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REGULARITY PROPERTIES FOR GENERAL HJB EQUATIONS: A BACKWARD STOCHASTIC DIFFERENTIAL EQUATION METHOD*

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Abstract. In this work we investigate regularity properties of a large class of Hamilton–Jacobi–Bellman (HJB) equations with or without obstacles, which can be stochastically interpreted in the form of a stochastic control system in which nonlinear cost functional is defined with the help of a backward stochastic differential equation (BSDE) or a reflected BSDE. More precisely, we prove that, first, the unique viscosity solution $V(t,x)$ of an HJB equation over the time interval $[0,T]$, with or without an obstacle, and with terminal condition at time T, is jointly Lipschitz in (t,x) for t running any compact subinterval of $[0,T)$. Second, for the case that V solves an HJB equation without an obstacle or with an upper obstacle it is shown under appropriate assumptions that $V(t,x)$ is jointly semiconcave in (t,x). These results extend earlier ones by Buckdahn, Cannarsa, and Quincampoix [*Nonlinear Differential Equations Appl.*, 17 (2010), pp. 715–728]. Our approach embeds their idea of time change into a BSDE analysis. We also provide an elementary counterexample which shows that, in general, for the case that V solves an HJB equation with a lower obstacle the semiconcavity doesn't hold true.

Key words. backward stochastic differential equation, HJB equation, Lipschitz continuity, reflected backward stochastic differential equations, semiconcavity, value function

AMS subject classifications. 93E20, 35D40, 60H10, 60H30, 93E05, 90C39, 35K55, 35K65

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1. Introduction.

We are interested in regularity properties of possibly degenerate Hamilton–Jacobi–Bellman (HJB) equations with or without obstacles. More precisely, we consider the HJB equation

$$\frac{\partial}{\partial t}V(t,x) + \inf_{u\in U} H(t,x,V(t,x),\nabla V(t,x),D^2V(t,x),u) = 0 \tag{1.1}$$

and the HJB equation with either a lower obstacle

(1.2)
$$\min\left\{V(t,x)-\varphi(t,x), -\frac{\partial}{\partial t}V(t,x) - \inf_{u\in U} H(t,x,V(t,x),\nabla V(t,x),D^2V(t,x),u)\right\} = 0$$

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or an upper obstacle

(1.3)
$$\max\left\{V(t,x)-\varphi(t,x),-\frac{\partial}{\partial t}V(t,x)-\inf_{u\in U}H(t,x,V(t,x),\nabla V(t,x),D^2V(t,x),u)\right\}=0,$$

$(t,x)\in[0,T]\times\mathbf{R}^d$ with terminal condition $V(T,x)=\Phi(x)$, $x\in\mathbf{R}^d$, and with the Hamiltonian

$$H(t,x,y,p,A,u)=\frac{1}{2}\mathrm{tr}\,(\sigma\sigma^*(t,x,u)A)+b(t,x,u)p+f(t,x,y,p\sigma(t,x,u),u),$$

$(t,x,y,p,A,u)\in[0,T]\times\mathbf{R}^d\times\mathbf{R}\times\mathbf{R}^d\times\mathbf{S}^d\times U$, where $\mathbf{S}^d$ denotes the space of all symmetric $d\times d$ matrices, and U is a compact metric control state space. If $\sigma\sigma^*(t,x,u)\geq\alpha I_{\mathbf{R}^d}$ $(\alpha>0)$, the regularity of the solution of the HJB equation (1.1) is well studied (see, e.g., Krylov [10]). Here we are interested in the case of possible degeneracy of $\sigma\sigma^*$.

It is well-known that under continuity and growth assumptions on the coefficients, the HJB equations (1.1), (1.2), and (1.3) have a unique viscosity solution $V\in C_p([0,T]\times\mathbf{R}^d)$, respectively; see, e.g., Buckdahn and Li [2], [3], Wu and Yu [16], and Crandall, Ishii, and Lions [5] (readers more interested in the viscosity solution are referred to the latter reference). Moreover, if the coefficients $b,\ \sigma,\ f$ are continuous and of linear growth, and if $b(t,.,u),\ \sigma(t,.,u),\ f(t,.,.,.,u)$ are Lipschitz, uniformly with respect to t,u, then

(1.4)
$$\begin{aligned}&\text{(i) } |V(t,x)-V(t,x')|\leq C|x-x'|,\\&\text{(ii) } |V(t,x)-V(t',x)|\leq C(1+|x|)\sqrt{|t-t'|},\end{aligned}$$

$(t,x),\ (t',x')\in[0,T]\times\mathbf{R}^d$, for some constant $C\in\mathbf{R}^+$; see, e.g., Lemma 3.5 and Theorem 3.10 in Buckdahn and Li [2] or Peng [13] for the HJB equation (1.1); or see Lemma 3.1 and Theorem 3.2 in Buckdahn and Li [3] or (ii) from the proof of Proposition 3.12 in Wu and Yu [16] for the HJB equations (1.2) and (1.3).

Remark 1.1. Indeed, in [2] and [3] stochastic differential games and the viscosity solutions of the associated HJB–Isaacs equations with and without obstacles are studied, but stochastic control problems and associated HJB equations with and without obstacles can be regarded as a special case, in which the control state space of one of the players is a singleton. Therefore, here we can use the results from [2] and [3].

However, here we are interested in regularity properties of $V(t,x)$ in (t,x). These regularity properties concern the joint Lipschitz property of V in (t,x), but also the semiconcavity of V in (t,x), where the semiconcavity is understood in the following sense (see [1] or [4]).

DEFINITION 1.1. *Let $A\subset\mathbf{R}^d$ be an open set and let $f:[0,T]\times A\to\mathbf{R}^n$. We say that f is $(C_\delta-)$ semiconcave (with linear modulus) on $[0,T-\delta]\times A$ if for all $\delta>0$, there exists a constant $C_\delta\geq 0$ such that for all $x,\ x'\in A,\ t,\ t'\in[0,T-\delta]$, and for all $\lambda\in[0,1]$,*

(1.5)
$$\lambda f(t,x)+(1-\lambda)f(t',x')\leq f(\lambda(t,x)+(1-\lambda)(t',x'))+C_\delta\lambda(1-\lambda)(|t-t'|^2+|x-x'|^2).$$

Any constant C_δ satisfying (1.5) *is called a semiconcavity constant for f on* $[0,T-\delta]\times A$.

However, one has to be careful here. It turns out, and will be pointed out by counterexamples, that the joint Lipschitz continuity and the semiconcavity hold not on $[0,T]\times\mathbf{R}^d$ but only on $[0,T-\delta]\times\mathbf{R}^d$ for any $\delta>0$. We emphasize the importance of the semiconcavity of V on $[0,T-\delta]\times\mathbf{R}^d$ for any $\delta>0$, which has, due to Alexandrov's theorem, the immediate consequence that V admits a second order expansion with respect to (t,x), in almost every $(t,x)\in[0,T]\times\mathbf{R}^d$. Cannarsa and Sinestrari [4] (for $\sigma=0$) showed that these regularity properties are the best ones which can be expected for Hamilton–Jacobi equations. The Lipschitz continuity and semiconcavity of $V(t,x)$ in x, uniformly with respect to $t\in[0,T]$, have been well-known for a long time. They are the result of straightforward computations; see, for instance, Fleming and Soner [8], Ishii and Lions [9], Ma and Yong [11], and Yong and Zhou [15]. Buckdahn, Cannarsa, and Quincampoix [1] studied the joint Lipschitz continuity and semiconcavity of solutions $V(t,x)$ of the HJB equation without obstacle when $f(t,x,y,z,u)=f(t,x,u)$ doesn't depend on (y,z). They used a new technique which is a method of time change in the associated stochastic control problem. In this paper we adapt their method to more general HJB equations and to HJB equations with obstacle by developing an associated approach using backward stochastic differential equations (BSDEs). To be more precise, let $(t,x)\in[0,T]\times\mathbf{R}^d$, and let $W=(W_s)_{s\in[t,T]}$ be an m-dimensional Brownian motion with $W_t=0$. By $\mathbf{F}^W=\{\mathcal{F}^W_s=\sigma\{W_r,r\le s\}\bigvee\mathcal{N}_P\}_{s\in[t,T]}$ we denote the filtration generated by W and augmented by all P-null sets. We consider the forward stochastic differential equation (SDE)

$$
\begin{cases} dX^{t,x,u}_s=\sigma(s,X^{t,x,u}_s,u_s)dW_s+b(s,X^{t,x,u}_s,u_s)ds, & s\in[t,T],\\ X^{t,x,u}_t=x, \end{cases} \tag{1.6}
$$

which we associate with the reflected BSDE (RBSDE) with a lower obstacle,

$$
\begin{cases} dY^{t,x,u}_s=-f(s,X^{t,x,u}_s,Y^{t,x,u}_s,Z^{t,x,u}_s,u_s)ds+Z^{t,x,u}_s dW_s-dK^{t,x,u}_s,\\ Y^{t,x,u}_T=\Phi(X^{t,x,u}_T),\ K^{t,x,u}\text{continuous, increasing, } K^{t,x,u}_t=0,\\ Y^{t,x,u}_s\ge\varphi(s,X^{t,x,u}_s),\quad (Y^{t,x,u}_s-\varphi(s,X^{t,x,u}_s))dK^{t,x,u}_s=0,\quad s\in[t,T], \end{cases} \tag{1.7}
$$

where the admissible controls u belong to the space $\mathcal{U}^W_{t,T}:=L^0_{\mathbf{F}^W}(t,T;U)$ of $\mathbf{F}^W$-adapted U-valued processes, and U is a compact metric space.

The coefficients

$$
\sigma:[0,T]\times\mathbf{R}^d\times U\to\mathbf{R}^{d\times m},\ b:[0,T]\times\mathbf{R}^d\times U\to\mathbf{R}^d,
$$
$$
f:[0,T]\times\mathbf{R}^d\times\mathbf{R}\times\mathbf{R}^m\times U\to\mathbf{R},\ \Phi:\mathbf{R}^d\to\mathbf{R}\text{ and }\varphi:[0,T]\times\mathbf{R}^d\to\mathbf{R}
$$

are continuous functions which we suppose to satisfy the following standard conditions:

(H1) The functions $\sigma(.,.,u),b(.,.,u),f(.,.,.,.,u),\varphi(.,.)$ are Lipschitz in $(t,x,y,z)\in[0,T]\times\mathbf{R}^d\times\mathbf{R}\times\mathbf{R}^m$, uniformly with respect to $u\in U$, and $\Phi:\mathbf{R}^d\to\mathbf{R}$ is Lipschitz in $x\in\mathbf{R}^d$.

(H2) The functions σ,b,f,φ, and Φ are bounded.

(H3) $\Phi(x)\ge\varphi(T,x),x\in\mathbf{R}^d$.

The above RBSDE was introduced in El Karoui et al. [7]. It extends the notion of BSDE, which was first studied in its general form by Pardoux and Peng [12], by endowing it with a lower or an upper barrier.

Then from [10] and [7] we know that SDE (1.6) and RBSDE (1.7) have a unique $\mathbf{F}^W$-adapted, square integrable solution $X^{t,x,u}$, and $(Y^{t,x,u},Z^{t,x,u},K^{t,x,u})$, respectively. From [3] (or [16]) we know that the deterministic function

$$
V(t,x):=\inf_{u\in\mathcal{U}^W_{t,T}}Y^{t,x,u}_t,\ (t,x)\in[0,T]\times\mathbf{R}^d \tag{1.8}
$$

belongs to $C_l([0,T]\times\mathbf{R}^d)$, and is the unique viscosity solution (unique in $C_p([0,T]\times\mathbf{R}^d)$) of HJB equation (1.2) with obstacle.

Remark 1.2. By $C_l([0,T]\times\mathbf{R}^d)$ (respectively, $C_p([0,T]\times\mathbf{R}^d)$) we denote the space of continuous real functions over $[0,T]\times\mathbf{R}^d$ which have at most linear (respectively, polynomial) growth.

For the proof that V is deterministic, see Proposition 3.3 in [2] or Proposition 3.1 in [3]. Using the time change method in the above control problem for SDE (1.6) and RBSDE (1.7) we get our main results.

THEOREM 1.1. *Under the assumptions* (H1)–(H3), $V(t,x)$ *is joint Lipschitz continuous in* $(t,x)\in[0,T-\delta]\times\mathbf{R}^d$ *for all* $\delta>0$, *i.e., there exists* $C_\delta>0$ *such that for any* $(t,x),(t',x')\in[0,T-\delta]\times\mathbf{R}^d$,

$$|V(t,x)-V(t',x')|\le C_\delta(|t-t'|+|x-x'|). \tag{1.9}$$

In fact, we will even show more: the value functions $V_n, n\ge1$, of the associated stochastic control problem in which the reflected BSDE is replaced by the penalized one (see (2.7) and (2.9)) satisfy (1.9), uniformly with respect to $n\ge1$.

Remark 1.3. A symmetric argument shows that the continuous viscosity solution $V(t,x)$ of (1.3) with an upper obstacle also satisfies the joint Lipschitz property as that stated in Theorem 1.1 for the viscosity solution of (1.2) with a lower obstacle. For the stochastic interpretation of the solution V of (1.3) see (3.2).

Concerning the joint semiconcavity which is our second main result, we will give a counterexample which shows that the viscosity solution V of HJB (1.2) with a lower obstacle is, in general, not semiconcave on $[0,T-\delta]\times\mathbf{R}^d$ ($\delta>0$), even if the lower obstacle is constant. However, if V is the viscosity solution of HJB equation (1.3) with an upper obstacle, then V has the joint semiconcavity property in $(t,x)\in[0,T-\delta]\times\mathbf{R}^d$ for all $\delta>0$. For this we need the following assumptions:

(H3') $\Phi(x)\le\varphi(T,x), x\in\mathbf{R}^d$.

(H4) $f(t,x,y,z,u)$ is semiconcave in $(t,x,y,z)\in[0,T]\times\mathbf{R}^d\times\mathbf{R}\times\mathbf{R}^m$, uniformly with respect to $u\in U$ (i.e., the semiconcavity constant C_δ doesn't depend on u); $\Phi(x)$ is semiconcave.

(H5) The first order derivatives $\nabla_{(t,x)}b,\ \nabla_{(t,x)}\sigma$ of b and σ with respect to (t,x) exist and are continuous in (t,x,u) and Lipschitz continuous in (t,x), uniformly with respect to $u\in U$.

(H6) $f(t,x,y,z,u)=f(t,x,y,u)$ is independent of z; φ is semiconcave in $(t,x)\in[0,T]\times\mathbf{R}^d$.

(H7) $\varphi(t,x)=\varphi\in\mathbf{R}$, $(t,x)\in[0,T]\times\mathbf{R}^d$.

THEOREM 1.2. *In addition to* (H1), (H2), *and* (H3')*, we assume that* (H4), (H5)*, as well as either* (H6) *or* (H7) *hold. Then, the value function V which is the viscosity solution of HJB equation* (1.3), *is* (C_δ)*-semiconcave on* $[0,T-\delta]\times\mathbf{R}^d$ *for all* $\delta>0$.

Remark 1.4. A standard transformation allows us to replace the assumption (H7) of constancy of φ by that of $\varphi\in C_b^{3,4}([0,T]\times\mathbf{R}^d)$. For simplicity we restrict ourselves to (H7). However, also here for the case of semiconcavity we will prove even more: under the assumptions of the theorem the value functions $V_n,\ n\ge1$, of the associated stochastic control problem, in which the reflected BSDE is replaced by penalized ones (see (3.6) and (3.7)), are C_δ-semiconcave on $[0,T-\delta]\times\mathbf{R}^d$ uniformly with respect to $n\ge1$ for all $\delta>0$.

Remark 1.5. (1) The boundedness assumption on the coefficients is made to simplify the computations and to emphasize the main arguments.

(2) The above two theorems remain valid for HJB equations (1.1) without obstacle. Indeed, all coefficients are bounded, and so the viscosity solution $V(t,x)$ of the

HJB equation without obstacle is $|V(t,x)| \le C,\ (t,x) \in [0,T] \times \mathbf{R}^d,$ for some $C \in \mathbf{R}$ depending only on the bounds of $\sigma,\ b,\ f$ and Φ. It suffices to suppose that the obstacle φ is sufficiently large, i.e., $|\varphi(t,x)| \ge C,\ (t,x) \in [0,T] \times \mathbf{R}^d$, in order to interpret V as a solution of HJB equation with obstacle. On the other hand, the associated BSDE becomes an RBSDE with a lower obstacle or an upper one; see Remark 2.1. Therefore, we only need to study HJB (1.2) or (1.3).

Our paper is organized as follows. In section 2, we study the joint Lipschitz continuity for the HJB equations with or without obstacles with the help of the associated stochastic control problems which cost functionals are given by BSDEs or by RBSDEs. For this end, a special BSDE method based on a time change is developed. Section 3 studies the semiconcavity for the HJB equations with or without obstacles. We prove that under some appropriate assumptions, the viscosity solution V also satisfies the semiconcavity property, but only if it is the solution of an HJB equation (1.3) with an upper obstacle. Our analysis is based on the combination of two time changes and the development of appropriate BSDE estimates under time change. Concerning the viscosity solution of an HJB equation (1.2) with a lower obstacle, we show with a simple counterexample that semiconcavity is, in general, not satisfied. For the purpose of readability some basics on BSDEs and RBSDEs are given but are postponed to the appendix (section 4).

2. The joint Lipschitz continuity of the value function. Given a compact metric control state space U we consider the HJB equation with a lower obstacle

$$\min\left\{V(t,x)-\varphi(t,x), -\frac{\partial}{\partial t}V(t,x)-\inf_{u\in U}H(t,x,V(t,x),\nabla V(t,x),D^2V(t,x),u)\right\}=0, \tag{2.1}$$

$(t,x) \in [0,T] \times \mathbf{R}^d$, with terminal condition $V(T,x) = \Phi(x)$, $x \in \mathbf{R}^d$, and with the Hamiltonian

$$H(t,x,y,p,A,u)=\frac{1}{2}\mathrm{tr}\,(\sigma\sigma^*(t,x,u)A)+b(t,x,u)p+f(t,x,y,p\sigma(t,x,u),u),$$

$(t,x,y,p,A,u) \in [0,T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d \times S^d \times U$.

The coefficients

$$\sigma : [0,T] \times \mathbf{R}^d \times U \to \mathbf{R}^{d\times m},\ b : [0,T] \times \mathbf{R}^d \times U \to \mathbf{R}^d,$$
$$f : [0,T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^m \times U \to \mathbf{R},\ \Phi : \mathbf{R}^d \to \mathbf{R} \text{ and } \varphi : [0,T] \times \mathbf{R}^d \to \mathbf{R}$$

are continuous functions which satisfy (H1)–(H3).

It is by now well-known (see, for instance, [3], [16]) that the above HJB equation with the obstacle possesses a continuous viscosity solution $V \in C_b([0,T] \times \mathbf{R}^d)$ (the space of bounded continuous functions over $[0,T] \times \mathbf{R}^d$) which is unique in the class of viscosity solutions with polynomial growth. It can be stochastically interpreted by the following controlled stochastic system.

Let $(t,x) \in [0,T]\times\mathbf{R}^d$. Given an m-dimensional Brownian motion $W = (W_s)_{s\in[t,T]}$ with $W_t = 0$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with the filtration $\mathbf{F}^W = (\mathcal{F}_s^W)_{s\in[t,T]}$ generated by the Brownian motion W and completed by all P-null sets. We introduce the following spaces which will be used frequently in what follows: $\mathcal{S}^2_{\mathbf{F}^W}(t,T;\mathbf{R}^d) := \{(\psi_s)_{t\le s\le T}\ \mathbf{R}^d\text{-valued } \mathbf{F}^W\text{-adapted continuous process} : E[\sup_{t\le s\le T}|\psi_s|^2] < +\infty\}$; $L^2_{\mathbf{F}^W}(t,T;\mathbf{R}^d) := \{(\psi_s)_{t\le s\le T}\ \mathbf{R}^d\text{-valued } \mathbf{F}^W\text{-progressively measurable process} : E[\int_t^T|\psi_s|^2ds] < +\infty\}$; $A^2_{\mathbf{F}^W}(t,T) := \{(\psi_s)_{t\le s\le T} \in \mathcal{S}^2_{\mathbf{F}^W}(t,T;\mathbf{R}) : \psi \text{ is an increasing process},\ \psi_t = 0\}$.

We consider the forward SDE

$$\begin{cases} dX_s^{t,x,u} = \sigma(s, X_s^{t,x,u}, u_s)dW_s + b(s, X_s^{t,x,u}, u_s)ds, & s \in [t,T], \\ X_t^{t,x,u} = x, \end{cases} \tag{2.2}$$

which we associate with the RBSDE with a lower barrier

$$\begin{cases} dY_s^{t,x,u} = -f(s, X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)ds + Z_s^{t,x,u}dW_s - dK_s^{t,x,u}, \\ Y_T^{t,x,u} = \Phi(X_T^{t,x,u}), \\ Y_s^{t,x,u} \geq \varphi(s, X_s^{t,x,u}), \quad (Y_s^{t,x,u} - \varphi(s, X_s^{t,x,u}))dK_s^{t,x,u} = 0, \quad s \in [t,T]. \end{cases} \tag{2.3}$$

The control process u runs the set of admissible controls $\mathcal{U}_{t,T}^W := L^0_{\mathbf{F}^W}(t,T;U)$, defined as the set of all $\mathbf{F}^W$-progressively measurable processes over $(\Omega, \mathcal{F}, P)$, taking their values in U. Then, from [10] and [7] we know SDE (2.2) and RBSDE (2.3) have a unique solution $(X^{t,x,u}, (Y^{t,x,u}, Z^{t,x,u}, K^{t,x,u})) \in \mathcal{S}^2_{\mathbf{F}^W}(t,T;\mathbf{R}^d) \times \mathcal{S}^2_{\mathbf{F}^W}(t,T;\mathbf{R}) \times L^2_{\mathbf{F}^W}(t,T;\mathbf{R}^m) \times A^2_{\mathbf{F}^W}(t,T)$.

In order to emphasize that we have to deal with the solution of a decoupled forward-backward system driven by the Brownian motion W, we also write

$$(X^{t,x,u}(W), (Y^{t,x,u}(W), Z^{t,x,u}(W), K^{t,x,u}(W))) = (X^{t,x,u}, (Y^{t,x,u}, Z^{t,x,u}, K^{t,x,u})).$$

Observe that $Y_t^{t,x,u}$ is $\mathcal{F}_t^W$-measurable and hence can be identified with the deterministic real value $E[Y_t^{t,x,u}]$. Moreover, from [3] or [16] we know

$$V(t,x) := \inf_{u \in \mathcal{U}_{t,T}^W} Y_t^{t,x,u}(W),\ (t,x) \in [0,T] \times \mathbf{R}^d \tag{2.4}$$

belongs to $C_b([0,T] \times \mathbf{R}^d)$ and is the unique viscosity solution (unique in $C_p([0,T] \times \mathbf{R}^d)$) of HJB equation (2.1) with the obstacle. Standard SDE and BSDE estimates allow us to show (see, e.g., [3] or [16]) that for all $t,\ t' \in [0,T]$, $x,\ x' \in \mathbf{R}^d$,

$$\begin{aligned} &\text{(i) } |V(t,x)| \leq C, \\ &\text{(ii) } |V(t,x) - V(t,x')| \leq C|x - x'|, \\ &\text{(iii) } |V(t,x) - V(t',x)| \leq C(1+|x|)\sqrt{|t-t'|}. \end{aligned} \tag{2.5}$$

Remark 2.1. The above constant C depends only on the bounds and the Lipschitz constants of the functions σ, b, f, φ, and Φ. We also observe that if the coefficients f and Φ are bounded by $C_0 \in \mathbf{R}^+$ and $\varphi \leq -C_0(1+T)$ on $[0,T] \times \mathbf{R}^d$, then from the comparison theorem Lemma 4.4 in section 4

$$|Y_s^{t,x,u}| \leq C_0(1+T) \text{ and } K_s^{t,x,u} = 0,\ s \in [t,T],\ (t,x) \in [0,T] \times \mathbf{R}^d.$$

This also shows that by choosing $\overline{\varphi}(t,x) = -C_0(1+T)$, $(t,x) \in [0,T] \times \mathbf{R}^d$, a BSDE with the coefficients f and Φ bounded by C_0 can be regarded as a reflected BSDE with a lower barrier $\overline{\varphi}$, which coefficients satisfy our standard assumptions of boundedness and Lipschitz continuity. Similarly, by choosing $\widehat{\varphi}(t,x) = C_0(1+T)$, $(t,x) \in [0,T] \times \mathbf{R}^d$, a BSDE with the coefficients f and Φ bounded by C_0 can be regarded as a reflected BSDE with an upper barrier $\widehat{\varphi}$, which coefficients satisfy our standard assumptions of boundedness and Lipschitz continuity. That means the value function defined by (2.4) is the unique viscosity solution of HJB equation (1.1) without obstacle. So our studies of the regularity properties of the solutions of HJB equations with obstacles include in particular those without obstacles.

Unlike (2.5) our objective here is to study the joint Lipschitz continuity of $V(t,x)$ in (t,x). This joint Lipschitz property of the solution V of such HJB equations was somewhat expected; see Krylov [10]. However, it doesn't hold on $[0,T]\times\mathbf{R}^d$, as the following example shows.

Example 2.1. We let the dimension $m=d=1$, and we choose the coefficients $b=0,\ \sigma=1,\ f=0$, and $\Phi(x)=|x|,\ x\in\mathbf{R}^d$. Then

$$V(t,0)=E[\Phi(X_T^{t,0})]=E[|W_T-W_t|]=\sqrt{\frac{2}{\pi}}\sqrt{T-t},\ t\in[0,T].$$

It's obvious that V is not Lipschitz in t and hence not jointly Lipschitz in (t,x) for t around $t=T$; however, V is jointly Lipschitz on $[0,T-\delta]\times\mathbf{R}$ for all $\delta>0$.

Our objective in this section is to investigate the joint Lipschitz property of the value function V. More precisely, we have the following.

THEOREM 2.1. *Under our standard assumptions* (H1)–(H3) *the value function* $V(.,.)$ *is jointly Lipschitz continuous on* $[0,T-\delta]\times\mathbf{R}^d$ *for all* $\delta>0$.

The proof of this theorem will be split into a sequel of statements. It formalizes and generalizes the method of time change for the underlying Brownian motion introduced into the frame of stochastic control problems with classical cost functional in [1].

In order to estimate the reflected BSDEs (2.3) driven by W, we approximate them by penalized BSDEs. More precisely, we approximate (2.3) with its unique solution $(Y^{t,x,u},Z^{t,x,u},K^{t,x,u})$ by the following penalized BSDEs:

(2.6)
$$\begin{cases} dY_s^{t,x,u;n}=-[f(s,X_s^{t,x,u},Y_s^{t,x,u;n},Z_s^{t,x,u;n},u_s)+n\left(Y_s^{t,x,u;n}-\varphi(s,X_s^{t,x,u})\right)^-]ds\\ \qquad\qquad +Z_s^{t,x,u;n}dW_s,\ s\in[t,T],\\ Y_T^{t,x,u;n}=\Phi(X_T^{t,x,u}),\ n\geq 1.\end{cases}$$

For all $n\geq1$, BSDE (2.6) has a unique solution $(Y^{t,x,u;n},Z^{t,x,u;n})\in\mathcal{S}^2_{\mathbf{F}^W}(t,T)\times L^2_{\mathbf{F}^W}(t,T;\mathbf{R}^m)$. We define

$$V_n(t,x):=\inf_{u\in\mathcal{U}^W_{t,T}}Y_t^{t,x,u;n},\ (t,x)\in[0,T]\times\mathbf{R}^d. \tag{2.7}$$

PROPOSITION 2.1. *Under our standard assumptions* (H1) *and* (H3) *we have*

(2.8)

(i) $Y_s^{t,x,u;1}\leq Y_s^{t,x,u;2}\leq\cdots\leq Y_s^{t,x,u;n}\to Y_s^{t,x,u}$, *as* $n\to+\infty,\ s\in[t,T],\ P$-*a.s.*, $u\in\mathcal{U}^W_{t,T}$;

(ii) $E\Bigg[\sup_{s\in[t,T]}\left|Y_s^{t,x,u;n}-Y_s^{t,x,u}\right|^2+\int_t^T\left|Z_s^{t,x,u;n}-Z_s^{t,x,u}\right|^2ds$
$\quad\sup_{s\in[t,T]}\left|K_s^{t,x,u}-n\int_t^s(Y_r^{t,x,u;n}-\varphi(r,X_r^{t,x,u}))^-dr\right|^2\Bigg]\to0$, *as* $n\to+\infty,\ u\in\mathcal{U}^W_{t,T}$;

(iii) $V_1(t,x)\leq V_2(t,x)\leq\cdots\leq V_n(t,x)\to V(t,x)$, *as* $n\to+\infty,\ (t,x)\in[0,T]\times\mathbf{R}^d$.

For the proof of these classical results, in particular those of (i) and (ii), see section 6 of [7]. The result (iii) can be consulted, for instance, in Theorem 4.2 in [3] or in Lemma 4.3 in [16].

Theorem 2.1 follows from the following theorem combined with Proposition 2.1 (iii).

THEOREM 2.2. *Under the assumptions* (H1)–(H3), *for all* $\delta > 0$, $V_n(t,x)$ *is jointly Lipschitz continuous in* $(t,x) \in [0,T-\delta]\times\mathbf{R}^d$, *uniformly with respect to* $n \geq 1$, *i.e., for all* $\delta > 0$, *there exists* $C_\delta > 0$ *such that for any* $n \geq 1$, $(t_0,x_0),\ (t_1,x_1) \in [0,T-\delta]\times\mathbf{R}^d$,

$$|V_n(t_0,x_0) - V_n(t_1,x_1)| \leq C_\delta(|t_0-t_1| + |x_0-x_1|). \tag{2.9}$$

The proof is based on the method of time change and split into several steps. Let us arbitrarily fix $\delta > 0$, $(t_0,x_0),\ (t_1,x_1) \in [0,T-\delta]\times\mathbf{R}^d$. Moreover, let $W^0 = (W^0_s)_{s\in[t_0,T]}$ be an m-dimensional Brownian motion with $W^0_{t_0} = 0$, and let $u^0 \in \mathcal{U}^{W^0}_{t_0,T}$. With the notation introduced above we put

$$(X^0,Y^0,Z^0,K^0) = (X^{t_0,x_0,u^0}(W^0), Y^{t_0,x_0,u^0}(W^0), Z^{t_0,x_0,u^0}(W^0), K^{t_0,x_0,u^0}(W^0)) \tag{2.10}$$

(the unique solution of SDE (2.2) and RBSDE (2.3) driven by the Brownian motion W^0 and with initial data (t_0,x_0)), and

$$(Y^{0,n}, Z^{0,n}) = (Y^{t_0,x_0,u^0;n}(W^0), Z^{t_0,x_0,u^0;n}(W^0)) \tag{2.11}$$

(the unique solution of BSDE (2.6) driven by the Brownian motion W^0 and with (t_0,x_0,u^0) instead of (t,x,u)).

We introduce the linear time change $\tau : [t_1,T] \to [t_0,T]$ by setting

$$\tau(s) = t_0 + \frac{T-t_0}{T-t_1}(s-t_1), \quad s \in [t_1,T], \tag{2.12}$$

and we remark that $\dot{\tau}(= \frac{d}{ds}\tau(s)) = \frac{T-t_0}{T-t_1}$. Consequently,

$$W^1_s := W^0_{\tau(s)}\frac{1}{\sqrt{\dot{\tau}}},\ s \in [t_1,T], \tag{2.13}$$

defines a (m-dimensional) Brownian motion with $W^1_{t_1} = 0$. Then, obviously, the time transformed control process $u^1_s := u^0_{\tau(s)}$, $s \in [t_1,T]$, is an admissible control process with respect to the natural filtration $\mathbf{F}^{W^1} = (\mathcal{F}^{W^1}_s)_{s\in[t_1,T]}$ generated by the Brownian motion W^1 over the time interval $[t_1,T]$: $u^1 = (u^1_s)_{s\in[t_1,T]} \in \mathcal{U}^{W^1}_{t_1,T} (= L^0_{\mathbf{F}^{W^1}}(t_1,T;U))$.

Having a Brownian motion $W^1 = (W^1_s)_{s\in[t_1,T]}$ over the time interval $[t_1,T]$ and an associated admissible control $u^1 \in \mathcal{U}^{W^1}_{t_1,T}$ we can solve the corresponding system (2.2)–(2.3), but now driven by the Brownian motion W^1, with $((t_1,x_1),W^1,u^1)$ at the place of $((t_0,x_0),W^0,u^0)$, and we denote its unique solution by

$$(X^1,Y^1,Z^1,K^1) = (X^{t_1,x_1,u^1}(W^1), Y^{t_1,x_1,u^1}(W^1), Z^{t_1,x_1,u^1}(W^1), K^{t_1,x_1,u^1}(W^1)). \tag{2.14}$$

Correspondingly, the solution of the penalized BSDE (2.6) driven by the Brownian motion W^1 is denoted by

$$(Y^{1,n}, Z^{1,n}) = (Y^{t_1,x_1,u^1;n}(W^1), Z^{t_1,x_1,u^1;n}(W^1)), \tag{2.15}$$

while the associated solution of the forward equation is again $X^1 = X^{t_1,x_1,u^1}(W^1)$.

Therefore, the above procedure has provided two different forward equations, that for $X^0 = X^{t_0,x_0,u^0}(W^0)$ and $X^1 = X^{t_1,x_1,u^1}(W^1)$, i.e.,

$$dX_s^0 = \sigma(s, X_s^0, u_s^0)dW_s^0 + b(s, X_s^0, u_s^0)ds,\ s \in [t_0, T],\ X_{t_0}^0 = x_0, \tag{2.16}$$
$$dX_s^1 = \sigma(s, X_s^1, u_s^1)dW_s^1 + b(s, X_s^1, u_s^1)ds,\ s \in [t_1, T],\ X_{t_1}^1 = x_1, \tag{2.17}$$

which we associate with the respective RBSDEs

$$\begin{cases} dY_s^0 = -f(s, X_s^0, Y_s^0, Z_s^0, u_s^0)ds + Z_s^0 dW_s^0 - dK_s^0, \\ Y_T^0 = \Phi(X_T^0), \\ Y_s^0 \ge \varphi(s, X_s^0), \quad (Y_s^0 - \varphi(s, X_s^0))dK_s^0 = 0, \quad s \in [t_0, T], \end{cases} \tag{2.18}$$

and

$$\begin{cases} dY_s^1 = -f(s, X_s^1, Y_s^1, Z_s^1, u_s^1)ds + Z_s^1 dW_s^1 - dK_s^1, \\ Y_T^1 = \Phi(X_T^1), \\ Y_s^1 \ge \varphi(s, X_s^1), \quad (Y_s^1 - \varphi(s, X_s^1))dK_s^1 = 0, \quad s \in [t_1, T]. \end{cases} \tag{2.19}$$

On the other hand, RBSDE (2.18) with its unique solution (Y^0, Z^0, K^0) is approximated by the following penalized BSDEs:

$$\begin{cases} dY_s^{0,n} = -[f(s, X_s^0, Y_s^{0,n}, Z_s^{0,n}, u_s^0) + n\left(Y_s^{0,n} - \varphi(s, X_s^0)\right)^-]ds + Z_s^{0,n} dW_s^0, \\ Y_T^{0,n} = \Phi(X_T^0),\ s \in [t_0, T],\ n \ge 1. \end{cases} \tag{2.20}$$

And RBSDE (2.19) with its unique solution (Y^1, Z^1, K^1) is approximated by the following penalized equations:

$$\begin{cases} dY_s^{1,n} = -[f(s, X_s^1, Y_s^{1,n}, Z_s^{1,n}, u_s^1) + n\left(Y_s^{1,n} - \varphi(s, X_s^1)\right)^-]ds + Z_s^{1,n} dW_s^1, \\ Y_T^{1,n} = \Phi(X_T^1),\ s \in [t_1, T],\ n \ge 1. \end{cases} \tag{2.21}$$

To compare both SDEs (2.16) and (2.17), which are defined over different time intervals and driven by different Brownian motions, we have to make the inverse time change $\tau^{-1} : [t_0, T] \to [t_1, T]$, $\tau^{-1}(s) = t_1 + \frac{T - t_1}{T - t_0}(s - t_0)$, $s \in [t_0, T]$, in (2.17) in order to have two SDEs driven by the same Brownian motion $W^0 = (W_s^0)_{s \in [t_0, T]}$. For this we define

$$\widetilde{X}_s^1 := X_{\tau^{-1}(s)}^1,\ \widetilde{Y}_s^{1,n} := Y_{\tau^{-1}(s)}^{1,n},\ \widetilde{Z}_s^{1,n} := \frac{1}{\sqrt{\dot{\tau}}} Z_{\tau^{-1}(s)}^{1,n},\ s \in [t_0, T]. \tag{2.22}$$

By observing that

$$W_{\tau^{-1}(s)}^1 = \frac{1}{\sqrt{\dot{\tau}}} W_s^0, \text{ and } u_{\tau^{-1}(s)}^1 = u_s^0,\ s \in [t_0, T], \tag{2.23}$$

we deduce from (2.17) that $\widetilde{X}^1 = (\widetilde{X}_s^1)_{s \in [t_0, T]}$ is the unique continuous $\mathbf{F}^{W^0}$-adapted solution of the SDE

$$d\widetilde{X}_s^1 = \frac{1}{\sqrt{\dot{\tau}}} \sigma(\tau^{-1}(s), \widetilde{X}_s^1, u_s^0) dW_s^0 + \frac{1}{\dot{\tau}} b(\tau^{-1}(s), \widetilde{X}_s^1, u_s^0) ds,\ s \in [t_0, T], \quad \widetilde{X}_{t_0}^1 = x_1, \tag{2.24}$$

and from (2.21) we get that $(\widetilde{Y}^{1,n}, \widetilde{Z}^{1,n}) = (\widetilde{Y}_s^{1,n}, \widetilde{Z}_s^{1,n})_{s \in [t_0, T]}$ is the unique solution of the penalized BSDE

$$
\begin{cases} d\widetilde{Y}_s^{1,n} = -\frac{1}{\dot{\tau}}\left[f(\tau^{-1}(s), \widetilde{X}_s^1, \widetilde{Y}_s^{1,n}, \sqrt{\dot{\tau}}\widetilde{Z}_s^{1,n}, u_s^0) + n(\widetilde{Y}_s^{1,n} - \varphi(\tau^{-1}(s), \widetilde{X}_s^1))^-\right] ds \\ \qquad\qquad + \widetilde{Z}_s^{1,n} dW_s^0, \quad s \in [t_0, T], \\ \widetilde{Y}_T^{1,n} = \Phi(\widetilde{X}_T^1), \ n \geq 1. \end{cases} \tag{2.25}
$$

We will prove the following crucial result.

PROPOSITION 2.2. *There is some $C_\delta \in \mathbf{R}$ only depending on δ, and on the bounds and the Lipschitz constants of the coefficients such that, for all $n \geq 1$, $s \in [t_0, T]$, P-a.s.,*

(i)

$$
|\widetilde{Y}_s^{1,n} - Y_s^{0,n}| \leq C_\delta \left(|t_0 - t_1| + \sup_{r \in [t_0, s]} |X_r^0 - \widetilde{X}_r^1| \right). \tag{2.26}
$$

In particular,

$$
|Y_{t_1}^{1,n} - Y_{t_0}^{0,n}| = |\widetilde{Y}_{t_0}^{1,n} - Y_{t_0}^{0,n}| \leq C_\delta(|t_0 - t_1| + |x_0 - x_1|). \tag{2.27}
$$

(ii) *If in addition $\varphi(t, x) = \varphi \in \mathbf{R}$, $(t, x) \in [0, T] \times \mathbf{R}^d$, then for all $p \geq 1$ there is some constant $C_{\delta,p}$ such that for all $n \geq 1$, $s \in [t_0, T]$, P-a.s.,*

$$
E\left[\left(\int_s^T |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr\right)^p \Big| \mathcal{F}_s^{W^0}\right] \leq C_{\delta,p}\left(|t_0 - t_1|^2 + \sup_{r \in [t_0, s]} |X_r^0 - \widetilde{X}_r^1|^2\right)^p. \tag{2.28}
$$

Let us begin by showing that Proposition 2.2 allows us to prove Theorem 2.2.

Proof of Theorem 2.2. Let $n \geq 1$, and recall that

$$
V_n(t_0, x_0) := \inf_{u^0 \in \mathcal{U}_{t_0,T}^{W^0}} Y_{t_0}^{t_0, x_0, u^0; n} = \inf_{u^0 \in \mathcal{U}_{t_0,T}^{W^0}} Y_{t_0}^{0,n}.
$$

Thus, choosing $\epsilon > 0$ arbitrarily small we can find some control $u^0 \in \mathcal{U}_{t_0,T}^{W^0}$ (depending on $\epsilon > 0$ and on $n \geq 1$) such that

$$
Y_{t_0}^{0,n} \leq V_n(t_0, x_0) + \epsilon.
$$

On the other hand,

$$
\widetilde{Y}_{t_0}^{1,n} = Y_{t_1}^{1,n} \geq V_n(t_1, x_1).
$$

Hence, from Proposition 2.2 we get

$$
\begin{aligned} V_n(t_1, x_1) - V_n(t_0, x_0) \leq \widetilde{Y}_{t_0}^{1,n} - Y_{t_0}^{0,n} + \epsilon &\leq |\widetilde{Y}_{t_0}^{1,n} - Y_{t_0}^{0,n}| + \epsilon \\ &\leq C_\delta(|t_0 - t_1| + |X_{t_0}^0 - \widetilde{X}_{t_0}^1|) + \epsilon \\ &= C_\delta(|t_0 - t_1| + |x_0 - x_1|) + \epsilon. \end{aligned} \tag{2.29}
$$

Then, the arbitrariness of $\epsilon > 0$ yields $V_n(t_1, x_1) - V_n(t_0, x_0) \leq C_\delta(|t_0 - t_1| + |x_0 - x_1|)$, and from the symmetry of the argument we obtain

$$
|V_n(t_1, x_1) - V_n(t_0, x_0)| \leq C_\delta(|t_1 - t_0| + |x_1 - x_0|).
$$

Finally, by recalling that the constant C_δ from Proposition 2.2 is independent of (t_0, x_0), (t_1, x_1), and $n \geq 1$, we complete the proof. ∎

The proof of Proposition 2.2 is split into a sequel of lemmas. The first one concerns the comparison of the SDEs (2.16) and (2.24), i.e.,

$$dX_s^0 = \sigma(s, X_s^0, u_s^0)dW_s^0 + b(s, X_s^0, u_s^0)ds,\ s \in [t_0, T],\ X_{t_0}^0 = x_0,$$

and,

$$d\widetilde{X}_s^1 = \frac{1}{\sqrt{\dot{\tau}}}\sigma(\tau^{-1}(s), \widetilde{X}_s^1, u_s^0)dW_s^0 + \frac{1}{\dot{\tau}}b(\tau^{-1}(s), \widetilde{X}_s^1, u_s^0)ds,\ s \in [t_0, T],\quad \widetilde{X}_{t_0}^1 = x_1.$$

To estimate the difference of solutions of these SDEs, the following lemma turns out to be useful. It can be achieved by a straightforward computation (see also [1]).

LEMMA 2.1. *For the above introduced time change $\tau : [t_1, T] \to [t_0, T]$ we have*

$$|\tau^{-1}(s) - s| + \left|\frac{1}{\dot{\tau}} - 1\right| + \left|\frac{1}{\sqrt{\dot{\tau}}} - 1\right| \leq C_\delta |t_0 - t_1|,\ s \in [t_0, T], \tag{2.30}$$

where the constant C_δ only depends on T and $\delta > 0$ but not on t_0, $t_1 \in [0, T - \delta]$.

The above lemma combined with SDE standard estimates allows us to get the following result.

LEMMA 2.2. *There is some $C_{\delta,p} \in \mathbf{R}^+$ only depending on the bounds of σ, b, their Lipschitz constants, and on δ, $p \geq 1$, such that for all $s \in [t_0, T]$,*

$$E\left[\sup_{r\in[s,T]} |X_r^0 - \widetilde{X}_r^1|^p | \mathcal{F}_s^{W^0}\right] \leq C_{\delta,p}(|t_0 - t_1|^p + |X_s^0 - \widetilde{X}_s^1|^p),\quad P\text{-a.s.} \tag{2.31}$$

In particular, for $s = t_0$,

$$E\left[\sup_{r\in[t_0,T]} |X_r^0 - \widetilde{X}_r^1|^p\right] \leq C_{\delta,p}(|t_0 - t_1|^p + |x_0 - x_1|^p). \tag{2.32}$$

Proof. Taking the difference between the SDEs (2.16) and (2.24) we obtain

$$\begin{aligned} d(X_s^0 - \widetilde{X}_s^1) &= \left(\sigma(s, X_s^0, u_s^0) - \frac{1}{\sqrt{\dot{\tau}}}\sigma(\tau^{-1}(s), \widetilde{X}_s^1, u_s^0)\right) dW_s^0 \\ &\quad + \left(b(s, X_s^0, u_s^0) - \frac{1}{\dot{\tau}}b(\tau^{-1}(s), \widetilde{X}_s^1, u_s^0)\right) ds,\ s \in [t_0, T], \\ X_{t_0}^0 - \widetilde{X}_{t_0}^1 &= x_0 - x_1. \end{aligned} \tag{2.33}$$

Thus, taking into account that b and σ are bounded, SDE standard estimates yield that for all $p \geq 1$ there is some constant C_p only depending on the bounds and the Lipschitz coefficients of σ and b as well as of T and p, such that

$$\begin{aligned} &\mathbf{E}\left[\sup_{s\leq r\leq T} |\widetilde{X}_r^1 - X_r^0|^p | \mathcal{F}_s^{W^0}\right] \\ &\leq C_p \cdot \left(\left|\frac{1}{\dot{\tau}} - 1\right|^p + \left|\frac{1}{\sqrt{\dot{\tau}}} - 1\right|^p + \int_s^T |\tau^{-1}(r) - r|^p dr + |\widetilde{X}_s^1 - X_s^0|^p\right), \end{aligned} \tag{2.34}$$

$t_0 \leq s \leq T$, $p \geq 1$. Finally, by applying the preceding lemma we complete the proof. ∎

We will also need the following lemma.

LEMMA 2.3. (i) *There exists some constant* C *only depending on the bounds of* f, Φ, *and* φ *such that*

$$\left|Y_s^{i,n}\right| \le C,\ s\in[t_i,T],\ n\ge 1,\ i=0,1,\ P\text{-a.s.} \tag{2.35}$$

(ii) *For all* $p\ge 1$ *there is some constant* C_p *only depending on the bounds of the coefficients* f, Φ, *and* φ *and on* p *such that* $s\in[t_i,T]$, $n\ge 1$, $i=0,\ 1$,

$$E\left[\left(\int_s^T \left|Z_r^{i,n}\right|^2 dr\right)^p + \left(n\int_s^T (Y_r^{i,n}-\varphi(r,X_r^i))^- dr\right)^{2p} |\mathcal{F}_s^{W^i}\right] \le C_p,\ P\text{-a.s.} \tag{2.36}$$

Proof. Assertion (i) follows directly from Proposition 2.1(i), the comparison theorem for reflected BSDEs (Lemma 4.4 in section 4), and the boundedness of the coefficients f, Φ, and φ.

(ii) From the penalized BSDEs (2.20) and (2.21), (i), and the boundedness of the coefficients f and Φ we have, for some constant C_p,

$$n\int_s^T (Y_r^{i,n}-\varphi(r,X_r^i))^- dr \le C_p + \int_s^T Z_r^{i,n} dW_r^i,\ s\in[t_i,T],\ n\ge 1.$$

Hence,

$$\begin{aligned} &E\left[\left(n\int_s^T (Y_r^{i,n}-\varphi(r,X_r^i))^- dr\right)^{2p} |\mathcal{F}_s^{W^i}\right] \\ &\quad\le C_p + C_p E\left[\left(\int_s^T |Z_r^{i,n}|^2 dr\right)^p |\mathcal{F}_s^{W^i}\right],\ s\in[t_i,T]. \end{aligned} \tag{2.37}$$

On the other hand, from Itô's formula,

$$\begin{aligned} |Y_s^{i,n}|^2 + \int_s^T |Z_r^{i,n}|^2 dr = |\Phi(X_T^i)|^2 &+ 2\int_s^T Y_r^{i,n} f(r,X_r^i,Y_r^{i,n},Z_r^{i,n},u_r^i)dr \\ &+ 2n\int_s^T Y_r^{i,n}(Y_r^{i,n}-\varphi(r,X_r^i))^- dr \\ &- 2\int_s^T Y_r^{i,n} Z_r^{i,n} dW_r^i,\ s\in[t_i,T],\ n\ge 1. \end{aligned} \tag{2.38}$$

From (2.38) together with (i),

$$\begin{aligned} E\left[\left(\int_s^T |Z_r^{i,n}|^2 dr\right)^{2p} |\mathcal{F}_s^{W^i}\right] &\le C_p + C_p E\left[\left(n\int_s^T (Y_r^{i,n}-\varphi(r,X_r^i))^- dr\right)^{2p} |\mathcal{F}_s^{W^i}\right] \\ &\quad + C_p E\left[\left(\int_s^T |Z_r^{i,n}|^2 dr\right)^p |\mathcal{F}_s^{W^i}\right],\ s\in[t_i,T],\ n\ge 1. \end{aligned} \tag{2.39}$$

The result (ii) for $p=1$ (see pp. 719–720 in section 6 in [7]) combined with (2.37) and (2.39) yields the general result (ii). Indeed, we have (ii) for $p=1$. Let us suppose now that we have (ii) for some integer $p \geq 1$. Then, due to (2.37) and (2.39),

$$
\begin{aligned}
&E\left[\left(\int_s^T |Z_r^{i,n}|^2 dr\right)^{2p} |\mathcal{F}_s^{W^i}\right] + E\left[\left(n\int_s^T (Y_r^{i,n} - \varphi(r, X_r^i))^- dr\right)^{4p} |\mathcal{F}_s^{W^i}\right] \\
&\quad\leq C_{2p} + C_{2p} E\left[\left(\int_s^T |Z_r^{i,n}|^2 dr\right)^{2p} |\mathcal{F}_s^{W^i}\right] \\
&\quad\leq C_p + C_p E\left[\left(\int_s^T |Z_r^{i,n}|^2 dr\right)^{p} |\mathcal{F}_s^{W^i}\right] \\
&\qquad + C_p E\left[\left(n\int_s^T (Y_r^{i,n} - \varphi(r, X_r^i))^- dr\right)^{2p} |\mathcal{F}_s^{W^i}\right] \\
&\quad\leq C_p,\ P\text{-a.s.},\ s \in [t_i, T],\ n \geq 1.
\end{aligned} \tag{2.40}
$$

Consequently, (ii) holds true for all integers of the form $p = 2^k$, $k \geq 0$, and hence also for any real $p \geq 1$. ▯

For the proof of Proposition 2.2 we have to compare the BSDEs (2.20) and (2.25), i.e.,

$$
\begin{cases} dY_s^{0,n} = -[f(s, X_s^0, Y_s^{0,n}, Z_s^{0,n}, u_s^0) + n\left(Y_s^{0,n} - \varphi(s, X_s^0)\right)^-]ds + Z_s^{0,n} dW_s^0, \\ Y_T^{0,n} = \Phi(X_T^0), \qquad s \in [t_0, T], \end{cases} \tag{2.41}
$$

and

$$
\begin{cases} d\widetilde{Y}_s^{1,n} = -\frac{1}{\dot{\tau}}\left[f(\tau^{-1}(s), \widetilde{X}_s^1, \widetilde{Y}_s^{1,n}, \sqrt{\dot{\tau}}\widetilde{Z}_s^{1,n}, u_s^0) + n(\widetilde{Y}_s^{1,n} - \varphi(\tau^{-1}(s), \widetilde{X}_s^1))^-\right] ds \\ \qquad\qquad + \widetilde{Z}_s^{1,n} dW_s^0, \quad s \in [t_0, T], \\ \widetilde{Y}_T^{1,n} = \Phi(\widetilde{X}_T^1). \end{cases} \tag{2.42}
$$

But, the different structure of the penalization terms and different obstacles don't allow a direct estimate to get Proposition 2.2, so intermediate steps are necessary.

Let us first compare BSDE (2.42) with the following BSDE (2.43):

$$
\begin{cases} d\widehat{Y}_s^{1,n} = -\left[f(s, X_s^0, \widehat{Y}_s^{1,n} - A_s, \widehat{Z}_s^{1,n}, u_s^0) + n(\widehat{Y}_s^{1,n} - \varphi(s, X_s^0) - A_s)^-\right. \\ \qquad\qquad \left. + C_\delta|t_0 - t_1|(1 + |\widehat{Z}_s^{1,n}|) + CA_s\right]ds + \widehat{Z}_s^{1,n} dW_s^0,\ s \in [t_0, T], \\ \widehat{Y}_T^{1,n} = \Phi(X_T^0) + A_T, \end{cases} \tag{2.43}
$$

where $C_\delta,\ C \geq 1$ are constants which are large enough (their precise choice becomes clear from the proof of the lemma below), and

$$
A_s := C_\delta \sup_{r \in [t_0, s]} (|t_0 - t_1| + |\widetilde{X}_r^1 - X_r^0|),\ s \in [t_0, T]. \tag{2.44}
$$

Note that $A = (A_s)_{s \in [t_0, T]}$ is an $\mathbf{F}^{W^0}$-adapted, continuous increasing process, $A_{t_0} = C_\delta(|t_0 - t_1| + |x_0 - x_1|)$, and from Lemma 2.2 we see that for all $q \geq 1$,

$$
\mathbf{E}[A_T^q - A_s^q | \mathcal{F}_s^{W^0}] \leq C_{\delta,q} \cdot \left(|t_0 - t_1|^q + |\widetilde{X}_s^1 - X_s^0|^q\right), \quad s \in [t_0, T]. \tag{2.45}
$$

LEMMA 2.4. *Under our standard assumptions* (H1)–(H3) *we have*

$$\widetilde{Y}_s^{1,n} \le \widehat{Y}_s^{1,n},\ s \in [t_0, T],\ n \ge 1,\ P\text{-a.s.} \tag{2.46}$$

Proof. The proof is based on the comparison theorem for BSDEs (Lemma 4.2 in section 4). For this we note that since φ is bounded and Lipschitz,

$$\begin{aligned} \left|\frac{1}{\dot\tau}\varphi(\tau^{-1}(s), \widetilde{X}_s^1) - \varphi(s, X_s^0)\right| &\le C\left(\left|1 - \frac{1}{\dot\tau}\right| + |\tau^{-1}(s) - s| + |\widetilde{X}_s^1 - X_s^0|\right) \\ &\le C_\delta(|t_1 - t_0| + |\widetilde{X}_s^1 - X_s^0|),\ s \in [t_0, T] \end{aligned} \tag{2.47}$$

(recall Lemma 2.1). Thus, recalling that $\widetilde{Y}^{1,n}$ s bounded, uniformly w.r.t. $n \ge 1$, we get from Lemma 2.1 that

$$\begin{aligned} \widetilde{Y}_s^{1,n} - \varphi(s, X_s^0) &\le \frac{1}{\dot\tau}(\widetilde{Y}_s^{1,n} - \varphi(\tau^{-1}(s), \widetilde{X}_s^1)) + C_\delta(|t_1 - t_0| + |\widetilde{X}_s^1 - X_s^0|) \\ &\le \frac{1}{\dot\tau}(\widetilde{Y}_s^{1,n} - \varphi(\tau^{-1}(s), \widetilde{X}_s^1)) + A_s,\ s \in [t_0, T]. \end{aligned} \tag{2.48}$$

Then, from (2.48),

$$\left(\frac{1}{\dot\tau}(\widetilde{Y}_s^{1,n} - \varphi(\tau^{-1}(s), \widetilde{X}_s^1))\right)^- \le (\widetilde{Y}_s^{1,n} - \varphi(s, X_s^0) - A_s)^-,\ s \in [t_0, T]. \tag{2.49}$$

Moreover, from

$$\begin{aligned} &\left|\frac{1}{\dot\tau}f(\tau^{-1}(s), \widetilde{X}_s^1, \widetilde{Y}_s^{1,n}, \sqrt{\dot\tau}\widetilde{Z}_s^{1,n}, u_s^0) - f(s, X_s^0, \widetilde{Y}_s^{1,n}, \widetilde{Z}_s^{1,n}, u_s^0)\right| \\ &\quad \le C\left(\left|\frac{1}{\dot\tau} - 1\right| + |\tau^{-1}(s) - s| + |1 - \sqrt{\dot\tau}||\widetilde{Z}_s^{1,n}|\right) + C|\widetilde{X}_s^1 - X_s^0| \\ &\quad \le C_\delta|t_0 - t_1|(1 + |\widetilde{Z}_s^{1,n}|) + C|\widetilde{X}_s^1 - X_s^0| \\ &\quad \le C_\delta|t_0 - t_1|(1 + |\widetilde{Z}_s^{1,n}|) + A_s, \quad s \in [t_0, T], \end{aligned} \tag{2.50}$$

we have

$$\begin{aligned} &\frac{1}{\dot\tau}f(\tau^{-1}(s), \widetilde{X}_s^1, \widetilde{Y}_s^{1,n}, \sqrt{\dot\tau}\widetilde{Z}_s^{1,n}, u_s^0) \\ &\quad \le f(s, X_s^0, \widetilde{Y}_s^{1,n}, \widetilde{Z}_s^{1,n}, u_s^0) + C_\delta|t_0 - t_1|(1 + |\widetilde{Z}_s^{1,n}|) + A_s \\ &\quad \le f(s, X_s^0, \widetilde{Y}_s^{1,n} - A_s, \widetilde{Z}_s^{1,n}, u_s^0) + C_\delta|t_0 - t_1|(1 + |\widetilde{Z}_s^{1,n}|) + CA_s,\ s \in [t_0, T]. \end{aligned} \tag{2.51}$$

We also observe that thanks to the Lipschitz property of Φ,

$$\Phi(\widetilde{X}_T^1) \le \Phi(X_T^0) + A_T,\ \text{P-a.s.} \tag{2.52}$$

The relations (2.49), (2.51), and (2.52) allow us to apply the comparison theorem (Lemma 4.2 in section 4) to both BSDEs and thus to conclude that

$$\widetilde{Y}_s^{1,n} \le \widehat{Y}_s^{1,n},\ s \in [t_0, T],\ n \ge 1,\ \text{P-a.s.} \qquad \square$$

The statement of the above lemma can be strengthened as follows.

LEMMA 2.5. *Under the standard assumptions* (H1)–(H3) *the following holds true:*

$$\begin{aligned} &\text{(i) } -C \le \widetilde{Y}_s^{1,n} \le \widehat{Y}_s^{1,n} \le C_\delta + C_\delta A_s,\ s\in[t_0,T],\ n\ge 1,\ P\text{-a.s.};\\ &\text{(ii) } E\left[\int_s^T |\widehat{Z}_r^{1,n}|^2 dr|\mathcal{F}_s^{W^0}\right] \le C_\delta(1+A_s^2),\ s\in[t_0,T],\ n\ge 1,\ P\text{-a.s.} \end{aligned} \tag{2.53}$$

Proof. (i) First, from Lemma 2.3 we know that $|\widetilde{Y}_s^{1,n}|\le C,\ s\in[t_0,T],\ n\ge 1$, P-a.s. Second, thanks to the boundedness of f and φ, for some constant C' large enough, we have

(2.54)
$$\begin{aligned} &f(s,X_s^0,\widehat{Y}_s^{1,n}-A_s,\widehat{Z}_s^{1,n},u_s^0)+n(\widehat{Y}_s^{1,n}-(\varphi(s,X_s^0)+A_s))^- + C_\delta|t_0-t_1|(1+|\widehat{Z}_s^{1,n}|)\\ &\quad + CA_s \le C' + n(\widehat{Y}_s^{1,n}-(C'+C'A_s))^- + C_\delta(1+|\widehat{Z}_s^{1,n}|)+C'A_s, \end{aligned}$$

and $\Phi(X_T^0)+A_T\le \Phi(X_T^0)+C'A_T$. Hence, we can compare (2.43) with the BSDE (2.55):

(2.55)
$$\begin{cases} d\overline{Y}_s^{1,n} = -\big(C'+n(\overline{Y}_s^{1,n}-(C'+C'A_s))^- + C_\delta(1+|\overline{Z}_s^{1,n}|)+C'A_s\big)ds + \overline{Z}_s^{1,n}dW_s^0,\\ \overline{Y}_T^{1,n} = \Phi(X_T^0)+C'A_T,\ s\in[t_0,T]. \end{cases}$$

From the comparison theorem for BSDEs (Lemma 4.2 in section 4) we get that

$$\widehat{Y}_s^{1,n} \le \overline{Y}_s^{1,n},\ s\in[t_0,T],\ n\ge 1,\ \text{P-a.s.} \tag{2.56}$$

On the other hand, putting $\overline{Y}_s^{2,n} := \overline{Y}_s^{1,n} - C'A_s,\ s\in[t_0,T]$, we get

(2.57)
$$\begin{cases} d\overline{Y}_s^{2,n} = -\big(C'+n(\overline{Y}_s^{2,n}-C')^- + C_\delta(1+|\overline{Z}_s^{1,n}|)+C'A_s\big)ds - C'dA_s + \overline{Z}_s^{1,n}dW_s^0,\\ \overline{Y}_T^{2,n} = \Phi(X_T^0),\ s\in[t_0,T]. \end{cases}$$

By observing that

$$(\overline{Y}_s^{2,n}-C')(\overline{Y}_s^{2,n}-C')^- \le 0, \tag{2.58}$$

thanks to Itô's formula and the boundedness of Φ, for arbitrary $\gamma>0$,

(2.59)
$$\begin{aligned} &e^{\gamma s}|\overline{Y}_s^{2,n}-C'|^2 + E\left[\int_s^T e^{\gamma r}(\gamma|\overline{Y}_r^{2,n}-C'|^2+|\overline{Z}_r^{1,n}|^2)dr|\mathcal{F}_s^{W^0}\right]\\ &\quad\le C_{\delta,\gamma} + E\left[\int_s^T e^{\gamma r}\left(C_\delta|\overline{Y}_r^{2,n}-C'|^2+\frac{1}{2}|\overline{Z}_r^{1,n}|^2\right)dr|\mathcal{F}_s^{W^0}\right]\\ &\qquad + C_\gamma E[A_T^2|\mathcal{F}_s^{W^0}] + 2E\left[\int_s^T e^{\gamma r}(\overline{Y}_r^{2,n}-C')C'dA_r|\mathcal{F}_s^{W^0}\right],\ s\in[t_0,T],\ n\ge 1. \end{aligned}$$

Hence, for $\gamma\ge C_\delta+1$ large enough,

$$\begin{aligned} &|\overline{Y}_s^{2,n}-C'|^2 + E\left[\int_s^T |\overline{Z}_r^{1,n}|^2dr|\mathcal{F}_s^{W^0}\right]\\ &\quad\le C_{\delta,\gamma}+C_{\delta,\gamma}A_s^2+\widehat{C}_\gamma E\left[\sup_{r\in[s,T]}|\overline{Y}_r^{2,n}-C'|A_T|\mathcal{F}_s^{W^0}\right],\ s\in[t_0,T],\ n\ge 1, \end{aligned} \tag{2.60}$$

where $\widehat{C}_\gamma$ only depends on the coefficients in (H1)–(H3) and on δ, $\gamma \geq 0$. Let $1 < p < 2$ and $q > 2$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and let us choose $\varepsilon > 0$ such that $\widehat{C}_\gamma \varepsilon (\frac{2}{2-p})^{\frac{2}{p}} < 1$. Then,

$$
\begin{aligned}
&E\left[\sup_{r\in[s,T]} |\overline{Y}^{2,n}_r - C'|A_T|\mathcal{F}^{W^0}_s\right] \\
&\leq \left(E\left[\sup_{r\in[s,T]} |\overline{Y}^{2,n}_r - C'|^p|\mathcal{F}^{W^0}_s\right]\right)^{\frac{1}{p}} (E[A_T^q|\mathcal{F}^{W^0}_s])^{\frac{1}{q}} \\
&\leq \varepsilon M_{s,t}^{\frac{2}{p}} + \frac{1}{\varepsilon}(E[A_T^q|\mathcal{F}^{W^0}_s])^{\frac{2}{q}} \\
&\leq \varepsilon M_{s,t}^{\frac{2}{p}} + \frac{1}{\varepsilon} C_{\delta,q} A_s^2, \quad t_0 \leq t \leq s \leq T, \quad n \geq 1 \text{ (see (2.44) and (2.45))},
\end{aligned}
\tag{2.61}
$$

where

$$
M_{s,t} := E\left[\sup_{r\in[t,T]} |\overline{Y}^{2,n}_r - C'|^p|\mathcal{F}^{W^0}_s\right].
$$

From Doob's martingale inequality, since $\frac{2}{p} > 1$,

$$
\begin{aligned}
E\left[\sup_{s\in[t,T]} M_{s,t}^{\frac{2}{p}}|\mathcal{F}^{W^0}_t\right] &\leq \left(\frac{2}{2-p}\right)^{\frac{2}{p}} E\left[M_{T,t}^{\frac{2}{p}}|\mathcal{F}^{W^0}_t\right] \\
&\leq \left(\frac{2}{2-p}\right)^{\frac{2}{p}} E\left[\sup_{s\in[t,T]} |\overline{Y}^{2,n}_s - C'|^2|\mathcal{F}^{W^0}_t\right], \ t \in [t_0, T].
\end{aligned}
\tag{2.62}
$$

Hence, from (2.60), (2.61), and (2.62),

$$
\begin{aligned}
&E\left[\sup_{s\in[t,T]} |\overline{Y}^{2,n}_s - C'|^2|\mathcal{F}^{W^0}_t\right] \\
&\leq C_{\delta,\gamma} + C_{\delta,\varepsilon} A_t^2 + \widehat{C}_\gamma \varepsilon \left(\tfrac{2}{2-p}\right)^{\frac{2}{p}} E\left[\sup_{s\in[t,T]} |\overline{Y}^{2,n}_s - C'|^2|\mathcal{F}^{W^0}_t\right], \ t \in [t_0, T],
\end{aligned}
\tag{2.63}
$$

and since $\widehat{C}_\gamma \varepsilon (\frac{2}{2-p})^{\frac{2}{p}} < 1$, we get

$$
E\left[\sup_{s\in[t,T]} |\overline{Y}^{2,n}_s - C'|^2|\mathcal{F}^{W^0}_t\right] \leq C'_\delta(1 + A_t^2), \ t \in [t_0, T], \ n \geq 1, \ \text{P-a.s.} \tag{2.64}
$$

Then we get that $(\overline{Y}^{2,n}_s - C')^2 \leq C'_\delta(1 + A_s^2)$, i.e.,

$$
|\overline{Y}^{2,n}_s| \leq C'_\delta(1 + A_s), \ s \in [t_0, T]. \tag{2.65}
$$

Consequently,

$$
\widetilde{Y}^{1,n}_s \leq \widehat{Y}^{1,n}_s \leq \overline{Y}^{1,n}_s = \overline{Y}^{2,n}_s + C' A_s \leq C_\delta(1 + A_s), \ s \in [t_0, T], \ n \geq 1, \ \text{P-a.s.}
$$

(ii) Let C_0 be a bound of φ. Then, from the BSDE (2.43), we have

$$
\begin{aligned}
&d(\widehat{Y}_s^{1,n}-(C_0+A_s))^2\\
&=-2(\widehat{Y}_s^{1,n}-(C_0+A_s))\{f(s,X_s^0,\widehat{Y}_s^{1,n}-A_s,\widehat{Z}_s^{1,n},u_s^0)+n(\widehat{Y}_s^{1,n}-(\varphi(s,X_s^0)+A_s))^-\\
&\quad+C_\delta|t_0-t_1|(1+|\widehat{Z}_s^{1,n}|)+CA_s\}ds+|\widehat{Z}_s^{1,n}|^2ds\\
&\quad+2(\widehat{Y}_s^{1,n}-(C_0+A_s))\widehat{Z}_s^{1,n}dW_s^0-2(\widehat{Y}_s^{1,n}-(C_0+A_s))dA_s,\ s\in[t_0,T].
\end{aligned}\tag{2.66}
$$

Furthermore, from the above result (i), $(\widehat{Y}_T^{1,n}-(C_0+A_T))^2\le C(1+A_T^2)$, $(\widehat{Y}_s^{1,n}-(C_0+A_s))(\widehat{Y}_s^{1,n}-(\varphi(s,X_s^0)+A_s))^-\le 0$, and $|\widehat{Y}_s^{1,n}-(C_0+A_s)|\le C_\delta(1+A_s)$, $s\in[t_0,T]$, $n\ge 1$, we get by standard estimates

$$
E\left[\int_s^T|\widehat{Z}_r^{1,n}|^2dr|\mathcal{F}_s^{W^0}\right]\le C_\delta+C_\delta E[A_T^2|\mathcal{F}_s^{W^0}]\le C_\delta+C_\delta(1+A_s^2),\ s\in[t_0,T],\ n\ge 1.\tag{2.67}
$$

The proof of the lemma is complete. □

Let us now put

$$
Y_s^{2,n}:=\widehat{Y}_s^{1,n}-A_s,\ s\in[t_0,T],\ n\ge 1.\tag{2.68}
$$

Then, from BSDE (2.43) with solution $(\widehat{Y}^{1,n},\widehat{Z}^{1,n})$ we get

$$
\begin{cases}
dY_s^{2,n}=\ -(f(s,X_s^0,Y_s^{2,n},\widehat{Z}_s^{1,n},u_s^0)+n(Y_s^{2,n}-\varphi(s,X_s^0))^-\\
\qquad\qquad+C_\delta|t_0-t_1|(1+|\widehat{Z}_s^{1,n}|)+CA_s)ds+\widehat{Z}_s^{1,n}dW_s^0-dA_s,\ s\in[t_0,T],\\
Y_T^{2,n}=\Phi(X_T^0).
\end{cases}\tag{2.69}
$$

BSDE (2.69) has the advantage that its penalization term is exactly of the same form as that in BSDE (2.41). This fact together with both latter lemmas allow us to prove the next lemma.

LEMMA 2.6. *Let us assume* (H1)–(H3). *Then, there is some constant C_δ such that*

$$
E\left[\sup_{r\in[s,T]}|Y_r^{0,n}-Y_r^{2,n}|^2+\int_s^T|Z_r^{0,n}-Z_r^{2,n}|^2dr|\mathcal{F}_s^{W^0}\right]\le C_\delta A_s^2,\tag{2.70}
$$

and, in particular, $|Y_s^{0,n}-Y_s^{2,n}|\le C_\delta A_s$, $s\in[t_0,T]$, $n\ge 1$, P-a.s.

Proof. We have to compare BSDE (2.69) with BSDE (2.41), i.e., with the equation

$$
\begin{cases}
dY_s^{0,n}=-(f(s,X_s^0,Y_s^{0,n},Z_s^{0,n},u_s^0)+n\left(Y_s^{0,n}-\varphi(s,X_s^0)\right)^-)ds\\
\qquad\qquad+Z_s^{0,n}dW_s^0,\ s\in[t_0,T],\\
Y_T^{0,n}=\Phi(X_T^0).
\end{cases}
$$

The proof uses ideas similar to that of (2.64). However, in view of the importance of the result we prefer to give the proof for the reader's convenience. Taking into account that

$$
(Y_s^{0,n}-Y_s^{2,n})((Y_s^{0,n}-\varphi(s,X_s^0))^--(Y_s^{2,n}-\varphi(s,X_s^0))^-)\le 0,\tag{2.71}
$$

we get from standard BSDE estimates that for arbitrary $\gamma > 0$,

$$\begin{aligned}
&e^{\gamma s}|Y_s^{0,n} - Y_s^{2,n}|^2 + E\left[\int_s^T e^{\gamma r}(\gamma|Y_r^{0,n} - Y_r^{2,n}|^2 + |Z_r^{0,n} - \widehat{Z}_r^{1,n}|^2)dr|\mathcal{F}_s^{W^0}\right]\\
&\le E\left[\int_s^T e^{\gamma r}\left(C_\delta|Y_r^{0,n} - Y_r^{2,n}|^2 + \frac{1}{2}|Z_r^{0,n} - \widehat{Z}_r^{1,n}|^2\right)dr|\mathcal{F}_s^{W^0}\right]\\
&\quad + C_{\delta,\gamma}|t_0 - t_1|^2 E\left[\int_s^T (1 + |\widehat{Z}_r^{1,n}|^2)dr|\mathcal{F}_s^{W^0}\right] + C_{\delta,\gamma}E[A_T^2|\mathcal{F}_s^{W^0}]\\
&\quad + 2E\left[\int_s^T e^{\gamma r}(Y_r^{0,n} - Y_r^{2,n})dA_r|\mathcal{F}_s^{W^0}\right],\ s \in [t_0, T],\ n \ge 1.
\end{aligned} \tag{2.72}$$

Hence, for $\gamma \ge C_\delta + 1$ large enough,

$$\begin{aligned}
&|Y_s^{0,n} - Y_s^{2,n}|^2 + E\left[\int_s^T |Z_r^{0,n} - \widehat{Z}_r^{1,n}|^2 dr|\mathcal{F}_s^{W^0}\right]\\
&\le C_{\delta,\gamma}A_s^2 + \widehat{C}_{\delta,\gamma}E\left[\sup_{r\in[s,T]}|Y_r^{0,n} - Y_r^{2,n}|A_T|\mathcal{F}_s^{W^0}\right],\ s \in [t_0, T],\ n \ge 1,
\end{aligned} \tag{2.73}$$

where $\widehat{C}_{\delta,\gamma}$ only depends on the coefficients in (H1)–(H3) and on $\delta,\ \gamma \ge 0$. Let $1 < p < 2$ and $q > 2$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and let us choose $\varepsilon > 0$ be such that $\widehat{C}_{\delta,\gamma}\varepsilon(\frac{2}{2-p})^{\frac{2}{p}} < 1$. Then,

$$\begin{aligned}
&E\left[\sup_{r\in[s,T]}|Y_r^{0,n} - Y_r^{2,n}|A_T|\mathcal{F}_s^{W^0}\right]\\
&\le \left(E\left[\sup_{r\in[s,T]}|Y_r^{0,n} - Y_r^{2,n}|^p|\mathcal{F}_s^{W^0}\right]\right)^{\frac{1}{p}}(E[A_T^q|\mathcal{F}_s^{W^0}])^{\frac{1}{q}}\\
&\le \varepsilon M_{s,t}^{\frac{2}{p}} + \tfrac{1}{\varepsilon}(E[A_T^q|\mathcal{F}_s^{W^0}])^{\frac{2}{q}}\\
&\le \varepsilon M_{s,t}^{\frac{2}{p}} + \tfrac{1}{\varepsilon}C_{\delta,q}A_s^2,\quad t_0 \le t \le s \le T,\quad n \ge 1 \text{ (see: (2.44) and (2.45))},
\end{aligned} \tag{2.74}$$

where

$$M_{s,t} := E\left[\sup_{r\in[t,T]}|Y_r^{0,n} - Y_r^{2,n}|^p|\mathcal{F}_s^{W^0}\right].$$

From Doob's martingale inequality, since $\frac{2}{p} > 1$,

$$\begin{aligned}
E\left[\sup_{s\in[t,T]}M_{s,t}^{\frac{2}{p}}|\mathcal{F}_t^{W^0}\right] &\le \left(\frac{2}{2-p}\right)^{\frac{2}{p}}E[M_{T,t}^{\frac{2}{p}}|\mathcal{F}_t^{W^0}]\\
&\le \left(\frac{2}{2-p}\right)^{\frac{2}{p}}E\left[\sup_{s\in[t,T]}|Y_s^{0,n} - Y_s^{2,n}|^2|\mathcal{F}_t^{W^0}\right],\ t \in [t_0, T].
\end{aligned} \tag{2.75}$$

Hence, from (2.73), (2.74), and (2.75),

$$
\begin{aligned}
&E\left[\sup_{s\in[t,T]}|Y_s^{0,n}-Y_s^{2,n}|^2|\mathcal{F}_t^{W^0}\right]\\
&\le C_{\delta,\varepsilon}A_t^2+\ \widehat{C}_{\delta,\gamma}\varepsilon\left(\frac{2}{2-p}\right)^{\frac{2}{p}}E\left[\sup_{s\in[t,T]}|Y_s^{0,n}-Y_s^{2,n}|^2|\mathcal{F}_t^{W^0}\right],\ t\in[t_0,T],
\end{aligned}
\tag{2.76}
$$

and since $\widehat{C}_{\delta,\gamma}\varepsilon(\frac{2}{2-p})^{\frac{2}{p}}<1$, we get

$$
E\left[\sup_{s\in[t,T]}|Y_s^{0,n}-Y_s^{2,n}|^2|\mathcal{F}_t^{W^0}\right]\le C_{\delta,\varepsilon}A_t^2,\ t\in[t_0,T],\ n\ge 1,\ \text{P-a.s.}
\tag{2.77}
$$

Consequently, from (2.73),

$$
E\left[\int_t^T|Z_r^{0,n}-\widehat{Z}_r^{1,n}|^2dr|\mathcal{F}_t^{W^0}\right]\le C_{\delta,\varepsilon}A_t^2,\ t\in[t_0,T]. \qquad \square
\tag{2.78}
$$

We now can prove Proposition 2.2.

Proof of Proposition 2.2.

(1) We begin by proving assertion (i). For this we note that for all $s\in[t_0,T]$, $n\ge 1$,

$$
\begin{aligned}
\widetilde{Y}_s^{1,n}-Y_s^{0,n}&\le\widehat{Y}_s^{1,n}-Y_s^{0,n} && \text{(Lemma 2.4)}\\
&=A_s+(Y_s^{2,n}-Y_s^{0,n}) && \text{(definition of } Y^{2,n})\\
&\le A_s+|Y_s^{2,n}-Y_s^{0,n}| && \\
&\le A_s+C_\delta A_s && \text{(Lemma 2.6)}\\
&\le C_\delta\left(|t_0-t_1|+\sup_{r\in[t_0,s]}|X_r^0-\widetilde{X}_r^1|\right) && \text{(definition of } A).
\end{aligned}
\tag{2.79}
$$

The same argument, slightly adapted, allows us to show

$$
Y_s^{0,n}-\widetilde{Y}_s^{1,n}\le C_\delta\left(|t_0-t_1|+\sup_{r\in[t_0,s]}|X_r^0-\widetilde{X}_r^1|\right),\ s\in[t_0,T],\quad n\ge 1.
\tag{2.80}
$$

Thus, it only remains to prove the estimate (ii) for $\widetilde{Z}^{1,n}-Z^{0,n}$, when

$$
\varphi(t,x)=\varphi\in\mathbf{R},\quad (t,x)\in[0,T]\times\mathbf{R}^d.
$$

(2) For this we observe that from Itô's formula applied to $(\widetilde{Y}_t^{1,n}-Y_t^{0,n})^2$ it follows that

$$
\begin{aligned}
&|\widetilde{Y}_s^{1,n}-Y_s^{0,n}|^2+E\left[\int_s^T|\widetilde{Z}_r^{1,n}-Z_r^{0,n}|^2dr|\mathcal{F}_s^{W^0}\right]\\
&\quad=E[|\Phi(\widetilde{X}_T^1)-\Phi(X_T^0)|^2|\mathcal{F}_s^{W^0}]\\
&\qquad+2E\bigg[\int_s^T(\widetilde{Y}_r^{1,n}-Y_r^{0,n})\bigg(\frac{1}{\dot{\tau}}f(\tau^{-1}(r),\widetilde{X}_r^1,\widetilde{Y}_r^{1,n},\sqrt{\dot{\tau}}Z_r^{1,n},u_r^0)\\
&\qquad\qquad-f(r,X_r^0,Y_r^{0,n},Z_r^{0,n},u_r^0)\bigg)dr|\mathcal{F}_s^{W^0}\bigg]\\
&\qquad+2\frac{n}{\dot{\tau}}E\left[\int_s^T(\widetilde{Y}_r^{1,n}-Y_r^{0,n})((\widetilde{Y}_r^{1,n}-\varphi)^--(Y_r^{0,n}-\varphi)^-)dr|\mathcal{F}_s^{W^0}\right]\\
&\hspace{25em}(\le 0)\\
&\qquad+2n\left(\frac{1}{\dot{\tau}}-1\right)E\left[\int_s^T(\widetilde{Y}_r^{1,n}-Y_r^{0,n})(Y_r^{0,n}-\varphi)^-dr|\mathcal{F}_s^{W^0}\right]\\
&\quad\le C_\delta A_s^2\\
&\hspace{25em}(\text{Lemma 2.2})\\
&\qquad+C_\delta E\bigg[\int_s^TA_r(|t_0-t_1|+|\widetilde{X}_r^1-X_r^0|+|\widetilde{Y}_r^{1,n}-Y_r^{0,n}|+|\widetilde{Z}_r^{1,n}-Z_r^{0,n}|\\
&\qquad\qquad+|t_0-t_1||Z_r^{0,n}|)dr|\mathcal{F}_s^{W^0}\bigg]\\
&\hspace{18em}(\text{Lemma 2.1 and Proposition 2.2(i)})\\
&\qquad+C_\delta|t_0-t_1|(E[A_T^2|\mathcal{F}_s^{W^0}])^{\frac12}\left(E\left[\left(n\int_s^T(Y_r^{0,n}-\varphi)^-dr\right)^2|\mathcal{F}_s^{W^0}\right]\right)^{\frac12}.\\
&\hspace{22em}(\text{Proposition 2.2(i)})
\end{aligned}
$$

Thus, again from Proposition 2.2(i) and Lemma 2.2,

(2.81)

$$
\begin{aligned}
&|\widetilde{Y}_s^{1,n}-Y_s^{0,n}|^2+E\left[\int_s^T|\widetilde{Z}_r^{1,n}-Z_r^{0,n}|^2dr|\mathcal{F}_s^{W^0}\right]\\
&\quad\le C_\delta A_s^2+\frac12E\left[\int_s^T|\widetilde{Z}_r^{1,n}-Z_r^{0,n}|^2dr|\mathcal{F}_s^{W^0}\right]\\
&\qquad+C_\delta|t_0-t_1|A_s\left(E\left[\int_s^T|Z_r^{0,n}|^2dr|\mathcal{F}_s^{W^0}\right]\right)^{\frac12}\\
&\qquad+C_\delta|t_0-t_1|A_s\left(E\left[\left(n\int_s^T(Y_r^{0,n}-\varphi)^-dr\right)^2|\mathcal{F}_s^{W^0}\right]\right)^{\frac12},\ s\in[t_0,T],\ n\ge1.
\end{aligned}
$$

Note that due to Lemma 2.3, we have

$$
E\left[\int_s^T|Z_r^{0,n}|^2dr+\left(n\int_s^T(Y_r^{0,n}-\varphi)^-dr\right)^2|\mathcal{F}_s^{W^0}\right]\le C,\ s\in[t_0,T].
$$

Consequently, P-a.s., for all $n \ge 1,\ s \in [t_0, T]$,

(2.82)
$$E\left[\int_s^T |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr|\mathcal{F}_s^{W^0}\right] \le C_\delta A_s^2 \le C_\delta\left(|t_0 - t_1|^2 + \sup_{r\in[t_0,s]}|\widetilde{X}_r^1 - X_r^0|^2\right).$$

On the other hand, recalling that φ is constant, from Itô's formula, Lemma 2.2, and Proposition 2.2(i) we deduce

(2.83)
$$\begin{aligned}
&|\widetilde{Y}_s^{1,n} - Y_s^{0,n}|^2 + \int_s^T |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr\\
&\quad = (\Phi(\widetilde{X}_T^1) - \Phi(X_T^0))^2\\
&\qquad + 2\int_s^T (\widetilde{Y}_r^{1,n} - Y_r^{0,n})\bigg(\frac{1}{\dot\tau} f\left(\tau^{-1}(r), \widetilde{X}_r^1, \widetilde{Y}_r^{1,n}, \sqrt{\dot\tau}\widetilde{Z}_r^{1,n}, u_r^0\right)\\
&\qquad\qquad - f(r, X_r^0, Y_r^{0,n}, Z_r^{0,n}, u_r^0)\bigg)dr\\
&\qquad + 2\frac{n}{\dot\tau}\int_s^T (\widetilde{Y}_r^{1,n} - Y_r^{0,n})((\widetilde{Y}_r^{1,n} - \varphi)^- - (Y_r^{0,n} - \varphi)^-)dr \quad (\le 0)\\
&\qquad + 2n\left(\frac{1}{\dot\tau} - 1\right)\int_s^T (\widetilde{Y}_r^{1,n} - Y_r^{0,n})(Y_r^{0,n} - \varphi)^- dr\\
&\qquad - 2\int_s^T (\widetilde{Y}_r^{1.n} - Y_r^{0,n})(\widetilde{Z}_r^{1.n} - Z_r^{0,n})dW_r^0\\
&\quad \le CA_T^2 + C_\delta\int_s^T A_r(A_r + |t_0 - t_1||Z_r^{0,n}| + |Z_r^{0,n} - \widetilde{Z}_r^{1,n}|)dr\\
&\qquad + C_\delta|t_0 - t_1|A_T\left(n\int_s^T (Y_r^{0,n} - \varphi)^- dr\right) - 2\int_s^T (\widetilde{Y}_r^{1,n} - Y_r^{0,n})(\widetilde{Z}_r^{1,n} - Z_r^{0,n})dW_r^0\\
&\quad \le C_\delta A_T^2 + \frac{1}{2}\int_s^T |Z_r^{0,n} - \widetilde{Z}_r^{1,n}|^2 dr + |t_0 - t_1|^2\int_s^T |Z_r^{0,n}|^2 dr\\
&\qquad + C_\delta|t_0 - t_1|A_T\left(n\int_s^T (Y_r^{0,n} - \varphi)^- dr\right) - 2\int_s^T (\widetilde{Y}_r^{1,n} - Y_r^{0,n})(\widetilde{Z}_r^{1,n} - Z_r^{0,n})dW_r^0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_s^T |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr\\
&\quad \le C_\delta A_T^2 + 2|t_0 - t_1|^2\left(\int_s^T |Z_r^{0,n}|^2 dr + \left(n\int_s^T (Y_r^{0,n} - \varphi)^- dr\right)^2\right) \qquad (2.84)\\
&\qquad - 4\int_s^T (\widetilde{Y}_r^{1,n} - Y_r^{0,n})(\widetilde{Z}_r^{1,n} - Z_r^{0,n})dW_r^0,\ s \in [t_0, T],\ n \ge 1,
\end{aligned}$$

and, consequently, for $p \geq 1$,

$$
\begin{aligned}
&E\left[\left(\int_s^T |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr\right)^{2p} |\mathcal{F}_s^{W^0}\right] \\
&\quad \leq C_p C_\delta A_s^{4p} \\
&\qquad + C_p |t_0 - t_1|^{4p} \left(E\left[\left(\int_s^T |Z_r^{0,n}|^2 dr\right)^{2p} |\mathcal{F}_s^{W^0}\right] \right. \\
&\qquad\qquad \left. + E\left[\left(n\int_s^T (Y_r^{0,n} - \varphi)^- dr\right)^{4p} |\mathcal{F}_s^{W^0}\right]\right) \\
&\qquad + C_p E\left[\left(\int_s^T |\widetilde{Y}_r^{1,n} - Y_r^{0,n}|^2 |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr\right)^p |\mathcal{F}_s^{W^0}\right].
\end{aligned}
\tag{2.85}
$$

We recall that due to Lemma 2.3,

$$
E\left[\left(\int_s^T |Z_r^{0,n}|^2 dr\right)^{2p} |\mathcal{F}_s^{W^0}\right] \leq C_p;
$$

$$
E\left[\left(n\int_s^T (Y_r^{0,n} - \varphi)^- dr\right)^{4p} |\mathcal{F}_s^{W^0}\right] \leq C_p,\ s \in [t_0, T],\ n \geq 1,\ \text{P-a.s.}
$$

Thus, due to Proposition 2.2(i), $|\widetilde{Y}_r^{1,n} - Y_r^{0,n}|^2 \leq C_\delta A_r^2 \leq C_\delta A_T^2$, P-a.s. Hence, we get

$$
\begin{aligned}
&E\left[\left(\int_s^T |\widetilde{Y}_r^{1,n} - Y_r^{0,n}|^2 |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr\right)^p |\mathcal{F}_s^{W^0}\right] \\
&\quad \leq C_\delta E\left[A_T^{2p}\left(\int_s^T |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr\right)^p |\mathcal{F}_s^{W^0}\right] \\
&\quad \leq C_\delta (E[A_T^{6p}|\mathcal{F}_s^{W^0}])^{\frac{1}{3}} \left(E\left[\left(\int_s^T |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr\right)^{\frac{3}{2}p} |\mathcal{F}_s^{W^0}\right]\right)^{\frac{2}{3}} \\
&\quad \leq C_\delta A_s^{2p} \left(E\left[\left(\int_s^T |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr\right)^{\frac{3}{2}p} |\mathcal{F}_s^{W^0}\right]\right)^{\frac{2}{3}}.
\end{aligned}
\tag{2.86}
$$

Hence, from (2.85) and (2.86) it follows that

$$
\begin{aligned}
&E\left[\left(\int_s^T |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr\right)^{4p} |\mathcal{F}_s^{W^0}\right] \\
&\quad \leq C_{\delta,p} A_s^{8p} + C_{\delta,p} A_s^{4p} \left(E\left[\left(\int_s^T |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr\right)^{3p} |\mathcal{F}_s^{W^0}\right]\right)^{\frac{2}{3}},
\end{aligned}
\tag{2.87}
$$

from where we get the announced result for $p \geq \frac{1}{4}$, which means

(2.88)
$$E\left[\left(\int_s^T |\widetilde{Z}_r^{1,n} - Z_r^{0,n}|^2 dr\right)^p |\mathcal{F}_s^{W^0}\right] \leq C_{\delta,p} A_s^{2p},\ \text{P-a.s.},\ s \in [t_0, T],\ n \geq 1,\ p \geq 1. \qquad \square$$

3. The semiconcavity of the value function. In this section we consider V as a value function of a stochastic control problem in which the cost functional is defined by a BSDE reflected at an upper barrier. Indeed, if it is reflected at a lower barrier, V can, in general, not be semiconcave. Let us illustrate this by an easy example.

Example 3.1. We consider the controlled system (2.2) endowed with RBSDE (2.3) reflected at a lower barrier φ. $T > 1$. We let the dimension $m = d = 1$ and consider the case of no control (U is a singleton) and with the coefficients $b \equiv 0,\ \sigma \equiv 0,$ $f \equiv -1,\ \varphi \equiv 0$, and $\Phi \equiv 1$.

Then, obviously, $X_s^{t,x} = x,\ s \in [t, T]$, and the solution of RBSDE (2.3) is given by

$$Y_s^{t,x} = (1-(T-s))^+,\ Z_s^{t,x} = 0,\ K_s^{t,x} = (1-(T-t))^- - (1-(T-s))^-,\ s \in [t, T].$$

Consequently,

$$V(t,x) = Y_t^{t,x} = (1-(T-t))^+,\ (t,x) \in [0,T] \times \mathbf{R}.$$

However, although the coefficients satisfy our assumptions, it can be easily seen that the function V is not semiconcave on $[0, T-\delta] \times \mathbf{R}$ for all $0 < \delta < 1 < T$.

For this reason, for $(t,x) \in [0,T] \times \mathbf{R}^d,\ W = (W_s)_{s\in[t,T]}$ m-dimensional Brownian motion with $W_t = 0$, and $u \in \mathcal{U}_{t,T}^W$, we associate SDE (2.2) with the RBSDE reflected at an upper barrier φ:

$$\text{(3.1)} \qquad \begin{cases} dY_s^{t,x,u} = \ -f(s, X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)ds + Z_s^{t,x,u} dW_s + dK_s^{t,x,u}, \\ Y_T^{t,x,u} = \Phi(X_T^{t,x,u}), \\ Y_s^{t,x,u} \leq \varphi(s, X_s^{t,x,u}),\ (Y_s^{t,x,u} - \varphi(s, X_s^{t,x,u}))dK_s^{t,x,u} = 0,\ s \in [t,T]. \end{cases}$$

Under the assumptions (H1) and (H3') it has a unique solution

$$(Y^{t,x,u}, Z^{t,x,u}, K^{t,x,u}) \in \mathcal{S}^2_{\mathbf{F}^W}(t,T) \times L^2_{\mathbf{F}^W}(t,T;\mathbf{R}^m) \times A^2_{\mathbf{F}^W}(t,T).$$

In order to emphasize the dependence on W, we also write

$$(Y^{t,x,u}(W), Z^{t,x,u}(W), K^{t,x,u}(W)) = (Y^{t,x,u}, Z^{t,x,u}, K^{t,x,u}).$$

We define

$$\text{(3.2)} \qquad V(t,x) = \inf_{u \in \mathcal{U}_{t,T}^W} Y_t^{t,x,u},\ (t,x) \in [0,T] \times \mathbf{R}^d,$$

and we recall that $V \in C_b([0,T] \times \mathbf{R}^d)$ is the unique (uniqueness in $C_p([0,T] \times \mathbf{R}^d)$) viscosity solution of the HJB equation with an upper obstacle

$$\text{(3.3)} \qquad \begin{cases} \max\Big\{V(t,x) - \varphi(t,x), -\dfrac{\partial}{\partial t} V(t,x) \\ \qquad\qquad - \inf\limits_{u \in U} H(t,x,V(t,x),\nabla V(t,x), D^2 V(t,x), u)\Big\} = 0, \\ V(T,x) = \Phi(x), \qquad (t,x) \in [0,T) \times \mathbf{R}^d. \end{cases}$$

The main result of this section is the following one.

THEOREM 3.1. *We assume that the conditions* (H1), (H2), (H3'), (H4), *and* (H5) *are satisfied, as well as* (H6) *or* (H7). *Then, for all $\delta > 0$, there is some $C_\delta > 0$ such that for any $(t_0, x_0),\ (t_1, x_1) \in [0, T-\delta] \times \mathbf{R}^d$, and $\lambda \in [0,1]$,*

$$\begin{aligned}\lambda V(t_1,x_1) + (1-\lambda)V(t_0,x_0) \le{}& V(\lambda(t_1,x_1) + (1-\lambda)(t_0,x_0)) \\ &+ C_\delta \lambda(1-\lambda)(|t_0-t_1|^2 + |x_0-x_1|^2).\end{aligned} \tag{3.4}$$

As in section 2, the proof will be based on the approximation of the reflected BSDE (3.1) by penalized BSDEs:

(3.5)
$$\begin{cases} dY_s^{t,x,u;n} = & -[f(s, X_s^{t,x,u}, Y_s^{t,x,u;n}, Z_s^{t,x,u;n}, u_s) - n(Y_s^{t,x,u;n} - \varphi(s, X_s^{t,x,u}))^+]ds \\ & + Z_s^{t,x,u;n} dW_s, \\ Y_T^{t,x,u;n} = \Phi(X_T^{t,x,u}),\ s \in [t,T],\ n \ge 1. \end{cases}$$

For every $n \ge 1$, BSDE (3.5) admits a unique solution $(Y^{t,x,u;n}, Z^{t,x,u;n})$, and we define

$$V_n(t,x) := \inf_{u \in \mathcal{U}_{t,T}^W} Y_t^{t,x,u;n},\quad (t,x) \in [0,T] \times \mathbf{R}^d. \tag{3.6}$$

In analogy to Proposition 2.1 we have the next proposition.

PROPOSITION 3.1. *Under the assumptions* (H1) *and* (H3') *the following assertions hold true:*

(i) $Y_s^{t,x,u;n} \downarrow Y_s^{t,x,u}$, *as* $n \to \infty$, *P-a.s.*, $s \in [t,T]$, $u \in \mathcal{U}_{t,T}^W$.

(ii) $E\big[\sup_{s\in[t,T]} |Y_s^{t,x,u;n} - Y_s^{t,x,u}|^2 + \int_t^T |Z_s^{t,x,u;n} - Z_s^{t,x,u}|^2 ds + \sup_{s\in[t,T]} |K_s^{t,x,u} - n\int_s^T (Y_r^{t,x,u;n} - \varphi(r, X_r^{t,x,u}))^+ dr|^2\big] \to 0$, *as* $n \to \infty$, $u \in \mathcal{U}_{t,T}^W$.

(iii) $V_n(t,x) \downarrow V(t,x)$, *as* $n \to \infty$, $(t,x) \in [0,T] \times \mathbf{R}^d$.

Theorem 3.1 is an immediate consequence of the following theorem combined with assertion (iii) of Proposition 3.1.

THEOREM 3.2. *Under the assumptions of Theorem* 3.1, *for all $\delta > 0$, there is some $C_\delta \in \mathbf{R}$ such that for all $n \ge 1$, $(t_0,x_0),\ (t_1,x_1) \in [0,T-\delta] \times \mathbf{R}^d$, and $\lambda \in (0,1)$,*

$$\begin{aligned}\lambda V_n(t_1,x_1) + (1-\lambda)V_n(t_0,x_0) \le{}& V_n(\lambda(t_1,x_1) + (1-\lambda)(t_0,x_0)) \\ &+ C_\delta \lambda(1-\lambda)(|t_0-t_1|^2 + |x_0-x_1|^2).\end{aligned} \tag{3.7}$$

As in section 2, our proof is based on the method of time change.

Let $\delta > 0$, $(t_i,x_i) \in [0,T-\delta] \times \mathbf{R}^d$, $i = 0,\ 1$, and $\lambda \in (0,1)$, and let us put $(t_\lambda, x_\lambda) := \lambda(t_1,x_1) + (1-\lambda)(t_0,x_0)$. Moreover, let $W^\lambda = (W_s^\lambda)_{s\in[t_\lambda,T]}$ be an m-dimensional Brownian motion with $W_{t_\lambda}^\lambda = 0$ and let $u^\lambda \in \mathcal{U}_{t_\lambda,T}^{W^\lambda}$ be an admissible control associated with W^λ.

Using the notation introduced in the preceding section, we put

$$\begin{aligned} X^\lambda &:= X^{t_\lambda,x_\lambda,u^\lambda}(W^\lambda); \\ (Y^\lambda, Z^\lambda, K^\lambda) &:= (Y^{t_\lambda,x_\lambda,u^\lambda}(W^\lambda), Z^{t_\lambda,x_\lambda,u^\lambda}(W^\lambda), K^{t_\lambda,x_\lambda,u^\lambda}(W^\lambda)); \\ (Y^{\lambda,n}, Z^{\lambda,n}) &:= (Y^{t_\lambda,x_\lambda,u^\lambda;n}(W^\lambda), Z^{t_\lambda,x_\lambda,u^\lambda;n}(W^\lambda)),\quad n \ge 1. \end{aligned}$$

We use the method of time change again. But since we have to compare the stochastic control system with initial data (t_λ, x_λ) with those of initial data (t_0,x_0) and (t_1,x_1), we have to define a separate time change for each $i = 0,\ 1$:

$$\tau_i : [t_i,T] \to [t_\lambda,T],\quad \tau_i(s) = t_\lambda + \tfrac{T-t_\lambda}{T-t_i}(s-t_i),\ s \in [t_i,T]. \tag{3.8}$$

We observe that $\dot{\tau}_i(=\frac{d}{ds}\tau_i(s))=\frac{T-t_\lambda}{T-t_i},\ i=0,\ 1$, and so $W^i_s:=\frac{1}{\sqrt{\dot{\tau}_i}}W^\lambda_{\tau_i(s)},\ s\in[t_i,T]$, is an m-dimensional Brownian motion with $W^i_{t_i}=0$, and $u^i_s:=u^\lambda_{\tau_i(s)},\ s\in[t_i,T]$, defines an admissible control belonging to $\mathcal{U}^{W^i}_{t_i,T},\ i=0,\ 1$.

For $i=0,1$, we consider the solution $X^i:=X^{t_i,x_i,u^i}(W^i)$ of SDE (2.2) governed by the Brownian motion W^i and the control u^i, as well as the solution $(Y^i,Z^i,K^i)=(Y^{t_i,x_i,u^i}(W^i),\ Z^{t_i,x_i,u^i}(W^i),K^{t_i,x_i,u^i}(W^i))$ of the associated reflected BSDE (3.1), and the solution $(Y^{i,n},Z^{i,n})=(Y^{t_i,x_i,u^i;n}(W^i),\ Z^{t_i,x_i,u^i;n}(W^i))$ of the associated penalized BSDE (3.5).

We have to work with the triples $(X^\lambda,Y^{\lambda,n},Z^{\lambda,n}),\ (X^i,Y^{i,n},Z^{i,n}),\ i=0,1,\ n\geq 1$. However, in order to make them comparable, we need equations driven by the same Brownian motion. For this end we consider the inverse time changes,

$$\tau_i^{-1}:[t_\lambda,T]\to[t_i,T],\quad \tau_i^{-1}(s)=t_i+\frac{T-t_i}{T-t_\lambda}(s-t_\lambda),\quad s\in[t_\lambda,T],\ i=0,\ 1, \tag{3.9}$$

and we introduce the time changed processes

$$\widetilde{X}^i_s:=X^i_{\tau_i^{-1}(s)},\ \widetilde{Y}^{i,n}_s:=Y^{i,n}_{\tau_i^{-1}(s)},\quad \widetilde{Z}^{i,n}_s:=\frac{1}{\sqrt{\dot{\tau}_i}}Z^{i,n}_{\tau_i^{-1}(s)},\quad s\in[t_\lambda,T],\quad i=0,\ 1. \tag{3.10}$$

By observing that

$$W^i_{\tau_i^{-1}(s)}=\frac{1}{\sqrt{\dot{\tau}_i}}W^\lambda_s,\quad u^i_{\tau_i^{-1}(s)}=u^\lambda_s,\quad s\in[t_\lambda,T],\quad i=0,\ 1, \tag{3.11}$$

we see that

$$\begin{cases} d\widetilde{X}^i_s=\ \dfrac{1}{\dot{\tau}_i}b(\tau_i^{-1}(s),\widetilde{X}^i_s,u^\lambda_s)ds+\dfrac{1}{\sqrt{\dot{\tau}_i}}\sigma(\tau_i^{-1}(s),\widetilde{X}^i_s,u^\lambda_s)dW^\lambda_s,\quad s\in[t_\lambda,T];\\ \widetilde{X}^i_{t_\lambda}=x_i, \end{cases} \tag{3.12}$$

and

(3.13)
$$\begin{cases} d\widetilde{Y}^{i,n}_s=\ -\left(\dfrac{1}{\dot{\tau}_i}f(\tau_i^{-1}(s),\widetilde{X}^i_s,\widetilde{Y}^{i,n}_s,\sqrt{\dot{\tau}_i}\widetilde{Z}^{i,n}_s,u^\lambda_s)-\dfrac{n}{\dot{\tau}_i}(\widetilde{Y}^{i,n}_s-\varphi(\tau_i^{-1}(s),\widetilde{X}^i_s))^+\right)ds\\ \qquad\qquad +\widetilde{Z}^{i,n}_s dW^\lambda_s,\\ \widetilde{Y}^{i,n}_T=\Phi(\widetilde{X}^i_T),\quad s\in[t_\lambda,T],\ i=0,\ 1. \end{cases}$$

With the same, only slightly adapted, arguments as those for Lemma 2.2 and Proposition 2.2, we can show the following statement.

LEMMA 3.1. *Let us suppose the assumptions* (H1), (H2), *and* (H3'). *Then,*

(i) *for all $p\geq 1$ there is some constant $C_{\delta,p}$ such that for all $t\in[t_\lambda,T],\quad n\geq 1$, P-a.s.,*

(1) $E\left[\sup\limits_{s\in[t,T]}|\widetilde{X}^0_s-\widetilde{X}^1_s|^p|\mathcal{F}^{W^\lambda}_t\right]\leq C_{\delta,p}(|t_0-t_1|^p+|\widetilde{X}^0_t-\widetilde{X}^1_t|^p)$;

(2) $|\widetilde{Y}^{0,n}_t-\widetilde{Y}^{1,n}_t|\leq C_{\delta,p}\left(|t_0-t_1|+\sup\limits_{s\in[t_\lambda,t]}|\widetilde{X}^0_s-\widetilde{X}^1_s|\right)$;

(ii) *if, moreover,* $\varphi(t,x) \equiv \varphi \in \mathbf{R}$, $(t,x) \in [0,T] \times \mathbf{R}^d$, *then for all* $p \geq 1$ *there is some constant* $C_{\delta,p}$ *such that for all* $t \in [t_\lambda, T]$, $n \geq 1$, *P-a.s.,*

$$E\left[\left(\int_t^T |\widetilde{Z}_r^{0,n} - \widetilde{Z}_r^{1,n}|^2 dr\right)^p |\mathcal{F}_t^{W^\lambda}\right] \leq C_{\delta,p}\left(|t_0 - t_1|^2 + \sup_{s\in[t_\lambda,t]} |\widetilde{X}_s^0 - \widetilde{X}_s^1|^2\right)^p.$$

We also shall introduce the process $\widetilde{X}_s := \lambda\widetilde{X}_s^1 + (1-\lambda)\widetilde{X}_s^0$, $\widetilde{Y}_s^n := \lambda\widetilde{Y}_s^{1,n} + (1-\lambda)\widetilde{Y}_s^{0,n}$, and $\widetilde{Z}_s^n := \lambda\widetilde{Z}_s^{1,n} + (1-\lambda)\widetilde{Z}_s^{0,n}$, $s \in [t_\lambda, T]$. Recall the definition of the processes X^λ, $(Y^{\lambda,n}, Z^{\lambda,n})$, and

$$A_t := \sup_{s\in[t_\lambda,t]} (|t_0 - t_1| + |\widetilde{X}_s^1 - \widetilde{X}_s^0|), \quad t \in [t_\lambda, T], \tag{3.14}$$

and introduce the continuous increasing process

$$B_t := \sup_{s\in[t_\lambda,t]} |\widetilde{X}_s - X_s^\lambda|, \quad t \in [t_\lambda, T]; \tag{3.15}$$

we have the next proposition.

PROPOSITION 3.2. *Under the assumption of Theorem* 3.1 *there is some* $C_\delta \in \mathbf{R}$ *only depending on* $\delta > 0$ *and on the bounds and the Lipschitz constants of* σ, b, f, Φ, φ, $\nabla_{(t,x)}\sigma$, *and* $\nabla_{(t,x)}b$, *such that*

$$\widetilde{Y}_t^n \leqslant Y_t^{\lambda,n} + C_\delta(B_t + \lambda(1-\lambda)A_t^2),\ t \in [t_\lambda, T],\ n \geqslant 1,\ P\text{-a.s.}$$

Before proving Proposition 3.2 let us show that Theorem 3.2 holds true.

Proof of Theorem 3.2. We recall that $\delta > 0$, and (t_0, x_0), $(t_1, x_1) \in [0, T-\delta] \times \mathbf{R}^d$ are arbitrarily chosen, and $(t_\lambda, x_\lambda) = \lambda(t_1, x_1) + (1-\lambda)(t_0, x_0)$. For an arbitrary $\lambda \in (0,1)$, $n \geqslant 1$, we choose $\varepsilon > 0$ small enough and we let $u^\lambda \in \mathcal{U}_{t_\lambda,T}^{W^\lambda}$ be such that

$$V_n(t_\lambda, x_\lambda) = \inf_{u\in\mathcal{U}_{t_\lambda,T}^{W^\lambda}} Y_{t_\lambda}^{t_\lambda,x_\lambda,u;n} \geqslant Y_{t_\lambda}^{t_\lambda,x_\lambda,u^\lambda;n} - \varepsilon = Y_{t_\lambda}^{\lambda,n} - \varepsilon. \tag{3.16}$$

As $V_n(t_i, x_i) \leqslant Y_{t_i}^{i,n} = \widetilde{Y}_{t_\lambda}^{i,n}$, $i = 0, 1$, we have from Proposition 3.2 (note that $B_{t_\lambda} = 0$ and $A_{t_\lambda} = |t_0 - t_1| + |x_0 - x_1|$)

$$\begin{aligned}
\lambda V_n(t_1,x_1) + (1-\lambda)V_n(t_0,x_0) &\leqslant \lambda\widetilde{Y}_{t_\lambda}^{1,n} + (1-\lambda)\widetilde{Y}_{t_\lambda}^{0,n} = \widetilde{Y}_{t_\lambda}^n \\
&\leqslant Y_{t_\lambda}^{\lambda,n} + C_\delta\lambda(1-\lambda)(|t_0-t_1|^2 + |x_0-x_1|^2) \\
&\leqslant V_n(t_\lambda, x_\lambda) + \varepsilon + C_\delta\lambda(1-\lambda)(|t_0-t_1|^2 + |x_0-x_1|^2).
\end{aligned} \tag{3.17}$$

Finally, from the arbitrariness of $\varepsilon > 0$,

$$\lambda V_n(t_1,x_1) + (1-\lambda)V_n(t_0,x_0) - V_n(t_\lambda,x_\lambda) \leqslant C_\delta\lambda(1-\lambda)(|t_0-t_1|^2 + |x_0-x_1|^2). \tag{3.18}$$

Note that C_δ does neither depend on λ, (t_0, x_0), and (t_1, x_1) nor on $n \geqslant 1$. ∎

The proof of Proposition 3.2 is split into a sequel of lemmas. The following lemma will be crucial for our computations.

LEMMA 3.2. *For all* $p \geqslant 1$ *there is some* $C_{p,\delta} \in \mathbf{R}$ *only depending on* δ, p *and on the bounds and the Lipschitz constants of* σ *and* b, *such that,* $t \in [t_\lambda, T]$, *P-a.s.,*

$$E\left[\sup_{s\in[t,T]} |\widetilde{X}_s - X_s^\lambda|^p |\mathcal{F}_t^{W^\lambda}\right] \leqslant C_p|\widetilde{X}_t - X_t^\lambda|^p + C_{p,\delta}(\lambda(1-\lambda))^p(|t_0-t_1|^2 + |\widetilde{X}_t^1 - \widetilde{X}_t^0|^2)^p. \tag{3.19}$$

Proof. For $s \in [t_\lambda, T]$, we have to estimate the equation

(3.20)
$$\begin{aligned} d(\widetilde{X}_s - X_s^\lambda) &= \left(\frac{\lambda}{\dot{\tau}_1} b\big(\tau_1^{-1}(s), \widetilde{X}_s^1, u_s^\lambda\big) + \frac{1-\lambda}{\dot{\tau}_0} b\big(\tau_0^{-1}(s), \widetilde{X}_s^0, u_s^\lambda\big) - b(s, X_s^\lambda, u_s^\lambda)\right) ds \\ &\quad + \left(\frac{\lambda}{\sqrt{\dot{\tau}_1}}\sigma\big(\tau_1^{-1}(s), \widetilde{X}_s^1, u_s^\lambda\big) + \frac{1-\lambda}{\sqrt{\dot{\tau}_0}}\sigma\big(\tau_0^{-1}(s), \widetilde{X}_s^0, u_s^\lambda\big)\right. \\ &\qquad \left. - \sigma(s, X_s^\lambda, u_s^\lambda)\right) dW_s^\lambda, \\ \widetilde{X}_{t_\lambda} - X_{t_\lambda}^\lambda &= \lambda \widetilde{X}_{t_\lambda}^1 + (1-\lambda)\widetilde{X}_{t_\lambda}^0 - X_{t_\lambda}^\lambda = 0. \end{aligned}$$

For this let us begin with

(1) estimating $|(\frac{\lambda}{\sqrt{\dot{\tau}_1}}\sigma(\tau_1^{-1}(s), \widetilde{X}_s^1, u_s^\lambda) + \frac{1-\lambda}{\sqrt{\dot{\tau}_0}}\sigma(\tau_0^{-1}(s), \widetilde{X}_s^0, u_s^\lambda)) - \sigma(s, X_s^\lambda, u_s^\lambda)|$. From a straightforward computation we get

$$\lambda \left|1 - \frac{1}{\sqrt{\dot{\tau}_1}}\right| \leqslant \frac{1}{2\delta}\lambda(1-\lambda)|t_0 - t_1|; \quad (1-\lambda)\left|1 - \frac{1}{\sqrt{\dot{\tau}_0}}\right| \leqslant \frac{1}{2\delta}\lambda(1-\lambda)|t_0 - t_1|;$$

and

$$\left|\lambda\left(1 - \frac{1}{\sqrt{\dot{\tau}_1}}\right) + (1-\lambda)\left(1 - \frac{1}{\sqrt{\dot{\tau}_0}}\right)\right| \leqslant \frac{1}{\delta^2}\lambda(1-\lambda)|t_1 - t_0|^2.$$

We also observe that $|\tau_1^{-1}(s) - \tau_0^{-1}(s)| \leqslant |t_1 - t_0|, \quad s \in [t_\lambda, T]$.

Consequently,

(3.21)
$$\begin{aligned} &\left|\lambda(1 - \frac{1}{\sqrt{\dot{\tau}_1}})\sigma\left(\tau_1^{-1}(s), \widetilde{X}_s^1, u_s^\lambda\right) + (1-\lambda)\left(1 - \frac{1}{\sqrt{\dot{\tau}_0}}\right)\sigma\left(\tau_0^{-1}(s), \widetilde{X}_s^0, u_s^\lambda\right)\right| \\ &\quad \leqslant \lambda\left|1 - \tfrac{1}{\sqrt{\dot{\tau}_1}}\right| |\sigma(\tau_1^{-1}(s), \widetilde{X}_s^1, u_s^\lambda) - \sigma(\tau_0^{-1}(s), \widetilde{X}_s^0, u_s^\lambda)| \\ &\qquad + C\left|\lambda\left(1 - \frac{1}{\sqrt{\dot{\tau}_1}}\right) + (1-\lambda)\left(1 - \frac{1}{\sqrt{\dot{\tau}_0}}\right)\right| \\ &\quad \leqslant C_\delta \lambda(1-\lambda)(|t_0 - t_1|^2 + |\widetilde{X}_s^0 - \widetilde{X}_s^1|^2), \quad s \in [t_\lambda, T]. \end{aligned}$$

Also note that thanks to assumption (H5) the functions $\sigma(\cdot,\cdot,u)$, $(-\sigma)(\cdot,\cdot,u)$, $b(\cdot,\cdot,u)$, $(-b)(\cdot,\cdot,u)$ are semiconcave, uniformly with respect to $u \in U$. Thus, from the latter estimate

(3.22)
$$\begin{aligned} &\left|\frac{\lambda}{\sqrt{\dot{\tau}_1}}\sigma\big(\tau_1^{-1}(s), \widetilde{X}_s^1, u_s^\lambda\big) + \frac{1-\lambda}{\sqrt{\dot{\tau}_0}}\sigma\big(\tau_0^{-1}(s), \widetilde{X}_s^0, u_s^\lambda\big) - \sigma(s, X_s^\lambda, u_s^\lambda)\right| \\ &\quad \leqslant \big|\lambda\sigma\big(\tau_1^{-1}(s), \widetilde{X}_s^1, u_s^\lambda\big) + (1-\lambda)\sigma\big(\tau_0^{-1}(s), \widetilde{X}_s^0, u_s^\lambda\big) - \sigma(s, X_s^\lambda, u_s^\lambda)\big| \\ &\qquad + C_\delta\lambda(1-\lambda)(|t_1 - t_0|^2 + |\widetilde{X}_s^1 - \widetilde{X}_s^0|^2) \\ &\quad \leqslant \big|\lambda\sigma\big(\tau_1^{-1}(s), \widetilde{X}_s^1, u_s^\lambda\big) + (1-\lambda)\sigma\big(\tau_0^{-1}(s), \widetilde{X}_s^0, u_s^\lambda\big) \\ &\qquad - \sigma\big(\lambda\tau_1^{-1}(s) + (1-\lambda)\tau_0^{-1}(s), \widetilde{X}_s, u_s^\lambda\big)\big| \\ &\qquad + \big|\sigma\big(\lambda\tau_1^{-1}(s) + (1-\lambda)\tau_0^{-1}(s), \widetilde{X}_s, u_s^\lambda\big) - \sigma(s, X_s^\lambda, u_s^\lambda)\big| \\ &\qquad + C_\delta\lambda(1-\lambda)(|t_1 - t_0|^2 + |\widetilde{X}_s^1 - \widetilde{X}_s^0|^2) \\ &\quad \leqslant C_\delta\lambda(1-\lambda)(|t_1 - t_0|^2 + |\widetilde{X}_s^1 - \widetilde{X}_s^0|^2) \\ &\qquad + C(|\lambda\tau_1^{-1}(s) + (1-\lambda)\tau_0^{-1}(s) - s| + |\widetilde{X}_s - X_s^\lambda|), \quad s \in [t_\lambda, T]. \end{aligned}$$

However, $\lambda\tau_1^{-1}(s)+(1-\lambda)\tau_0^{-1}(s)-s\equiv 0, s\in[t_\lambda,T]$, so that

$$\begin{aligned}(3.23)\qquad &\left|\frac{\lambda}{\sqrt{\dot\tau_1}}\sigma\big(\tau_1^{-1}(s),\widetilde X_s^1,u_s^\lambda\big)+\frac{1-\lambda}{\sqrt{\dot\tau_0}}\sigma\big(\tau_0^{-1}(s),\widetilde X_s^0,u_s^\lambda\big)-\sigma(s,X_s^\lambda,u_s^\lambda)\right|\\ &\quad\leqslant C_\delta\lambda(1-\lambda)(|t_1-t_0|^2+|\widetilde X_s^1-\widetilde X_s^0|^2)+C|\widetilde X_s-X_s^\lambda|,\quad s\in[t_\lambda,T].\end{aligned}$$

(2) By now using

$$\lambda\left(1-\frac{1}{\dot\tau_1}\right)=\lambda(1-\lambda)\frac{t_1-t_0}{T-t_\lambda},\quad (1-\lambda)\left(1-\frac{1}{\dot\tau_0}\right)=\lambda(1-\lambda)\frac{t_0-t_1}{T-t_\lambda},$$

we get, similar to (3.23),

$$\begin{aligned}(3.24)\qquad &\left|\frac{\lambda}{\dot\tau_1}b\big(\tau_1^{-1}(s),\widetilde X_s^1,u_s^\lambda\big)+\frac{1-\lambda}{\dot\tau_0}b\big(\tau_0^{-1}(s),\widetilde X_s^0,u_s^\lambda\big)-b(s,X_s^\lambda,u_s^\lambda)\right|\\ &\quad\leqslant C_\delta\lambda(1-\lambda)(|t_0-t_1|^2+|\widetilde X_s^1-\widetilde X_s^0|^2)+C|\widetilde X_s-X_s^\lambda|,\quad s\in[t_\lambda,T].\end{aligned}$$

From (3.23), (3.24), Lemma 3.1, and standard SDE estimates we then get the desired result. □

Now we have still to prepare the proof of Proposition 3.2. For this we recall that

$$\begin{aligned}&(3.25)\\ &d\widetilde Y_s^n=-\Big\{\frac{\lambda}{\dot\tau_1}f(\tau_1^{-1}(s),\widetilde X_s^1,\widetilde Y_s^{1,n},\sqrt{\dot\tau_1}\widetilde Z_s^{1,n},u_s^\lambda)\\ &\qquad\qquad+\frac{1-\lambda}{\dot\tau_0}f(\tau_0^{-1}(s),\widetilde X_s^0,\widetilde Y_s^{0,n},\sqrt{\dot\tau_0}\widetilde Z_s^{0,n},u_s^\lambda)\\ &\qquad\qquad-\Big(\lambda\tfrac{n}{\dot\tau_1}\big(\widetilde Y_s^{1,n}-\varphi(\tau_1^{-1}(s),\widetilde X_s^1)\big)^++(1-\lambda)\tfrac{n}{\dot\tau_0}\big(\widetilde Y_s^{0,n}-\varphi(\tau_0^{-1}(s),\widetilde X_s^0)\big)^+\Big)\Big\}ds\\ &\qquad+\widetilde Z_s^n dW_s^\lambda,\\ &\widetilde Y_T^n=\lambda\Phi(\widetilde X_T^1)+(1-\lambda)\Phi(\widetilde X_T^0),\end{aligned}$$

and we compare this equation with the BSDE

$$\begin{aligned}&(3.26)\\ &d\widehat Y_s^n=-\Big(f(s,X_s^\lambda,\widehat Y_s^n-CB_s-C_\delta\lambda(1-\lambda)A_s^2,\widehat Z_s^n,u_s^\lambda)+C(CB_s+C_\delta\lambda(1-\lambda)A_s^2)\\ &\qquad\quad+C_\delta^0\lambda(1-\lambda)\big(|t_0-t_1|^2(1+|\widetilde Z_s^{0,n}|^2)+|\widetilde Z_s^{1,n}-\widetilde Z_s^{0,n}|^2\big)\\ &\qquad\quad-n\big(\widehat Y_s^n-\varphi(s,X_s^\lambda)-CB_s-C_\delta\lambda(1-\lambda)A_s^2\big)^+\Big)ds+\widehat Z_s^n dW_s^\lambda,\\ &\widehat Y_T^n=\Phi(X_T^\lambda)+CB_T+C_\delta\lambda(1-\lambda)A_T^2,\end{aligned}$$

where $C_\delta^0=0$, if f is independent of z.

Remark 3.1. We point out that due to Lemma 2.3,

$$E\left[\left(\int_s^T|\widetilde Z_r^{i,n}|^2dr\right)^p|\mathcal F_s^{W^\lambda}\right]\leqslant C_{\delta,p},\ s\in[t_\lambda,T],\quad p\geqslant 1.$$

This shows that the above BSDE (3.26) is well-posed.

LEMMA 3.3. *Under the assumptions for Theorem* 3.1 *we have*

$$\widetilde{Y}_t^n \leqslant \widehat{Y}_t^n, \quad t \in [t_\lambda, T],\ n \geqslant 1,\ P\text{-a.s.}$$

Proof. The proof is based on the comparison theorem (Lemma 4.2 in section 4). We prepare for the application of this comparison theorem by the following three steps.

Step 1. Using that $-a^+ - b^+ \leqslant -(a+b)^+,\ a,\ b \in \mathbf{R}$, we have

$$\begin{aligned}
&-\lambda \tfrac{n}{\dot{\tau}_1}\big(\widetilde{Y}_t^{1,n} - \varphi(\tau_1^{-1}(t), \widetilde{X}_t^1)\big)^+ - (1-\lambda)\tfrac{n}{\dot{\tau}_0}\big(\widetilde{Y}_t^{0,n} - \varphi(\tau_0^{-1}(t), \widetilde{X}_t^0)\big)^+ \\
&\leqslant -n\left(\frac{\lambda}{\dot{\tau}_1}\widetilde{Y}_t^{1,n} + \frac{1-\lambda}{\dot{\tau}_0}\widetilde{Y}_t^{0,n} - \left(\frac{\lambda}{\dot{\tau}_1}\varphi(\tau_1^{-1}(t), \widetilde{X}_t^1) + \frac{1-\lambda}{\dot{\tau}_0}\varphi(\tau_0^{-1}(t), \widetilde{X}_t^0)\right)\right)^+ \\
&= -n\Bigg\{\widetilde{Y}_t^n - \left(\lambda\left(1 - \frac{1}{\dot{\tau}_1}\right)\widetilde{Y}_t^{1,n} + (1-\lambda)\left(1 - \frac{1}{\dot{\tau}_0}\right)\widetilde{Y}_t^{0,n}\right) \\
&\qquad - \big(\lambda\varphi(\tau_1^{-1}(t), \widetilde{X}_t^1) + (1-\lambda)\varphi(\tau_0^{-1}(t), \widetilde{X}_t^0)\big) + \lambda\left(1 - \frac{1}{\dot{\tau}_1}\right)\varphi(\tau_1^{-1}(t), \widetilde{X}_t^1) \\
&\qquad + (1-\lambda)\left(1 - \frac{1}{\dot{\tau}_0}\right)\varphi(\tau_0^{-1}(t), \widetilde{X}_t^0)\Bigg\}^+ \\
&= -n\Bigg\{\widetilde{Y}_t^n - \frac{\lambda(1-\lambda)}{T - t_\lambda}(t_1 - t_0)(\widetilde{Y}_t^{1,n} - \widetilde{Y}_t^{0,n}) \\
&\qquad - \big(\lambda\varphi(\tau_1^{-1}(t), \widetilde{X}_t^1) + (1-\lambda)\varphi(\tau_0^{-1}(t), \widetilde{X}_t^0)\big) \\
&\qquad + \frac{\lambda(1-\lambda)}{T - t_\lambda}(t_1 - t_0)\big(\varphi(\tau_1^{-1}(t), \widetilde{X}_t^1) - \varphi(\tau_0^{-1}(t), \widetilde{X}_t^0)\big)\Bigg\}^+ \\
&\leqslant -n\big\{\widetilde{Y}_t^n - \big(\lambda\varphi(\tau_1^{-1}(t), \widetilde{X}_t^1) + (1-\lambda)\varphi(\tau_0^{-1}(t), \widetilde{X}_t^0)\big) - C_\delta\lambda(1-\lambda)|t_1 - t_0|A_t\big\}^+,
\end{aligned} \tag{3.27}$$

where Lemma 3.1 was applied for the latter inequality.

Hence, from the semiconcavity of φ, and since $\lambda\tau_1^{-1}(t) + (1-\lambda)\tau_0^{-1}(t) = t$,

$$\begin{aligned}
&-\lambda\frac{n}{\dot{\tau}_1}\big(\widetilde{Y}_t^{1,n} - \varphi(\tau_1^{-1}(t), \widetilde{X}_t^1)\big)^+ - (1-\lambda)\frac{n}{\dot{\tau}_0}\big(\widetilde{Y}_t^{0,n} - \varphi(\tau_0^{-1}(t), \widetilde{X}_t^0)\big)^+ \\
&\quad \leqslant -n\big(\widetilde{Y}_t^n - \varphi(t, \widetilde{X}_t) - C_\delta\lambda(1-\lambda)A_t^2\big)^+ \\
&\quad \leqslant -n\big(\widetilde{Y}_t^n - \varphi(t, X_t^\lambda) - CB_t - C_\delta\lambda(1-\lambda)A_t^2\big)^+, \quad t \in [t_\lambda, T], \quad n \geq 1,
\end{aligned} \tag{3.28}$$

where $B_t := \sup_{s\in[t_\lambda,t]}|\widetilde{X}_s - X_s^\lambda|$.

We recall that from Lemma 3.2

$$E[B_T^p|\mathcal{F}_t^{W^\lambda}] \leqslant C_pB_t^p + C_{p,\delta}(\lambda(1-\lambda))^p(|t_1-t_0|^2 + |\widetilde{X}_t^1 - \widetilde{X}_t^0|^2)^p,\ t \in [t_\lambda, T],\ p \geqslant 1,\ \text{P-a.s.} \tag{3.29}$$

Hence,

$$\begin{aligned}
&-\lambda\frac{n}{\dot{\tau}_1}\big(\widetilde{Y}_t^{1,n} - \varphi(\tau_1^{-1}(t), \widetilde{X}_t^1)\big)^+ - (1-\lambda)\frac{n}{\dot{\tau}_0}\big(\widetilde{Y}_t^{0,n} - \varphi(\tau_0^{-1}(t), \widetilde{X}_t^0)\big)^+ \\
&\quad \leqslant -n\big(\widetilde{Y}_t^n - \varphi(t, X_t^\lambda) - CB_t - C_\delta\lambda(1-\lambda)A_t^2\big)^+,\ t \in [t_\lambda, T],\ n \geqslant 1.
\end{aligned} \tag{3.30}$$

Note that, if φ is a constant independent of (t,x), then

$$
\begin{aligned}
&-\lambda\frac{n}{\dot\tau_1}(\widetilde Y_t^{1,n}-\varphi)^+-(1-\lambda)\frac{n}{\dot\tau_0}(\widetilde Y_t^{0,n}-\varphi)^+\\
&\quad\leqslant -n\big(\widetilde Y_t^{n}-\varphi-C_\delta\lambda(1-\lambda)A_t^2\big)^+,\ t\in[t_\lambda,T],\ n\geqslant 1.
\end{aligned}\tag{3.31}
$$

Step 2. From the semiconcavity of f and standard arguments similar to those used in Step 1 we obtain

$$
\begin{aligned}
&\frac{\lambda}{\dot\tau_1}f(\tau_1^{-1}(s),\widetilde X_s^1,\widetilde Y_s^{1,n},\sqrt{\dot\tau_1}\widetilde Z_s^{1,n},u_s^\lambda)+\frac{1-\lambda}{\dot\tau_0}f(\tau_0^{-1}(s),\widetilde X_s^0,\widetilde Y_s^{0,n},\sqrt{\dot\tau_0}\widetilde Z_s^{0,n},u_s^\lambda)\\
&\quad=\lambda f(\tau_1^{-1}(s),\widetilde X_s^1,\widetilde Y_s^{1,n},\sqrt{\dot\tau_1}\widetilde Z_s^{1,n},u_s^\lambda)+(1-\lambda)f(\tau_0^{-1}(s),\widetilde X_s^0,\widetilde Y_s^{0,n},\sqrt{\dot\tau_0}\widetilde Z_s^{0,n},u_s^\lambda)\\
&\qquad-\lambda(1-\lambda)\frac{t_1-t_0}{T-t_\lambda}\{f(\tau_1^{-1}(s),\widetilde X_s^1,\widetilde Y_s^{1,n},\sqrt{\dot\tau_1}\widetilde Z_s^{1,n},u_s^\lambda)\\
&\qquad\qquad-f(\tau_0^{-1}(s),\widetilde X_s^0,\widetilde Y_s^{0,n},\sqrt{\dot\tau_0}\widetilde Z_s^{0,n},u_s^\lambda)\}\\
&\quad\leqslant f(s,\widetilde X_s,\widetilde Y_s^n,\lambda\sqrt{\dot\tau_1}\widetilde Z_s^{1,n}+(1-\lambda)\sqrt{\dot\tau_0}\widetilde Z_s^{0,n},u_s^\lambda)\\
&\qquad+C_\delta\lambda(1-\lambda)(|t_1-t_0|^2+|\widetilde X_s^1-\widetilde X_s^0|^2+|\widetilde Y_s^{1,n}-\widetilde Y_s^{0,n}|^2+|\widetilde Z_s^{1,n}\\
&\qquad\qquad-\widetilde Z_s^{0,n}|^2+|t_1-t_0|^2|\widetilde Z_s^{0,n}|^2),\quad s\in[t_\lambda,T].
\end{aligned}\tag{3.32}
$$

Since, on the other hand,

$$
\begin{aligned}
&|\lambda\sqrt{\dot\tau_1}\widetilde Z_s^{1,n}+(1-\lambda)\sqrt{\dot\tau_0}\widetilde Z_s^{0,n}-\widetilde Z_s^n|\\
&\quad=|\lambda(1-\sqrt{\dot\tau_1})\widetilde Z_s^{1,n}+(1-\lambda)(1-\sqrt{\dot\tau_0})\widetilde Z_s^{0,n}|\\
&\quad\leqslant\lambda|1-\sqrt{\dot\tau_1}||\widetilde Z_s^{1,n}-\widetilde Z_s^{0,n}|+|\lambda(1-\sqrt{\dot\tau_1})+(1-\lambda)(1-\sqrt{\dot\tau_0})||\widetilde Z_s^{0,n}|\\
&\quad\leqslant C_\delta\lambda(1-\lambda)(|t_1-t_0||\widetilde Z_s^{1,n}-\widetilde Z_s^{0,n}|+|t_1-t_0|^2|\widetilde Z_s^{0,n}|)
\end{aligned}\tag{3.33}
$$

(see the proof of Lemma 3.2), we have

$$
\begin{aligned}
&\frac{\lambda}{\dot\tau_1}f(\tau_1^{-1}(s),\widetilde X_s^1,\widetilde Y_s^{1,n},\sqrt{\dot\tau_1}\widetilde Z_s^{1,n},u_s^\lambda)+\frac{1-\lambda}{\dot\tau_0}f(\tau_0^{-1}(s),\widetilde X_s^0,\widetilde Y_s^{0,n},\sqrt{\dot\tau_0}\widetilde Z_s^{0,n},u_s^\lambda)\\
&\quad\leqslant f(s,\widetilde X_s,\widetilde Y_s^n,\widetilde Z_s^n,u_s^\lambda)+C_\delta\lambda(1-\lambda)\big(|t_0-t_1|^2(1+|\widetilde Z_s^{0,n}|^2)+|\widetilde X_s^1-\widetilde X_s^0|^2\\
&\qquad\qquad+|\widetilde Y_s^{1,n}-\widetilde Y_s^{0,n}|^2+|\widetilde Z_s^{1,n}-\widetilde Z_s^{0,n}|^2\big),\\
&\qquad s\in[t_\lambda,T],\quad n\geqslant 1.
\end{aligned}\tag{3.34}
$$

Thus, from Lemmas 3.1 and 3.2

$$
\begin{aligned}
&\frac{\lambda}{\dot\tau_1}f(\tau_1^{-1}(s),\widetilde X_s^1,\widetilde Y_s^{1,n},\sqrt{\dot\tau_1}\widetilde Z_s^{1,n},u_s^\lambda)+\frac{1-\lambda}{\dot\tau_0}f(\tau_0^{-1}(s),\widetilde X_s^0,\widetilde Y_s^{0,n},\sqrt{\dot\tau_0}\widetilde Z_s^{0,n},u_s^\lambda)\\
&\quad\leqslant f(s,X_s^\lambda,\widetilde Y_s^n-CB_s-C_\delta\lambda(1-\lambda)A_s^2,\widetilde Z_s^n,u_s^\lambda)+C'B_s+C'_\delta\lambda(1-\lambda)A_s^2\\
&\qquad+C_\delta^0\lambda(1-\lambda)\big(|t_0-t_1|^2(1+|\widetilde Z_s^{0,n}|^2)+|\widetilde Z_s^{1,n}-\widetilde Z_s^{0,n}|^2\big),\quad s\in[t_\lambda,T],\quad n\geqslant 1.
\end{aligned}\tag{3.35}
$$

Note also that if f does not depend on z, the constant C_δ^0 in (3.35) can be chosen to be zero.

Step 3. We also note that thanks to the semiconcavity and the Lipschitz condition on Φ,

$$\lambda\Phi(\widetilde{X}_T^1) + (1-\lambda)\Phi(\widetilde{X}_T^0) \leqslant \Phi(X_T^\lambda) + C_\delta\lambda(1-\lambda)A_T^2 + CB_T. \tag{3.36}$$

The above three steps allow us to conclude. Indeed, taking into account (3.30), (3.35), and (3.36), it follows from the comparison theorem, Lemma 4.2, that

$$\widetilde{Y}_t^n \leqslant \widehat{Y}_t^n,\ t\in[t_\lambda, T],\ n\geqslant 1. \tag{3.37}$$

The proof is complete. ∎

Let us now introduce the process

$$\overline{Y}_t^n = \widehat{Y}_t^n - CB_t - C_\delta\lambda(1-\lambda)A_t^2, \quad t\in[t_\lambda, T].$$

Then, for $D_t := CB_t + C_\delta\lambda(1-\lambda)A_t^2, \quad t\in[t_\lambda,T]$, we have

$$\begin{cases} d\overline{Y}_s^n = -\big\{f(s, X_s^\lambda, \overline{Y}_s^n, \widehat{Z}_s^n, u_s^\lambda) + CD_s \\ \qquad\qquad + C_\delta^0\lambda(1-\lambda)\big(|t_0-t_1|^2(1+|\widetilde{Z}_s^{0,n}|^2) + |\widetilde{Z}_s^{1,n} - \widetilde{Z}_s^{0,n}|^2\big) \\ \qquad\qquad - n\big(\overline{Y}_s^n - \varphi(s, X_s^\lambda)\big)^+\big\}ds - dD_s + \widehat{Z}_s^n dW_s^\lambda, \quad s\in[t_\lambda,T], \\ \overline{Y}_T^n = \Phi(X_T^\lambda). \end{cases} \tag{3.38}$$

Recalling that

$$\begin{cases} dY_s^{\lambda,n} = -\big\{f(s, X_s^\lambda, Y_s^{\lambda,n}, Z_s^{\lambda,n}, u_s^\lambda) - n\big(Y_s^{\lambda,n} - \varphi(s, X_s^\lambda)\big)^+\big\}ds \\ \qquad\qquad + Z_s^{\lambda,n} dW_s^\lambda, \quad s\in[t_\lambda,T], \\ Y_T^{\lambda,n} = \Phi(X_T^\lambda). \end{cases} \tag{3.39}$$

We can establish the following statement.

LEMMA 3.4. *Under the assumptions of Theorem* 3.1

$$E\left[\sup_{s\in[t,T]}|\overline{Y}_s^n - Y_s^{\lambda,n}|^2 + \int_t^T |\widehat{Z}_s^n - Z_s^{\lambda,n}|^2 ds \,|\mathcal{F}_t^{W^\lambda}\right] \leqslant C_\delta D_t^2, \quad t\in[t_\lambda,T].$$

Proof. Taking into account that

$$-(\overline{Y}_s^n - Y_s^{\lambda,n})\big((\overline{Y}_s^n - \varphi(s,X_s^\lambda))^+ - (Y_s^{\lambda,n} - \varphi(s,X_s^\lambda))^+\big) \leqslant 0,\ s\in[t_\lambda,T],$$

we see that for $\gamma > 0$,

(3.40)

$$
\begin{aligned}
&e^{\gamma t}(\overline{Y}_t^n - Y_t^{\lambda,n})^2 + E\left[\int_t^T e^{\gamma s}\big(\gamma|\overline{Y}_s^n - Y_s^{\lambda,n}|^2 + |\widehat{Z}_s^n - Z_s^{\lambda,n}|^2\big)ds|\mathcal{F}_t^{W^\lambda}\right]\\
&\quad\leqslant 2E\bigg[\int_t^T e^{\gamma s}(\overline{Y}_s^n - Y_s^{\lambda,n})\big(f(s,X_s^\lambda,\overline{Y}_s^n,\widehat{Z}_s^n,u_s^\lambda)\\
&\qquad\qquad - f(s,X_s^\lambda,Y_s^{\lambda,n},Z_s^{\lambda,n},u_s^\lambda)\big)ds|\mathcal{F}_t^{W^\lambda}\bigg]\\
&\qquad + 2E\left[\int_t^T e^{\gamma s}(\overline{Y}_s^n - Y_s^{\lambda,n})dD_s|\mathcal{F}_t^{W^\lambda}\right]\\
&\qquad + 2E\bigg[\int_t^T e^{\gamma s}(\overline{Y}_s^n - Y_s^{\lambda,n})\Big(CD_s + C_\delta^0\lambda(1-\lambda)\big(|t_0-t_1|^2(1+|\widetilde{Z}_s^{0,n}|^2)\\
&\qquad\qquad\qquad + |\widetilde{Z}_s^{1,n} - \widetilde{Z}_s^{0,n}|^2\big)\Big)ds|\mathcal{F}_t^{W^\lambda}\bigg]\\
&\quad\leqslant CE\left[\int_t^T e^{\gamma s}|\overline{Y}_s^n - Y_s^{\lambda,n}|^2 ds|\mathcal{F}_t^{W^\lambda}\right] + \frac{1}{2}E\left[\int_t^T e^{\gamma s}|\widehat{Z}_s^n - Z_s^{\lambda,n}|^2 ds|\mathcal{F}_t^{W^\lambda}\right]\\
&\qquad + C_\gamma E[D_T^2|\mathcal{F}_t^{W^\lambda}]\\
&\qquad + C_{\gamma,\delta}E\Bigg[\sup_{s\in[t,T]}|\overline{Y}_s^n - Y_s^{\lambda,n}|\Bigg(\lambda(1-\lambda)\Bigg(|t_0-t_1|^2\left(1+\int_t^T|\widetilde{Z}_s^{0,n}|^2ds\right)\\
&\qquad\qquad\qquad + \int_t^T|\widetilde{Z}_s^{1,n} - \widetilde{Z}_s^{0,n}|^2ds\Bigg) + D_T\Bigg)|\mathcal{F}_t^{W^\lambda}\Bigg].
\end{aligned}
$$

Let

$$
\widetilde{D}_{t,T} := \lambda(1-\lambda)\left(|t_0-t_1|^2\left(1+\int_t^T|\widetilde{Z}_s^{0,n}|^2ds\right) + \int_t^T|\widetilde{Z}_s^{1,n} - \widetilde{Z}_s^{0,n}|^2ds\right) + D_T.
$$

Recall that $C_\delta^0 = 0$ if f does not depend on z. If f depends on z, we have, thanks to assumption (H7), that φ is constant (see Lemma 3.1(ii)),

$$
E\left[\left(\int_t^T|\widetilde{Z}_s^{0,n} - \widetilde{Z}_s^{1,n}|^2ds\right)^p|\mathcal{F}_t^{W^\lambda}\right] \leqslant C_{\delta,p}A_t^{2p},\ p\geqslant 1,\ \text{P-a.s.}
$$

On the other hand, from Lemma 2.3(ii) we know that

$$
E\left[\left(\int_t^T|Z_s^{0,n}|^2ds\right)^p|\mathcal{F}_t^{W^0}\right] \leqslant C_p,\ t\in[t_0,T],\ n\geqslant 1,\ p\geqslant 1,\ \text{P-a.s.},
$$

and hence

$$
E\left[\left(\int_t^T|\widetilde{Z}_s^{0,n}|^2ds\right)^p|\mathcal{F}_t^{W^\lambda}\right] \leqslant C_{\delta,p},\ t\in[t_\lambda,T],\ n\geqslant 1,\ p\geqslant 1,\ \text{P-a.s.}
$$

Consequently, considering that from Lemmas 3.1(i) and 3.2 it follows that $E[D_T^2|\mathcal{F}_t^{W^\lambda}] \leqslant C_\delta D_t^2$, we get for $\gamma \geqslant C+1$

$$(3.41)\quad \begin{aligned} &(\overline{Y}_t^n - Y_t^{\lambda,n})^2 + E\left[\int_t^T |\widehat{Z}_s^n - Z_s^{\lambda,n}|^2 ds|\mathcal{F}_t^{W^\lambda}\right] \\ &\leqslant C_\delta D_t^2 + C_\gamma E\left[\sup_{s\in[t,T]} |\overline{Y}_s^n - Y_s^{\lambda,n}|\widetilde{D}_{t,T}|\mathcal{F}_t^{W^\lambda}\right], \quad t\in[t_\lambda,T]. \end{aligned}$$

Finally, applying the argument used for (2.64) in the proof of Lemma 2.5 (or (2.73) in the proof of Lemma 2.6) it follows that

$$E\left[\sup_{s\in[t,T]} |\overline{Y}_s^n - Y_s^{\lambda,n}|^2 + \int_t^T |\widehat{Z}_s^n - Z_s^{\lambda,n}|^2 ds|\mathcal{F}_t^{W^\lambda}\right] \leqslant C_\delta D_t^2, \quad t\in[t_\lambda,T],\ n\geq 1.$$

The proof is complete. □

Lemmas 3.3 and 3.4 allow us to give the proof of Proposition 3.2.

Proof of Proposition 3.2. From Lemmas 3.3 and 3.4 we can conclude that

$$(3.42)\quad \begin{aligned} \widetilde{Y}_t^n \quad (=\quad & \lambda\widetilde{Y}_t^{1,n} + (1-\lambda)\widetilde{Y}_t^{0,n}) \leqslant \widehat{Y}_t^n \\ &= \overline{Y}_t^n + D_t \leqslant Y_t^{\lambda,n} + C_\delta D_t \\ &= Y_t^{\lambda,n} + C_\delta(CB_t + C_\delta\lambda(1-\lambda)A_t^2),\ t\in[t_\lambda,T],\ n\geqslant 1,\ \text{P-a.s.} \end{aligned}$$

Thus, the proof is complete. □

4. Appendix.

4.1. BSDEs. The objective of this section is to recall some basic results concerning backward and reflected backward SDEs, which are frequently used in our paper. Let $(\Omega,\mathcal{F},P)$ be a probability space endowed with a d-dimensional Brownian motion and let $T>0$ be a finite time horizon. By $\mathbf{F}^W = \{\mathcal{F}_s^W,\ 0\leq s\leq T\}$ we denote the natural filtration generated by the Brownian motion W and augmented by all P-null sets, i.e.,

$$\mathcal{F}_s^W = \sigma\{W_r, r\leq s\}\vee\mathcal{N}_P, \quad s\in[0,T].$$

Here $\mathcal{N}_P$ is the set of all P-null sets.

A measurable function $g:\Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}^d\to\mathbf{R}$ satisfies that $(g(t,y,z))_{t\in[0,T]}$ is $\mathbf{F}^W$-progressively measurable for all (y,z) in $\mathbf{R}\times\mathbf{R}^d$ and also satisfies the following standard assumptions:

(A1) There is some real $C\geq 0$ such that, P-a.s., for all $t\in[0,T]$, $y_1,y_2\in\mathbf{R}$, $z_1,z_2\in\mathbf{R}^d$,

$$|g(t,y_1,z_1) - g(t,y_2,z_2)| \leq C(|y_1-y_2| + |z_1-z_2|).$$

(A2) $g(\cdot,0,0)\in L^2_{\mathbf{F}^W}(0,T;\mathbf{R})$.

The following result on BSDEs is well-known; for its proof, see the pioneering paper by Pardoux and Peng [12].

LEMMA 4.1. *Let the function g satisfy assumptions* (A1) *and* (A2). *Then, for any random variable $\xi \in L^2(\Omega, \mathcal{F}_T^W, P)$, the BSDE*

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s\, dW_s, \quad 0 \le t \le T, \tag{4.1}$$

has a unique adapted solution

$$(Y_t, Z_t)_{t\in[0,T]} \in \mathcal{S}^2_{\mathbf{F}^W}(0,T) \times L^2_{\mathbf{F}^W}(0,T;\mathbf{R}^d).$$

Besides the above existence and uniqueness result we also recall the important comparison theorem for BSDEs. (See, e.g., Theorem 2.2 in [6] or Proposition 2.4 in [14].)

LEMMA 4.2 (comparison theorem). *Given two coefficients g_1 and g_2 satisfying* (A1) *and* (A2) *and two terminal values $\xi_1,\ \xi_2 \in L^2(\Omega, \mathcal{F}_T^W, P)$, we denote by (Y^1, Z^1) and (Y^2, Z^2) the solution of the BSDE with the data (ξ_1, g_1) and (ξ_2, g_2), respectively. Then we have the following:*

(i) *(Monotonicity) If $\xi_1 \ge \xi_2$ and $g_1 \ge g_2$, a.s., then $Y_t^1 \ge Y_t^2$, for all $t \in [0,T]$, a.s.*

(ii) *(Strict monotonicity) If in addition to* (i) *we also assume that $P\{\xi_1 > \xi_2\} > 0$, then $P\{Y_t^1 > Y_t^2\} > 0$ for all $0 \le t \le T$, and in particular, $Y_0^1 > Y_0^2$.*

4.2. RBSDEs. After the above very short recall on BSDEs let us come now to RBSDES. Here we introduce only RBSDEs with lower barriers; the results on RBSDEs with upper barriers are symmetric. An RBSDE is connected with a terminal value $\xi \in L^2(\Omega, \mathcal{F}_T^W, P)$, a generator g, and a "barrier" process $\{S_t\}_{0\le t\le T}$. We shall make the following condition on the barrier process:

(A3) $\{S_t\}_{0\le t\le T}$ is a continuous process such that $\{S_t\}_{0\le t\le T} \in \mathcal{S}^2_{\mathbf{F}^W}(0,T)$.

A solution of an RBSDE is a triple (Y, Z, K) which is $\mathbf{F}^W$-progressively measurable processes; take its values in $\mathbf{R} \times \mathbf{R}^d \times \mathbf{R}_+$, and satisfy the following conditions:

(i) $Y \in \mathcal{S}^2(0,T;\mathbf{R})$, $Z \in \mathcal{H}^2(0,T;\mathbf{R}^d)$, and $K_T \in L^2(\Omega, \mathcal{F}_T^W, P)$;

(ii)

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s, \quad t \in [0,T]; \tag{4.2}$$

(iii) $Y_t \ge S_t$ a.s. for any $t \in [0,T]$;

(iv) $\{K_t\}$ is continuous and increasing, $K_0 = 0$ and $\int_0^T (Y_t - S_t)dK_t = 0$.

The following two lemmas refer to Theorems 5.2 and 4.1 in [7], respectively.

LEMMA 4.3. *Assume that g satisfies* (A1) *and* (A2), *ξ belongs to $L^2(\Omega, \mathcal{F}_T^W, P)$, $\{S_t\}_{0\le t\le T}$ satisfies* (A3), *and $S_T \le \xi$ a.s. Then RBSDE* (4.2) *has a unique solution $(Y, Z, K) \in \mathcal{S}^2_{\mathbf{F}^W}(0,T) \times L^2_{\mathbf{F}^W}(0,T;\mathbf{R}^d) \times A^2_{\mathbf{F}^W}(0,T)$.*

Remark 4.1. For simplicity, a given triple (ξ, g, S) is said to satisfy the standard assumptions if the coefficient g satisfies (A1) and (A2), the terminal condition ξ belongs to $L^2(\Omega, \mathcal{F}_T^W, P)$, the barrier process S satisfies (A3), and $S_T \le \xi$, a.s.

LEMMA 4.4 (comparison theorem). *Assume that two triples (ξ_1, g_1, S^1) and (ξ_2, g_2, S^2) satisfy the standard assumptions and one of both generators g_1 and g_2 is Lipschitz. Furthermore, we make the following assumptions:*

(i) *$\xi_1 \le \xi_2$, a.s.;*

(ii) *$g_1(t,y,z) \le g_2(t,y,z)$, a.s., for $(t,y,z) \in [0,T] \times \mathbf{R} \times \mathbf{R}^d$;*

(iii) *$S_t^1 \le S_t^2$, a.s., for $t \in [0,T]$.*

Let (Y^1, Z^1, K^1) *and* (Y^2, Z^2, K^2) *be solutions of RBSDEs* (4.2) *with data* (ξ_1, g_1, S^1) *and* (ξ_2, g_2, S^2)*, respectively. Then* $Y_t^1 \le Y_t^2$ *a.s. for* $t \in [0, T]$.

LEMMA 4.5. *Let* (Y, Z, K) *be the solution of the above RBSDE* (4.2) *with data* (ξ, g, S) *satisfying the above standard assumptions. Then there exists a constant* C *such that*

$$E\left[\sup_{t\le s\le T}|Y_s|^2 + \int_t^T |Z_s|^2 ds + |K_T - K_t|^2 |\mathcal{F}_t\right] \le CE[\xi^2 + \left(\int_t^T g(s,0,0)ds\right)^2 + \sup_{t\le s\le T} S_s^2 |\mathcal{F}_t].$$

The constant C *depends only on the Lipschitz constant of* g.

LEMMA 4.6. *Let* (ξ, g, S) *and* (ξ', g', S') *be two triples satisfying the above standard assumptions.* (Y, Z, K) *and* (Y', Z', K') *are the solutions of RBSDE* (4.2) *with the data* (ξ, g, S) *and* (ξ', g', S')*, respectively. We define*

$$\Delta\xi = \xi - \xi', \qquad \Delta g = g - g', \qquad \Delta S = S - S';$$
$$\Delta Y = Y - Y', \qquad \Delta Z = Z - Z', \qquad \Delta K = K - K'.$$

Then there exists a constant C *such that*

$$E\left[\sup_{t\le s\le T}|\Delta Y_s|^2 + \int_t^T |\Delta Z_s|^2 ds + |\Delta K_T - \Delta K_t|^2 |\mathcal{F}_t\right] \le CE[|\Delta\xi|^2 + \left(\int_t^T |\Delta g(s, Y_s, Z_s)|ds\right)^2 |\mathcal{F}_t] + C\left(E[\sup_{t\le s\le T}|\Delta S_s|^2|\mathcal{F}_t]\right)^{1/2} \Psi_{t,T}^{1/2},$$

where

$$\Psi_{t,T} = E[|\xi|^2 + \left(\int_t^T |g(s,0,0)|ds\right)^2 + \sup_{t\le s\le T}|S_s|^2 + |\xi'|^2 + \left(\int_t^T |g'(s,0,0)|ds\right)^2 + \sup_{t\le s\le T}|S_s'|^2 |\mathcal{F}_t].$$

The constant C *depends only on the Lipschitz constant of* g'.

The Lemmas 4.5 and 4.6 refer to the Propositions 3.5 and 3.6 in [7], and their generalizations can be consulted in [16, Propositions 2.1 and 2.2], respectively.

Remark 4.2. For the Markovian case where the barrier process is a deterministic function of the solution of the associated forward equation, Lemma 4.6 has been considerably improved. Indeed, Proposition 6.1 in [3] shows that Y is Lipschitz with respect to the possibly random initial condition of the driving forward SDE which solution governs the RBSDE as well as its barrier.

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