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SWITCHING GAMES OF STOCHASTIC DIFFERENTIAL SYSTEMS*

SHANJIAN TANG† AND SHUI-HUNG HOU‡

Abstract. A two-player, zero-sum, switching game is formulated for general stochastic differential systems and is studied using a combined dynamic programming and viscosity solution approach. The existence of the game value is proved. For the proof of the related dynamic programming principle (DDP) for the lower and upper value functions, the measurability problem, of the same kind as mentioned in the paper of Fleming and Souganidis, is also encountered, and we are able to get around it via a delicate adaptation of their technique. Moreover, the traditional direct method to prove the time continuity of lower and upper value functions also gives rise to a serious measurability problem. To get around the new difficulty, a subtle dynamic programming argument is developed to obtain the time continuity, which in return is used to derive the DDP for random intermediate times from the DDP with deterministic intermediate times.

Key words. stochastic differential games, dynamic programming inequalities, switching strategies, value function, viscosity solution

AMS subject classifications. 49N70, 49L25, 60H30, 49L20, 90C39, 93E20

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1. Introduction.

Consider a differential game of the following stochastic differential system on Wiener space $(\Omega, \mathcal{F}, P)$:

$$\begin{cases} dy(t) = f(t, y(t), a(t), b(t))\, dt + g(t, y(t), a(t), b(t))\, dw(t), & t \in (s, 1], \\ y(s) = x \end{cases} \tag{1.1}$$

with the cost functional

$$\begin{aligned} J_{s,x}(a(\cdot), b(\cdot)) = E_{sx}\bigg\{ \int_s^1 f^0(t, y(t), a(t), b(t))\, dt + h(y(1)) \\ + \sum_{i\geq 1} k(\theta_i, a_{i-1}, a_i) - \sum_{j\geq 1} l(\tau_j, b_{j-1}, b_j) \bigg\}. \end{aligned} \tag{1.2}$$

Here f, g, f^0, and h are given maps; $w(\cdot)$ is the coordinate process in Ω, and its natural filtration is denoted by $\mathcal{F}_t$. The subscript sx of the expectation operator E indicates that the underlying mathematical expectation is taken under the condition that the underlying system state process $y(\cdot)$ takes the value x at time s. The first player chooses the control a from a given finite set A to minimize the payoff (1.2), and each

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†Department of Finance and Control Sciences, School of Mathematical Sciences, Fudan University, and Key Laboratory of Mathematics for Nonlinear Sciences (Fudan University), Ministry of Education, Shanghai 200433, China (sjtang@fudan.edu.cn).

‡Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong (mahough@polyu.edu.hk).

of his instantaneous actions is related with one positive cost k, while the second player chooses the control b from a given finite set B to maximize the payoff (1.2), and each of his instantaneous actions is associated with the other positive cost l.

For differential switching games, a key point in connecting value functions with the corresponding Isaacs' equations is to prove the following fact: It is the best way for a player to keep his underlying switching position for some time of a positive length, whenever he is not on his switching set. In the deterministic case, such an assertion is easy to understand from the following almost obvious fact: If he is not on his switching set, a player will keep away from the set for some deterministic time interval of a positive length, as the system state evolves continuously. See Yong [10] for details. In the stochastic case, the situation becomes complicated due to the nature of diffusion: Even if he is not on his switching set, it is possible for a player to arrive at his switching set in an arbitrarily short time. That is, if he is not on his switching set, although the system state still evolves continuously, a player can keep away from the switching set only for some *random* (rather than deterministic, in general) time interval, *almost surely* (rather than uniformly, in general) of a positive length. Then the intuition of the dynamic programming principle for the underlying switching game suggests that if he is not on his switching set, the optimal action of a player has to vary with different events, even within a very short deterministic time period. We show in section 3 by using arguments quite different from the deterministic case that, whenever he is not on his switching set, a player's best action is to keep his underlying switching position, *before he escapes from a sufficiently small ball centered at the current state*, within some deterministic time interval of a positive length.

It has been widely recognized that the dynamic programming method is both easy and efficient for the study of deterministic optimal control and differential games within the framework of viscosity solutions. The general nonsmooth feature of inf-sup functions is no longer a difficulty in view of the notion of viscosity solutions. However, applications of dynamic programming to optimal stochastic controls and stochastic differential games still encounter difficulties; the reader is referred to Bertsekas and Shreve [1] and Fleming and Souganidis [6] for detail. It was noticed by Fleming and Souganidis [6], in the study of classical stochastic differential games, that the conventional proof of the dynamic programming principle for the lower and upper value functions encounters a serious measurability issue. In this paper, we observe that the traditional direct approach to show the time continuity of lower and upper value functions also gives rise to a serious measurability problem. The difficulty is circulated using a dynamic programming argument.

In this paper, the coefficients of differential games are allowed to grow linearly, and a powerful simple test function is given to prove the uniqueness of unbounded viscosity solutions for the associated Isaacs' system of variational inequalities.

The rest of our paper is organized as follows. Section 2 is devoted to the formulation of our stochastic switching game, the definitions of some restrictive class of admissible controls, and strategies to be used in the following sections. Several useful dynamic programming results and the time continuous properties are established in section 3. The existence of the value is proved for our game in section 4.

There are some related papers which remain to be mentioned. For the optimal switching problem, the reader is referred to Capuzzo Dolcetta and Evans in [2] in the deterministic case, and to Evans and Friedman [4], Tang and Yong [7] and the references therein in the stochastic case. For the switching game, the reader is referred to Yong [10] in the deterministic case with the dynamic programming approach and the notion of viscosity solution, and to Yamada [8, 9] in the stochastic and infinite time-horizon case with an analytical approach rather than the dynamic programming approach.

2. Preliminaries. Let $A = \{1, 2, \ldots, m\}, B = \{1, 2, \ldots, n\}$, and X be a finite-dimensional Euclidean space. Let $f : [0, 1] \times X \times A \times B \to X, g : [0, 1] \times X \times A \times B \to X \times W, f^0 : [0, 1] \times X \times A \times B \to \mathbb{R}, k : [0, 1] \times A \times A \to \mathbb{R}_+ \equiv [0, \infty)$, and $l : [0, 1] \times B \times B \to \mathbb{R}_+$ be continuous functions satisfying the following hypotheses.

Hypothesis 1. There exists a constant $L > 0$ such that for all $x, \widehat{x} \in X, t \in [0, 1], a \in A$, and $b \in B$,

$$
\begin{aligned}
&|f(t, x, a, b) - f(t, \widehat{x}, a, b)| + |g(t, x, a, b) - g(t, \widehat{x}, a, b)| \leq L|x - \widehat{x}|,\\
&|f(t, x, a, b)| + |g(t, x, a, b)| \leq L(1 + |x|),\\
&|f^0(t, x, a, b) - f^0(t, \widehat{x}, a, b)| + |h(x) - h(\widehat{x})| \leq L|x - \widehat{x}|,\\
&|f^0(t, 0, a, b)| + |h(0)| \leq L.
\end{aligned}
$$

Hypothesis 2. For all $a, \widehat{a}, \widetilde{a} \in A, a \neq \widehat{a} \neq \widetilde{a}$, and $0 \leq s \leq t \leq 1$,

$$
\begin{aligned}
&k(t, a, \widetilde{a}) < k(t, a, \widehat{a}) + k(t, \widehat{a}, \widetilde{a}),\\
&k(t, a, \widehat{a}) > 0, k(t, a, a) = 0,\\
&k(t, a, \widetilde{a}) \leq k(s, a, \widetilde{a}).
\end{aligned}
$$

Hypothesis 3. For all $b, \widehat{b}, \widetilde{b} \in B, b \neq \widehat{b} \neq \widetilde{b}$, and $0 \leq s \leq t \leq 1$,

$$
\begin{aligned}
&l(t, b, \widetilde{b}) < l(t, b, \widehat{b}) + l(t, \widehat{b}, \widetilde{b}),\\
&l(t, b, \widehat{b}) > 0, l(t, b, b) = 0,\\
&l(t, b, \widetilde{b}) \leq l(s, b, \widetilde{b}).
\end{aligned}
$$

For $s, \hat{s} \in [0, 1]$ such that $s < \hat{s}$, let

$$\Omega_{s,\hat{s}} = \{\omega \in C([s, \hat{s}]; \mathbb{R}^d) : \omega(s) = 0\}. \tag{2.1}$$

Denote by $\mathcal{F}_{s,\hat{s}}$ the topological σ-field of $\Omega_{s,\hat{s}}$ and consider the Wiener space $(\Omega_{s,\hat{s}}, \mathcal{F}_{s,\hat{s}}, P_{s,\hat{s}})$. Let

$$\Omega_s = \Omega_{s,1}, \quad P_s = P_{s,1}, \quad \mathcal{F}_s = \mathcal{F}_{s,1}, \tag{2.2}$$

and

$$
\begin{cases}
\omega_1 = \omega|_{[s,\hat{s}]},\\
\omega_2 = (\omega - \omega(\hat{s}))|_{[\hat{s},1]},\\
\Pi\omega = (\omega_1, \omega_2).
\end{cases} \tag{2.3}
$$

We see that the map $\Pi : \Omega_s \to \Omega_{s,\hat{s}} \times \Omega_{\hat{s}}$ induces an identification

$$\Omega_s = \Omega_{s,\hat{s}} \times \Omega_{\hat{s}}. \tag{2.4}$$

Moreover, the inverse of Π is defined in an evident way: $\Omega_s = \Pi^{-1}(\Omega_{s,\hat{s}}, \Omega_{\hat{s}})$. Finally, we have

$$P_s = P_{s,\hat{s}} \otimes P_{\hat{s}}.$$

Define

$$w(r,\omega)=\omega(r), \quad (\omega,r)\in\Omega_s\times[s,1]. \tag{2.5}$$

Then $\{w(r), r\in[s,1]\}$ is a standard Wiener process.

DEFINITION 2.1. *An admissible switching process for player* I *(resp.,* II*) on* $[s,1]$ *with initial value* a_0 *(resp.,* b_0*) is defined to be a pair of sequences* $\{a_i,\theta_i\}_{i\geq 0}$ *(resp.,* $\{b_i,\tau_i\}_{i\geq 0}$*) such that each* θ_i *(resp.,* τ_i*) is an* $\mathcal{F}_{s,\cdot}$*-stopping time, with*

$$s=\theta_0\leq\theta_1\leq\cdots\leq 1 \quad \text{a.s.}$$
$$(\text{resp., } s=\tau_0\leq\tau_1\leq\cdots\leq 1 \quad \text{a.s.}),$$

each a_i *(resp.,* b_i*) is* $\mathcal{F}_{s,\theta_i}$*- (resp.,* $\mathcal{F}_{s,\tau_i}$*-) measurable with values in* A *(resp.,* B*), and*

$$E\sum_{i\geq 1}k(\theta_i,a_{i-1},a_i)<\infty \quad \left(\text{resp., } E\sum_{j\geq 1}l(\tau_j,b_{j-1},b_j)<\infty\right).$$

Denote by $\mathcal{A}^a[s,\hat{s}]$ *(resp.,* $\mathcal{B}^b[s,\hat{s}]$*) the totality of the admissible switchings for player* I *(resp.,* II*) on* $[s,\hat{s}]$ *with the initial value* a *(resp.,* b*).*

We shall identify $\{a_i,\theta_i\}_{i\geq 0}\in\mathcal{A}^a[s,1]$ with

$$a(r)=\sum_{i\geq 1}a_{i-1}\chi_{[\theta_{i-1},\theta_i)}(r), \quad r\in[s,1].$$

Note that in the case of $\theta_1=\theta_2$ the term $a_1\chi_{[\theta_1,\theta_2)}(r)$ will be void, but we still keep it in the above expression. This is due to the fact that the sequence $\{a_i,\theta_i\}$ with or without (a_1,θ_1) represents two different switching controls and their costs are different. A similar identification will also be used for $\{b_i,\tau_i\}\in\mathcal{B}^b[t,1]$.

Following Elliott and Kalton [3] and Fleming and Souganidis [6], we define in the switching game an admissible strategy as follows.

DEFINITION 2.2. *For* $s\in[0,1]$ *and* $a\in A$ *(resp.,* $b\in B$*), an admissible strategy* $\alpha^{a,s}$ *(resp.,* $\beta^{b,s}$*) with the initial value* a *(resp.,* b*) for player* I *(resp.,* II*) on* $[s,1]$ *is a mapping* $\alpha^{a,s}:\cup_{b\in B}\mathcal{B}^b[s,1]\to\mathcal{A}^a[s,1]$ *(resp.,* $\beta^{b,s}:\cup_{a\in A}\mathcal{A}^a[s,1]\to\mathcal{B}^b[s,1]$*) such that*

$$b(r)=\widehat{b}(r) \quad (\text{resp., } a(r)=\widehat{a}(r)) \quad \text{a.s. } \forall r\in[s,\hat{s}]$$

implies

$$\alpha^{a,t}[b(\cdot)](r)=\alpha^{a,t}[\widehat{b}(\cdot)](r) \quad (\text{resp., } \beta^{b,t}[a(\cdot)](r)=\beta^{b,t}[\widehat{a}(\cdot)](r))$$

for $r\in[s,\hat{s}]$.

We denote all admissible strategies with the initial value a (resp., b) for player I (resp., II) on $[s,1]$ by $\Gamma^a[s,1]$ (resp., $\Delta^b[s,1]$). We adopt the convention that

$$\begin{aligned}&\mathcal{A}^a[1,1]=a, \quad \Gamma^a[1,1]=a,\\ &\mathcal{B}^b[1,1]=b, \quad \Delta^b[1,1]=b.\end{aligned} \tag{2.6}$$

Set for $(s,x)\in[0,1]\times X$,

(2.7)

$$V_{a,b}(s,x)=\inf_{\alpha\in\Gamma^a[s,1]}\sup_{b(\cdot)\in\mathcal{B}^b[s,1]}J_{s,x}(\alpha(b(\cdot)),b(\cdot)), \quad V(s,x)=(V_{a,b}(s,x))_{a\in A,b\in B};$$

$$U^{a,b}(s,x)=\sup_{\beta\in\Delta^b[s,1]}\inf_{a(\cdot)\in\mathcal{A}^a[s,1]}J_{s,x}(a(\cdot),\beta(a(\cdot))), \quad U(s,x)=(U^{a,b}(s,x))_{a\in A,b\in B}.$$

The matrix-valued functions V and U are called the lower and the upper value function, respectively. If $V = U$, we say that the above stochastic switching game has a value. Our aim is to study the existence of the value of our stochastic switching game. U and V should satisfy the dynamic programming principle. However, the conventional proof leads to serious technical problems related to measurability issues, which have been noticed by Fleming and Souganidis [6] in the study of classical stochastic differential games . To circumvent these problems, we borrow the techniques of Fleming and Souganidis [6] and introduce in the following the concepts of restrictive class of admissible strategies, π-admissible switching processes, and π-admissible strategies.

Consider $s \in [0, 1], \hat{s} \in (s, 1)$, and $b(\cdot) \in \mathcal{B}^b[s, 1]$. For $P_{s,\hat{s}}$-a.s. $\omega_1 \in \Omega_{s,\hat{s}}$, the map $b(\omega_1) : [\hat{s}, 1] \times \Omega_{\hat{s}} \to B$ defined by

$$b(\omega_1)(\omega_2)(r) = b(\omega_1, \omega_2)(r), \quad r \in [\hat{s}, 1],$$

is an admissible control for player II, i.e., $b(\omega_1) \in \mathcal{B}^{b(\hat{s})}[\hat{s}, 1]$.

DEFINITION 2.3. *Let $\alpha \in \Gamma^a[\hat{s}, 1]$. If for $\forall s \in (0, \hat{s})$ and $\forall b(\cdot) \in \mathcal{B}^b[s, 1], b \in B$, the map $(\tau, \omega) \mapsto \alpha[b(\omega_1)](\omega_2)(\tau)$ is $\mathcal{B}([\hat{s}, \tau]) \otimes \mathcal{F}_{s,\tau}$-measurable for every $\tau \in [\hat{s}, 1]$, then α is called an r-strategy with initial value a for player* I *on $[s, 1]$. The set of r-strategies with initial value a on $[s, 1]$ of player* I *is denoted by $\Gamma_1^a[s, 1]$.*

Similarly, we define r-strategies with initial value $b \in B$ on $[s, 1]$ for player II and denote their collection by $\Delta_1^b[s, 1]$.

Set

$$V_{a,b}^1(s, x) = \inf_{\alpha \in \Gamma_1^a[s,1]} \sup_{b(\cdot) \in \mathcal{B}^b[s,1]} J_{s,x}(\alpha(b(\cdot)), b(\cdot)), \quad V^1(s, x) = (V_{a,b}^1(s, x))_{a \in A, b \in B};$$
$$U_1^{a,b}(s, x) = \sup_{\beta \in \Delta_1^b[s,1]} \inf_{a(\cdot) \in \mathcal{A}^a[s,1]} J_{s,x}(a(\cdot), \beta(a(\cdot))), \quad U_1(s, x) = (U_1^{a,b}(s, x))_{a \in A, b \in B}. \tag{2.8}$$

The matrix-valued functions V^1 and U_1 are called the r-lower and the r-upper value function, respectively.

Let $\pi_s = \{s = t_0 < t_1 < \cdots < t_M = 1\}$ be a partition of $[s, 1]$, and denote by $||\pi_s|| = \max_{1 \le i \le M}(t_i - t_{i-1})$ its mesh. The notions of π-admissible switching processes and π-admissible strategies are then defined as follows.

DEFINITION 2.4. *Let $a(\cdot) = \{a_i, \theta_i\}_{i \ge 0} \in \mathcal{A}^a[s, 1]$. If each θ_i is a π_s-valued stopping time, then it is called a π-admissible switching process with initial value $a \in A$ on $[s, 1]$ for player* I*. The set of π-admissible switching processes with initial value $a \in A$ on $[s, 1]$ for player* I *is denoted by $\mathcal{A}_\pi^a[s, 1]$. The π-admissible switching processes with initial value $b \in B$ on $[s, 1]$ for player* II *are defined in a similar way, and their collection is denoted by $\mathcal{B}_\pi^b[s, 1]$.*

DEFINITION 2.5. $\alpha \in \Gamma^a[s, 1]$ *is called a π-admissible strategy with initial value $a \in A$ on $[s, 1]$ for player* I*, if it satisfies the following properties:* (1) $\forall b(\cdot) \in \mathcal{B}^b[s, 1], b \in B$, $\alpha[b(\cdot)] \in \mathcal{A}_\pi^a[s, 1]$. (2) *Fix $b \in B$. If $s \in [t_{i_0}, t_{i_0+1})$, then $\alpha[b_1(\cdot)]|_{[s,t_{i_0+1})} = \alpha[b_2(\cdot)]|_{[s,t_{i_0+1})}$ $\forall b_1(\cdot), b_2(\cdot) \in \mathcal{B}^b[s, 1]$.* (3) *If $b(\cdot) = \bar{b}(\cdot)$ on $[s, t_i)$, then $\alpha[b(\cdot)](t_i) = \alpha[\bar{b}(\cdot)](t_i), P_s$-a.s. for $i \in \{i_0 + 1, \ldots, M\}$. The collection of π-admissible strategies with initial value $a \in A$ on $[s, 1]$ for player I is denoted by $\Gamma_\pi^a[s, 1]$. The π-admissible strategies with initial value $b \in B$ on $[s, 1]$ for player* II *are defined in a similar way, and their collection is denoted by $\Delta_\pi^b[s, 1]$.*

It is crucial, in our case of the switching game, that $\alpha[b(\cdot)](t_i)$ in Definition 2.5 is required to be independent of $b(t_i)$ for $\alpha \in \Gamma_\pi^a[s, 1]$ and $i = i_0 + 1, \ldots, M$. Definition 2.5 differs from Fleming and Souganidis' in that $\alpha[b(\cdot)]|_{[s,t_{i_0+1})}$ may depend on $b(s-)$

even if $\alpha \in \Gamma_\pi^a[s,1]$, and this is due to the fact that the initial position of a player is crucial in our switching game.

According to the definitions of $V^1_{a,b}(s,x), U_1^{a,b}(s,x), V_{a,b}(s,x)$, and $U^{a,b}(s,x)$, we have immediately the following two relations:

$$V_{a,b}(s,x) \le V^1_{a,b}(s,x) \text{ and } U_1^{a,b}(s,x) \le U^{a,b}(s,x). \tag{2.9}$$

Next, let us introduce some operators. For any $m \times n$ matrix-valued function $W(\cdot,\cdot) = (W^{a,b}(\cdot,\cdot))$ defined on $[0,1]\times X$, we define for $(a,b,s,x) \in A \times B \times [0,1] \times X$

$$\begin{aligned} M^{a,b}[W](s,x) &= \min_{\hat{a}\ne a}\{W^{\hat{a},b}(s,x) + k(s,a,\hat{a})\}, \\ M_{a,b}[W](s,x) &= \max_{\hat{b}\ne b}\{W^{a,\hat{b}}(s,x) - l(s,b,\hat{b})\}. \end{aligned} \tag{2.10}$$

The two operators are called obstacle operators. According to the definitions, for any $(a,b,s,x) \in A \times B \times [0,1] \times X$, the following are true:

$$\begin{aligned} M_{a,b}[V](s,x) &\le V_{a,b}(s,x) \le M^{a,b}[V](s,x), \\ M_{a,b}[V^1](s,x) &\le V^1_{a,b}(s,x) \le M^{a,b}[V^1](s,x), \\ M_{a,b}[U](s,x) &\le U^{a,b}(s,x) \le M^{a,b}[U](s,x), \\ M_{a,b}[U_1](s,x) &\le U_1^{a,b}(s,x) \le M^{a,b}[U_1](s,x). \end{aligned} \tag{2.11}$$

Before closing this section, we state without proof the following result on the continuity in the space variable of the costs and the value functions.

PROPOSITION 2.1. (1) *For any* $a(\cdot) \in \mathcal{A}^a[s,1], b(\cdot) \in \mathcal{B}^b[s,1], \alpha \in \Gamma^a[s,1]$, *and* $\beta \in \Delta^b[s,1]$, *the functions* $J_{sx}(\alpha[b(\cdot)], b(\cdot))$ *and* $J_{sx}(a(\cdot), \beta[a(\cdot)])$, $(s,x) \in [0,T]\times X$, *grow linearly, are Lipschitz continuous in the space variable* x, *and are Hölder-continuous in the time variable* s, *uniformly with respect to the other variable* s *and* x, *respectively, and uniformly as well with respect to the four parameters:* $\alpha, a(\cdot), \beta$, *and* $b(\cdot)$.

(2) *The functions* V, V^1, U, *and* U_1 *grow linearly and are Lipschitz continuous in the space variable* x, *uniformly with respect to the time variable* s.

The time continuity of value functions turns out to be a measurability issue and will be considered in the next section.

3. Dynamic programming and time continuity of various value functions. In this section, we use the Bellman dynamic programming principle to study the time continuity and the dynamics of various value functions related to our game.

PROPOSITION 3.1. (1) *The lower value function* $V^1(\cdot,\cdot)$ *satisfies the following suboptimality condition: For any* $(a,b) \in A \times B$, $x \in X$, *and* $0 \le s < \hat{s} \le 1$,

(3.1a)

$$\begin{aligned} V^1_{a,b}(s,x) \;\le \inf_{\alpha\in\Gamma_1^a[s,1]} \sup_{b(\cdot)\in\mathcal{B}^b[s,1]} E_{sx}\biggl\{ &\int_s^{\hat{s}} f^0(r,y(r),\alpha[b(\cdot)](r),b(r))\,dr \\ &+ \sum_{\theta_i\le\hat{s}} k(\theta_i,a_{i-1},a_i) - \sum_{\tau_j\le\hat{s}} l(\tau_j,b_{j-1},b_j) + V^1_{\alpha[b(\cdot)](\hat{s}),b(\hat{s})}(\hat{s},y(\hat{s}))\biggr\}, \end{aligned}$$

where $\{a_i,\theta_i\}$ *and* $\{b_j,\tau_j\}$ *are associated with* $\alpha[b(\cdot)]$ *and* $b(\cdot)$, *respectively, and* $\alpha[b(\cdot)](\hat{s}) = \alpha[b(\cdot)](\hat{s}+0), b(\hat{s}) = b(\hat{s}+0)$.

(2) *The upper value function* $U_1(\cdot,\cdot)$ *satisfies the following superoptimality condition: For any* $(a,b) \in A \times B$, $x \in X$, *and* $0 \le s < \hat{s} \le 1$,

(3.1b)
$$U_1^{a,b}(s,x) \ \geq \sup_{\beta\in\Delta_1^b[s,1]} \inf_{a(\cdot)\in\mathcal{A}^a[s,1]} E_{sx}\bigg\{\int_s^{\hat{s}} f^0(r,y(r),a(\cdot),\beta[a(\cdot)](r))\,dr + \sum_{\theta_i\leq\hat{s}} k(\theta_i,a_{i-1},a_i) - \sum_{\tau_j\leq\hat{s}} l(\tau_j,b_{j-1},b_j) + U_1^{a(\hat{s}),\beta[a(\cdot)](\hat{s})}(\hat{s},y(\hat{s}))\bigg\},$$

where $\{a_i,\theta_i\}$ *and* $\{b_j,\tau_j\}$ *are associated with* $a(\cdot)$ *and* $\beta[a(\cdot)]$*, respectively, and* $\beta[a(\cdot)](\hat{s}) = \beta[a(\cdot)](\hat{s}+0), b(\hat{s}) = b(\hat{s}+0)$*.*

Proof of Proposition 3.1. We prove only the inequality (3.1a); the inequality (3.1b) can be proved in the same manner.

Let (s,x,a,b) be fixed, and let $W_{a,b}(s,x)$ be the right-hand side of (3.1a). Then, $\forall\varepsilon > 0$, there exists $\alpha\in\Gamma_1^a[s,1]$ such that

(3.2)
$$W_{a,b}(s,x) \ \geq \ E_{sx}\bigg\{\int_s^{\hat{s}} f^0(r,y(r),\alpha[b(\cdot)](r),b(r))\,dr + \sum_{\theta_i\leq\hat{s}} k(\theta_i,a_{i-1},a_i) - \sum_{\tau_j\leq\hat{s}} l(\tau_j,b_{j-1},b_j) + V^1_{\alpha[b(\cdot)](\hat{s}),b(\hat{s})}(\hat{s},y(\hat{s}))\bigg\} - \varepsilon$$

for every $b(\cdot)\in\mathcal{B}^b[s,1]$. Also, for each $\hat{a}\in A, \hat{b}\in B, \xi\in X$,

$$V^1_{\hat{a},\hat{b}}(\hat{s},\xi) \ = \inf_{\alpha\in\Gamma_1^{\hat{a}}[\hat{s},1]} \sup_{b(\cdot)\in\mathcal{B}^{\hat{b}}[\hat{s},1]} J_{\hat{s}\xi}(\alpha[b(\cdot)],b(\cdot)); \tag{3.3}$$

thus there exists $\alpha_\xi^{\hat{a},\hat{b}}\in\Gamma_1^{\hat{a}}[\hat{s},1]$ for which

$$V^1_{\hat{a},\hat{b}}(\hat{s},\xi) \ \geq \sup_{b(\cdot)\in\mathcal{B}^{\hat{b}}[\hat{s},1]} J_{\hat{s}\xi}(\alpha_\xi^{\hat{a},\hat{b}}[b(\cdot)],b(\cdot)) - \varepsilon. \tag{3.4}$$

Next let $\{A_i : i = 1,2,\dots\}$ be a partition of X by Borel sets, and choose $\xi_i\in A_i (i = 1,2,\dots)$. If the diameter of the A_i's is sufficiently small, then for $i = 1,2,\dots$ and $w\in A_i$,

(3.5)
$$|J_{\hat{s}w}(\alpha[b(\cdot)],b(\cdot)) - J_{\hat{s}\xi_i}(\alpha[b(\cdot)],b(\cdot))| < \varepsilon \quad \text{for any } b(\cdot)\in\mathcal{B}^{\hat{b}}[\hat{s},1] \text{ and } \alpha\in\Gamma_1^{\hat{a}}[\hat{s},1],$$

and

$$|V^1_{\hat{a},\hat{b}}(\hat{s},w) - V^1_{\hat{a},\hat{b}}(\hat{s},\xi_i)| < \varepsilon. \tag{3.6}$$

Now we use the strategies α and $\alpha_{\xi_i}^{\hat{a},\hat{b}}, i = 1,\dots,\hat{a}\in A,\hat{b}\in B$, to construct a new admissible strategy $\widetilde{\alpha}\in\Gamma_1^a[s,1]$ as follows: For $(r,\omega)\in[s,1]\times\Omega_s$ and $b(\cdot)\in\mathcal{B}^b[s,1]$, we define

(3.7)
$$\alpha[b(\cdot)](\omega)(r) = \begin{cases} \alpha[b(\cdot)](\omega)(r) & \text{if } r < \hat{s}, \\ \sum_{i=1}^\infty \sum_{\hat{a}\in A,\hat{b}\in B} \chi_{\{b(\hat{s})=\hat{b},\alpha[b(\cdot)](\hat{s})=\hat{a}\}}\chi_{A_i}(y_{sx}(\hat{s}))\alpha_{\xi_i}^{\hat{a},\hat{b}}[b(\omega_1)](\omega_2)(r) & \text{if } r \geq \hat{s}, \end{cases}$$

where $\omega = (\omega_1,\omega_2)\in\Omega_{t,\hat{s}}\times\Omega_{\hat{s}}$ and $b(\omega_1)(\cdot)\in\mathcal{B}^{b(\hat{s})}[\hat{s},1]$ is given by $b(\omega_1)(\omega_2)(r) = b(\omega_1,\omega_2)(r)$.

Consequently for any $b(\cdot) \in \mathcal{B}^b[s,1]$, using (3.2), (3.4), and (3.6), we obtain

$$
\begin{aligned}
W_{a,b}(s,x) \geq & E_{sx}\bigg\{\int_s^{\hat{s}} f^0(r,y(r),\alpha[b(\cdot)](r),b(r))\,dr \\
& + \sum_{\theta_i \leq \hat{s}} k(\theta_i,a_{i-1},a_i) - \sum_{\tau_j \leq \hat{s}} l(\tau_j,b_{j-1},b_j) \\
& + \sum_{i=1}^{\infty} \sum_{\hat{a}\in A,\hat{b}\in B} \chi_{\{\alpha[b(\cdot)](\hat{s})=\hat{a},b(\hat{s})=\hat{b}\}} \chi_{A_i}(y_{s,x}(\hat{s})) V^1_{\hat{a},\hat{b}}(\hat{s},y(\hat{s}))\bigg\} - \varepsilon \\
\geq & E_{sx}\bigg\{\int_s^{\hat{s}} f^0(r,y(r),\alpha[b(\cdot)](r),b(r))\,dr \\
& + \sum_{\theta_i \leq \hat{s}} k(\theta_i,a_{i-1},a_i) - \sum_{\tau_j \leq \hat{s}} l(\tau_j,b_{j-1},b_j) \\
& + \sum_{i=1}^{\infty} \sum_{\hat{a}\in A,\hat{b}\in B} \chi_{\{\alpha[b(\cdot)](\hat{s})=\hat{a},b(\hat{s})=\hat{b}\}} \chi_{A_i}(y_{s,x}(\hat{s})) V^1_{\hat{a},\hat{b}}(\hat{s},\xi_i)\bigg\} - 2\varepsilon.
\end{aligned} \tag{3.8}
$$

On the other hand, for $y_{sx}(\hat{s}) \in A_i, i = 1,2,\ldots$ and $\forall\, b(\cdot) \in \mathcal{B}^{\hat{b}}[\hat{s},1]$, we derive from (3.4) and (3.5) that

$$
V^1_{\hat{a},\hat{b}}(\hat{s},\xi_i) \geq J_{\hat{s}\xi_i}(\alpha^{\hat{a},\hat{b}}_{\xi_i}[b(\cdot)],b(\cdot)) - \varepsilon \geq J_{\hat{s}y_{sx}(\hat{s})}(\alpha^{\hat{a},\hat{b}}_{\xi_i}[b(\cdot)],b(\cdot)) - 2\varepsilon. \tag{3.9}
$$

Combining the above inequalities, we have

$$
\begin{aligned}
W_{a,b}(s,x) \geq & E_{sx}\bigg\{\int_s^{\hat{s}} f^0(r,y(r),\alpha[b(\cdot)](r),b(r))\,dr \\
& + \sum_{\theta_i \leq \hat{s}} k(\theta_i,a_{i-1},a_i) - \sum_{\tau_j \leq \hat{s}} l(\tau_j,b_{j-1},b_j) \\
& + \sum_{i=1}^{\infty} \chi_{A_i}(y_{s,x}(\hat{s})) E_{\hat{s}y_{sx}(\hat{s})}\bigg\{\int_{\hat{s}}^1 f^0(r,y(r),\widetilde{\alpha}[b(\cdot)](r),b(r))\,dr \\
& + \sum_{\theta_i > \hat{s}} k(\theta_i,a_{i-1},a_i) - \sum_{\tau_j > \hat{s}} l(\tau_j,b_{j-1},b_j) + h(y(1))\bigg\}\bigg\} - 4\varepsilon.
\end{aligned} \tag{3.10}
$$

Therefore,

$$W_{a,b}(s,x) \geq \ J_{sx}(\widetilde{\alpha},b(\cdot)) - 4\varepsilon,$$

which in turn implies

$$W_{a,b}(s,x) \geq \ V^1_{a,b}(s,x) - 4\varepsilon,$$

and the result now follows. ∎

From Proposition 3.1, we can obtain the following time continuity of V^1 and U_1.

PROPOSITION 3.2. *There exists $L > 0$ such that for any $a \in A, b \in B, x \in X$, and $s,t \in [0,1]$*

$$
\begin{aligned}
|V^1_{a,b}(s,x) - V^1_{a,b}(t,x)| \ &\leq L(1+|x|)\sqrt{|s-t|}, \\
|U_1^{a,b}(s,x) - U_1^{a,b}(t,x)| \ &\leq L(1+|x|)\sqrt{|s-t|}.
\end{aligned} \tag{3.11}
$$

Proof of Proposition 3.2. We prove only the $\frac{1}{2}$-Hölder continuity of the r-lower value function V^1 in the time variable the $\frac{1}{2}$-Hölder continuity of the r-upper value function U_1 in the time variable can be proved in the same way.

Suppose that $s < t$. First, we prove the following:

$$V^1_{a,b}(s,x) - V^1_{a,b}(t,x) \le L(1+|x|)\sqrt{t-s}. \tag{3.12}$$

From Proposition 3.1 and Hypothesis 3, we derive

$$\begin{aligned}
&V^1_{a,b}(s,x)\\
&\le \sup_{b(\cdot)\in\mathcal{B}^b[s,1]} E_{sx}\left\{\int_s^t f^0(r,y(r),a,b(r))\,dr - \sum_{\tau_j\le t} l(\tau_j,b_{j-1},b_j) + V^1_{a,b(t)}(t,y(t))\right\}\\
&\le \sup_{b(\cdot)\in\mathcal{B}^b[s,1]} E_{sx}\left\{\int_s^t f^0(r,y(r),a,b(r))\,dr - \sum_{\tau_j\le t} l(t,b_{j-1},b_j) + V^1_{a,b(t)}(t,y(t))\right\}\\
&\le \sup_{b(\cdot)\in\mathcal{B}^b[s,1]} E_{sx}\left\{\int_s^t f^0(r,y(r),a,b(r))\,dr - l(t,b,b(t)) + V^1_{a,b(t)}(t,y(t))\right\}\\
&\le \sup_{b(\cdot)\in\mathcal{B}^b[s,1]} E_{sx}\left\{\int_s^t f^0(r,y(r),a,b(r))\,dr + V^1_{a,b}(t,y(t))\right\}.
\end{aligned} \tag{3.13}$$

Note that in the last step, we have used the relation (2.11). We then have

$$\begin{aligned}
&V^1_{a,b}(s,x) - V^1_{a,b}(t,x)\\
&\le \sup_{b(\cdot)\in\mathcal{B}^b[s,1]} E_{sx}\left\{\int_s^t f^0(r,y(r),a,b(r))\,dr + V^1_{a,b}(t,y(t)) - V^1_{a,b}(t,x)\right\},
\end{aligned} \tag{3.14}$$

which proves (3.12) by the uniformly Lipschitz continuity of $V^1_{a,b}(t,x)$ in x and the following estimate:

$$E|y_{sx}(t) - x| \le L(1+|x|)\sqrt{t-s}.$$

Second, we prove the following:

$$V^1_{a,b}(s,x) - V^1_{a,b}(t,x) \ge -L(1+|x|)\sqrt{t-s}. \tag{3.15}$$

In fact, for any $\widehat{b}(\cdot) \in \mathcal{B}^b[t,1]$ and $\alpha \in \Gamma^a_1[s,1]$, we define

$$b(r) = \begin{cases} b, & r \in [s,t),\\ \widehat{b}(r), & r \in [t,1], \end{cases}$$

and

$$\begin{aligned}
&\widehat{\alpha}(\omega_1)[\widehat{b}(\cdot)](r) = \alpha[b(\cdot)](\omega_1)(r), \quad r \in [t,1],\\
&\widehat{\alpha}(\omega_1)[\widehat{b}(\cdot)](t-0) = a.
\end{aligned} \tag{3.16}$$

Then we see that $b(\cdot) \in \mathcal{B}^b[s,1]$ and $\widehat{\alpha}(\omega_1) \in \Gamma_1^a[s,1]$, a.s. It follows that

$$
\begin{aligned}
&J_{sx}(\alpha[b(\cdot)], b(\cdot)) \\
&\geq E_{sx}\biggl\{\int_s^t f^0(r, y(r), \alpha[b(\cdot)](r), b(r))\, dr + l(t, a, a(t)) + J_{tx}(\alpha(\omega_1)[b(\cdot)], \widehat{b}(\cdot)) \\
&\quad + \int_t^1 [f^0(r, y(r), \widehat{\alpha}(\omega_1)[\widehat{b}(\cdot)](r), \widehat{b}(r)) - f^0(r, y(r), \widehat{\alpha}(\omega_1)[\widehat{b}(\cdot)](r), b(r))]\, dr\biggr\}.
\end{aligned}
\tag{3.17}
$$

Here we have used Hypothesis 2. Then we see that

$$
\begin{aligned}
&\sup_{b(\cdot)\in\mathcal{B}^b[s,1]} J_{sx}(\alpha[b(\cdot)], b(\cdot)) \\
&\geq \sup_{b(\cdot)\in\mathcal{B}^b[s,1]} E_{sx}[V^1_{\alpha[b(\cdot)](t),b}(t,x) + l(t, a, \alpha[b(\cdot)](t))] - L(1+|x|)\sqrt{t-s} \\
&\geq V^1_{a,b}(t,x) - L(1+|x|)|s-t|^{1/2},
\end{aligned}
\tag{3.18}
$$

which implies (3.15). ∎

Remark 3.1. It is still true to replace in Proposition 3.1 the deterministic time $\hat{s} \in (s,1]$ with a stopping time τ which takes its values in $(s,1]$. In fact, in this version of Proposition 3.2, it is sufficient to note that, for any $(x,a,b) \in X \times A \times B$, the two random variables $V^1_{a,b}(\tau,x)$ and $U^{a,b}_1(\tau,x)$ may be sufficiently approximated by

$$
\sum_{i=0}^{N-1} V^1_{a,b}(t_i,x)\chi_{[t_i,t_{i+1})}(\tau) \quad \text{and} \quad \sum_{i=0}^{N-1} U^{a,b}_1(t_i,x)\chi_{[t_i,t_{i+1})}(\tau),
$$

respectively, by letting N be sufficiently large. Here we have used the following notation:

$$
t_i = \frac{i(1-s)}{N}, \quad i = 0, 1, \ldots, N.
$$

For $(s,x,\delta) \in [0,1] \times X \times (0,\infty)$ and $(a(\cdot), b(\cdot)) \in \mathcal{A}^a[s,1] \times \mathcal{B}^b[s,1]$, define

$$
\tau^\delta_{s,x}(a(\cdot), b(\cdot)) := \inf\Bigl\{t \in [s,T] : |y^{a(\cdot),b(\cdot)}_{s,x}(t) - x| \geq \delta\Bigr\} \wedge T,
$$

where $y^{a(\cdot),b(\cdot)}_{s,x}$ is the solution of the system (1.1) corresponding to $(a(\cdot), b(\cdot)) \in \mathcal{A}^a[s,1] \times \mathcal{B}^b[s,1]$, which will occasionally be abbreviated as y_{sx} or simply y to simplify the notation. It is easy to see that $\tau^\delta_{s,x}(a(\cdot), b(\cdot))$ is a stopping time for any triplet $(s,x,\delta) \in [0,1] \times X \times [0,\infty)$ and any pair $(a(\cdot), b(\cdot)) \in \mathcal{A}^a[s,1] \times \mathcal{B}^b[s,1]$. To simplify the notation, we shall simply write τ^δ for $\tau^\delta_{s,x}(a(\cdot), b(\cdot))$ when the dependence on $(s, x, a(\cdot), b(\cdot))$ is not confused from the context. We also have

$$
|y^{a(\cdot),b(\cdot)}_{s,x}(t \wedge \tau^d) - x| \leq \delta
$$

for $s \leq t \leq 1, a(\cdot) \in \mathcal{A}^a[s,1]$, and $b(\cdot) \in \mathcal{B}^b[s,1]$. Moreover, we have

$$
\lim_{\hat{s}\to s+} \sup_{a(\cdot)\in\mathcal{A}^a[s,1], b(\cdot)\in\mathcal{B}^b[s,1]} \frac{P(\{\hat{s} \geq \tau^\delta\})}{\hat{s} - s} = 0.
$$

In fact, we have

$$P(\{\tau^\delta \le \hat{s}\}) = P\left(\left\{\sup_{s\le t\le \hat{s}} |y_{s,x}^{a(\cdot),b(\cdot)}(t) - x| \ge \delta\right\}\right) \le \sum_{i=1}^{N} P_{s,x}^{\delta,i}(a(\cdot), b(\cdot)),$$

where N is the dimension of the state space X, for $i = 1, \dots, N$, e_i is the unit vector of X whose ith component is one, and

$$P_{s,x}^{\delta,i}(a(\cdot), b(\cdot)) := P\left(\left\{\sup_{s\le t\le \hat{s}} \langle e_i, y_{s,x}^{a(\cdot),b(\cdot)}(t) - x\rangle \ge \delta N^{-\frac{1}{2}}\right\}\right).$$

Define

$$f_{s,x}^{\delta} := \sup\{|f(t, y, a, b)| : (t, a, b) \in [s, 1] \times A \times B, |y - x| \le \delta\}$$

and

$$g_{s,x}^{\delta} := \sup\{|g(t, y, a, b)| : (t, a, b) \in [s, 1] \times A \times B, |y - x| \le \delta\}.$$

For $\theta \in X$, from Itô's formula, it follows that the process

$$\begin{aligned} &Z_{s,x}^{\delta,\theta}(t; a(\cdot), b(\cdot)) \\ := &\exp\left\{\left\langle \theta, y_{s,x}(t\wedge\tau^\delta) - x - \int_s^{t\wedge\tau^\delta} f(r, y_{s,x}(r), a(r), b(r))\,dr\right\rangle\right. \\ &\left.-\frac{1}{2}\int_s^{t\wedge\tau^\delta} |g^*(r, y_{s,x}(r), a(r), b(r))e_i|^2\,dr\right\}, \quad t \in [s, 1], \end{aligned}$$

is a continuous martingale, and $E[Z_{s,x}^{\delta,\theta}(t; a(\cdot), b(\cdot))] = 1$ for any $(t, a(\cdot), b(\cdot), \theta) \in [s, T] \times \mathcal{A}^a[s, 1] \times \mathcal{B}^b[s, 1] \times X$. Therefore, using Doob's inequality, we have for $h := \hat{s} - s, \lambda > 0, \delta_0 \ge \delta > 0, a(\cdot) \in \mathcal{A}^a[s, 1]$, and $b(\cdot) \in \mathcal{B}^b[s, 1]$

$$\begin{aligned} &P_{s,x}^{\delta,i}(a(\cdot), b(\cdot)) \\ \le\ &P\left(\left\{\sup_{s\le t\le \hat{s}} Z_{s,x}^{\delta,\lambda e_i}(t; a(\cdot), b(\cdot)) \ge \exp\left[\lambda(\delta N^{-\frac{1}{2}} - hf_{s,x}^{\delta_0}) - \frac{1}{2}\lambda^2 h|g_{s,x}^{\delta_0}|^2\right]\right\}\right) \\ \le\ &\exp\left[-\lambda(\delta N^{-\frac{1}{2}} - hf_{s,x}^{\delta_0}) + \frac{1}{2}\lambda^2 h|g_{s,x}^{\delta_0}|^2\right]. \end{aligned}$$

As h is sufficiently small, take

$$\lambda = \frac{\delta N^{-\frac{1}{2}} - hf_{s,x}^{\delta_0}}{h|g_{s,x}^{\delta_0}|^2},$$

and we further have

$$P_{s,x}^{\delta,i}(a(\cdot), b(\cdot)) \le \exp\left\{\frac{-|\delta N^{-\frac{1}{2}} - hf_{s,x}^{\delta_0}|^2}{2h|g_{s,x}^{\delta_0}|^2}\right\}.$$

Hence,

$$\begin{aligned}
&\lim_{\hat{s}\to s+} \sup_{a(\cdot)\in\mathcal{A}^a[s,1],b(\cdot)\in\mathcal{B}^b[s,1]} \frac{P(\{\tau^\delta \le \hat{s}\})}{\hat{s}-s} \\
&\le \lim_{\hat{s}\to s+} \sup_{a(\cdot)\in\mathcal{A}^a[s,1],b(\cdot)\in\mathcal{B}^b[s,1]} h^{-1}\sum_{i=1}^N P_{s,x}^{\delta,i}(a(\cdot),b(\cdot)) \\
&\le \lim_{h\to 0} Nh^{-1}\exp\left\{\frac{-|\delta N^{-\frac{1}{2}} - hf_{s,x}^{\delta_0}|^2}{2h|g_{s,x}^{\delta_0}|^2}\right\} = 0.
\end{aligned}$$

The desired result then follows.

PROPOSITION 3.3. (1) *The r-lower value function $V^1(\cdot,\cdot)$ satisfies the following: Suppose at $(a,b,s,x) \in A\times B\times[0,1]\times X$*

$$V_{a,b}^1(s,x) > M_{a,b}[V^1](s,x). \tag{3.19a}$$

Then there exist a deterministic time $s_0 > s$ and a sufficiently small number $\delta_0 > 0$, such that for all $\hat{s}\in[s,s_0]$ and $\delta\in(0,\delta_0]$,

$$V_{a,b}^1(s,x) \le E_{sx}\left\{\int_s^{\hat{s}\wedge\tau^\delta} f^0(r,y^{a,b}(r),a,b)\,dr + V_{a,b}^1(\hat{s}\wedge\tau^\delta, y^{a,b}(\hat{s}\wedge\tau^\delta))\right\}. \tag{3.19b}$$

Here we have abbreviated $\tau_{s,x}^\delta(a,b)$ as τ^δ.

(2) *The r-upper value function $U_1(\cdot,\cdot)$ satisfies the following: Suppose at $(a,b,s,x)\in A\times B\times[0,1]\times X$*

$$U_1^{a,b}(s,x) < M^{a,b}[U_1](s,x). \tag{3.20a}$$

Then there exist a deterministic time $s_0 > s$ and a sufficiently small number $\delta_0 > 0$, such that for all $\hat{s}\in[s,s_0]$ and $\delta\in(0,\delta_0]$,

$$U_1^{a,b}(s,x) \ge E_{sx}\left\{\int_s^{\hat{s}\wedge\tau^\delta} f^0(r,y^{a,b}(r),a,b)\,dr + U_1^{a,b}(\hat{s}\wedge\tau^\delta, y^{a,b}(\hat{s}\wedge\tau^\delta))\right\}. \tag{3.20b}$$

Here we have abbreviated $\tau_{s,x}^\delta(a,b)$ as τ^δ.

Remark 3.2. Proposition 3.3 can be viewed as a stochastic version of Theorem 3.2 by Yong [10]. However, it is by no means trivial and is of stochastic nature in its formulation. The upper limits of the integrals in (3.19b) and (3.20b) are more complicated than the deterministic counterparts: The former are a deterministic time $\hat{s} > s$ which is sufficiently close to the initial time s, stopped by the first time of the system state process $y_{s,x}^{a,b}$ (steered by both players I and II with constant actions $a\in A$ and $b\in B$, respectively) escaping from a sufficiently small ball centered at the initial state x, while the latter are simply a deterministic time $\hat{s} > s$ which is sufficiently close to the initial time s. Obviously, both coincide. Our proof below is quite different from the deterministic case and is of stochastic nature; it includes a delicate analysis.

Proof of Proposition 3.3. We prove only statement (1); the proof of statement (2) is similar.

If statement (1) were not true, then there would exist sequences $\hat{s}\to s, \delta\to 0+$, and $\varepsilon\to 0+$ such that

$$V_{a,b}^1(s,x)-\varepsilon > E_{sx}\left\{\int_s^{\hat{s}\wedge\tau^\delta} f^0(s,y(s;a,b),a,b)\,ds + V_{a,b}^1(\hat{s}\wedge\tau^\delta, y(\hat{s}\wedge\tau^\delta;a,b))\right\}. \tag{3.21}$$

On the other hand, using Proposition 3.2 and the idea exposed in Remark 3.1, we can show the following analogy to Proposition 3.1 (1):

$$\begin{aligned} V^1_{a,b}(s,x) \le \inf_{\alpha\in\Gamma_1^a[s,1]} \sup_{b(\cdot)\in\mathcal{B}^b[s,1]} E_{sx}\bigg\{ \int_s^{\hat{s}\wedge\tau^\delta_{a(\cdot),b(\cdot)}} & f^0(r,y(r),\alpha[b(\cdot)](r),b(r))\,dr \\ & + \sum_{\theta_i\le\hat{s}\wedge\tau^\delta_{a(\cdot),b(\cdot)}} k(\theta_i,a_{i-1},a_i) - \sum_{\tau_j\le\hat{s}\wedge\tau^\delta_{a(\cdot),b(\cdot)}} l(\tau_j,b_{j-1},b_j) \\ & + V^1_{\alpha[b(\cdot)](\hat{s}),b(\hat{s})}(\hat{s}\wedge\tau^\delta_{a(\cdot),b(\cdot)},y(\hat{s}\wedge\tau^\delta_{a(\cdot),b(\cdot)}))\bigg\}, \end{aligned}$$

where $\{a_i,\theta_i\}$ and $\{b_j,\tau_j\}$ are associated with $\alpha[b(\cdot)]$ and $b(\cdot)$, respectively; $\alpha[b(\cdot)](\hat{s}) = \alpha[b(\cdot)](\hat{s}+0), b(\hat{s}) = b(\hat{s}+0)$, and $\tau^\delta_{a(\cdot),b(\cdot)} := \tau^\delta_{s,x}(a(\cdot),b(\cdot))$. Therefore, we have

$$\begin{aligned} V^1_{a,b}(s,x) \le \sup_{b(\cdot)\in\mathcal{B}^{b,s}} E_{sx}\bigg\{ \int_s^{\hat{s}\wedge\tau^\delta_{a,b(\cdot)}} & f^0(r,y(r),a,b(r))\,dr \\ & - \sum_{\tau_j\le\hat{s}\wedge\tau^\delta_{a,b(\cdot)}} l(\tau_j,b_{j-1},b_j) + V^1_{a,b(\hat{s})}(\hat{s}\wedge\tau^\delta_{a,b(\cdot)},y(\hat{s}\wedge\tau^\delta_{a,b(\cdot)}))\bigg\}. \end{aligned} \tag{3.22}$$

Furthermore, by definition, we conclude that there exists $b^\varepsilon(\cdot)\in\mathcal{B}^b[s,1]$ such that

$$\begin{aligned} V^1_{a,b}(s,x) - \varepsilon \le E_{sx}\bigg\{ \int_s^{\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)}} & f^0(r,y(r),a,b^\varepsilon(r))\,dr \\ & - \sum_{\tau^\varepsilon_j\le\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)}} l(\tau^\varepsilon_j,b^\varepsilon_{j-1},b^\varepsilon_j) + V^1_{a,b^\varepsilon(\hat{s})}(\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)},y(\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)}))\bigg\}, \end{aligned} \tag{3.23}$$

where $\{\tau^\varepsilon_j, b^\varepsilon_j\} = b^\varepsilon(\cdot)$.

Set

$$\begin{aligned} B_1 &:= \{\omega : b^\varepsilon(r\wedge\tau^\delta_{a,b^\varepsilon(\cdot)},\omega) \ne b \text{ for some } r\in[s,\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)}]\}, \\ B_1^c &:= \Omega\backslash B_1 = \{\omega : b^\varepsilon(r\wedge\tau^\delta_{a,b^\varepsilon(\cdot)},\omega) = b \ \forall\, r\in[s,\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)}]\}. \end{aligned}$$

Note that B_1 and B_1^c depend on $(\delta,\hat{s})$. Then the two inequalities (3.21) and (3.23) yield

$$E[\chi_{B_1}] > 0 \text{ for sufficiently small positive } \delta \text{ and } \hat{s}. \tag{3.24}$$

Combining (3.21) and (3.23), we have

$$(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) > 0, \tag{3.25}$$

where

(3.26)

$$\begin{aligned}
\text{(I)} &= E_{sx}\bigg\{\int_s^{\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)}} f^0(r,y(r;a,b^\varepsilon(\cdot)),a,b^\varepsilon(r))\,dr - \int_s^{\hat{s}\wedge\tau^\delta_{a,b}} f^0(r,y(r;a,b),a,b)\,dr\bigg\}\\
&\le C((\hat{s}-s)(1+|x|+\delta))E\chi_{B_1},\\
\text{(II)} &= E_{sx}\bigg\{-\sum_{\tau^\varepsilon_j\le\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)}} l(\tau^\varepsilon_j,b^\varepsilon_{j-1},b^\varepsilon_j)\bigg\} \le -E_{sx}\bigg\{\sum_{\tau^\varepsilon_j\le\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)}} l(\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)},b^\varepsilon_{j-1},b^\varepsilon_j)\bigg\},\\
\text{(III)} &= E_{sx}[V^1_{a,b^\varepsilon(\hat{s})}(\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)},y(\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)};a,b^\varepsilon(\cdot))) - V^1_{a,b}(\hat{s}\wedge\tau^\delta_{a,b},y(\hat{s}\wedge\tau^\delta_{a,b};a,b))]\\
&= E_{sx}\Big\{[V^1_{a,b^\varepsilon(\hat{s})}(\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)},y(\hat{s}\wedge\tau^\delta_{a,b^\varepsilon(\cdot)};a,b^\varepsilon(\cdot)))\\
&\qquad - V^1_{a,b}(\hat{s}\wedge\tau^\delta_{a,b},y(\hat{s}\wedge\tau^\delta_{a,b};a,b))]\chi_{B_1}\Big\}.
\end{aligned}$$

Hence, noting Propositions 2.1 and 3.2, we have

$$\begin{aligned}
&0 \le \text{(I)} + \text{(II)} + \text{(III)}\\
&\quad\le \{M_{a,b}[V^1](s,x) - V^1_{a,b}(s,x) + C[\sqrt{\hat{s}-s}(1+|x|)+\delta]\}E\chi_{B_1}
\end{aligned}\tag{3.27}$$

for some positive constant C, which implies that

$$M_{a,b}[V^1](s,x) - V^1_{a,b}(s,x) \ge -C\sqrt{\hat{s}-s}(1+|x|)+\delta.\tag{3.28}$$

Letting $\delta\to 0+$ and $\varepsilon\to 0+$, we have

$$M_{a,b}[V^1](s,x) \ge V^1_{a,b}(s,x),$$

which contradicts (3.19a). ∎

Note that the time continuity of V^1 and U_1 given by Proposition 3.2 is used in the proof of Proposition 3.3.

Denote by $C^{0,1}(X,\mathbb{R}^{m\times n})$ the totality of $\mathbb{R}^{m\times n}$-valued uniformly Lipschitz continuous functions on X. For $\varphi(\cdot) = (\varphi_{a,b}(\cdot))_{a\in A,b\in B} \in C^{0,1}(X,\mathbb{R}^{m\times n})$, define

(3.29a)

$$\begin{aligned}
F_{a,b}(s,\hat{s})\varphi(x) = &\sup_{\hat{b}\in B}\inf_{a(\cdot)\in\mathcal{A}^a[s,\hat{s}]} E_{sx}\bigg\{\varphi_{a(\hat{s}-),\hat{b}}(y(\hat{s})) + \int_s^{\hat{s}} f^0(r,y(r),a(r),\hat{b})\,dr\\
&+\sum_{s\le\theta_i<\hat{s}} k(\theta_i,a_{i-1},a_i) - l(s,b,\hat{b})\bigg\},\quad (a,b,x)\in A\times B\times X;\\
F(s,\hat{s})\varphi = &(F_{a,b}(s,\hat{s})\varphi)_{a\in A,b\in B}
\end{aligned}$$

and

(3.29b)

$$\begin{aligned}
G^{a,b}(s,\hat{s})\varphi(x) = &\inf_{\hat{a}\in A}\sup_{b(\cdot)\in\mathcal{B}^b[s,\hat{s}]} E_{sx}\bigg\{\varphi_{\hat{a},b(\hat{s}-)}(y(\hat{s})) + \int_s^{\hat{s}} f^0(r,y(r),\hat{a},b(r))\,dr\\
&+k(s,a,\hat{a}) - \sum_{s\le\tau_j<\hat{s}} l(\tau_j,a_{j-1},a_j)\bigg\},\quad (a,b,x)\in A\times B\times X;\\
G(s,\hat{s})\varphi = &(G^{a,b}(s,\hat{s})\varphi)_{a\in A,b\in B}.
\end{aligned}$$

It is easily shown that $F(s,\hat{s})$ and $G(s,\hat{s})$ are self-mappings on $C^{0,1}(X,\mathbb{R}^{m\times n})$. Therefore, the function $V^\pi:[0,1]\times X\to C^{0,1}(X,\mathbb{R}^{m\times n})$ given by

(3.30a)
$$V^\pi(1,x)=(V^\pi_{a,b}(1,x))_{a\in A,b\in B},\quad V^\pi_{a,b}(1,x)\equiv h(x) \text{ for } (a,b)\in A\times B, \text{ with } x\in X;$$
$$V^\pi(s,x)=F(s,t_{i_0+1})\prod_{i=i_0+2}^{M}F(t_{i-1},t_i)h(x),\qquad x\in X,\qquad \text{if } s\in[t_{i_0},t_{i_0+1}),$$

is well defined. Let $V^\pi_{a,b}(s,x)$ be the (a,b)-component of the matrix $V^\pi(s,x)$. Similarly, define $U_\pi=(U^{a,b}_\pi)_{a\in A,b\in B}:[0,1]\times X\to C^{0,1}(X,\mathbb{R}^{m\times n})$ as follows:

(3.30b)
$$U_\pi(1,x)=(U^{a,b}_\pi(1,x))_{a\in A,b\in B},\quad U^{a,b}_\pi(1,x)\equiv h(x) \text{ for } (a,b)\in A\times B, \text{ with } x\in X;$$
$$U_\pi(s,x)=G(s,t_{i_0+1})\prod_{i=i_0+2}^{M}G(t_{i-1},t_i)h(x),\ x\in X,\qquad \text{if } s\in[t_{i_0},t_{i_0+1}).$$

We have the following.

PROPOSITION 3.4. *For* $(a,b,s,x)\in A\times B\times[0,1]\times X$ *and* $\hat{s}\in\pi\cap[s,1]$, *we have*

$$\begin{aligned}V^\pi_{a,b}(s,x)=\inf_{\alpha\in\Gamma^a[s,1]}\sup_{b(\cdot)\in\mathcal{B}^b_\pi[s,1]}&E_{sx}\biggl\{\int_s^{\hat{s}}f^0(r,y(r),\alpha[b(\cdot)](r),b(r))\,dr\\&+\sum_{s\le\theta_i<\hat{s}}k(\theta_i,a_{i-1},a_i)-\sum_{s\le\tau_j<\hat{s}}l(\tau_j,b_{j-1},b_j)\\&+V^\pi_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))\biggr\},\end{aligned}\tag{3.31a}$$

where $b(\cdot)=\{b_j,\tau_j\}$ *and* $a(\cdot)=\alpha[b(\cdot)]=\{a_i,\theta_i\}$, *and*

$$\begin{aligned}U^{a,b}_\pi(s,x)=\sup_{\beta\in\Delta^b[s,1]}\inf_{a(\cdot)\in\mathcal{A}^a_\pi[s,1]}&E_{sx}\biggl\{\int_s^{\hat{s}}f^0(r,y(r),a(r),\beta[a(\cdot)](r))\,dr\\&+\sum_{s\le\theta_i<\hat{s}}k(\theta_i,a_{i-1},a_i)-\sum_{s\le\tau_j<\hat{s}}l(\tau_j,b_{j-1},b_j)\\&+U_\pi^{a(\hat{s}-),\beta[a(\cdot)](\hat{s}-)}(\hat{s},y(\hat{s}))\biggr\},\end{aligned}\tag{3.31b}$$

where $a(\cdot)=\{a_i,\theta_i\}$ *and* $b(\cdot)=\beta[a(\cdot)]=\{b_j,\tau_j\}$.

Proof of Proposition 3.4. We prove only (3.31a) here; the proof of (3.31b) is identical and therefore will be omitted.

For $a(\cdot)\in\mathcal{A}^a[s,1]$ and $b(\cdot)\in\mathcal{B}^b[s,1]$, set

$$\begin{aligned}\widehat{J}_{sx}(a(\cdot),b(\cdot))=&E_{sx}\biggl\{\int_s^{\hat{s}}f^0(r,y(r),a(r),b(r))\,dr\\&+\sum_{s\le\theta_i<\hat{s}}k(\theta_i,a_{i-1},a_i)-\sum_{s\le t_j<\hat{s}}l(t_j,b_{j-1},b_j)\\&+V^\pi_{a(\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))\biggr\}.\end{aligned}\tag{3.32}$$

The desired result can be derived from the following assertion: For $(a,b,s,x)\in A\times B\times[0,1]\times X$ and $\forall\varepsilon>0$, there exist $\beta_\varepsilon\in\Delta^b_\pi[s,1]$ and $\alpha_\varepsilon\in\Gamma^a[s,1]$ such that

$$V^\pi_{a,b}(s,x)\ge\widehat{J}_{sx}(\alpha_\varepsilon[b(\cdot)],b(\cdot))-\varepsilon\quad\forall b(\cdot)\in\mathcal{B}^b_\pi[s,1]\tag{3.33}$$

and

$$V^{\pi}_{a,b}(s,x) \le \widehat{J}_{sx}(a(\cdot),\beta_\varepsilon[a(\cdot)]) + \varepsilon \quad \forall a(\cdot) \in \Gamma^a[s,1]. \tag{3.34}$$

In fact, the inequality (3.33) implies (3.31a) with the equality sign replaced by "≥." On the other hand, for any $\alpha \in \Gamma^a[s,1]$, the pair of strategies of $\beta_\varepsilon \in \Delta^b_\pi[s,1]$ and $\alpha \in \Gamma^a[s,1]$ define a pair of switching processes $a_\varepsilon(\cdot) \in \mathcal{A}^a[s,1]$ and $b_\varepsilon \in \mathcal{B}^b_\pi[s,1]$ such that

$$\widehat{J}_{sx}(a_\varepsilon(\cdot),\beta_\varepsilon) = \widehat{J}_{sx}(\alpha, b_\varepsilon(\cdot)), \tag{3.35}$$

and this gives the other inequality in (3.31a). We invite the reader to see Fleming and Souganidis [6] for the details of the proof.

We conclude the proof by establishing (3.33) and (3.34). For $\varphi \in C^{0,1}(X,\mathbb{R}^{m\times n})$, define

$$\begin{aligned}\psi_{a,b}(s,x,\hat{s},\varphi,\widetilde{b}) = \inf_{a(\cdot)\in\mathcal{A}^a[s,\hat{s})} E_{sx}\bigg\{&\varphi_{a(\hat{s}-),\widetilde{b}}(y(\hat{s})) + \int_s^{\hat{s}} f^0(r,y(r),a(r),\widetilde{b})\,dr \\ &+ \sum_{s\le\theta_i<\hat{s}} k(\theta_i,a_{i-1},a_i) - l(s,b,\widetilde{b})\bigg\}.\end{aligned} \tag{3.36}$$

Here $y_{sx}(\cdot)$ is the solution of (1.1) with $b(r) = \widetilde{b}, r \in [s,\hat{s}]$.

$$F_{a,b}(s,\hat{s})\varphi(x) = \sup_{\widetilde{b}\in B} \psi_{a,b}(s,x,\hat{s},\varphi,\widetilde{b}). \tag{3.37}$$

If $s \in [t_{i_0}, t_{i_0+1})$ for $i_0 \in \{0,1,\dots,M-1\}$, let $D_M = h, D_j = F(t_j,t_{j+1})D_{j+1}$, for $j = i_0+1,\dots,M-1$, and $D_{s,i_0} = F(t,t_{i_0+1})D_{i_0+1}$. Thus,

$$D_{s,i_0}(x) = V^{\pi}_{a,b}(s,x), \tag{3.38}$$

and, in particular,

$$D_{i_0}(x) = V^{\pi}_{a,b}(t_{i_0},x). \tag{3.39}$$

We partition X into Borel sets $\{A_i : i = 1,2,\dots\}$ of diameter less than δ, where δ is to be specified later, and we choose $x_i \in A_i$. Given $\gamma > 0$, we can choose δ small enough and $\widetilde{b}^a_{i(j-1)} \in B$ for $i = 1,2,\dots$ and $j = i_0+1,\dots,M$ such that

$$\psi_{a,b}(t_{j-1},x_i,t_j,D_j,\widetilde{b}^a_{i(j-1)}) > F_{a,b}(t_{j-1},t_j)D_j(x_i) - \gamma, \tag{3.40a}$$

and thus

(3.40b)
$$\begin{aligned}&E_{t_{j-1}x_i}\bigg\{D_j^{a(t_j-),\widetilde{b}^a_{i(j-1)}}(y(t_j)) + \int_{t_{j-1}}^{t_j} f^0(r,y(r),a(r),\widetilde{b}^a_{i(j-1)})\,dr \\ &+ \sum_{t_{j-1}\le\theta_i<t_j} k(\theta_i,a_{i-1},a_i) - l(t_{j-1},b,\widetilde{b})\bigg\} > D^{a,b}_{j-1}(x_i) - \gamma \quad \forall a(\cdot) \in \mathcal{A}^a[t_{j-1},t_j].\end{aligned}$$

We also choose $a^{\tilde{b}}_{i(j-1)}(\cdot) \in \mathcal{A}^a(t_{j-1}, t_j)$ such that, for $a(\cdot) = a^{\tilde{b}}_{ij}(\cdot)$ and $b(r) = \widetilde{b}, r \in (t_{j-1}, t_j]$,

$$\begin{aligned} &E_{t_{j-1}x_i}\Big\{D_j^{a(t_j-),\widetilde{b}}(y(t_j; a^{\tilde{b}}_{i(j-1)}(\cdot), \widetilde{b})) + \int_{t_{j-1}}^{t_j} f^0(r, y(r), a^{\tilde{b}}_{i(j-1)}(r), \widetilde{b})\, dr \\ &\qquad + \sum_{t_{j-1}\le\theta_i<t_j} k(\theta_i, a_{i-1}, a_i) - l(t_{j-1}, b, \widetilde{b})\Big\} \\ &\quad < \psi_{a,b}(t_{j-1}, x_i, t_j, D_j, \widetilde{b}) + \gamma = D_j^{a,b}(x_i) + \gamma, \end{aligned} \tag{3.41}$$

where for $j = i_0 + 1$ we replace t_{i_0} by s. Here $y_{t_jx_i}(\cdot; a^{\tilde{b}}_{ij}(\cdot), \widetilde{b})$ is the solution of (1.1) with the initial data (t_{j-1}, x_i) and on the switchings $a(\cdot) = a^{\tilde{b}}_{ij}(\cdot)$ and $b(\cdot) \equiv \widetilde{b}$.

We need to introduce more notations. As before, we identify $\omega \in \Omega_s$ with the pair $(\omega_{1j}, \omega_{2j})$ for $j = i_0 + 2, \ldots, M$, where $\omega_{1j} = \omega|_{[s,t_{j-1}]}$ and $\omega_{2j} = \omega - \omega_{t_{j-1}}|_{[t_{j-1},1]}$. With this identification, the Wiener measure P_s on Ω_s can be regarded as the product measure $P_{1j} \otimes P_{2j}$ of the two probability measures P_{1j} and P_{2j}, which are defined on the two measure spaces $(\Omega_{s,t_{j-1}}, \mathcal{F}_{s,t_{j-1}})$ and $(\Omega_{t_{j-1}}, \mathcal{F}_{t_{j-1}})$, respectively. In view of this identification, we will be writing

$$E^{P_{2j}} \equiv E_{t_{j-1}x_i}. \tag{3.42}$$

The strategies $\alpha_\varepsilon \in \Gamma^a[s, 1]$ and $\beta_\varepsilon \in \Delta^b_\pi[s, 1]$ are defined as follows. Let $(a, b, s, x) \in A \times B \times [0, 1) \times X$ be fixed. For $a(\cdot) \in \mathcal{A}^a[s, 1]$, we define

$$\begin{cases} \beta_\varepsilon[a(\cdot)](r) = b\chi_{[s,s)} + \chi_{[s,t_{i_0+1})}(r) \displaystyle\sum_{i=1}^{\infty} \widetilde{b}^a_{ii_0}\chi_{A_i}(x) \\ \qquad\qquad + \displaystyle\sum_{j=i_0+1}^{M-1} \chi_{[t_j,t_{j+1})}(r) \sum_{\substack{\tilde{a}\in A\\ i=1}}^{\infty} \widetilde{b}^{\tilde{a}}_{ij}\chi_{A_i}(y_{sx}(t_j))\chi_{\{a(t_j-)=\tilde{a}\}}, r \in [s, 1), \\ \beta_\varepsilon[a(\cdot)](1) = \beta_\varepsilon[a(\cdot)](1-), \end{cases} \tag{3.43}$$

where the random variable $y_{sx}(\cdot)$ is defined successively on intervals $[s, t_{i_0+1}]$, $[t_j, t_{j+1}]$, $j = i_0 + 1, \ldots, M - 1$, as the solution to (1.1) with $b(r) = \beta_\varepsilon[a(\cdot)](r)$. Note that $\forall a(\cdot) \in \mathcal{A}^a[s, 1]$ and $r \in (s, 1)$, $\beta_\varepsilon[a(\cdot)](r)$ depends only on $a(\cdot)|_{[s,r)}$ and is independent of $a(r)$. For $b(\cdot) \in \mathcal{B}^b[s, 1]$, we define

$$\begin{cases} \alpha_\varepsilon[b(\cdot)](r) = \chi_{[s,s)} \displaystyle\sum_{i=1}^{\infty} \widetilde{a}^b_{ii_0}\chi_{A_i}(x) + \chi_{[s,t_{i_0+1})}(r) \sum_{\tilde{b}\in B, i=1}^{\infty} \widetilde{a}^{\tilde{b}}_{ii_0}(r)\chi_{A_i}(x)\chi_{\{b(s)=\tilde{b}\}} \\ \qquad\qquad + \displaystyle\sum_{j=i_0+1}^{M-1} \chi_{[t_j,t_{j+1})}(r) \sum_{\tilde{b}\in B, i=1}^{\infty} \widetilde{a}^{\tilde{b}}_{ij}(r)\chi_{A_i}(y_{sx}(t_j))\chi_{\{b(r)=\tilde{b}\}}, r \in [s, 1); \\ \alpha_\varepsilon[b(\cdot)](1) = \alpha_\varepsilon[b(\cdot)](1-), \end{cases} \tag{3.44}$$

where again $y_{sx}(\cdot)$ is defined successively on intervals $[s, t_{i_0+1}]$, $[t_j, t_{j+1}]$, $j = i_0 + 1, \ldots, M - 1$, as the solution to (1.1) with $a(r) = \alpha_\varepsilon[b(\cdot)](r)$. Note that for any $b(\cdot) \in \mathcal{B}^b[s, 1]$ and $r \in [s, 1]$, $\alpha_\varepsilon[b(\cdot)](r)$ depends on $b(r)$.

For either $a(\cdot) \in \mathcal{A}^a[s,1]$ and $b(\cdot) = \beta_\varepsilon[a(\cdot)]$ or $b(\cdot) \in \mathcal{B}^b_\pi[s,1]$ and $a(\cdot) = \alpha_\varepsilon[b(\cdot)]$, we have

$$
\begin{aligned}
&V^\pi_{a,b}(s,x) - \widehat{J}_{sx}(a(\cdot),b(\cdot)) \\
&= E^{P_s} \sum_{t_{i_0+1} \le t_j \le \hat{s}} \Bigg\{ D^{a(t_{j-1}-),b(t_{j-1}-)}_{j-1}(y_{sx}(t_{j-1})) \\
&\quad + E^{P_s}\Bigg[-\int_{t_{j-1}}^{t_j} f^0(r,y_{sx}(r),a(r),b(r))\,dr - \sum_{t_{j-1}\le\theta_i<t_j} k(\theta_i,a_{i-1},a_i) \\
&\quad + \sum_{t_{j-1}\le\tau_j<t_j} l(\tau_j,b_{j-1},b_j) - D^{a(t_j-),b(t_j-)}_j(y_{sx}(t_j)) \,\Bigg|\, \mathcal{F}_{s,t_{j-1}} \Bigg] \Bigg\}.
\end{aligned}
\tag{3.45}
$$

To obtain (3.33) and (3.34), it suffices to show that the following statements hold:

$$
\begin{aligned}
&D^{a(t_{j-1}-),b(t_{j-1}-)}_{j-1}(y_{sx}(t_{j-1})) \ \ge \\
&\quad E^{P_s}\Bigg[\int_{t_{j-1}}^{t_j} f^0(r,y_{sx}(r),a(r),b(r))\,dr + \sum_{t_{j-1}\le\theta_i<t_j} k(\theta_i,a_{i-1},a_i) \\
&\quad - \sum_{t_{j-1}\le\tau_j<t_j} l(\tau_j,b_{j-1},b_j) + D^{a(t_j-),b(t_j-)}_j(y_{sx}(t_j)) \,\Bigg|\, \mathcal{F}_{s,t_{j-1}} \Bigg] - \varepsilon(t_j - t_{j-1}), \\
&\qquad P_s\text{-a.s.} \quad \forall\, b(\cdot) \in \mathcal{B}^b_\pi[s,1] \text{ and } a(\cdot) = \alpha_\varepsilon[b(\cdot)]
\end{aligned}
\tag{3.46}
$$

and

$$
\begin{aligned}
&D^{a(t_{j-1}-),b(t_{j-1}-)}_{j-1}(y_{sx}(t_{j-1})) \\
&\le E^{P_s}\Bigg[\int_{t_{j-1}}^{t_j} f^0(r,y_{sx}(r),a(r),b(r))\,dr + \sum_{t_{j-1}\le\theta_i<t_j} k(\theta_i,a_{i-1},a_i) \\
&\quad - \sum_{t_{j-1}\le\tau_j<t_j} l(\tau_j,b_{j-1},b_j) + D^{a(t_j-),b(t_j-)}_j(y_{sx}(t_j)) \,\Bigg|\, \mathcal{F}_{s,t_{j-1}} \Bigg] + \varepsilon(t_j - t_{j-1}), \\
&\qquad P_s\text{-a.s.} \quad \forall\, a(\cdot) \in \mathcal{A}^a[s,1] \text{ and } b(\cdot) = \beta_\varepsilon[a(\cdot)].
\end{aligned}
\tag{3.47}
$$

They can be derived from (3.41) and (3.40b), separately. ◻

It is easy to see that $V^\pi_{a,b}(s,x)$ and $U^{a,b}_\pi(s,x)$ grow in a linear way in the state variable $x \in X$, uniformly with respect to $(a,b,s) \in A \times B \times [0,T]$ and the partition π. Analogous to the first part of the proof of Proposition 3.2, we can also show some time continuity of V^π and U_π. These properties are summarized into the following.

Proposition 3.5. *There is a positive constant L such that for $s \in [0,1], t \in \pi \cap [s,1]$, $a \in A, b \in B, x \in X$, and $y \in X$ we have*

$$
\begin{aligned}
|V^\pi_{a,b}(s,x)| + |U^{a,b}_\pi(s,x)| &\le L(1+|x|), \\
|V^\pi_{a,b}(s,x) - V^\pi_{a,b}(s,y)| + |U^{a,b}_\pi(s,x) - U^{a,b}_\pi(s,y)| &\le L|x-y|, \\
|V^\pi_{a,b}(s,x) - V^\pi_{a,b}(t,x)| + |U^{a,b}_\pi(s,x) - U^{a,b}_\pi(t,x)| &\le L(1+|x|)\sqrt{t-s}.
\end{aligned}
\tag{3.48}
$$

Next we assume that $\pi_i = \{j/2^i\}_{j=0}^{2^i}, i = 0,1,2,\ldots$. Then $||\pi_i|| = 1/2^i$, and, from the definition of V^{π_i} and $U_{\pi_i}, i = 0,1,2,\ldots$, and Proposition 3.4, we have

$$
V^{\pi_0}_{a,b} \le V^{\pi_1}_{a,b} \le \cdots \le V^{\pi_i}_{a,b} \le \cdots \le V_{a,b}, \quad (a,b) \in A \times B
\tag{3.49a}
$$

and

$$U^{a,b}_{\pi_0} \geq U^{a,b}_{\pi_1} \geq \cdots \geq U^{a,b}_{\pi_i} \geq \cdots \geq U^{a,b}, \quad (a,b) \in A \times B. \tag{3.49b}$$

PROPOSITION 3.6. *For $(a,b) \in A\times B$, let $v_{a,b} = \lim_{i\to\infty} V^{\pi_i}_{a,b}$ and $v := (v_{a,b})_{a\in A, b\in B}$. Then, for $s,t \in [0,1]$ and $x, y \in X$, we have*

$$\begin{aligned} v_{a,b}(s,x) &\leq V_{a,b}(s,x), \\ |v_{a,b}(s,x)| &\leq L(1+|x|), \\ |v_{a,b}(s,x) - v_{a,b}(t,x)| &\leq L(1+|x|)\sqrt{|t-s|}, \\ |v_{a,b}(s,x) - v_{a,b}(s,y)| &\leq L|x-y|. \end{aligned} \tag{3.50a}$$

Similarly, let $u^{a,b} = \lim_{i\to\infty} U^{a,b}_{\pi_i}$ and $u := (u^{a,b})_{a\in A, b\in B}$. Then, for $s,t \in [0,1]$ and $x,y \in X$,

$$\begin{aligned} u^{a,b}(s,x) &\geq U^{a,b}(s,x), \\ |u^{a,b}(s,x)| &\leq L(1+|x|), \\ |u^{a,b}(s,x) - u^{a,b}(t,x)| &\leq L(1+|x|)\sqrt{|t-s|}, \\ |u^{a,b}(s,x) - u^{a,b}(s,y)| &\leq L|x-y|. \end{aligned} \tag{3.50b}$$

Proof of Proposition 3.6. First, we prove (3.50a). Assume, without loss of generality, that $s < t$. Let $\{t_i\}_{i=1}^{\infty} \subset \cup_{i=0}^{\infty}\pi_i \cap [t,1]$ and $\lim_{i\to\infty} t_i = t$. Then we have from Proposition 3.5 that

$$\begin{aligned} &|v_{a,b}(s,x) - v_{a,b}(t,x)| \\ &\leq |v_{a,b}(s,x) - v_{a,b}(t_i,x)| + |v_{a,b}(t_i,x) - v_{a,b}(t,x)| \\ &\leq L(1+|x|)(\sqrt{t_i - s} + \sqrt{t_i - t}), \quad i = 1,2,\ldots. \end{aligned} \tag{3.51}$$

This concludes the $\frac{1}{2}$-Hölder continuity in the time variable of $v_{a,b}$. Its linear growth and uniform Lipschitz continuity in the space variable x is straightforward.

In an identical way, we can show (3.50b). ∎

Passing to the limit $\|\pi\| \to 0$ in Proposition 3.4, we obtain that two functions v and u satisfy the following dynamic programming principle.

PROPOSITION 3.7. *For $(a,b,s,x) \in A\times B\times[0,1]\times X$ and $\hat{s} \in \cup_{i=0}^{\infty}\pi_i \cap [s,1]$, we have*

$$\begin{aligned} v_{a,b}(s,x) = \lim_{i\to\infty} \inf_{\alpha\in\Gamma^a[s,1]} \sup_{b(\cdot)\in\mathcal{B}^b_{\pi_i}[s,1]} &E_{sx}\bigg\{\int_s^{\hat{s}} f^0(r,y(r),\alpha[b(\cdot)](r),b(r))\,dr \\ &+ \sum_{s\leq\theta_j<\hat{s}} k(\theta_j,a_{j-1},a_j) - \sum_{s\leq\tau_j<\hat{s}} l(\tau_j,b_{j-1},b_j) \\ &+ v_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))\bigg\}, \end{aligned} \tag{3.52a}$$

where $b(\cdot) = \{b_j,\tau_j\}$ and $a(\cdot) = \alpha[b(\cdot)] = \{a_j,\theta_j\}$, and

$$\begin{aligned} u^{a,b}(s,x) = \lim_{i\to\infty} \sup_{\beta\in\Delta^b[s,1]} \inf_{a(\cdot)\in\mathcal{A}^a_{\pi_i}[s,1]} &E_{sx}\bigg\{\int_s^{\hat{s}} f^0(r,y(r),a(r),\beta[a(\cdot)](r))\,dr \\ &+ \sum_{s\leq\theta_j<\hat{s}} k(\theta_j,a_{j-1},a_j) - \sum_{s\leq\tau_j<\hat{s}} l(\tau_j,b_{j-1},b_j) \\ &+ u^{a(\hat{s}-),\beta[a(\cdot)](\hat{s}-)}(\hat{s},y(\hat{s}))\bigg\}, \end{aligned} \tag{3.52b}$$

where $a(\cdot) = \{a_j,\theta_j\}$ and $b(\cdot) = \beta[a(\cdot)] = \{b_j,\tau_j\}$.

Proof of Proposition 3.7. We only derive the equality (3.52a) from the equality (3.31a) in Proposition 3.4. The proof of the equality (3.52b) is similar.

Since $\hat{s} \in \cup_{i=0}^{\infty} \pi_i \cap [s,1]$, we have $\hat{s} \in \pi_i \cap [s,1]$ when i is sufficiently large. From Proposition 3.4, we have that, when i is sufficiently large,

$$\begin{aligned} V_{a,b}^{\pi_i}(s,x) = \inf_{\alpha \in \Gamma^a[s,1]} \sup_{b(\cdot)\in \mathcal{B}^b_{\pi_i}[s,1]} E_{sx}\bigg\{ &\int_s^{\hat{s}} f^0(r,y(r),\alpha[b(\cdot)](r),b(r))\,dr \\ &+ \sum_{s\le\theta_j<\hat{s}} k(\theta_j,a_{j-1},a_j) - \sum_{s\le\tau_j<\hat{s}} l(\tau_j,b_{j-1},b_j) \\ &+ V^{\pi_i}_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))\bigg\}, \end{aligned} \tag{3.53}$$

where $b(\cdot) = \{b_j, \tau_j\}$ and $a(\cdot) = \alpha[b(\cdot)] = \{a_j, \theta_j\}$.

Set, for any $C > 0$,

$$O_C(x) := \{y : |y-x| \le C\}, \quad O_C^c(x) := \{y : |y-x| > C\}.$$

It is easy to see from Propositions 3.5 and 3.6 that

$$\begin{aligned} &E_{sx}|V^{\pi_i}_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s})) - v_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))| \\ &\le E_{sx}\{\chi_{O_C^c(x)}(y(\hat{s}))|V^{\pi_i}_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s})) - v_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))|\} \\ &\quad + E_{sx}\{\chi_{O_C(x)}(y(\hat{s}))|V^{\pi_i}_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s})) - v_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))|\} \\ &\le \{E_{sx}|V^{\pi_i}_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s})) - v_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))|^2\}^{\frac{1}{2}}\{P(O_C^c(x))\}^{\frac{1}{2}} \\ &\quad + E_{sx}\{\chi_{O_C(x)}(y(\hat{s}))|V^{\pi_i}_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s})) - v_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))|\}. \end{aligned} \tag{3.54}$$

From Propositions 3.5 and 3.6, we have that, for any given positive constant C,

$$\lim_{i\to\infty} V^{\pi_i}(\hat{s},y) = v(\hat{s},y) \quad \text{uniformly in } y \in O_C(x),$$

which implies that

$$\lim_{i\to\infty} E_{sx}\{\chi_{O_C(x)}(y(\hat{s}))|V^{\pi_i}_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s})) - v_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))|\} = 0. \tag{3.55}$$

Moreover,

$$E_{sx}|V^{\pi_i}_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s})) - v_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))|^2 \le L(1+E_{sx}|y(\hat{s})|^2). \tag{3.56}$$

Since

$$\begin{aligned} &E_{sx}|y(\hat{s})|^2 \le L(1+|x|^2), \\ &P(O_C^c(x)) \le C^{-2}E_{sx}|y(\hat{s})-x|^2 \le LC^{-2}(1+|x|^2), \end{aligned} \tag{3.57}$$

we see that

$$0 \le \overline{\lim_{i\to\infty}} \sup_{\alpha,b(\cdot)} E_{sx}|V^{\pi_i}_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s})) - v_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))| \le LC^{-2}(1+|x|^2)$$

for an arbitrary sufficiently large positive number C, and therefore

$$\lim_{i\to\infty} \sup_{\alpha,b(\cdot)} E_{sx}|V^{\pi_i}_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s})) - v_{\alpha[b(\cdot)](\hat{s}-),b(\hat{s}-)}(\hat{s},y(\hat{s}))| = 0. \tag{3.58}$$

The last equality implies (3.52a) immediately. ∎

Note that

$$v_{a,b}(1,x) = u^{a,b}(1,x) = h(x), \quad (a,b,x) \in A \times B \times X. \tag{3.59}$$

It follows from Proposition 3.7 that, for $(a,b,s,x) \in A \times B \times [0,1] \times X$,

$$\begin{aligned} v_{a,b}(s,x) = \lim_{i\to\infty} \inf_{\alpha\in\Gamma^a[s,1]} \sup_{b(\cdot)\in\mathcal{B}^b_{\pi_i}[s,1]} E_{sx}\bigg\{ &\int_s^1 f^0(r,y(r),\alpha[b(\cdot)](r),b(r))\,dr \\ &+ \sum_{s\le\theta_j<1} k(\theta_j,a_{j-1},a_j) - \sum_{s\le\tau_j<1} l(\tau_j,b_{j-1},b_j) + h(y(1))\bigg\}, \end{aligned} \tag{3.60a}$$

where $b(\cdot) = \{b_j, \tau_j\}$ and $a(\cdot) = \alpha[b(\cdot)] = \{a_j, \theta_j\}$, and

$$\begin{aligned} u^{a,b}(s,x) = \lim_{i\to\infty} \sup_{\beta\in\Delta^b[s,1]} \inf_{a(\cdot)\in\mathcal{A}^a_{\pi_i}[s,1]} E_{sx}\bigg\{ &\int_s^1 f^0(r,y(r),a(r),\beta[a(\cdot)](r))\,dr \\ &+ \sum_{s\le\theta_j<1} k(\theta_j,a_{j-1},a_j) - \sum_{s\le\tau_j<1} l(\tau_j,b_{j-1},b_j) + h(y(1))\bigg\}, \end{aligned} \tag{3.60b}$$

where $a(\cdot) = \{a_j, \theta_j\}$ and $b(\cdot) = \beta[a(\cdot)] = \{b_j, \tau_j\}$. From the above two formulas, we have

$$M_{a,b}[v](s,x) \le v_{a,b}(s,x) \le M^{a,b}[v](s,x), \quad (a,b,s,x) \in A \times B \times [0,1] \times X, \tag{3.61a}$$

and

$$M_{a,b}[u](s,x) \le u^{a,b}(s,x) \le M^{a,b}[u](s,x), \quad (a,b,s,x) \in A \times B \times [0,1] \times X. \tag{3.61b}$$

In view of the time continuity given by Proposition 3.6, the deterministic time $\hat{s}$ may be replaced in Proposition 3.7 with an arbitrary stopping time which takes values in $[s,1]$. That is, we have the following.

PROPOSITION 3.8. *For $(a,b,s,x) \in A \times B \times [0,1] \times X$ and any stopping time τ which take values in $[s,1]$, we have*

$$\begin{aligned} v_{a,b}(s,x) = \lim_{i\to\infty} \inf_{\alpha\in\Gamma^a[s,1]} \sup_{b(\cdot)\in\mathcal{B}^b_{\pi_i}[s,1]} E_{sx}\bigg\{ &\int_s^\tau f^0(r,y(r),\alpha[b(\cdot)](r),b(r))\,dr \\ &+ \sum_{s\le\theta_j<\tau} k(\theta_j,a_{j-1},a_j) - \sum_{s\le\tau_j<\tau} l(\tau_j,b_{j-1},b_j) \\ &+ v_{\alpha[b(\cdot)](\tau-),b(\tau-)}(\tau,y(\tau))\bigg\}, \end{aligned} \tag{3.62a}$$

where $b(\cdot) = \{b_j, \tau_j\}$ and $a(\cdot) = \alpha[b(\cdot)] = \{a_j, \theta_j\}$, and

$$\begin{aligned} u^{a,b}(s,x) = \lim_{i\to\infty} \sup_{\beta\in\Delta^b[s,1]} \inf_{a(\cdot)\in\mathcal{A}^a_{\pi_i}[s,1]} E_{sx}\bigg\{ &\int_s^\tau f^0(r,y(r),a(r),\beta[a(\cdot)](r))\,dr \\ &+ \sum_{s\le\theta_j<\tau} k(\theta_j,a_{j-1},a_j) - \sum_{s\le\tau_j<\tau} l(\tau_j,b_{j-1},b_j) \\ &+ u^{a(\tau-),\beta[a(\cdot)](\tau-)}(\tau,y(\tau))\bigg\}, \end{aligned} \tag{3.62b}$$

where $a(\cdot) = \{a_j, \theta_j\}$ and $b(\cdot) = \beta[a(\cdot)] = \{b_j, \tau_j\}$.

Proceeding similarly as in the proof of Proposition 3.3, we derive from Proposition 3.8 the following.

PROPOSITION 3.9. (1) *The lower value function* $v(\cdot,\cdot) := (v_{a,b})_{a\in A, b\in B}$ *satisfies the following: Suppose at* $(a,b,s,x) \in A \times B \times [0,1] \times X$,

$$v_{a,b}(s,x) > M_{a,b}[v](s,x) \quad (\text{resp., } v_{a,b}(s,x) < M^{a,b}[v](s,x)). \tag{3.63a}$$

Then there exist a deterministic time $s_0 > s$ *and a sufficiently small number* $\delta_0 > 0$, *such that for all* $\hat{s} \in [s, s_0]$ *and* $\delta \in (0, \delta_0]$,

(3.63b)

$$v_{a,b}(s,x) \leq \ (\text{resp., } \geq)\ E_{sx}\left\{\int_s^{\hat{s}\wedge\tau^\delta} f^0(r, y^{a,b}(r), a, b)\,dr + v_{a,b}(\hat{s}\wedge\tau^\delta, y^{a,b}(\hat{s}\wedge\tau^\delta))\right\}.$$

Here we have abbreviated $\tau^\delta_{s,x}(a,b)$ *as* τ^δ.

(2) *The upper value function* $u(\cdot,\cdot) := (u^{a,b})_{a\in A, b\in B}$ *satisfies the following: Suppose at* $(a,b,s,x) \in A \times B \times [0,1] \times X$,

$$u^{a,b}(s,x) < M^{a,b}[u](s,x) \quad (\text{resp., } u^{a,b}(s,x) > M_{a,b}[u](s,x)). \tag{3.64a}$$

Then there exist a deterministic time $s_0 > s$ *and a sufficiently small number* $\delta_0 > 0$, *such that for all* $\hat{s} \in [s, s_0]$ *and* $\delta \in (0, \delta_0]$,

(3.64b)

$$u^{a,b}(s,x) \geq \ (\text{resp., } \leq)\ E_{sx}\left\{\int_s^{\hat{s}\wedge\tau^\delta} f^0(r, y^{a,b}(r), a, b)\,dr + u^{a,b}(\hat{s}\wedge\tau^\delta, y^{a,b}(\hat{s}\wedge\tau^\delta))\right\}.$$

Here we have abbreviated $\tau^\delta_{s,x}(a,b)$ *as* τ^δ.

4. Viscosity solutions, uniqueness result, dynamic programming equations, and existence of the game value. In this section, we shall introduce the generalized notion of viscosity solution for our Isaacs' system of variational inequalities. The value functions defined in sections 2 and 3 turn out to be its viscosity sub- or supersolutions. We then prove the uniqueness of the viscosity solution and establish the existence of the value of our stochastic switching game.

Define for $(a,b,t,x,q,Q) \in A \times B \times [0,1] \times X \times X \times \mathcal{S}$,

$$H^{a,b}(t,x,q,Q) := \ f^0(t,x,a,b) + < q, f(t,x,a,b) > + \frac{1}{2}\mathrm{tr}(Qgg^*(t,x,a,b)). \tag{4.1}$$

Here $\mathcal{S}$ is the set of all real symmetric transformations in X. Let $C^{1,2}([0,1) \times X)$ be the set of all continuous functions which are continuously differentiable in t and twice continuously differentiable in x.

Associated with our stochastic switching game is the following Isaacs' system of quasi-variational inequalities where W is to be solved:

(1) For $(a,b,t,x) \in A \times B \times [0,1] \times X$,

$$M_{a,b}[W](t,x) \leq W_{a,b}(t,x) \leq M^{a,b}[W](t,x); \tag{4.2}$$

(2) for $(a,b,t,x) \in A \times B \times [0,1] \times X$ such that $W_{a,b}(t,x) > M_{a,b}[W](t,x)$,

$$\frac{\partial}{\partial t} W_{a,b}(t,x) + H^{a,b}(t,x,\nabla W_{a,b}(t,x), \nabla^2 W_{a,b}(t,x)) \geq 0; \tag{4.3}$$

(3) for $(a,b,t,x) \in A \times B \times [0,1] \times X$ such that $W_{a,b}(t,x) < M^{a,b}[W](t,x)$,

$$\frac{\partial}{\partial t} W_{a,b}(t,x) + H^{a,b}(t,x,\nabla W_{a,b}(t,x), \nabla^2 W_{a,b}(t,x)) \leq 0; \tag{4.4}$$

(4) the terminal condition

$$W_{a,b}(1,x) = h(x), \quad (a,b,x) \in A \times B \times X. \tag{4.5}$$

DEFINITION 4.1. *An $\mathbb{R}^{m\times n}$-valued continuous function $W = (W_{a,b})_{a\in A, b\in B}$ on $[0,T] \times X$ is called a viscosity sub- (resp., super-) solution of* (4.2)–(4.5) *if it satisfies* (4.2) *and* (4.5), *and moreover, for any $\varphi(\cdot,\cdot) \in C^{1,2}([0,1) \times X)$ and $(a,b) \in A \times B$, whenever $W_{a,b}(\cdot,\cdot) - \varphi(\cdot,\cdot)$ attains a local maximum (resp., minimum) at $(t_0,x_0) \in [0,1) \times X$ and*

$$W_{a,b}(t_0,x_0) > M_{a,b}[W](t_0,x_0) \qquad (\text{resp., } W_{a,b}(t_0,x_0) < M^{a,b}[W](t_0,x_0)),$$

we have

$$\frac{\partial}{\partial t}\varphi(t_0,x_0) + H^{a,b}(t_0,x_0,\nabla\varphi(t_0,x_0), \nabla^2\varphi(t_0,x_0)) \geq 0$$
$$\left(\textit{resp., } \frac{\partial}{\partial t}\varphi(t_0,x_0) + H^{a,b}(t_0,x_0,\nabla\varphi(t_0,x_0), \nabla^2\varphi(t_0,x_0)) \leq 0 \right).$$

An $\mathbb{R}^{m\times n}$-valued function $W = (W_{a,b})_{a\in A, b\in B}$ on $[0,T] \times X$ is called a viscosity solution of (4.2)–(4.5) *if it is both a viscosity sub- and supersolution of* (4.2)–(4.5).

Propositions 3.3 and 3.9 imply the following result.

PROPOSITION 4.1. (1) *The r-lower and r-upper value functions V^1 and U_1 are viscosity sub- and supersolutions of* (4.2)–(4.5), respectively.

(2) The functions $v = (v_{a,b})$ and $u = (u^{a,b})$ defined in Proposition 3.6 *are viscosity solutions of* (4.2)–(4.5).

Proof of Proposition 4.1. We now prove that the r-lower value function V^1 is a viscosity subsolution of (4.2)–(4.5). From the definition, it follows that $V^1_{a,b} = h$ for $(a,b) \in A \times B$. In view of (2.11), we see that V^1 satisfies (4.2).

Consider $\varphi(\cdot,\cdot) \in C^{1,2}([0,1)\times X)$ and $(a,b) \in A\times B$. Assume that $V^1_{a,b}(\cdot,\cdot) - \varphi(\cdot,\cdot)$ attains a local maximum at $(s,x) \in [0,1) \times X$ and

$$V^1_{a,b}(s,x) > M_{a,b}[V^1](s,x).$$

From Proposition 3.3, we see that there exist a deterministic time $s_0 > s$ and a sufficiently small number $\delta_0 > 0$, such that for all $\hat{s} \in (s,s_0]$ and $\delta \in (0,\delta_0]$, we have

$$V^1_{a,b}(s,x) \leq E_{sx}\bigg\{ \int_s^{\hat{s}\wedge\tau^\delta} f^0(r,y^{a,b}(r),a,b)\,dr + V^1_{a,b}(\hat{s}\wedge\tau^\delta, y^{a,b}(\hat{s}\wedge\tau^\delta)) \bigg\}.$$

Here we have abbreviated $\tau^\delta_{s,x}(a,b)$ as τ^δ. For sufficiently small $\hat{s} \in [s,s_0]$ and $\delta \in (0,\delta_0]$, we have

$$V^1_{a,b}(\hat{s}\wedge\tau^\delta, y^{a,b}(\hat{s}\wedge\tau^\delta)) - \varphi(\hat{s}\wedge\tau^\delta, y^{a,b}(\hat{s}\wedge\tau^\delta)) \leq V^1(s,x) - \varphi(s,x). \tag{4.6}$$

Therefore,

$$E_{sx}\bigg\{ \int_s^{\hat{s}\wedge\tau^\delta} f^0(r,y^{a,b}(r),a,b)\,dr + \varphi(\hat{s}\wedge\tau^\delta, y^{a,b}(\hat{s}\wedge\tau^\delta)) - \varphi(s,x) \bigg\} \geq 0.$$

From Itô's formula, we conclude that

$$E_{sx}\int_s^{\hat{s}\wedge\tau^\delta}\left[\frac{\partial}{\partial t}\varphi(r,y^{a,b}(r))+H^{a,b}(r,y^{a,b}(r),\nabla\varphi(r,y^{a,b}(r)),\nabla^2\varphi(r,y^{a,b}(r)))\right]dr\geq 0.$$

Noting (see the arguments following Remark 3.1) that

$$\lim_{\hat{s}\to s+}\frac{P(\{\tau^\delta\leq\hat{s}\})}{\hat{s}-s}=0,$$

we have

$$\begin{aligned}0\leq&\lim_{\hat{s}\to s+}(\hat{s}-s)^{-1}E_{sx}\left\{\int_s^{\hat{s}\wedge\tau^\delta}\left[\frac{\partial}{\partial t}\varphi(r,y^{a,b}(r))+H^{a,b}(\cdots)\right]dr\right\}\\ =&\lim_{\hat{s}\to s+}(\hat{s}-s)^{-1}E_{sx}\left\{\chi_{\{\hat{s}\leq\tau^\delta\}}\int_s^{\hat{s}}\left[\frac{\partial}{\partial t}\varphi(r,y^{a,b}(r))+H^{a,b}(\cdots)\right]dr\right\}\\ =&\frac{\partial}{\partial t}\varphi(s,x)+H^{a,b}(s,x,\nabla\varphi(s,x),\nabla^2\varphi(t,x)).\end{aligned}$$

Concluding the above, we see that V^1 is a viscosity subsolution.

Noting (3.59), (3.61a), and (3.61b), we can prove all other assertions in Proposition 4.1 in an identical way. ∎

Let us introduce the following sets, which are adopted from Evans and Ishii [5]. For function $v:[0,1]\times X\to[-\infty,+\infty]$ and $(s,z)\in[0,1)\times X$, define

$$\begin{aligned}\wp^{2,+}v(s,z):=&\Big\{(p,q,Q)\in\mathbb{R}\times X\times\mathcal{S}:v(t,x)\\ &\leq v(s,z)+p(t-s)+\langle q,x-z\rangle+\tfrac{1}{2}\langle Q(x-z),x-z\rangle\\ &+o(|t-s|+|x-z|^2)\text{ as }[0,1]\times X\ni(t,x)\to(s,z)\Big\},\end{aligned}\tag{4.7}$$

$$\begin{aligned}\bar{\wp}^{2,+}v(s,z):=&\Big\{(p,q,Q)\in\mathbb{R}\times X\times\mathcal{S}:\exists(t_i,x_i)\in[0,1]\times X,\\ &(p_i,q_i,Q_i)\in\wp^{2,+}v(t_i,x_i),\\ &(t_i,x_i,v(t_i,x_i),p_i,q_i,Q_i)\to(s,z,v(s,z),p,q,Q)\Big\}.\end{aligned}\tag{4.8}$$

Define for $(s,z)\in[0,1)\times X$

$$\wp^{2,-}v(s,z)=-\wp^{2,+}(-v)(s,z)\text{ and }\bar{\wp}^{2,-}v(s,z)=-\bar{\wp}^{2,+}(-v)(s,z).\tag{4.9}$$

The following result is standard.

PROPOSITION 4.2. *An $\mathbb{R}^{m\times n}$-valued function $W=(W_{a,b})_{a\in A,b\in B}$ on $[0,T]\times X$ is a viscosity sub- (resp., super-) solution of (4.2)–(4.5) if and only if it satisfies (4.2) and (4.5), and moreover, for any $(t,x,a,b)\in[0,1)\times X\times A\times B$, whenever $(p,q,Q)\in\bar{\wp}^{2,+}W_{a,b}(t,x)$ (resp., $\bar{\wp}^{2,-}W_{a,b}(t,x)$) and*

$$W_{a,b}(t_0,x_0)>M_{a,b}[W](t_0,x_0)\qquad(\text{resp., }W_{a,b}(t_0,x_0)<M^{a,b}[W](t_0,x_0)),$$

we have

$$p+H^{a,b}(t,x,q,Q)\geq 0\text{ (resp., }p+H^{a,b}(t,x,q,Q)\leq 0).\tag{4.10}$$

Now let us make a further assumption that will play an important role in the proof of the uniqueness result.

Hypothsis 4. For any loop $\{a_i, b_i\}_{i=1}^{j+1} \subset A \times B$, with the properties that

$$\begin{aligned} &j \le mn, \quad a_{j+1} = a_1, \quad b_{j+1} = b_1, \\ &\text{and either} \quad a_{i+1} = a_i \quad \text{or} \quad b_{i+1} = b_i \quad \forall\, 1 \le i \le j, \end{aligned} \tag{4.11}$$

we have

$$\sum_{i=1}^{j} k(s, a_i, a_{i+1}) - \sum_{i=1}^{j} l(s, b_i, b_{i+1}) \neq 0 \quad \forall s \in [0,1]. \tag{4.12}$$

THEOREM 4.1. *Assume Hypotheses* 1–4. *If W and $\widehat{W}$ are continuous viscosity sub- and supersolutions of* (4.2)–(4.5), *respectively, and satisfy for* $(t, x, y, a, b) \in [0,1] \times X \times X \times A \times B$ *the following:*

$$\begin{aligned} &|W_{a,b}(t,x)| + |\widehat{W}_{a,b}(t,x)| \le C(1 + |x|), \\ &|W_{a,b}(t,x) - W_{a,b}(t,y)| + |\widehat{W}_{a,b}(t,x) - \widehat{W}_{a,b}(t,y)| \le C|x-y|, \end{aligned} \tag{4.13}$$

then

$$W_{a,b}(t,x) \le \widehat{W}_{a,b}(t,x) \quad \forall (t,x,a,b) \in [0,1] \times X \times A \times B. \tag{4.14}$$

Proof of Theorem 4.1. We prove the theorem by contradiction. So suppose that $\exists (\bar a, \bar b, \bar t, \bar x) \in A \times B \times (0,1) \times X$ such that

$$W_{\bar a,\bar b}(\bar t, \bar x) - \widehat{W}_{\bar a,\bar b}(\bar t, \bar x) = \eta > 0. \tag{4.15}$$

Consider the following test function:

$$\psi(t,x,y) = \frac{|x-y|^2}{2\varepsilon} + \alpha e^{-\beta t}(1 + |x|^2 + |y|^2), \quad (t,x,y) \in [0,1] \times X \times X, \tag{4.16}$$

with parameters $\alpha > 0$ and $\beta > 0$. We choose a sufficiently small $\alpha > 0$ such that it does not depend on the parameter $\beta > 0$ and that it satisfies the following:

$$\psi(\bar t, \bar x, \bar x) < \frac{\eta}{2} \quad \forall \beta > 0. \tag{4.17}$$

Now consider the function

$$\Psi^{a,b}(t,x,y) = W_{a,b}(t,x) - \widehat{W}_{a,b}(t,y) - \psi(t,x,y), \quad (a,b,t,x,y) \in A \times B \times [0,1] \times X \times X. \tag{4.18}$$

From (4.13), (4.15), and (4.17), we see that there is a point $(a_0, b_0, t_0, x_0, y_0) \in A \times B \times [0,1] \times X \times X$ such that

$$\Psi^{a_0,b_0}(t_0,x_0,y_0) = \max_{\substack{a\in A\\ b\in B}} \sup_{t,x,y} \Psi^{a,b}(t,x,y) \ge \Psi^{\bar a,\bar b}(\bar t,\bar x,\bar x) \ge \frac{\eta}{2}. \tag{4.19}$$

At this stage, we have the following two conclusions.

Conclusion 1. Following the arguments of Yamada [9, pp. 424–425], we can show the following assertion: Without loss of generality, we may assume that

$$M_{a_0,b_0}[W](t_0,x_0) < W_{a_0,b_0}(t_0,x_0), \quad M^{a_0,b_0}[\widehat{W}](t_0,y_0) > \widehat{W}_{a_0,b_0}(t_0,y_0). \tag{4.20}$$

Otherwise, we have

$$M_{a_0,b_0}[W](t_0,x_0) = W_{a_0,b_0}(t_0,x_0) \quad \text{or} \quad M^{a_0,b_0}[\widehat{W}](t_0,y_0) = \widehat{W}_{a_0,b_0}(t_0,y_0). \tag{4.21}$$

Consequently, there is $b_1 \in B$ or $a_1 \in A$ such that

$$W_{a_0,b_0}(t_0,x_0) = W_{a_0,b_1}(t_0,x_0) - l(t_0,b_0,b_1) \tag{4.22}$$

or

$$\widehat{W}_{a_0,b_0}(t_0,y_0) = \widehat{W}_{a_1,b_0}(t_0,y_0) + k(t_0,a_0,a_1). \tag{4.23}$$

On the other hand, from (4.19), we have

$$\Psi^{a_0,b_0}(t_0,x_0,y_0) \geq \Psi^{a_1,b_0}(t_0,x_0,y_0), \tag{4.24}$$

which implies immediately

$$W_{a_0,b_0}(t_0,x_0) - \widehat{W}_{a_0,b_0}(t_0,y_0) \geq W_{a_1,b_0}(t_0,x_0) - \widehat{W}_{a_1,b_0}(t_0,y_0). \tag{4.25}$$

Therefore, we have

$$\begin{aligned} 0 \geq & W_{a_0,b_0}(t_0,x_0) - W_{a_1,b_0}(t_0,x_0) - k(t_0,a_0,a_1) \\ \geq & \widehat{W}_{a_0,b_0}(t_0,y_0) - \widehat{W}_{a_1,b_0}(t_0,y_0) - k(t_0,a_0,a_1), \end{aligned} \tag{4.26}$$

which shows the following:

$$W_{a_0,b_0}(t_0,x_0) = W_{a_1,b_0}(t_0,x_0) + k(t_0,a_0,a_1) \tag{4.27}$$

if (4.23) is true. In summary, there is $b_1 \in B$ or $a_1 \in A$ such that either (4.22) or (4.27) is true. Moreover, we have

$$W_{a_0,b_0}(t_0,x_0) - \widehat{W}_{a_0,b_0}(t_0,y_0) = W_{a_1,b_0}(t_0,x_0) - \widehat{W}_{a_1,b_0}(t_0,y_0), \tag{4.28}$$

from which it follows that

$$\Psi^{a_0,b_0}(t_0,x_0,y_0) = \Psi^{a_1,b_0}(t_0,x_0,y_0) = \max_{\substack{a\in A\\ b\in B}} \sup_{t,x,y} \Psi^{a,b}(t,x,y). \tag{4.29}$$

Symmetrically, we have

$$\Psi^{a_0,b_0}(t_0,x_0,y_0) = \Psi^{a_0,b_1}(t_0,x_0,y_0) = \max_{\substack{a\in A\\ b\in B}} \sup_{t,x,y} \Psi^{a,b}(t,x,y). \tag{4.30}$$

Set $(\tilde{a}_1,\tilde{b}_1) := (a_0,b_0)$. We can repeat the above argument to start from the pair of parameters $(\tilde{a}_2,\tilde{b}_2)$—which is $(a_1,\tilde{b}_1)$ or $(\tilde{a}_1,b_1)$—to find a new pair of parameters (a_2,b_2) such that either

$$W_{\tilde{a}_2,\tilde{b}_2}(t_0,x_0) = W_{\tilde{a}_2,b_2}(t_0,x_0) - l(t_0,\tilde{b}_2,b_2) \tag{4.31}$$

or

$$W_{\tilde{a}_2,\tilde{b}_2}(t_0,x_0) = W_{a_2,\tilde{b}_2}(t_0,x_0) + k(t_0,\tilde{a}_2,a_2) \tag{4.32}$$

is true. Moreover,

$$\Psi^{\tilde{a}_2,b_2}(t_0,x_0,y_0)=\Psi^{a_2,\tilde{b}_2}(t_0,x_0,y_0)=\max_{\substack{a\in A\\ b\in B}}\sup_{t,x,y}\Psi^{a,b}(t,x,y). \tag{4.33}$$

Then we can continue the procedure until we find a loop $\{\tilde{a}_i,\tilde{b}_i\}_{i=1}^{j+1}$ which satisfies the properties (4.11). Summing up (4.31)–(4.32) for the loop, we get

$$\sum_{i=1}^{j}k(s,\tilde{a}_i,\tilde{a}_{i+1})-\sum_{i=1}^{j}l(s,\tilde{b}_i,\tilde{b}_{i+1})=0. \tag{4.34}$$

Then we get a contradiction to Hypothesis 4.

Conclusion 2. On the maximum point (a_0,b_0,t_0,x_0,y_0), we have the following properties:

(i) There is a constant $C_{\alpha,\beta}$, which depends on positive α,β such that $|x_0|+|y_0|\le C_{\alpha,\beta}$;

(ii) from (4.13) and the following inequality:

$$2\Psi^{a_0,b_0}(t_0,x_0,y_0)\ge\Psi^{a_0,b_0}(t_0,x_0,x_0)+\Psi^{a_0,b_0}(t_0,y_0,y_0),$$

we obtain $|x_0-y_0|\le\varepsilon C_{\alpha,\beta}$. Hence, $|x_0-y_0|\to 0$ as $\varepsilon\to 0$, while keeping α and β fixed;

(iii) since $\Psi^{a_0,b_0}(1,x_0,y_0)\le h(x_0)-h(y_0)\le C|x_0-y_0|$, we conclude from (4.19) that $t_0\in[0,1)$ whenever $\varepsilon>0$ is sufficiently small.

A simple computation gives rise to the following:

$$\begin{aligned}\partial_t\psi(t,x,y)&=-\beta\alpha e^{-\beta t}(1+|x|^2+|y|^2),\\ \partial_x\psi(t,x,y)&=\frac{(x-y)}{\varepsilon}+2\alpha e^{-\beta t}x,\\ \partial_y\psi(t,x,y)&=\frac{(y-x)}{\varepsilon}+2\alpha e^{-\beta t}y,\\ \partial^2_{(x,y)}\psi(t,x,y)&=\frac{1}{\varepsilon}\begin{pmatrix}I&-I\\-I&I\end{pmatrix}+2\alpha e^{-\beta t}\begin{pmatrix}I&0\\0&I\end{pmatrix}.\end{aligned} \tag{4.35}$$

Then, applying Theorem 9 of Evans and Ishii [5] to the function

$$W_{a_0,b_0}(t,x)+(-\widehat{W}_{a_0,b_0})(t,y)-\psi(t,x,y)$$

at the point (t_0,x_0,y_0), we can find $p_1,p_2\in\mathbb{R}$ and $Q_1,Q_2\in\mathcal{S}$ such that

$$\begin{aligned}&(p_1,\partial_x\psi(t_0,x_0,y_0),Q_1)\in\bar{\wp}^{2,+}W_{a_0,b_0}(t_0,x_0),\\ &(p_2,\partial_y\psi(t_0,x_0,y_0),Q_2)\in\bar{\wp}^{2,+}(-\widehat{W}_{a_0,b_0})(t_0,y_0),\\ &p_1+p_2=\partial_t\psi(t_0,x_0,y_0),\\ &\begin{pmatrix}Q_1&0\\0&Q_2\end{pmatrix}\le\partial^2_{(x,y)}\psi(t_0,x_0,y_0)+\varepsilon\Big(\partial^2_{(x,y)}\psi(t_0,x_0,y_0)\Big)^2.\end{aligned} \tag{4.36}$$

By the definitions of viscosity sub- and supersolutions, and in view of (4.20), we have

$$\begin{aligned}p_1+H^{a,b}\left(t_0,x_0,\frac{(x_0-y_0)}{\varepsilon}+2\alpha e^{-\beta t_0}x_0,Q_1\right)&\ge 0,\\ -p_2+H^{a,b}\left(t_0,y_0,-\frac{(y_0-x_0)}{\varepsilon}-2\alpha e^{-\beta t_0}y_0,-Q_2\right)&\le 0.\end{aligned} \tag{4.37}$$

Thus, we have (see (4.36))

$$\begin{aligned}
&\beta\alpha e^{-\beta t_0}(1+|x_0|^2+|y_0|^2)\\
\le\ & H^{a,b}\left(t_0,x_0,\frac{(x_0-y_0)}{\varepsilon}+2\alpha e^{-\beta t_0}x_0,Q_1\right)\\
&-H^{a,b}\left(t_0,y_0,-\frac{(y_0-x_0)}{\varepsilon}-2\alpha e^{-\beta t_0}y_0,-Q_2\right)\\
\le\ & \frac{1}{2}\mathrm{tr}[(g^*Q_1g)(t_0,x_0,a_0,b_0)+(g^*Q_2g)(t_0,y_0,a_0,b_0)]\\
&+\left[\left\langle\frac{x_0-y_0}{\varepsilon},f(t_0,x_0,a_0,b_0)-f(t_0,y_0,a_0,b_0)\right\rangle\right.\\
&\left.+2\alpha e^{-\beta t_0}\Big(\langle x_0,f(t_0,x_0,a_0,b_0)\rangle+\langle y_0,f(t_0,y_0,a_0,b_0)\rangle\Big)\right]\\
&+f^0(t_0,x_0,a_0,b_0)-f^0(t_0,y_0,a_0,b_0).
\end{aligned}$$

Set

$$\begin{aligned}
(\mathrm{I}) &:= \frac{1}{2}\mathrm{tr}[(g^*Q_1g)(t_0,x_0,a_0,b_0)+(g^*Q_2g)(t_0,y_0,a_0,b_0)]\\
(\mathrm{II}) &:= \left\langle\frac{x_0-y_0}{\varepsilon},f(t_0,x_0,a_0,b_0)-f(t_0,y_0,a_0,b_0)\right\rangle\\
&\quad+2\alpha e^{-\beta t_0}\left(\langle x_0,f(t_0,x_0,a_0,b_0)\rangle+\langle y_0,f(t_0,y_0,a_0,b_0)\rangle\right)\\
(\mathrm{III}) &:= f^0(t_0,x_0,a_0,b_0)-f^0(t_0,y_0,a_0,b_0).
\end{aligned}$$

We now estimate (I), (II), and (III) separately. It is immediate that

$$\begin{aligned}
&\partial^2_{(x,y)}\psi(t_0,x_0,y_0)+\varepsilon\Big(\partial^2_{(x,y)}\psi(t_0,x_0,y_0)\Big)^2\\
&\le\frac{3}{\varepsilon}\begin{pmatrix}I & -I\\ -I & I\end{pmatrix}+4\alpha e^{-\beta t_0}\begin{pmatrix}I & -I\\ -I & I\end{pmatrix}\\
&\quad+(4\varepsilon\alpha^2e^{-2\beta t_0}+2\alpha e^{-\beta t_0})\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}.
\end{aligned}\tag{4.38}$$

Then we have

$$\begin{aligned}
(\mathrm{I}) &\le \frac{C}{\varepsilon}|x_0-y_0|^2+C\alpha e^{-\beta t_0}|x_0-y_0|^2\\
&\quad+C(2\varepsilon\alpha^2e^{-2\beta t_0}+\alpha e^{-\beta t_0})(1+|x_0|^2+|y_0|^2),\\
(\mathrm{II}) &\le \frac{C}{\varepsilon}|x_0-y_0|^2+C\alpha e^{-\beta t_0}(1+|x_0|^2+|y_0|^2),\\
(\mathrm{III}) &\le C|x_0-y_0|.
\end{aligned}\tag{4.39}$$

Hence, we have

$$\begin{aligned}
&\beta\alpha e^{-\beta t_0}(1+|x_0|^2+|y_0|^2)\\
&\le\frac{C}{\varepsilon}|x_0-y_0|^2+C|x_0-y_0|+C\alpha e^{-\beta t_0}(|x_0-y_0|^2+1+|x_0|^2+|y_0|^2)\\
&\quad+C(2\varepsilon\alpha^2e^{-2\beta t_0}+\alpha e^{-\beta t_0})(1+|x_0|^2+|y_0|^2).
\end{aligned}\tag{4.40}$$

Letting $\varepsilon \to 0$, we get

$$\beta\alpha e^{-\beta t}(1+2|x|^2) \leq C\alpha e^{-\beta t}(1+2|x|^2) + C(2\varepsilon\alpha^2 e^{-2\beta t} + \alpha e^{-\beta t})(1+2|x|^2) \tag{4.41}$$

for some $(t,x) \in [0,1] \times X$, which immediately implies $\beta \leq C + C\alpha$. Since we can choose β to be sufficiently large so that $\beta > C + \alpha C$, we arrive at a contradiction. Hence, (4.14) is proved. ◻

Remark 4.1. Note that the stochastic nature leads to the corresponding Isaacs' system of variational inequalities involving a second-order differential operator, and thus the proof of the uniqueness of the viscosity solutions necessarily involves the computation of the second-order differentials of the chosen test function, say, ψ in our proof. Due to this feature, the test function used by Yong [10] does not seem to carry over to our case. Here we use a different test function. It is both simpler and more powerful in proving the uniqueness of unbounded viscosity solutions, as is shown in the above proof.

THEOREM 4.2. *Let Hypotheses* 1–4 *be satisfied. Then our stochastic differential switching game described by* (1.1) *and* (1.2) *has a value. The function* $V^1 = v = V = U = u = U^1$ *is the unique viscosity solution of* (4.2)–(4.5).

Proof of Theorem 4.2. From Proposition 4.1, we see that V^1 is a viscosity subsolution and v is a viscosity supersolution, while u is a viscosity subsolution and U^1 is a viscosity supersolution. From Theorem 4.1, it follows immediately that

$$V^1_{a,b}(t,x) \leq v_{a,b}(t,x) \text{ and } u^{a,b}(t,x) \leq U_1^{a,b}(t,x), \quad (t,x,a,b) \in [0,1] \times X \times A \times B.$$

In view of Proposition 3.6, we have

$$V^1_{a,b} \leq v_{a,b} \leq V_{a,b}, \quad U_1^{a,b} \geq u^{a,b} \geq U^{a,b}, \quad (a,b) \in A \times B.$$

Combining these inequalities with (2.9), we have

$$V^1_{a,b} = v_{a,b} = V_{a,b}, \quad U_1^{a,b} = u^{a,b} = U^{a,b}, \quad (a,b) \in A \times B.$$

In short form, we have

$$V^1 = v = V, \quad U_1 = u = U.$$

From Proposition 4.1, we also know that u and v are two viscosity solutions of (4.2)–(4.5). By Theorem 4.1, we have $u = v$.

Concluding the above, we have

$$V^1 = v = V = U_1 = u = U.$$

Therefore, our stochastic differential switching game described by (1.1) and (1.2) has a value, and the function $V^1 = v = V = U = u = U^1$ is the unique viscosity solution of (4.2)–(4.5). ◻

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