SIAM J. MATRIX ANAL. APPL.
Vol. 32, No. 1, pp. 23–43

HIGHER RANK NUMERICAL RANGES OF NORMAL MATRICES*

HWA-LONG GAU†, CHI-KWONG LI‡, YIU-TUNG POON§, AND NUNG-SING SZE¶

Abstract. The higher rank numerical range is closely connected to the construction of quantum error correction code for a noisy quantum channel. It is known that if a normal matrix $A \in M_n$ has eigenvalues $a_1, \ldots, a_n$, then its higher rank numerical range $\Lambda_k(A)$ is the intersection of convex polygons with vertices $a_{j_1}, \ldots, a_{j_{n-k+1}}$, where $1 \leq j_1 < \cdots < j_{n-k+1} \leq n$. In this paper, it is shown that the higher rank numerical range of a normal matrix with m distinct eigenvalues can be written as the intersection of no more than $\max\{m, 4\}$ closed half planes. In addition, given a convex polygon $\mathcal{P}$, a construction is given for a normal matrix $A \in M_n$ with minimum n such that $\Lambda_k(A) = \mathcal{P}$. In particular, if $\mathcal{P}$ has p vertices, with $p \geq 3$, there is a normal matrix $A \in M_n$ with $n \leq \max\{p + k - 1, 2k + 2\}$ such that $\Lambda_k(A) = \mathcal{P}$.

Key words. quantum error correction, higher rank numerical range, normal matrices, convex polygon

AMS subject classifications. 15A60, 15A90, 47N50, 81P68

DOI. 10.1137/09076430X

1. Introduction.

Let M_n be the algebra of $n \times n$ complex matrices regarded as linear operators acting on the n-dimensional Hilbert space $\mathbb{C}^n$. The *classical numerical range* of $A \in M_n$ is defined and denoted by

$$W(A) = \{x^*Ax \in \mathbb{C} : x \in \mathbb{C}^n \text{ with } x^*x = 1\},$$

which is a useful concept in studying matrices and operators; see [6].

In the context of quantum information theory, if the quantum states are represented as matrices in M_n, then a *quantum channel* is a trace preserving completely positive map $L : M_n \to M_n$ with the operator sum representation

$$L(A) = \sum_{j=1}^{r} E_j^* A E_j, \tag{1.1}$$

where $E_1, \ldots, E_r \in M_n$ satisfy $\sum_{j=1}^{r} E_j E_j^* = I_n$. The matrices $E_1, \ldots, E_r$ are known as the *error operators* of the quantum channel L. A subspace V of $\mathbb{C}^n$ is a *quantum error correction code* for the channel L if and only if the orthogonal projection $P \in M_n$ with range space V satisfies $PE_i^*E_jP = \gamma_{ij}P$ for all $i, j \in \{1, \ldots, r\}$; for example, see

*Received by the editors July 9, 2009; accepted for publication (in revised form) by H. J. Woerdeman October 27, 2010; published electronically January 6, 2011.
http://www.siam.org/journals/simax/32-1/76430.html

†Department of Mathematics, National Central University, Chung-Li 320, Taiwan (hlgau@math.ncu.edu.tw). This author's research was supported by the National Science Council of the Republic of China.

‡Department of Mathematics, College of William & Mary, Williamsburg, VA 23185 (ckli@math.wm.edu). This author's research was supported by the USA NSF and the William & Mary Plumeri Award. This author is an honorary professor of the University of Hong Kong and an honorary professor of the Taiyuan University of Technology.

§Department of Mathematics, Iowa State University, Ames, IA 50051 (ytpoon@iastate.edu). This author's research was supported by the USA NSF.

¶Corresponding author. Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong (raymond.sze@inet.polyu.edu.hk). This author's research was supported by a startup grant from The Hong Kong Polytechnic University.

[7, 8, 9, 15]. In this connection, for $1 \leq k < n$, researchers define the *rank-k numerical range* of $A \in M_n$ by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank-}k \text{ orthogonal projection } P\}$$

and the *joint rank-k numerical range* of $A_1, \ldots, A_m \in M_n$ by $\Lambda_k(A_1, \ldots, A_m)$ to be the collection of complex vectors $(a_1, \ldots, a_m) \in \mathbb{C}^{1\times m}$ such that $PA_jP = a_jP$ for a rank-k orthogonal projection $P \in M_n$. Evidently, there is a quantum error correction code V of dimension k for the quantum channel L described in (1.1) if and only if $\Lambda_k(A_1, \ldots, A_m)$ is nonempty for $(A_1, \ldots, A_m) = (E_1^*E_1, E_1^*E_2, \ldots, E_r^*E_r)$. Also it is easy to see that if $(a_1, \ldots, a_m) \in \Lambda_k(A_1, \ldots, A_m)$, then $a_j \in \Lambda_k(A_j)$ for $j = 1, \ldots, m$. When $k = 1$, $\Lambda_k(A)$ reduces to the classical numerical range $W(A)$.

Recently, interesting results have been obtained for the rank-k numerical range and the joint rank-k numerical range; see [1, 2, 3, 4, 5, 10, 11, 12, 13, 15]. In particular, an explicit description of the rank-k numerical range of $A \in M_n$ is given in [13], namely,

$$\Lambda_k(A) = \bigcap_{\xi\in[0,2\pi)} \{\mu \in \mathbb{C} : e^{-i\xi}\mu + e^{i\xi}\overline{\mu} \leq \lambda_k(e^{-i\xi}A + e^{i\xi}A^*)\}, \tag{1.2}$$

where $\lambda_k(X)$ is the kth largest eigenvalue of a Hermitian matrix X.

In the study of quantum error correction, there are channels such as the randomized unitary channels and Pauli channels whose error operators are commuting normal matrices. Thus, it is of interest to study the rank-k numerical ranges of normal matrices. Although the error operators of a generic quantum channel may not commute, a good understanding of the special case would lead to deeper insights and more proof techniques for the general case.

Given $S \subseteq \mathbb{C}$, let $\operatorname{conv} S$ denote the smallest convex subset of $\mathbb{C}$ containing S. For a normal matrix $A \in M_n$ with eigenvalues $a_1, \ldots, a_n$, it was conjectured in [3, 4] that

$$\Lambda_k(A) = \bigcap_{1\leq j_1<\cdots<j_{n-k+1}\leq n} \operatorname{conv}\{a_{j_1}, \ldots, a_{j_{n-k+1}}\}, \tag{1.3}$$

which is a convex polygon including its interior (if it is nonempty). This conjecture was confirmed in [13] using the description of $\Lambda_k(A)$ in (1.2). *In our discussion, a polygon would always mean a convex polygon with its interior.*

In this paper, we improve the description (1.3) of the rank-k numerical range of a normal matrix. In particular, in section 2 we show that for a normal matrix A with m distinct eigenvalues, $\Lambda_k(A)$ can be written as the intersection of no more than $\max\{m, 4\}$ closed half planes in $\mathbb{C}$. Moreover, if $\Lambda_k(A) \neq \emptyset$, then it is a polygon with no more than m vertices. We then consider the "inverse" problem, namely, for a given polygon $\mathcal{P}$, construct a normal matrix $A \in M_n$ with $\Lambda_k(A) = \mathcal{P}$. In other words, we study the necessary condition for the existence of quantum channels whose error operators have prescribed rank-k numerical ranges. It is easy to check that $\Lambda_k(\tilde{A}) = \mathcal{P}$ if $\tilde{A} = A \otimes I_k$ with $W(A) = \mathcal{P}$. Our goal is to find a normal matrix $\hat{A}$ with the smallest size so that $\Lambda_k(\hat{A}) = \mathcal{P}$. To achieve this, we give a necessary and sufficient condition for the existence of a normal matrix $A \in M_n$ so that $\Lambda_k(A) = \mathcal{P}$ in terms of k-regular sets in $\mathbb{C}$ (see Definition 3.3). Furthermore, we show that the problem of finding a desired normal matrix A is equivalent to a combinatorial problem of extending a given p element set of unimodular complex numbers to a k-regular set. We then give the solution of the problem in section 4. As a consequence of our results, if $\mathcal{P}$ is a polygon with p vertices, then there is a normal matrix $A \in M_n$ with

$$n \leq \max\{p+k-1, 2k+2\}$$

such that $\Lambda_k(A) = \mathcal{P}$. Moreover, this upper bound is best possible in the sense that there exists $\mathcal{P}$ so that there is no matrix of smaller dimension with rank-k numerical range equal to $\mathcal{P}$.

2. Construction of higher rank numerical ranges. By (1.2), $\Lambda_k(A)$ can be obtained as the intersection of infinitely many closed half planes for a given $A \in M_n$. Suppose A is normal. By (1.3), one can write $\Lambda_k(A)$ as the intersection of $\binom{n}{k-1}$ convex polygons so that $\Lambda_k(A)$ is a polygon. In particular, it is well known that $\Lambda_1(A) = \operatorname{conv}\{a_1, \ldots, a_m\}$, where $a_1, \ldots, a_m$ are the distinct eigenvalues of A.

There is a nice interplay between the algebraic properties of $A \in M_n$ and the geometric properties of $\Lambda_1(A) = W(A)$. For instance, $\Lambda_1(A)$ is always nonempty; $\Lambda_1(A)$ is a singleton if and only if A is a scalar matrix; $\Lambda_1(A)$ is a nondegenerate line segment if and only if A is a nonscalar normal matrix and its eigenvalues lie on a straight line. Unfortunately, these results have no analogs for $\Lambda_k(A)$ if $k > 1$. First, the set $\Lambda_k(A)$ may be empty (see [12]); there are nonscalar matrices A such that $\Lambda_k(A)$ is a singleton, and there are nonnormal matrices A such that $\Lambda_k(A)$ is a line segment. Even for a normal matrix A, it is not easy to determine whether $\Lambda_k(A)$ is empty, a point, or a line segment without actually constructing the set $\Lambda_k(A)$. Moreover, there is no easy way to express the vertices of the polygon $\Lambda_k(A)$ (if it is nonempty) in terms of the eigenvalues of the normal matrix A as in the case of $\Lambda_1(A)$. Of course, one can use (1.3) to construct $\Lambda_k(A)$ for the normal matrix A, but the number of polygons needed in the construction will grow exponentially for large n and k. In the following, we will study efficient ways to generate $\Lambda_k(A)$ for a normal matrix $A \in M_n$. While it is difficult to use the eigenvalues of A to determine the set $\Lambda_k(A)$, it turns out that we can use half planes determined by the eigenvalues to generate $\Lambda_k(A)$ efficiently. In the following, we will focus on the following problem.

PROBLEM 2.1. *Determine the minimum number of half planes needed to construct $\Lambda_k(A)$ using the eigenvalues of the normal matrix $A \in M_n$.*

We will show that, for a normal matrix A with m distinct eigenvalues, $\Lambda_k(A)$ either is empty or is a polygon with at most m vertices. In fact, by examining the location of the eigenvalues of A on the complex plane, one may further reduce the number of half planes needed to construct $\Lambda_k(A)$.

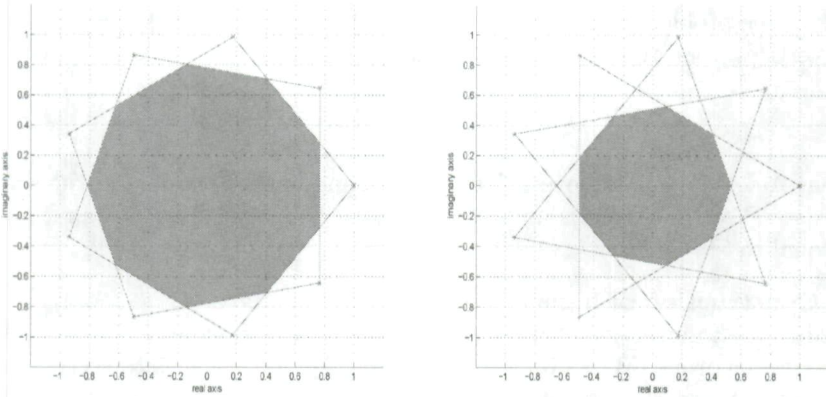
Suppose the eigenvalues of $A \in M_n$ are collinear. Then by a translation, followed by a rotation, we may assume that A is Hermitian with eigenvalues $a_1 \geq \cdots \geq a_n$. Then we have $\Lambda_k(A) = [a_{n-k+1}, a_k]$. So we focus on those normal matrices whose eigenvalues are not collinear.

Let us motivate our result with the following examples, which can be verified by using (1.3).

Example 2.2. Let $A = \operatorname{diag}(1, w, w^2, \ldots, w^{n-1})$ with $w = e^{2\pi i/n}$. Then, for $k \leq n/2$, we have $\Lambda_k(A) = \cap_{j=0}^{n-1} \mathcal{H}_j$, where

$$\mathcal{H}_j = \left\{ z \in \mathbb{C} : \operatorname{Re}\left(e^{-\frac{(2j+k)\pi i}{n}} z\right) \leq \cos\frac{k\pi}{n} \right\}$$

and only a small part of $\operatorname{conv}\{w^{j-1}, w^{j-1+k}\}$ lies in $\Lambda_k(A)$ (see Figure 1).



$\Lambda_2(A)$ with $n = 9$ in Example 2.2 $\quad$ $\Lambda_3(A)$ with $n = 9$ in Example 2.2

FIG. 1.

More generally, we have the following.

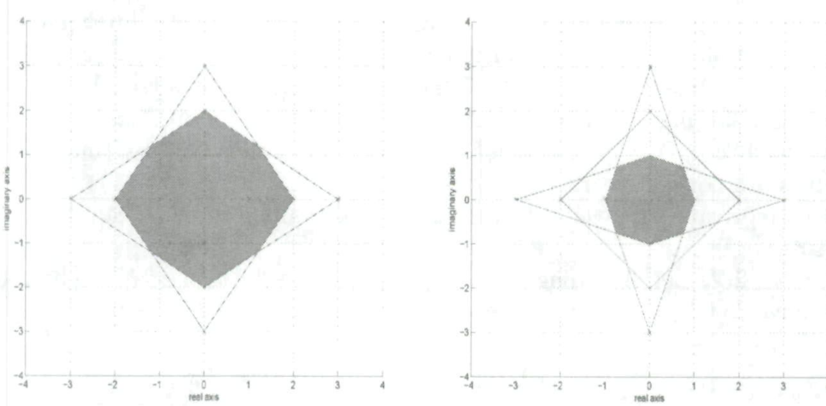
Example 2.3. Let $a_1, \ldots, a_n$ be the eigenvalues of $A \in M_n$ with $n \geq 3$. Suppose $\operatorname{conv}\{a_1, \ldots, a_n\} = \mathcal{P}$ is an n-sided convex polygon containing the origin in the interior. We may assume that $a_1, \ldots, a_n$ are arranged in the counterclockwise direction on the boundary of $\mathcal{P}$. For $j \in \{1, \ldots, n\}$, let L_j be the line passing through a_j and a_{j+k}, where $a_{j+k} = a_{j+k-n}$ if $j+k > n$, and let $\mathcal{H}_j$ be the closed half plane determined by L_j which does not contain a_ℓ for $j < \ell < j+k$. Then

$$\Lambda_k(A) = \bigcap_{j=1}^{n} \mathcal{H}_j.$$

Note that each $\mathcal{H}_j$ in Example 2.3 contains exactly $n-k+1$ eigenvalues of A.

The situation is more complicated if $\Lambda_1(A)$ is not an n-sided convex polygon for the normal matrix $A \in M_n$.

Example 2.4. Suppose $B = \operatorname{diag}(1, i, -1, -i, 2, 2i, -2, -2i, 3, 3i, -3, -3i)$. One can see from Figure 2 that the eigenvalues $1, i, -1, -i$ are interior points of $\Lambda_2(B)$ while these eigenvalues are the vertices of $\Lambda_3(B)$.



$\Lambda_2(B)$ $\quad$ $\Lambda_3(B)$

FIG. 2.

To deal with normal matrices $A \in M_n$ as in Example 2.4 that $\Lambda_1(A)$ is not an n-sided convex polygon, we need to construct some half spaces using the eigenvalues of the normal matrix A. To do this, we introduce the following. Given any two distinct complex numbers a and b, let $L(a,b)$ be the (directed) line passing through a and b. The closed half plane

$$H(a,b) = \{z \in \mathbb{C} : \mathrm{Im}\,\left((\bar{b}-\bar{a})(z-a)\right) \geq 0\}$$

is called the *left closed half plane* determined by $L(a,b)$. For example, $H(0,i) = \{z \in \mathbb{C} : \mathrm{Re}\,(z) \leq 0\}$ and $H(i,0) = \{z \in \mathbb{C} : \mathrm{Re}\,(z) \geq 0\}$. Note that in Example 2.2, the set $\mathcal{H}_j$ is, indeed, the closed half plane $H(w^j, w^{j+k})$. Note that $L(a,b) \neq L(b,a)$. In our discussion, it is sometimes convenient to write

$$H(a,b) = \{z \in \mathbb{C} : \mathrm{Re}\,(e^{-i\xi}z) \leq \mathrm{Re}\,(e^{-i\xi}a)\}$$

with $\xi = \arg(b-a) - \pi/2$. Also we use $H_0(a,b)$ to denote the *left open half plane* determined by $L(a,b)$, i.e., $H_0(a,b) = H(a,b) \setminus L(a,b)$.

We have the following result showing that, for a normal matrix $A \in M_n$ with m distinct eigenvalues, $\Lambda_k(A)$ can be written as the intersection of at most $\max\{m,4\}$ half spaces. Even without any knowledge about the final shape of the set $\Lambda_k(A)$, one can use $m(m-1)$ half spaces to generate $\Lambda_k(A)$. Evidently the construction is more efficient than the construction using (1.2) or (1.3). Furthermore, we can conclude that $\Lambda_k(A)$ is either an empty set, a singleton, a line segment, or a nondegenerate polygon with at most m vertices.

THEOREM 2.5. *Let $A \in M_n$ be normal with distinct eigenvalues $a_1, \ldots, a_m$ that are not collinear. Let $\mathcal{S}$ be the set of index pairs (r,s) such that $H(a_r, a_s)$ contains at least $n-k+1$ eigenvalues (counting multiplicities) of A and*

$$\mathcal{S}_0 = \{(r,s) \in \mathcal{S} : H_0(a_r,a_s) \text{ contains at most } n-k-1 \text{ eigenvalues (counting multiplicities) }\}.$$

Then

$$\Lambda_k(A) = \bigcap_{(r,s)\in\mathcal{S}} H(a_r,a_s) = \bigcap_{(r,s)\in\mathcal{S}_0} H(a_r,a_s). \tag{2.1}$$

Moreover, $\Lambda_k(A)$ can be written as the intersection of at most $\max\{m,4\}$ *half planes $H(a_r,a_s)$ with $(r,s) \in \mathcal{S}_0$.*

Proof. In the first part of the proof, we assume that $A \in M_n$ has n eigenvalues $a_1, \ldots, a_n$. For notational simplicity, we write $H(a_r,a_s) = H(r,s)$, $H_0(a_r,a_s) = H_0(r,s)$, and $L(a_r,a_s) = L(r,s)$ for any two distinct eigenvalues a_r and a_s of A. For each $(r,s) \in \mathcal{S}$, since $H(r,s)$ is convex and contains at least $n-k+1$ eigenvalues of A, by (1.3), we have

$$\Lambda_k(A) = \bigcap_{1\leq j_1<\cdots<j_{n-k+1}\leq n} \mathrm{conv}\,\{a_{j_1},\ldots,a_{j_{n-k+1}}\} \subseteq H(r,s).$$

It follows that

$$\Lambda_k(A) \subseteq \bigcap_{(r,s)\in\mathcal{S}} H(r,s). \tag{2.2}$$

To prove the reverse inclusion of (2.2), note that if z is a point not in $\Lambda_k(A)$, then z will lie outside a convex polygon which equals the convex hull of $n-k+1$

eigenvalues of A. So it suffices to show that the convex hull $\mathcal{W}$ of any $n-k+1$ eigenvalues of A can be written as an intersection of half planes, $\mathcal{W} = \cap_{j=1}^{\ell} H(r_j, s_j)$ for some $(r_1, s_1), \dots, (r_\ell, s_\ell) \in \mathcal{S}$. We consider the following three cases.

Case 1. Suppose $\mathcal{W}$ is a singleton. Then $\mathcal{W} = \{a_r\}$ for some eigenvalue a_r with multiplicity at least $n-k+1$. Since the eigenvalues of A are noncollinear, there are eigenvalues a_s and a_t such that a_r, a_s, and a_t are not collinear. Then

$$\mathcal{W} = H(r,s) \cap H(s,r) \cap H(r,t) \cap H(t,r).$$

Case 2. Suppose $\mathcal{W}$ is a nondegenerate line segment. In this case, $\mathcal{W} = \operatorname{conv}\{a_r, a_s\}$ for some eigenvalues a_r and a_s with $a_r \neq a_s$. Since the eigenvalues of A are noncollinear, there is another eigenvalue a_t such that a_r, a_s, and a_t are not collinear. Without loss of generality, we assume that $a_t \in H(r,s)$. Otherwise, we interchange a_r and a_s. Then

$$\mathcal{W} = H(r,s) \cap H(s,r) \cap H(s,t) \cap H(t,r).$$

Case 3. Suppose $\mathcal{W}$ is a nondegenerate polygonal disk. We may relabel the eigenvalues of A and assume that $\mathcal{W}$ has vertices $a_1, \dots, a_q$ arranged in the counterclockwise direction, where $q \geq 3$. For convenience of notation, we will let $a_{q+1} = a_1$ and $H(q, q+1) = H(q, 1)$. Then

$$\mathcal{W} = \bigcap_{1 \leq t \leq q} H(t, t+1).$$

Thus, the first equality in (2.1) is proved. To prove the second equality in (2.1), we claim the following.

Claim. For each $(r,s) \in \mathcal{S} \setminus \mathcal{S}_0$, there exist two ordered pairs (r_1, s_1) and (r_2, s_2) in $\mathcal{S}_0$ such that $H(r_1, s_1) \cap H(r_2, s_2) \subseteq H_0(r,s)$.

Once the claim is proved, all the half planes $H(r,s)$ with $(r,s) \in \mathcal{S} \setminus \mathcal{S}_0$ are not needed in the intersection $\bigcap_{(r,s) \in \mathcal{S}} H(a_r, a_s)$, and hence the second equality in (2.1) holds.

To prove the claim, suppose $(r,s) \in \mathcal{S} \setminus \mathcal{S}_0$. Then $H_0(r,s)$ contains at least $n - k$ eigenvalues of A. By a translation followed by a rotation, we may assume that $H(r,s) = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$, and we can relabel the index of eigenvalues so that, for $1 \leq j \leq n-1$, either $\operatorname{Im} a_j > \operatorname{Im} a_{j+1}$ or $\operatorname{Im} a_j = \operatorname{Im} a_{j+1}$ with $\operatorname{Re} a_j \geq \operatorname{Re} a_{j+1}$. Let

$$\mathcal{U} = \operatorname{conv}\{a_1, \dots, a_{n-k}\} \quad \text{and} \quad \mathcal{V} = \operatorname{conv}\{a_{n-k+1}, \dots, a_n\}.$$

Then $\mathcal{U}$ and $\mathcal{V}$ are disjoint if $a_{n-k} \neq a_{n-k+1}$ or $\mathcal{U} \cap \mathcal{V} = \{a_{n-k}\}$ if $a_{n-k} = a_{n-k+1}$. By the assumption, $\mathcal{U} \subseteq H_0(r,s)$ and $\{a_r, a_s\} \subseteq \mathcal{V}$. Define the set

$$\mathcal{W} = \operatorname{conv}\{a_i - a_j : 1 \leq i \leq n-k < j \leq n\} = \{u - v : u \in \mathcal{U} \text{ and } v \in \mathcal{V}\},$$

which is a convex polygon. Note that $\mathcal{W} \subseteq \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\}$ since $\operatorname{Im}(a_i - a_j) \geq 0$ for all $1 \leq i \leq n-k < j \leq n$. By the facts that $\mathcal{U}$ and $\mathcal{V}$ can intersect in at most one point and that the union $\mathcal{U} \cup \mathcal{V}$ cannot be contained in any line, the set $\mathcal{W}$ does not lie in any line that passes through the origin, and the point 0 can only either be an extreme point of $\mathcal{W}$ or is not in $\mathcal{W}$. Under these conditions, one can find two extreme points w_1 and w_2 in $\mathcal{W}$ with $\operatorname{Im}(\bar{w}_1 w_2) \neq 0$ such that

$$\operatorname{Im}(\bar{w}_1 w) \geq 0 \geq \operatorname{Im}(\bar{w}_2 w) \quad \text{for all} \quad w \in \mathcal{W}. \tag{2.3}$$

Since w_1 is an extreme point in $\mathcal{W}$, there are eigenvalues $a_{s_1} \in \mathcal{U}$ and $a_{r_1} \in \mathcal{V}$ such that $w_1 = a_{s_1} - a_{r_1}$. Then (2.3) gives

$$\mathrm{Im}\,(\bar{a}_{s_1} - \bar{a}_{r_1})(u - a_{r_1}) \geq 0 \quad \text{and} \quad \mathrm{Im}\,(\bar{a}_{r_1} - \bar{a}_{s_1})(v - a_{s_1}) \geq 0$$

for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$, and thus, $\mathcal{U} \subseteq H(r_1, s_1)$ and $\mathcal{V} \subseteq H(s_1, r_1)$. With the fact that a_{r_1} and a_{s_1} lie in the line $L(r_1, s_1)$, the closed half plane $H(r_1, s_1)$ contains at least $n-k+1$ eigenvalues of A while the open half plane $H_0(r_1, s_1)$ contains at most $n-k-1$ eigenvalues. Therefore, $(r_1, s_1) \in \mathcal{S}_0$. By a similar argument, one can show that there are eigenvalues $a_{r_2} \in \mathcal{U}$ and $a_{s_2} \in \mathcal{V}$ such that $w_2 = a_{r_2} - a_{s_2}$. Then (2.3) yields

$$\mathrm{Im}\,(\bar{a}_{r_2} - \bar{a}_{s_2})(u - a_{s_2}) \leq 0 \quad \text{and} \quad \mathrm{Im}\,(\bar{a}_{s_2} - \bar{a}_{r_2})(v - a_{r_2}) \leq 0$$

for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$. Thus, $\mathcal{U} \subseteq H(r_2, s_2)$ and $\mathcal{V} \subseteq H(s_2, r_2)$, and one can conclude that $(r_2, s_2) \in S_0$. Observe that the two lines $L(r_1, s_1)$ and $L(r_2, s_2)$ are not parallel as $\mathrm{Im}\,(\bar{a}_{r_1} - \bar{a}_{s_1})(a_{s_2} - a_{r_2}) = \mathrm{Im}\,(\bar{w}_1 w_2) \neq 0$. Using the fact that the two distinct eigenvalues a_r and a_s are in $\mathcal{V}$, which is contained in the intersection $H(s_1, r_1) \cap H(s_2, r_2)$, one can conclude that the intersection $H(r_1, s_1) \cap H(r_2, s_2)$ must lie in $H_0(r, s)$, the interior of $H(r, s)$. Therefore, the claim holds.

Next we turn to the last part of the theorem. It is trivial that if $\Lambda_k(A)$ is an empty set, a singleton, or a nondegenerate line segment, then at most four half planes are needed in the construction of $\Lambda_k(A)$.

Suppose A has m distinct eigenvalues $a_1, \ldots, a_m$ and $\Lambda_k(A)$ is a nondegenerate polygon. Let $\mathcal{T}$ be a minimal subset of $\mathcal{S}_0$ such that $\Lambda_k(A) = \cap_{(r,s)\in\mathcal{T}} H(r, s)$. Since $\mathcal{T}$ is minimal, the half planes $H(r, s)$, $(r, s) \in \mathcal{T}$, are all distinct. We may further assume that for all $(r, s) \in \mathcal{T}$, $\{a_1, \ldots, a_m\} \cap L(a_r, a_s) \subseteq \mathrm{conv}\,\{a_r, a_s\}$. Since $\Lambda_k(A)$ is a nondegenerate polygon, for each $1 \leq t \leq m$, there exist at most two pairs $(r, s) \in \mathcal{T}$ such that $t \in \{r, s\}$. Therefore, $\mathcal{T}$ contains at most m ordered pairs. $\square$

Example 2.6. Let $A = \mathrm{diag}\,(0, 0, 1, 1, i)$. Then

$$\Lambda_2(A) = [0, 1] = H(0, 1) \cap H(1, 0) \cap H(1, i) \cap H(i, 0),$$

$$\Lambda_3(A) = \emptyset = H(1, 0) \cap H(1, i) \cap H(i, 0),$$

and the intersection of any two half planes $H(a_r, a_s)$ is nonempty. This example also shows that one cannot replace $\max\{m, 4\}$ by m in the conclusion in Theorem 2.5.

Example 2.7. Let $A = \mathrm{diag}\,(1, -1, i, -i)$. Then

$$\Lambda_2(A) = \{0\} = H(1, -1) \cap H(-1, 1) \cap H(i, -i) \cap H(-i, i),$$

and $\Lambda_2(A)$ cannot be written as an intersection of less than four half planes $H(a_r, a_s)$.

COROLLARY 2.8. *Suppose $A \in M_n$ is normal such that $W(A)$ is an n-sided polygon containing the origin as its interior point. Let $v_1, \ldots, v_n$ be the vertices of $W(A)$ having arguments $0 \leq \xi_1 < \cdots < \xi_n < 2\pi$. If $k < n/2$, then $\Lambda_k(A)$ is an n-sided convex polygon obtained by joining v_j and v_{j+k}, where $v_{j+k} = v_{j+k-n}$ if $j + k > n$.*

By Theorem 2.5, it is easy to see that the boundary of $\Lambda_k(A)$ is subsets of the union of line segments of the form $\mathrm{conv}\,\{a_r, a_s\}$ such that a_r and a_s satisfy the $H(a_r, a_s)$ condition. However, it is not easy to determine which part of the line segment actually belongs to $\Lambda_k(A)$ as shown in Examples 2.2, 2.3, and 2.4. By Theorem 2.5, if the normal matrix $A \in M_n$ has m distinct eigenvalues, we need no more than $\max\{m, 4\}$ half planes $H(a_r, a_s)$ to generate $\Lambda_k(A)$. *Can one determine these half planes effectively?* We will answer this question by presenting an algorithm in section 5 based on the discussion in this section.

3. Matrices with prescribed higher rank numerical ranges. We study the following problem in this section.

PROBLEM 3.1. *Let $k > 1$ be a positive integer, and let $\mathcal{P}$ be a p-sided polygon in $\mathbb{C}$. Construct a normal matrix A with the smallest size (dimension) such that $\Lambda_k(A) = \mathcal{P}$.*

If $\mathcal{P}$ degenerates to a line segment joining two points a_1 and a_2, then the smallest n to get a normal matrix with $\Lambda_k(A) = \mathcal{P}$ is $n = 2k$ if a_1 and a_2 are distinct and $n = k$ if $a_1 = a_2$. So we focus on the case when the polygon $\mathcal{P}$ is nondegenerate.

A natural approach to Problem 3.1 is to reverse the construction of $\Lambda_k(A)$ in Example 2.3. Suppose we have a nondegenerate p-sided polygon $\mathcal{P}$ with vertices $v_1, \ldots, v_p$.

Without loss of generality, we may assume that 0 lies in the interior of $\mathcal{P}$ and that the arguments of v_j in $[0, 2\pi)$ are arranged in ascending order. Our goal is to use the support line L_j which passes through v_j, v_{j+1} for $j = 1, \ldots, p$, where $v_{p+1} = v_1$, to construct $A = \operatorname{diag}(a_1, \ldots, a_p)$ such that $\Lambda_k(A) = \mathcal{P}$. Note that if the desired values $a_1, \ldots, a_p$ exist and are arranged in a counterclockwise direction, then (by proper numbering) the line L_j will coincide with the line passing through a_j and a_{j+k}, where $a_{j+k} = a_{j+k-p}$ if $j+k > p$. Consequently, a_j will lie at the intersection of L_j and L_{j-k}, where $L_{j-k} = L_{j-k+p}$ if $j - k < 0$. Consequently, there exists $A = \operatorname{diag}(a_1, \ldots, a_p)$ satisfying $\Lambda_k(A) = \mathcal{P}$ if the following hold.

(1) $k < p/2$.

(2) There exist $a_1, \ldots, a_p \in \mathbb{C}$ such that

- (2a) $L_j \cap L_{j-k} = \{a_j\}$ for $j = 1, \ldots, p$,
- (2b) $a_1, \ldots, a_p$ have arguments $\xi_1 < \cdots < \xi_p$ in the interval $[\xi_1, \xi_1 + 2\pi)$ and 0 lies in the interior of their convex hull.

Note that by Theorem 2.5, A has the smallest dimension among all normal matrices B such that $\Lambda_k(B) = \mathcal{P}$.

Clearly conditions (1) and (2a) are necessary in the above construction. From the following example, one can see that the above construction also fails when condition (2b) is not satisfied.

Example 3.2. Let $\mathcal{P}$ be the 5-sided polygon with vertices $\{v_1, \ldots, v_5\} = \{2+i, 1+2i, -1+3i, -1-i, 3-i\}$; see Figure 3. Then, with $k = 2$, we have

$$\{a_1, \ldots, a_5\} = \{-1+4i, 7-i, -1+7i, 4-i, (5+5i)/3\},$$

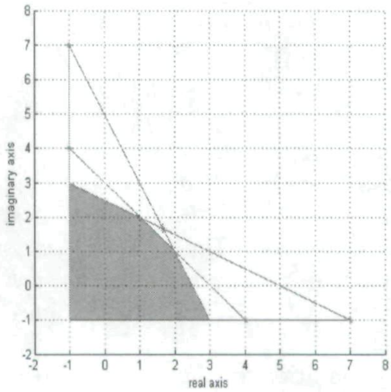
which does not satisfy the condition (2b). Clearly, for $A = \operatorname{diag}(a_1, \ldots, a_5)$, $\Lambda_2(A)$ lies in the convex hull of $\{a_1, \ldots, a_5\}$, which does not contain $\mathcal{P}$.

Conditions (2a) and (2b) motivate the following definition.

DEFINITION 3.3. *Let $\Omega = \{z \in \mathbb{C} : |z| = 1\}$. A subset $\Pi = \{\alpha_1, \ldots, \alpha_m\}$, with distinct $\alpha_1, \ldots, \alpha_m \in \Omega$, is k-regular if every semicircular arc of Ω without endpoints contains at least k elements in Π.*

Given distinct α_1, $\alpha_2 \in \Omega$, $\alpha_2/\alpha_1 = e^{i\theta}$ for a unique $0 < \theta < 2\pi$. Then $[\alpha_1, \alpha_2] = \{e^{it}\alpha_1 : 0 \le t \le \theta\}$ is the closed arc on Ω from α_1 to α_2 in the counterclockwise direction. Also define the open arc

$$(\alpha_1, \alpha_2) = [\alpha_1, \alpha_2] \setminus \{\alpha_1, \alpha_2\}.$$



The polygon $\mathcal{P}$

FIG. 3.

The value θ is called the *length* of these intervals. Suppose $1 \leq k \leq n$ and $\Pi \subseteq \Omega$. Then Π is k-regular if, for each $\alpha \in \Pi$, $(\alpha, -\alpha) \cap \Pi$ contains at least k elements.

Note that if $\Pi = \{e^{i\xi_j} : 1 \leq j \leq n\}$ with distinct $\xi_1, \ldots, \xi_n \in [0, 2\pi)$, then Π is k-regular if and only if, for each $r = 1, \ldots, n$, there are $1 \leq j_1 < \cdots < j_k \leq n$ such that

$$e^{i\xi_{j_1}}, \ldots, e^{i\xi_{j_k}} \in \left(e^{i\xi_r}, e^{i(\xi_r+\pi)}\right). \tag{3.1}$$

For this reason, a set $\{\xi_1, \ldots, \xi_n\} \subseteq [0, 2\pi)$ of n distinct numbers is also called k-regular if $\{e^{i\xi_j} : 1 \leq j \leq n\}$ is k-regular as defined in Definition 3.3. For $\xi, \xi' \in [0, 2\pi)$, $[\xi, \xi']$ will denote the subset $\{t \in [0, 2\pi) : e^{it} \in [e^{i\xi}, e^{i\xi'}]\}$; the intervals $[\xi, \xi')$, $(\xi, \xi']$ and (ξ, ξ') will also be defined similarly.

In Example 2.2, a direct computation shows that, for $1 \leq r, k \leq n$,

$$\xi_{r+k} - \xi_r = \begin{cases} 2k\pi/n & \text{if } r + k \leq n, \\ 2k\pi/n - 2\pi & \text{if } r + k > n. \end{cases}$$

Therefore, the set $\{\xi_1, \ldots, \xi_n\}$ is k-regular, and $\Lambda_k(A)$ is nonempty for $1 \leq k < n/2$. Otherwise, the set $\{\xi_1, \ldots, \xi_n\}$ is not k-regular, and $\Lambda_k(A)$ is either empty or a singleton.

In the following, we need an alternate formulation of (1.2). For any $d, \xi \in \mathbb{R}$, consider the closed half plane

$$\mathcal{H}(d, \xi) = \{\mu \in \mathbb{C} : \mathrm{Re}\,(e^{-i\xi}\mu) \leq d\} \tag{3.2}$$

and its boundary, which is the straight line

$$\mathcal{L}(d, \xi) = \partial\mathcal{H}(d, \xi) = \{\mu \in \mathbb{C} : \mathrm{Re}\,(e^{-i\xi}\mu) = d\}. \tag{3.3}$$

For $A \in M_n$, let $\mathrm{Re}A = (A + A^*)/2$. Then (1.2) is equivalent to

$$\Lambda_k(A) = \bigcap_{\xi \in [0, 2\pi)} \mathcal{H}(\lambda_k(\mathrm{Re}\,(e^{-i\xi}A)), \xi).$$

The following result is easy to verify.

PROPOSITION 3.4. *Let $A \in M_n$ and $\Lambda_k(A) = \cap_{j=1}^m \mathcal{H}(d_j, \xi_j) \neq \emptyset$, where $\mathcal{H}(d_j, \xi_j)$ is defined as in* (3.2) *for some $d_1, \ldots, d_m \in \mathbb{R}$ and distinct $\xi_1, \ldots, \xi_m \in [0, 2\pi)$.*

(a) *We have* $0 \in \Lambda_k(A)$ *if and only if* $d_1, \ldots, d_m \geq 0$, *and* 0 *is an interior point of* $\Lambda_k(A)$ *if and only if* $d_1, \ldots, d_m > 0$.

(b) *If* $\mu = re^{i\xi}$ *with* $r > 0$ *and* $\xi \in \mathbb{R}$, *then* $\Lambda_k(\mu A) = \cap_{j=1}^m \mathcal{H}(rd_j, \xi_j + \xi)$ *and* $\Lambda_k(A + \mu I) = \cap_{j=1}^m \mathcal{H}(d_j + r\cos(\xi - \xi_j), \xi_j)$.

In connection to Problem 3.1, we have the following.

THEOREM 3.5. *Suppose* $\mathcal{P} = \bigcap_{j=1}^p \mathcal{H}(d_j, \xi_j)$ *is a nondegenerate p-sided polygon, where* $\mathcal{H}(d_j, \xi_j)$ *is defined as in* (3.2) *with* $d_1, \ldots, d_p \in \mathbb{R}$ *and distinct* $\xi_1, \ldots, \xi_p \in [0, 2\pi)$. *Let* q *be a nonnegative integer. The following two statements are equivalent:*

(I) *There is a* $(p+q) \times (p+q)$ *normal matrix* A *such that* $\Lambda_k(A) = \mathcal{P}$.

(II) *There are distinct* $\xi_{p+1}, \ldots, \xi_{p+q} \in [0, 2\pi)$ *such that* $\{\xi_1, \ldots, \xi_{p+q}\}$ *is* k*-regular.*

Notice that a necessary condition for the set $\bigcap_{j=1}^p \mathcal{H}(d_j, \xi_j)$ to be a nondegenerate polygon is that

$$\{e^{i\xi_1}, \ldots, e^{i\xi_p}\} \text{ is 1-regular.} \tag{3.4}$$

By Proposition 3.4, one may assume that 0 lies in the interior of $\mathcal{P}$ in our proofs. However, it is equally convenient for us not to impose this assumption so that we need not verify $d_j > 0$ in $\mathcal{H}(d_j, \xi_j)$ in our proofs.

To prove Theorem 3.5, we need some lemmas.

LEMMA 3.6. *Let* $A = \operatorname{diag}(a_1, \ldots, a_n)$ *and* $1 \leq m < n$. *Suppose the eigenvalues* $a_{m+1}, \ldots, a_n$ *are in* $\Lambda_k(A)$ *but not extreme points of* $\Lambda_k(A)$. *Then*

$$\Lambda_k(\operatorname{diag}(a_1, \ldots, a_m)) = \Lambda_k(A).$$

Proof. It suffices to show that if a_n is in $\Lambda_k(A)$ but not an extreme point of $\Lambda_k(A)$, then $\Lambda_k(\operatorname{diag}(a_1, \ldots, a_{n-1})) = \Lambda_k(A)$.

Suppose a_n satisfies the above assumption. Clearly $\Lambda_k(\operatorname{diag}(a_1, \ldots, a_{n-1}))$ is a subset of $\Lambda_k(A)$. On the other hand, for any $1 \leq j_1 < \cdots < j_{n-k} \leq n-1$, $\Lambda_k(A) \subseteq \operatorname{conv}\{a_{j_1}, \ldots, a_{j_{n-k}}, a_n\}$. Since a_n is not an extreme point of $\Lambda_k(A)$, it follows that a_n lies in $\operatorname{conv}\{a_{j_1}, \ldots, a_{j_{n-k}}, a_n\}$ but is not its extreme point. Therefore,

$$\operatorname{conv}\{a_{j_1}, \ldots, a_{j_{n-k}}\} = \operatorname{conv}\{a_{j_1}, \ldots, a_{j_{n-k}}, a_n\}.$$

Thus,

$$\begin{aligned}
\Lambda_k(A) &= \bigcap \{\operatorname{conv}\{a_{j_1}, \ldots, a_{j_{n-k+1}}\} : 1 \leq j_1 < \cdots < j_{n-k} < j_{n-k+1} \leq n\} \\
&\subseteq \bigcap \{\operatorname{conv}\{a_{j_1}, \ldots, a_{j_{n-k}}, a_n\} : 1 \leq j_1 < \cdots < j_{n-k} \leq n-1\} \\
&= \bigcap \{\operatorname{conv}\{a_{j_1}, \ldots, a_{j_{n-k}}\} : 1 \leq j_1 < \cdots < j_{n-k} \leq n-1\} \\
&= \Lambda_k(\operatorname{diag}(a_1, \ldots, a_{n-1})). \qquad \square
\end{aligned}$$

The next lemma shows that if a convex polygon $\mathcal{P}$ is the intersection of half planes $\mathcal{H}(d_j, \zeta_j)$ for $j = 1, \ldots, m$, such that the set $\{\zeta_1, \ldots, \zeta_m\}$ is "almost" k-regular (in the sense that $\{\zeta_1, \ldots, \zeta_m\}$ is k-regular if we count the multiplicity of each element in the set), one may replace these half planes by n other half planes $\mathcal{H}(\tilde{d}_j, \tilde{\zeta}_j)$ for $j = 1, \ldots, n$, $n \geq m$, with $\tilde{\zeta}_i \neq \tilde{\zeta}_j$ for all $i \neq j$, such that $\{\tilde{\zeta}_1, \ldots, \tilde{\zeta}_n\}$ is k-regular and the boundary $\mathcal{L}(\tilde{d}_j, \tilde{\zeta}_j)$ of $\mathcal{H}(\tilde{d}_j, \tilde{\zeta}_j)$ touches the polygon $\mathcal{P}$ for each $j = 1, \ldots, n$.

LEMMA 3.7. *Suppose* $\mathcal{P} = \cap_{j=1}^m \mathcal{H}(d_j, \zeta_j)$ *such that* $0 \leq \zeta_1 \leq \cdots \leq \zeta_m < 2\pi$, *and, for each* $r = 1, \ldots, m$, *there are* $1 \leq j_1 < \cdots < j_k \leq m$ *such that* $\zeta_{j_1}, \ldots, \zeta_{j_k} \in (\zeta_r, \zeta_r + \pi)$. *For every* $n \geq m$, *there exist* $\tilde{d}_1, \ldots, \tilde{d}_n \in \mathbb{R}$ *and distinct* $\tilde{\zeta}_1, \ldots, \tilde{\zeta}_n \in [0, 2\pi)$ *with*

$\{\tilde{\zeta}_1,\ldots,\tilde{\zeta}_n\}$ *being* k*-regular such that* $\{\zeta_1,\ldots,\zeta_m\}\subseteq\{\tilde{\zeta}_1,\ldots,\tilde{\zeta}_n\}$, $\mathcal{P}=\cap_{j=1}^n\mathcal{H}(\tilde{d}_j,\tilde{\zeta}_j)$, *and* $\mathcal{P}\cap\mathcal{L}(\tilde{d}_j,\tilde{\zeta}_j)\neq\emptyset$ *for each* $j=1,\ldots,n$.

Proof. Set $\tilde{\zeta}_1=\zeta_1$, and for $s\in\{2,\ldots,m\}$, let $\tilde{\zeta}_s=\zeta_s$ if $\zeta_{s-1}<\zeta_s$. For the remaining values, we have $\zeta_{s-1}=\zeta_s$, and we can set

$$\tilde{\zeta}_{s-t_1}=\zeta_{s-t_1+1}=\cdots=\zeta_s=\cdots=\zeta_{s+t_2}<\tilde{\zeta}_{s+t_2+1}$$

for some $t_1\geq 1$ and $t_2\geq 0$. Let $\ell=\min\{j:\tilde{\zeta}_j>\tilde{\zeta}_{s-t_1}+\pi\}$. Then we can replace ζ_{s+j} by $\tilde{\zeta}_{s+j}=\zeta_{s+j}+\epsilon_j$ for sufficiently small $\epsilon_j>0$ for $j=-t_1+1,-t_1+2,\ldots,0,\ldots,t_2$ such that

$$\tilde{\zeta}_{s-t_1}<\tilde{\zeta}_{s-t_1+1}<\cdots<\tilde{\zeta}_s<\cdots<\tilde{\zeta}_{s+t_2}<\min\{\tilde{\zeta}_\ell-\pi,\tilde{\zeta}_{s+t_2+1}\}.$$

After this modification, $\tilde{\zeta}_1,\ldots,\tilde{\zeta}_m$ are distinct, and $\{\tilde{\zeta}_1,\ldots,\tilde{\zeta}_m\}$ is k-regular. If $n>m$, one can always pick distinct $\tilde{\zeta}_{m+1},\ldots,\tilde{\zeta}_n\in[0,2\pi)\setminus\{\tilde{\zeta}_1,\ldots,\tilde{\zeta}_m\}$ so that $\{\tilde{\zeta}_1,\ldots,\tilde{\zeta}_n\}$ forms a k-regular set. Finally, let $\tilde{d}_j=\max_{\mu\in\mathcal{P}}\ \mathrm{Re}\,(e^{-i\tilde{\zeta}_j}\mu)$ for $j=1,\ldots,n$. Clearly we have $\mathcal{P}\cap\mathcal{L}(\tilde{d}_j,\tilde{\zeta}_j)\neq\emptyset$ and $\mathcal{P}\subseteq\mathcal{H}(\tilde{d}_j,\tilde{\zeta}_j)$ for all j. By construction, $\{\zeta_1,\ldots,\zeta_m\}\subseteq\{\tilde{\zeta}_1,\ldots,\tilde{\zeta}_n\}$, and $\mathcal{P}=\cap_{j=1}^m\mathcal{H}(d_j,\zeta_j)=\cap_{j=1}^n\mathcal{H}(\tilde{d}_j,\tilde{\zeta}_j)$.

We can now present the proof of Theorem 3.5.

Proof of Theorem 3.5. Let $\mathcal{P}=\bigcap_{j=1}^p\mathcal{H}(d_j,\xi_j)$ be a nondegenerate p-sided polygon, where $d_1,\ldots,d_p\in\mathbb{R}$ and $\xi_1,\ldots,\xi_p\in[0,2\pi)$.

Suppose (I) holds. We may assume that $A=\mathrm{diag}\,(a_1,\ldots,a_{p+q})$ and $\Lambda_k(A)=\mathcal{P}$. By Lemma 3.6, one can remove the eigenvalues of A in $\Lambda_k(A)$ that are not extreme points of $\Lambda_k(A)$ to get $\tilde{A}\in M_n$ for some positive integer $n\leq p+q$ so that $\Lambda_k(A)=\Lambda_k(\tilde{A})$. We have the following.

Claim. There are $f_1,\ldots,f_n\in\mathbb{R}$ and $\zeta_1,\ldots,\zeta_n\in[0,2\pi)$ such that $\Lambda_k(\tilde{A})=\cap_{j=1}^n\mathcal{H}(f_j,\zeta_j)$. Furthermore, for each $r=1,\ldots,n$, there exist $1\leq j_1<\cdots<j_k\leq n$ such that $\zeta_{j_1},\ldots,\zeta_{j_k}\in(\zeta_r,\zeta_r+\pi)$.

Once the claim holds, Lemma 3.7 will ensure that $\Lambda_k(\tilde{A})=\cap_{j=1}^{p+q}\mathcal{H}(\tilde{d}_j,\tilde{\xi}_j)$ for some $\tilde{d}_1,\ldots,\tilde{d}_{p+q}\in\mathbb{R}$ and a k-regular set $\{\tilde{\xi}_1,\ldots,\tilde{\xi}_{p+q}\}$ with

$$\bigcap_{j=1}^p\mathcal{H}(d_j,\xi_j)=\mathcal{P}=\Lambda_k(A)=\Lambda_k(\tilde{A})=\cap_{j=1}^{p+q}\mathcal{H}(\tilde{d}_j,\tilde{\xi}_j)$$

and $\{\xi_1,\ldots,\xi_p\}\subseteq\{\tilde{\xi}_1,\ldots,\tilde{\xi}_{p+q}\}$. Thus, we can take $\xi_{p+1},\ldots,\xi_{p+q}\in[0,2\pi)$ so that $\{\xi_1,\ldots,\xi_{p+q}\}=\{\tilde{\xi}_1,\ldots,\tilde{\xi}_{p+q}\}$. Therefore, (II) holds.

For notational convenience, we assume that $A=\tilde{A}$ in the claim so that every eigenvalue of A is either an extreme point of $\Lambda_k(A)$ or does not lie in $\Lambda_k(A)$.

We first construct $\zeta_1,\ldots,\zeta_n\in[0,2\pi)$ and $f_1,\ldots,f_n\in\mathbb{R}$. For $r=1,\ldots,n$, let Γ_r be the set containing all $\xi\in[0,2\pi)$ such that the closed half plane $\mathcal{H}(\mathrm{Re}\,(e^{-i\xi}a_r),\xi)$ contains at least $n-k+1$ eigenvalues of A. As a_r is either an extreme point of $\Lambda_k(A)$ or not in $\Lambda_k(A)$, there is a $\zeta\in[0,2\pi)$ such that $\mathrm{Re}\,(e^{-i\zeta}a_r)\geq\lambda_k(\mathrm{Re}\,(e^{-i\zeta}A))$, and hence the half plane $\mathcal{H}(\mathrm{Re}\,(e^{-i\zeta}a_r),\zeta)$ contains at least $n-k+1$ eigenvalues. Then Γ_r is always nonempty. Furthermore, by the definition of Γ_r, the set Γ_r is a union of closed arcs of Ω. Clearly

$$\mathcal{P}=\Lambda_k(A)\subseteq\bigcap_{\xi\in\Gamma_r}\mathcal{H}(\mathrm{Re}\,(e^{-i\xi}a_r),\xi).$$

Also the above intersection, which contains $\mathcal{P}$, is a nondegenerate conical region. Then Γ_r is contained in some open semicircular arc of Ω; otherwise, the above intersection of half planes is equal to the singleton $\{a_r\}$. As Γ_r is a union of closed arcs in some open semicircular arc of Ω, there exists a unique $\zeta_r \in \Gamma_r$ such that

$$\Gamma_r \subseteq (\zeta_r - \pi, \zeta_r]. \tag{3.5}$$

Let $f_r = \mathrm{Re}\,(e^{-i\zeta_r} a_r)$ for $1 \le r \le n$. We show that $\Lambda_k(A) = \bigcap_{j=1}^n \mathcal{H}(f_j, \zeta_j)$. Suppose $\mathcal{T}$ is a minimal subset of $\mathcal{S}_0$ such that $\Lambda_k(A) = \bigcap_{(r,s)\in\mathcal{T}} H(a_r, a_s)$. We may further assume that, for all $(r,s) \in \mathcal{T}$, $\{a_1, \ldots, a_m\} \cap L(a_r, a_s) \subseteq \mathrm{conv}\,\{a_r, a_s\}$. For each $(r,s) \in \mathcal{T}$, write $H(a_r, a_s) = \mathcal{H}(\mathrm{Re}\,(e^{-i\zeta} a_r), \zeta)$ with $\zeta = \arg(a_s - a_r) - \pi/2$. Then $\zeta \in \Gamma_r$. We claim that $\zeta = \zeta_r$. Suppose not. By the above assumption on a_r, one can see that for a sufficiently small $\epsilon > 0$, the half plane $\mathcal{H}(\mathrm{Re}\,(e^{-i\hat{\zeta}} a_r), \hat{\zeta})$ with $\hat{\zeta} = \zeta - \epsilon$ will contain all eigenvalues of A that are in $H(a_r, a_s)$, i.e., $\hat{\zeta} \in \Gamma_r$. With (3.5), we have

$$\Lambda_k(A) \subseteq \mathcal{H}(\mathrm{Re}\,(e^{-i\zeta_r} a_r), \zeta_r) \cap \mathcal{H}(\mathrm{Re}\,(e^{-i\hat{\zeta}} a_r), \hat{\zeta}) \subseteq H_0(a_r, a_s) \cup \{a_r\}.$$

So $L(a_r, a_s) \cap \Lambda_k(A)$ contains at most one point. But this contradicts the fact that (r,s) is an element in the minimal subset $\mathcal{T}$. Therefore, $\zeta = \zeta_r$. Then, for each $(r,s) \in \mathcal{T}$, $H(a_r, a_s) = \mathcal{H}(f_r, \zeta_r)$, and so

$$\bigcap_{j=1}^n \mathcal{H}(f_j, \zeta_j) \subseteq \bigcap_{(r,s)\in\mathcal{T}} H(a_r, a_s) = \Lambda_k(A) \subseteq \bigcap_{j=1}^n \mathcal{H}(f_j, \zeta_j).$$

Thus, $\Lambda_k(A) = \bigcap_{j=1}^n \mathcal{H}(f_j, \zeta_j)$, and the first part of the claim holds.

To prove the second part of the claim, without loss of generality, we may assume that $a_r = 0$ and $\zeta_r = 0$. Then $f_r = 0$, and

$$\mathcal{H}(f_r, \zeta_r) = H = \{z \in \mathbb{C} : \mathrm{Re}\,(z) \le 0\}.$$

Thus, the closed left half plane contains at least $n - k + 1$ eigenvalues of A. Suppose that the closed right half plane $-H$ contains eigenvalues $a_{j_1}, \ldots, a_{j_h}$ of A with $\zeta_{j_t} \ne 0$ for $t = 1, \ldots, g$ and $\zeta_{j_t} = 0$ for $t = g+1, \ldots, h$ for some $g \le h$. Fix a sufficiently small $\epsilon > 0$. We choose $g + 1 \le \ell \le h$ so that

$$\mathrm{Re}\,(e^{-i\epsilon} a_{j_\ell}) = \max_{g+1\le t\le h} \mathrm{Re}\,(e^{-i\epsilon} a_{j_t}).$$

Then $\{a_{j_{g+1}}, \ldots, a_{j_h}\} \subseteq \mathcal{H}(\mathrm{Re}\,(e^{-i\epsilon} a_{j_\ell}), \epsilon)$. On the other hand, this closed half plane $\mathcal{H}(\mathrm{Re}\,(e^{-i\epsilon} a_{j_\ell}), \epsilon)$ also contains all eigenvalues of A that are in the left open half plane. Thus, this closed half plane $\mathcal{H}(\mathrm{Re}\,(e^{-i\epsilon} a_{j_\ell}), \epsilon)$ has at least $n - g$ eigenvalues of A. On the other hand, by (3.5), $\epsilon \notin \Gamma_{j_\ell}$, and so $\mathcal{H}(\mathrm{Re}\,(e^{-i\epsilon} a_{j_\ell}), \epsilon)$ can have at most $n - k$ eigenvalues. Thus, we have $g \ge k$.

Now for each $t = 1, \ldots, k$, let $\hat{d}_t = \mathrm{Re}\,(a_{j_t})$. Then $\hat{d}_t \ge 0$, and $H \subseteq \mathcal{H}(\hat{d}_t, 0)$. Thus, the closed half plane $\mathcal{H}(\hat{d}_t, 0)$ contains at least $n - k + 1$ eigenvalues of A, i.e., $\zeta_r = 0 \in \Gamma_{j_t}$. Recall that $\zeta_{j_t} \ne 0$. By (3.5), one see that

$$\zeta_r \in (\zeta_{j_t} - \pi, \zeta_{j_t}) \quad \text{for } t = 1, \ldots, k.$$

Equivalently $\zeta_{j_1}, \ldots, \zeta_{j_k} \in (\zeta_r, \zeta_r + \pi)$. Thus, our **claim** is proved, and (II) holds.

Suppose now (II) holds, namely, there are distinct $\xi_{p+1}, \ldots, \xi_{p+q}$ such that $\{\xi_1, \ldots, \xi_{p+q}\}$ is k-regular. For $j = p+1, \ldots, p+q$, define

$$d_j = \max_{\mu\in\mathcal{P}} \mathrm{Re}\,(e^{-i\xi_j} \mu).$$

Then $\mathcal{P} \subseteq \mathcal{H}(d_j, \xi_j)$, and so

$$\mathcal{P} = \bigcap_{j=1}^{p} \mathcal{H}(d_j, \xi_j) = \bigcap_{j=1}^{n} \mathcal{H}(d_j, \xi_j)$$

with $n = p + q$. By Lemma 3.7, we may assume that $\mathcal{P} \cap \mathcal{L}(d_j, \xi_j) \neq \emptyset$ for all $j = 1, \ldots, n$ and that $0 \leq \xi_1 < \cdots < \xi_n < 2\pi$ such that condition (3.1) holds. For each $r = 1, \ldots, n$, let

$$a_r = \frac{i}{\sin(\xi_{r+k} - \xi_r)} \left(e^{i\xi_r} d_{r+k} - e^{i\xi_{r+k}} d_r\right)$$

and $A = \operatorname{diag}(a_1, \ldots, a_n)$. Then

$$\operatorname{Re}(e^{-i\xi_r} a_r) = d_r \quad \text{and} \quad \operatorname{Re}(e^{-i\xi_{r+k}} a_r) = d_{r+k}.$$

Note that $a_r \in \mathcal{L}(d_r, \xi_r) \cap \mathcal{L}(d_{r+k}, \xi_{r+k})$ is the vertex of the conical region $\mathcal{H}(d_r, \xi_r) \cap \mathcal{H}(d_{r+k}, \xi_{r+k})$, which contains $\mathcal{P}$. Therefore,

$$\operatorname{Re}(e^{-i\xi_r}(a_r - \mu)) \geq 0, \quad \text{and} \quad \operatorname{Re}(e^{-i\xi_{r+k}}(a_r - \mu)) \geq 0$$

for all $\mu \in \mathcal{P}$. Since $\xi_{r+k} \in (\xi_r, \xi_r + \pi)$, we have

$$\operatorname{Re}(e^{-i\xi} a_r) \geq \max_{\mu \in \mathcal{P}} \operatorname{Re}(e^{-i\xi}\mu) \quad \text{for all } \xi \in [\xi_r, \xi_{r+k}]. \tag{3.6}$$

Let $\mu_j \in \mathcal{L}(d_j, \xi_j) \cap \mathcal{P}$ for $j = r, r+k$. As $\xi_{r+k} \in (\xi_r, \xi_r + \pi)$, we have $\mu_r = a_r - ie^{i\xi_r} b_r$ and $\mu_{r+k} = a_r + ie^{i\xi_{r+k}} c_r$ for some $b_r, c_r \geq 0$. Note that

$$\operatorname{Re}(e^{-i\xi}(\mu_r - a_r)) = b_r \sin(\xi_r - \xi) \geq 0 \quad \text{for all } \xi \in [\xi_r - \pi, \xi_r],$$

and

$$\operatorname{Re}(e^{-i\xi}(\mu_{r+k} - a_r)) = c_r \sin(\xi - \xi_{r+k}) \geq 0 \quad \text{for all } \xi \in [\xi_{r+k}, \xi_{r+k} + \pi].$$

Since $\{\xi_1, \ldots, \xi_n\}$ is k-regular, it is easily seen that

$$[0, 2\pi) \setminus [\xi_r, \xi_{r+k}] = [\xi_r - \pi, \xi_r) \cup (\xi_{r+k}, \xi_{r+k} + \pi].$$

Therefore, for $\xi \in [0, 2\pi) \setminus [\xi_r, \xi_{r+k}]$, we have

$$\max\{\operatorname{Re}(e^{-i\xi}(\mu_r - a_r)), \operatorname{Re}(e^{-i\xi}(\mu_{r+k} - a_r))\} \geq 0.$$

Moreover, we have

$$\max\{\operatorname{Re}(e^{-i\xi}\mu_r), \operatorname{Re}(e^{-i\xi}\mu_{r+k})\} \geq \operatorname{Re}(e^{-i\xi} a_r). \tag{3.7}$$

Let $\xi \in [0, 2\pi)$. Then $\xi \in [\xi_s, \xi_{s+1})$ for some $s \in \{1, \ldots, n\}$. It follows that $\xi \in [\xi_r, \xi_{r+k}]$ for $r = s - k + 1, \ldots, s$, and $\xi \in [0, 2\pi) \setminus [\xi_r, \xi_{r+k}]$ for other r. By (3.6) and (3.7),

$$\min_{r \in \{s-k+1, \ldots, s\}} \operatorname{Re}(e^{-i\xi} a_r) \geq \max_{\mu \in \mathcal{P}} \operatorname{Re}(e^{-i\xi}\mu) \geq \max_{r \notin \{s-k+1, \ldots, s\}} \operatorname{Re}(e^{-i\xi} a_r).$$

Thus, $\lambda_k(\mathrm{Re}\,(e^{-i\xi}A)) = \min_{r\in\{s-k+1,\ldots,s\}} \mathrm{Re}\,(e^{-i\xi}a_r)$, and so

$$\mathcal{P} \subseteq \mathcal{H}\left(\lambda_k(\mathrm{Re}\,(e^{-i\xi}A)), \xi\right).$$

Hence, $\mathcal{P} \subseteq \Lambda_k(A)$. Furthermore, if $\xi = \xi_s$, then $\mathrm{Re}\,(e^{-i\xi_s}a_s) = d_s$. Thus,

$$\lambda_k(\mathrm{Re}\,(e^{-i\xi_s}A)) = \min_{r\in\{s-k+1,\ldots,s\}} \mathrm{Re}\,(e^{-i\xi_s}a_r) \le d_s.$$

It follows that

$$\begin{aligned}\Lambda_k(A) &= \bigcap_{\xi\in[0,2\pi)} \mathcal{H}\left(\lambda_k(\mathrm{Re}\,(e^{-i\xi}A)), \xi\right)\\ &\subseteq \bigcap_{1\le s\le n} \mathcal{H}\left(\lambda_k(\mathrm{Re}\,(e^{-i\xi_s}A)), \xi_s\right) \subseteq \bigcap_{1\le s\le n} \mathcal{H}\left(d_s, \xi_s\right) = \mathcal{P}.\end{aligned}$$

Thus, $\mathcal{P} = \Lambda_k(A)$. □

By Theorem 3.5, Problem 3.1 is equivalent to the following combinatorial problem, whose solution will be given in the next section.

PROBLEM 3.8. *Suppose $\{\xi_1, \ldots, \xi_p\} \subseteq [0, 2\pi)$ is 1-regular. For $k > 1$, determine the smallest nonnegative integer q so that $\{\xi_1, \ldots, \xi_{p+q}\}$ is k-regular for some distinct $\xi_{p+1}, \ldots, \xi_{p+q} \in [0, 2\pi)$.*

4. Solutions for Problems 3.1 and 3.8. In this section, we give the solutions for Problems 3.1 and 3.8. Given a nonempty set $\Pi = \{\xi_1, \ldots, \xi_p\} \subseteq \Omega$, Problem 3.8 is equivalent to the study of the smallest nonnegative integer q so that $\{\xi_1, \ldots, \xi_{p+q}\}$ is k-regular for some distinct $\xi_{p+1}, \ldots, \xi_{p+q} \in \Omega$. We have the following.

THEOREM 4.1. *Let $k > 1$ be a positive integer and Π be a p element subset of Ω, including s pairs of antipodal points: $\{\beta_1, -\beta_1\}, \ldots, \{\beta_s, -\beta_s\}$, where $p \ge 3$ and $s \ge 0$. Suppose Π is 1-regular but not k-regular and q is the minimum number of points in Ω that one can add to Π to form a k-regular set.*
(a) *If $k \ge p - s$, then*

$$q = \begin{cases} 2k+1-p & \text{if } s = 0, \\ 2k+2-p & \text{if } s > 0. \end{cases} \tag{4.1}$$

(b) *If $k < p - s$, then q is the smallest nonnegative integer t such that one can remove t nonantipodal points from Π to get a $(k-t)$-regular set. More precisely,*

$$\begin{aligned} q = \min\{t \in \mathbb{N} : \Pi \setminus \{\beta_1, -\beta_1, \ldots, \beta_s, -\beta_s\} &\text{ has a } t\text{-element} \\ &\text{subset } T \text{ such that } \Pi \setminus T \text{ is } (k-t)\text{-regular}\}. \end{aligned} \tag{4.2}$$

Consequently

$$q \le \min\{2k+2-p, k-1\}. \tag{4.3}$$

The inequality in (4.3) becomes an equality if $\Pi = \{1, i, -1, \alpha_4, \ldots, \alpha_p\}$, where $\alpha_4, \ldots, \alpha_p$ lie in the open lower half plane.

Several remarks concerning Theorem 4.1 are in order. If condition (a) in the theorem holds, then the value q can be determined immediately. However, it is important to consider two cases depending on whether Π has pairs of antipodal points as illustrated by the following.

Example 4.2. Suppose $S_1 = \{1, w, w^2, w^3\}$ with $w = e^{2i\pi/5}$. Then $\alpha \neq -\beta$ for any two elements $\alpha, \beta \in S_1$, and adding w^4 to S_1 results in a 2-regular set. Suppose $S_2 = \{1, -1, i, -i\}$. Then we need to add at least two points, say, $z, -z \in \Omega \setminus S_2$, to get a 2-regular set.

Suppose condition (b) in the theorem holds. We can determine the value q by taking t nonantipodal elements away from Π at a time and checking whether the resulting set is $(k-t)$-regular. The value q can then be determined in no more than $\sum_{i=0}^{p-2s} \binom{p-2s}{i} = 2^{p-2s}$ steps. The success of reducing Problem 3.8 to a problem which is solvable in finite steps depends on Lemma 4.7 and Proposition 4.8.

It would be nice to have a simple formula for q in terms of p, k, s in case (b) of the theorem. However, the following example shows that the value q depends not only on the values p,k,s but also on the relative positions of the points in Π.

Example 4.3. Let $S_1 = \{1, w, w^2, w^3, w^4, w^5\}$ with $w = e^{2\pi i/7}$ and $S_2 = \{z^2, z^3, z^7, z^8, z^{12}, z^{13}\}$ with $z = e^{2\pi i/15}$. Notice that both S_1 and S_2 contain six elements and have no antipodal pairs. Furthermore, both of them are 2-regular but not 3-regular. Clearly, adding w^6 to S_1 results in a 3-regular set. However, as each of the open arcs $(z^3, -z^3)$, $(z^8, -z^8)$, and $(z^{13}, -z^{13})$ contains only two elements of S_2 while the intersection of these three open arcs is empty, at least two elements have to be added to S_2 to form a 3-regular set.

Note that our proofs are constructive; see Lemma 4.7 and Propositions 4.6 and 4.8. One can actually construct a subset $\Pi' \subseteq \Omega$ with q elements so that $\Pi \cup \Pi'$ is k-regular.

By Theorem 4.1, we can answer Problems 3.1 and 3.8 and obtain some additional information on their solutions. We will continue to use the notation $\mathcal{H}(d, \xi)$ defined in (3.2) in the following.

THEOREM 4.4. *For Problem* 3.1, *if a p-sided polygon $\mathcal{P}$ is expressed as $\mathcal{P} = \cap_{j=1}^{p} \mathcal{H}(d_j, \xi_j)$ for some $d_1, \ldots, d_p \in \mathbb{R}$ and $\xi_1, \ldots, \xi_p \in [0, 2\pi)$, then the minimum dimension n for the existence of a normal matrix $A \in M_n$ such that $\Lambda_k(A) = \mathcal{P}$ is equal to $p + q$, where q is determined in Theorem* 4.1. *Moreover,*

$$n \leq \max\{2k+2, p+k-1\}. \tag{4.4}$$

The inequality in (4.4) *becomes an equality if $(\xi_1, \xi_2, \xi_3) = (0, \pi/2, \pi)$ and $\xi_4, \ldots, \xi_p$ lie in $(\pi, 2\pi)$.*

We break down the proofs of Theorems 4.1 and 4.4 in several propositions. We first give a lower bound for the number of elements in a k-regular set.

PROPOSITION 4.5. *Suppose $S = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Omega$ is k-regular. Then $n \geq 2k+1$. Furthermore, if S contains a pair of antipodal points $\{\alpha, -\alpha\}$, then $n \geq 2k+2$.*

Proof. For any $r \in \{1, \ldots, n\}$, each of the open arcs $(\alpha_r, -\alpha_r)$ and $(-\alpha_r, \alpha_r)$ contains k elements of S. Thus, $n \geq 2k+1$. For the last statement, if we take $\alpha_r = \alpha$, then together with α and $-\alpha$, we see that $n \geq 2k+2$. The proof of the assertion is complete. □

As shown in Proposition 4.5, the existence of a pair of antipodal points $\{\alpha, -\alpha\}$ has implications on the size of a k-regular set Π. The next result together with Proposition 4.5 show that the lower bound in (4.1) is best possible.

PROPOSITION 4.6. *Let $k > 1$ and Π be a p element subset of Ω containing s pairs of antipodal points, where $p \geq 3$ and $s \geq 0$. If Π is 1-regular but not k-regular and $k \geq p - s$, then one can extend Π to a k-regular set by adding $2k+1-p$ or $2k+2-p$ elements, depending on whether s is zero.*

Proof. Assume $k \geq p - s$. Suppose first that $s > 0$. Let Π'' be a set containing $(k - p + s + 1)$ pairs of antipodal points such that $\Pi'' \cap \Pi$ is empty. Take

$$\Pi' = \Pi'' \cup -(\Pi \setminus \{\beta_1, -\beta_1, \ldots, \beta_s, -\beta_s\}).$$

Then Π' contains $(2k + 2 - p)$ elements. Furthermore, the set $\Pi \cup \Pi'$ contains exactly $k + 1$ pairs of antipodal points, and hence it is k-regular. Thus, the result follows if $s > 0$.

Next suppose $s = 0$. Without loss of generality, we may assume that $1 \in \Pi$. Hence, $-1 \in \Pi'$. We now modify Π'. We first delete the point -1 in Π'. Then for all other points $\alpha \in \Pi'$, we replace α by $e^{i\xi}\alpha$ if α lies in the upper open half plane $P = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and by $e^{-i\xi}\alpha$ if α lies in the lower open half plane $-P$ with sufficiently small $\xi > 0$. Then we see that for every $\alpha \in \Pi \cup \Pi'$, αP still contains exactly k elements. Thus, $\Pi \cup \Pi'$ is k-regular. Furthermore, the modified set Π' has one fewer point, i.e., Π' has only $2k + 1 - p$ elements. □

A referee pointed out that each $(k - 1)$-regular set can be enlarged to a k-regular set by adding in not more than two extra elements. The following result shows that sometimes two may not be the minimum number needed.

LEMMA 4.7. *Let $k > 1$ and Π be a subset of Ω containing at least one nonantipodal point. The following are equivalent:*

(a) *One can add a point $\beta \notin \Pi$ so that $\Pi \cup \{\beta\}$ is k-regular.*

(b) *One can delete a nonantipodal point $\gamma \in \Pi$ so that $\Pi \setminus \{\gamma\}$ is $(k - 1)$-regular.*

Here an element $\alpha \in \Pi$ is called a nonantipodal point of Π if $-\alpha \notin \Pi$.

Proof. Suppose first that (b) holds. Let $P = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. Without loss of generality, we may assume that $\gamma = 1$ is a nonantipodal point in Π. Suppose $\Pi \setminus \{\gamma\} = \{e^{i\theta_1}, \ldots, e^{i\theta_{p-1}}\}$ such that

$$0 < \theta_1 < \cdots < \theta_m < \pi < \theta_{m+1} < \cdots < \theta_{p-1} < 2\pi.$$

As $\Pi \setminus \{\gamma\}$ is $(k - 1)$-regular, by Proposition 4.5, $\Pi \setminus \{\gamma\}$ has $p - 1 \geq 2(k - 1) + 1$ elements. Therefore, for every $\alpha \in \Omega$, the open half plane αP contains at least $k - 1$ elements in $\Pi \setminus \{\gamma\}$, and either P or $-P$ contains at least k elements in $\Pi \setminus \{\gamma\}$. Hence, we have either $m = k - 1$ or $k \leq m \leq p - k$.

Choose $\beta = e^{i\theta}$, where

$$\theta = \begin{cases} \max\{\pi + \theta_m, \theta_{p-1}\}/2 & \text{if } m = k - 1, \\ \min\{2\pi + \theta_1, \pi + \theta_{m+1}\}/2 & \text{if } k \leq m \leq p - k. \end{cases}$$

Now for every $\alpha \neq \pm 1$, the open half plane αP contains at least $k - 1$ elements of $\Pi \setminus \{\gamma\}$ and either γ or β. Hence, αP contains at least k elements of $\Pi \cup \{\beta\}$. On the other hand, when $\alpha = \pm 1$, the open half plane αP contains either k elements of Π or $k - 1$ elements of Π and β. Again, αP contains at least k elements of $\Pi \cup \{\beta\}$. Thus, (a) holds.

Conversely, suppose (a) holds. If $-\beta \in \Pi$, then it is easy to see that the set $\Pi \setminus \{-\beta\}$ is $(k - 1)$-regular. From now onward, we assume that $-\beta \notin \Pi$. Without loss of generality, we may assume that $\beta = -1$. Furthermore, by replacing Π with the set $\{\bar{\xi} : \xi \in \Pi\}$, if necessary, we can assume that the number of elements in $\Pi \cap P$ is greater than or equal to the number of elements in $\Pi \cap (-P)$. Under this assumption, the upper open half plane must contain at least one nonantipodal point of Π.

Let γ be the nonantipodal point in Π such that $0 < \arg(\gamma) \leq \arg(\alpha)$ for all nonantipodal points $\alpha \in \Pi$. Then $\gamma \in P$. We show that $\Pi \setminus \{\gamma\}$ is $(k - 1)$-regular.

Take any $\alpha \in \Pi \setminus \{\gamma\}$. Suppose $\alpha \in \beta P \cup \gamma P$. Then the open half plane αP can contain at most one of points β and γ. As the open half plane αP contains at least k elements of $\Pi \cup \{\beta\}$, αP contains at least $k-1$ elements of $\Pi \setminus \{\gamma\}$. Thus, $\Pi \setminus \{\gamma\}$ is $(k-1)$-regular if $\Pi \setminus \{\gamma\} \subseteq \beta P \cup \gamma P$. Now suppose $(\Pi \setminus \{\gamma\}) \setminus (\beta P \cup \gamma P)$ is nonempty, and let $\omega_1, \ldots, \omega_t$ be the points in this set. Notice that all of them lie in the upper open half plane P. Therefore, we may assume that

$$0 < \arg(\omega_1) < \cdots < \arg(\omega_t) < \arg(\gamma) < \pi.$$

Also, by the choice of γ, $\omega_1, \ldots, \omega_t$ cannot be nonantipodal points, and hence the points $-\omega_1, \ldots, -\omega_t$ are in Π. Clearly each open half plane $\omega_j P$ contains at least k elements of $\Pi \cup \{\beta\}$. Notice that $(w_j P) \setminus P$ contains exactly j elements of $\Pi \cup \{\beta\}$, namely, $-\omega_1, \ldots, -\omega_{j-1}$ and β. Also the set $P \setminus (w_j P)$ contains exactly j elements of $\Pi \cup \{\beta\}$, namely, $\omega_1, \ldots, \omega_j$. It follows that the half plane $w_j P$ contains the same number of elements of $P \cup \{\beta\}$ as the upper half plane P. By Proposition 4.5, $\Pi \cup \{\beta\}$ contains at least $2k+2$ elements. Then, by assumption, the upper open half plane P contains at least $k+1$ elements of $\Pi \cup \{\beta\}$. Thus, every open half plane $w_j P$ contains at least $k+1$ elements of $\Pi \cup \{\beta\}$, and it contains at least $k-1$ elements of $\Pi \setminus \{\gamma\}$. Therefore, $\Pi \setminus \{\gamma\}$ is a $(k-1)$-regular set, and the assertion follows. □

Applying the above lemma inductively (repeatedly), we have the following.

PROPOSITION 4.8. *Let $k > 1$ and Π be a p element subset of Ω containing s pairs of antipodal points, where $p \geq 3$ and $s \geq 0$. Suppose $p > 2s$. For any positive $t \leq \min\{k, p-2s, p-1\}$, the following are equivalent:*

(a) *One can add t points $\beta_1, \ldots, \beta_t \notin \Pi$ so that $\Pi \cup \{\beta_1, \ldots, \beta_t\}$ is k-regular.*

(b) *One can delete t nonantipodal points $\gamma_1, \ldots, \gamma_t \in \Pi$ so that $\Pi \setminus \{\gamma_1, \ldots, \gamma_t\}$ is $(k-t)$-regular.*

Proof. Clearly the result holds for $t = 1$ by Lemma 4.7. Assume the statement holds for all $\ell < t$. Suppose $\Pi \cup \{\beta_1, \ldots, \beta_t\}$ is k-regular. Let $\Pi_1 = \Pi \cup \{\beta_1\}$. Then $\Pi_1 \cup \{\beta_2, \ldots, \beta_t\}$ is k-regular, and it follows from the assumption that one can find $t-1$ nonantipodal points $\gamma_1, \ldots, \gamma_{t-1} \in \Pi \cup \{\beta_1\}$ such that $\Pi_1 \setminus \{\gamma_1, \ldots, \gamma_{t-1}\}$ is $(k-t+1)$-regular. If $\beta_1 \notin \{\gamma_1, \ldots, \gamma_{t-1}\}$, by applying Lemma 4.7 to the set $(\Pi \setminus \{\gamma_1, \ldots, \gamma_{t-1}\}) \cup \{\beta_1\}$, one can find another nonantipodal point $\gamma_t \in \Pi_1 \setminus \{\gamma_1, \ldots, \gamma_{t-1}\}$ so that $\Pi \setminus \{\gamma_1, \ldots, \gamma_t\}$ is $(k-t)$-regular. On the other hand, if β_1 is one of the γ_j, say, $\beta_1 = \gamma_1$, then $\Pi \setminus \{\gamma_2, \ldots, \gamma_{t-1}\}$ is $(k-t+1)$-regular. In this case, take an arbitrary element $\gamma_t \in \Pi \setminus \{\gamma_2, \ldots, \gamma_{t-1}\}$, and apply Lemma 4.7 to the set $(\Pi \setminus \{\gamma_2, \ldots, \gamma_t\}) \cup \{\gamma_t\}$; one can find another nonantipodal point γ_{t+1} so that the set $\Pi \setminus \{\gamma_2, \ldots, \gamma_{t+1}\}$ is $(k-t)$-regular. Then (b) follows. The proof of (b) implying (a) can also be done by induction in a similar way. □

Suppose $k < p - s$. Given a p element subset Π of Ω containing s pairs of antipodal points, $\beta_1, -\beta_1, \ldots, \beta_s, -\beta_s$ with $s > 0$, which is not k-regular, the set obtained from Π by deleting all $p - 2s$ nonantipodal points is an $(s-1)$-regular set. On the other hand, if Π does not have any pair of antipodal points, then $k \leq p-1$, and one can always delete k elements to form a 0-regular set. In both cases, one sees that the following minimum always exists:

$$\begin{aligned} q = &\min\{t \in \mathbb{N} : \Pi \setminus \{\beta_1, -\beta_1, \ldots, \beta_s, -\beta_s\} \text{ has a } t\text{-element} \\ &\qquad\qquad \text{subset } T \text{ such that } \Pi \setminus T \text{ is } (k-t)\text{-regular}\}. \end{aligned}$$

By Proposition 4.8, one can always add this minimum number q of points to Π to form a k-regular set. Furthermore, this number q is optimal in the sense that one cannot add fewer than q elements to do so.

By definition, q is a positive integer bounded above by $\min\{k, p-2s\}$. The following proposition gives more information about the minimum value (4.2) in Theorem 4.1.

PROPOSITION 4.9. *Using the notation in Theorem* 4.1, *if $k < p-s$, then the value q in* (4.2) *exists and satisfies*

$$q \leq \begin{cases} k & \text{if } (p,s) = (k+1,0) \text{ or } (k+2,1), \\ \min\{k-1, p-2s\} & \text{otherwise.} \end{cases}$$

Also q is bounded below by $2k+1-p$ or $2k+2-p$, depending on whether s is zero. Furthermore, $q = 2k+1-p$ if $p \leq k+2$ with $s=0$, and $q = 2k+2-p$ if $p \leq k+3$ with $s > 0$.

Proof. The lower bound can be seen easily from Proposition 4.5. Also the case when $(p,s) = (k+1,0)$ or $(k+2,1)$ has already been discussed. Now we assume that $(p,s) \notin \{(k+1,0),(k+2,1)\}$. Consider the case when $s \geq 2$. Take $t = \min\{k-1, p-2s\}$, and delete t nonantipodal elements in Π. Then the resulting set is $(s-1)$-regular and hence $(k-t)$-regular as $k - t = \max\{1, k-p+2s\} \leq s-1$. Thus, $q \leq t$.

Next we consider the case when $s = 1$ and $p \geq k+3$. Let $\{\alpha, -\alpha\}$ be the pair of antipodal points in Π. Since Π is 1-regular, there are $\alpha_1 \in (\alpha, -\alpha) \cap \Pi$ and $\alpha_2 \in (-\alpha, \alpha) \cap \Pi$. Pick another $k-1$ nonantipodal points $\alpha_3, \ldots, \alpha_{k+1}$ in Π. The set $\Pi \setminus \{\alpha_3, \ldots, \alpha_{k+1}\}$ containing $\{\alpha_1, \alpha_2, \alpha, -\alpha\}$ is 1-regular. Then $q \leq k-1$.

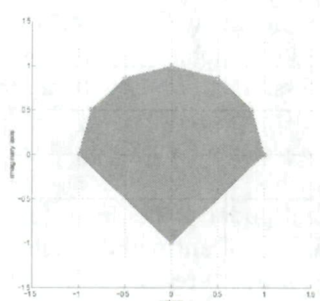
Finally consider the case when $s = 0$ and $p \geq k+2$. We may assume that $\Pi = \{e^{i\xi_j} : 1 \leq j \leq p\}$ with $0 = \xi_1 < \cdots < \xi_p < 2\pi$. Since Π is 1-regular, we can choose ℓ such that $\xi_\ell = \max\{\xi_j : 0 < \xi_j < \pi\}$. Then $S = \{\xi_1, \xi_\ell, \xi_{\ell+1}\}$ is 1-regular. Then any $p-k+1$ subset of Π containing S is 1-regular. Thus, $q \leq k-1$. □

Now we are ready to present the following proof.

Proof of Theorems 4.1 *and* 4.4. The assertions on q and n follow by Propositions 4.6 and 4.8. For the last assertion in Theorem 4.1, we see that, in order to get a k-regular set by adding q points to Π, we need to add at least $k-1$ points $e^{i\xi}$ with $0 < \xi < \pi$. If $2k+2-p > k-1$, then $p - 3 < k$, and we need to add extra $k-(p-3)$ points $e^{i\xi}$ with $\pi < \xi < 2\pi$, giving a total of $k-1+k-(p-3) = 2k+2-p$ points. This proves the equality in (4.3). The equality in (4.4) now follows readily. □

To close this section, let us illustrate our results by the following example.

Example 4.10. Let the polygon $\mathcal{P} = \operatorname{conv}\{1, w, w^2, w^3, w^4, w^5, w^6, w^9\}$ with $w = e^{2\pi i/12}$; see Figure 4.



The polygon $\mathcal{P}$

FIG. 4.

Then $\mathcal{P} = \bigcap_{j=1}^{8} \mathcal{H}(d_j, \xi_j)$ with $d_1 = \cdots = d_6 = \cos\frac{\pi}{12}$, $d_7 = d_8 = \cos\frac{\pi}{4}$, and

$$(\xi_1, \ldots, \xi_8) = \left(\frac{\pi}{12}, \frac{3\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \frac{11\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}\right).$$

Thus,

$$\Pi = \{\alpha_1, \ldots, \alpha_8\} = \left\{e^{\frac{\pi i}{12}}, e^{\frac{3\pi i}{12}}, e^{\frac{5\pi i}{12}}, e^{\frac{7\pi i}{12}}, e^{\frac{9\pi i}{12}}, e^{\frac{11\pi i}{12}}, e^{\frac{15\pi i}{12}}, e^{\frac{21\pi i}{12}}\right\}.$$

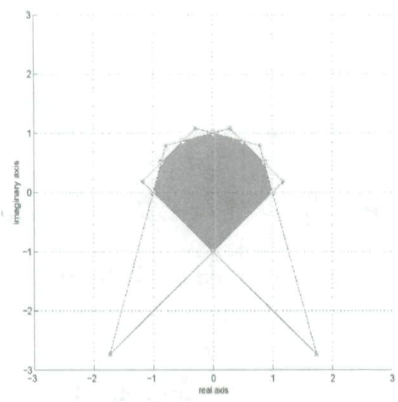
In particular, Π has two pairs of antipodal points, namely, $\{e^{\frac{3\pi i}{12}}, e^{\frac{15\pi i}{12}}\}$ and $\{e^{\frac{9\pi i}{12}}, e^{\frac{21\pi i}{12}}\}$, i.e., $p = 8$ and $s = 2$. By Theorem 4.4 and Proposition 4.9, for $k \geq 5$, a $(2k+2) \times (2k+2)$ normal matrix A can be constructed so that $\Lambda_k(A) = \mathcal{P}$.

It remains to consider the cases for $k \leq 4$. Clearly Π is 2-regular. Thus, an 8×8 normal matrix A_2 can be constructed so that $\Lambda_2(A_2) = \mathcal{P}$. However, Π is not k-regular for $k \geq 3$.

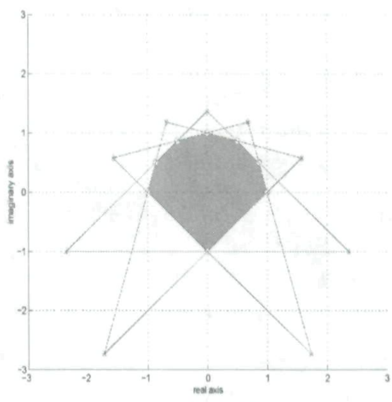
Now we consider the case $k = 3$. Clearly $\Pi \setminus \{e^{\frac{5\pi i}{12}}\}$ is 2-regular. Then Theorem 4.4 shows that there is a 9×9 normal matrix A_3 such that $\Lambda_3(A_3) = \mathcal{P}$. Indeed, following the proof of Lemma 4.7, we see that if $\Pi' = \{e^{\frac{18\pi i}{12}}\}$, $\Pi \cup \Pi'$ is 3-regular.

Finally we turn to the case when $k = 4$. Notice that $\Pi \setminus \{e^{\frac{5\pi i}{12}}, e^{\frac{7\pi i}{12}}\}$ is 2-regular. Thus, Theorem 4.4 shows that there is a 10×10 normal matrix A_4 such that $\Lambda_4(A_4) = \mathcal{P}$.

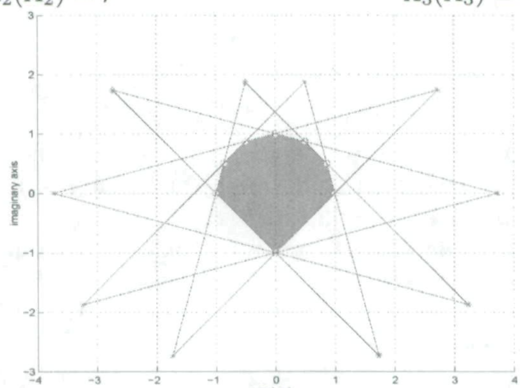
In Figure 5, we display the higher rank numerical ranges of A_2, A_3, and A_4. In the figure, the points "o" correspond to the vertices of the polygon, while the points "$*$" correspond to the eigenvalues of the normal matrices.



$\Lambda_2(A_2) = \mathcal{P}$



$\Lambda_3(A_3) = \mathcal{P}$



$\Lambda_4(A_4) = \mathcal{P}$

FIG. 5.

5. An algorithm. In this section, we further present a detailed procedure for constructing the rank-k numerical ranges of normal matrices based on the discussion in section 2.

Given a normal matrix A with m distinct eigenvalues $a_1, \ldots, a_m$, one can easily construct $\Lambda_k(A)$ through the following algorithms.

BASIC ALGORITHM. First construct the set $\mathcal{S}_0$. For each ordered pair (r, s) with $r < s$, count the number of eigenvalues of A (counting multiplicities) in the open planes $H_0(a_r, a_s)$ and $H_0(a_s, a_r)$.

1. If $H_0(a_r, a_s)$ has at most $n-k-1$ eigenvalues while $H_0(a_s, a_r)$ has at most $k-1$ eigenvalues, then collect the index pair (r, s) in $\mathcal{S}_0$.
2. If $H_0(a_s, a_r)$ has at most $n-k-1$ eigenvalues while $H_0(a_r, a_s)$ has at most $k-1$ eigenvalues, then collect the index pair (s, r) in $\mathcal{S}_0$.

Notice that one can already construct $\Lambda_k(A)$ by determining the intersection of all the half planes $H(a_r, a_s)$ with $(r, s) \in \mathcal{S}_0$. Nevertheless, one can perform the following additional steps to simplify the set $\mathcal{S}_0$ before constructing $\Lambda_k(A)$.

MODIFIED ALGORITHM 1. Suppose in the basic algorithm that there is an index pair (p, q) satisfying both (1) and (2), i.e., both pairs (p, q) and (q, p) are in $\mathcal{S}_0$. Then $\Lambda_k(A)$ is a subset of a line segment. In this case, $\Lambda_k(A)$ can be constructed as follows.

Set $\hat{a}_j = (a_j - a_p)/(a_q - a_p)$, and define $\mathcal{S}_1 = \{(r, s) \in \mathcal{S}_0 : \operatorname{Im}(\hat{a}_r) \neq \operatorname{Im}(\hat{a}_s)\}$. If $\mathcal{S}_1 = \emptyset$, then $\Lambda_k(A) = \emptyset$. Suppose $\mathcal{S}_1 \neq \emptyset$. For each $(r, s) \in \mathcal{S}_1$, compute

$$b_{rs} = \frac{\operatorname{Im}(\hat{a}_r)\operatorname{Re}(\hat{a}_s) - \operatorname{Im}(\hat{a}_s)\operatorname{Re}(\hat{a}_r)}{\operatorname{Im}(\hat{a}_r) - \operatorname{Im}(\hat{a}_s)}.$$

Take

$$\begin{aligned} b_1 &= \max\{b_{rs} : (r, s) \in \mathcal{S}_1,\ \operatorname{Im}(\hat{a}_r) \geq 0 \text{ and } \operatorname{Im}(\hat{a}_s) \leq 0\}, \\ b_2 &= \min\{b_{rs} : (r, s) \in \mathcal{S}_1,\ \operatorname{Im}(\hat{a}_r) \leq 0 \text{ and } \operatorname{Im}(\hat{a}_s) \geq 0\}. \end{aligned}$$

Then $\Lambda_k(A)$ is the line segment in $\mathbb{C}$ joining the points $(a_q - a_p)b_1 + a_p$ and $(a_q - a_p)b_2 + a_p$ if $b_1 \leq b_2$; otherwise, $\Lambda_k(A) = \emptyset$.

MODIFIED ALGORITHM 2. Assume the situation mentioned in Modified Algorithm 1 does not hold. Check if the set $\mathcal{S}_0$ satisfies the following:

There are $(r_1, s_1), \ldots, (r_\ell, s_\ell) \in \mathcal{S}_0$ with $\ell \geq 3$ such that

$$\{r_1, s_1\} \cap \{r_2, s_2\} \cap \cdots \cap \{r_\ell, s_\ell\} = \{t\} \quad \text{for some } 1 \leq t \leq m. \tag{5.1}$$

If yes, define

$$\theta_j = \begin{cases} \arg(a_{s_j} - t) & \text{if } r_j = t, \\ \arg(t - a_{r_j}) & \text{if } s_j = t. \end{cases}$$

Relabel the indices so that $0 \leq \theta_1 \leq \cdots \leq \theta_\ell < 2\pi$. Consider the following three cases:

1. If $\theta_\ell - \theta_1 < \pi$, remove all pairs (r_j, s_j) in $\mathcal{S}_0$ for $j \neq 1, \ell$. Then check again whether the modified set still satisfies (5.1).
2. If $\theta_{k+1} - \theta_k > \pi$ for some k, remove all pairs (r_j, s_j) in $\mathcal{S}_0$ for $j \neq k, k+1$. Then check again whether the modified set still satisfies (5.1).
3. If the above two items are not satisfied, then $\Lambda_k(A)$ is either the empty set or the singleton set $\{a_t\}$. In this case, check whether a_t lies in $H(a_r, a_s)$ for all $(r, s) \in \mathcal{S}_0$. If yes, $\Lambda_k(A)$ is the singleton set; otherwise, it is the empty set.

Finally, if the modified set $\mathcal{S}_0$ does not satisfy (5.1), then one can construct $\Lambda_k(A)$ by determining the intersection of all the half planes $H(a_r, a_s)$ with (r, s) in the modified set $\mathcal{S}_0$.

Acknowledgments. This research began at the 2008 Institute for Mathematics and Its Applications (IMA) PI Summer Program for Graduate Students, where the second author is a lecturer and the third author is a coorganizer. The support of IMA and the National Science Foundation for the program is graciously acknowledged. The hospitality of the colleagues at Iowa State University is deeply appreciated. The authors would also like to thank the referees for some helpful comments.

REFERENCES

[1] M. D. Choi, M. Giesinger, J. A. Holbrook, and D. W. Kribs, *Geometry of higher-rank numerical ranges*, Linear Multilinear Algebra, 56 (2008), pp. 53–64.

[2] M. D. Choi, J. A. Holbrook, D. W. Kribs, and K. Życzkowski, *Higher-rank numerical ranges of unitary and normal matrices*, Oper. Matrices, 1 (2007), pp. 409–426.

[3] M. D. Choi, D. W. Kribs, and K. Życzkowski, *Higher-rank numerical ranges and compression problems*, Linear Algebra Appl., 418 (2006), pp. 828–839.

[4] M. D. Choi, D. W. Kribs, and K. Życzkowski, *Quantum error correcting codes from the compression formalism*, Rep. Math. Phys., 58 (2006), pp. 77–91.

[5] H. L. Gau, C. K. Li, and P. Y. Wu, *Higher-rank numerical ranges and dilations*, J. Operator Theory, 63 (2010), pp. 181–189.

[6] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[7] E. Knill and R. Laflamme, *Theory of quantum error-correcting codes*, Phys. Rev. A, 55 (1997), pp. 900–911.

[8] E. Knill, R. Laflamme, and L. Viola, *Theory of quantum error correction for general noise*, Phys. Rev. Lett., 84 (2000), 2525–2528.

[9] D. W. Kribs, R. Laflamme, D. Poulin, and M. Lesosky, *Operator quantum error correction*, Quant. Inf. Comput., 6 (2006), pp. 383–399.

[10] C. K. Li and Y. T. Poon, *Generalized numerical ranges and quantum error correction*, J. Operator Theory, to appear.

[11] C. K. Li, Y. T. Poon, and N. S. Sze, *Higher rank numerical ranges and low rank perturbation of quantum channels*, J. Math. Anal. Appl., 348 (2008), pp. 843–855.

[12] C. K. Li, Y. T. Poon, and N. S. Sze, *Condition for the higher rank numerical range to be non-empty*, Linear Multilinear Algebra, 57 (2009), pp. 365–368.

[13] C. K. Li and N. S. Sze, *Canonical forms, higher rank numerical ranges, totally isotropic subspaces, and matrix equations*, Proc. Amer. Math. Soc., 136 (2008), pp. 3013–3023.

[14] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge, University Press, New York, 2000.

[15] H. Woerdeman, *The higher rank numerical range is convex*, Linear Multilinear Algebra, 56 (2008), pp. 65–67.