+e^{-(r-0.41)\beta})} + 1$, $\xi \in Q$, $k = 1, 2, 3$, where r is a radius vector in polar coordinates. Here we take $\beta = 10^3$. Let Θ_1^p, Θ_2^p, $p = 1, 2, 3$, be the weak solutions of cell problems (2.2) and (2.6) for piecewise constant coefficients, respectively. We denote by $\Theta_1^{p,(\beta)}$, $\Theta_2^{p,(\beta)}$, $p = 1, 2, 3$, the associated weak solutions of cell problems (2.2) and (2.6) for smooth coefficients. Set $\mathbf{e}_1^p = \Theta_1^p(\xi) - \Theta_1^{p,(\beta)}(\xi)$, $\mathbf{e}_2^p = \Theta_2^p(\xi) - \Theta_2^{p,(\beta)}(\xi)$, $p = 1, 2, 3$. For simplicity, let $\|\mathbf{u}\|_{(0)} = \|\mathbf{u}\|_{(L^2(\Omega))^3}$, $\|\mathbf{u}\|_{(1)} = \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl};\Omega)}$. The associated computational results are listed in Table B.1.

TABLE B.1
The computational results for Propositions 2.1 and 2.2 in Case 5.2.2.

$\frac{\|\mathbf{e}_1^1\|_{(0)}}{\|\Theta_1^{1,(\beta)}\|_{(0)}}$	$\frac{\|\mathbf{e}_1^2\|_{(0)}}{\|\Theta_1^{2,(\beta)}\|_{(0)}}$	$\frac{\|\mathbf{e}_1^3\|_{(0)}}{\|\Theta_1^{3,(\beta)}\|_{(0)}}$	$\frac{\|\mathbf{e}_1^1\|_{(1)}}{\|\Theta_1^{1,(\beta)}\|_{(1)}}$	$\frac{\|\mathbf{e}_1^2\|_{(1)}}{\|\Theta_1^{2,(\beta)}\|_{(1)}}$	$\frac{\|\mathbf{e}_1^3\|_{(1)}}{\|\Theta_1^{3,(\beta)}\|_{(1)}}$
0.00048	0.0024	0.0010	0.0014	0.0070	0.0031
$\frac{\|\mathbf{e}_2^1\|_{(0)}}{\|\Theta_2^{1,(\beta)}\|_{(0)}}$	$\frac{\|\mathbf{e}_2^2\|_{(0)}}{\|\Theta_2^{2,(\beta)}\|_{(0)}}$	$\frac{\|\mathbf{e}_2^3\|_{(0)}}{\|\Theta_2^{3,(\beta)}\|_{(0)}}$	$\frac{\|\mathbf{e}_2^1\|_{(1)}}{\|\Theta_2^{1,(\beta)}\|_{(1)}}$	$\frac{\|\mathbf{e}_2^2\|_{(1)}}{\|\Theta_2^{2,(\beta)}\|_{(1)}}$	$\frac{\|\mathbf{e}_2^3\|_{(1)}}{\|\Theta_2^{3,(\beta)}\|_{(1)}}$
0.0958	0.0936	0.0956	0.1009	0.1013	0.1010

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