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WEAK SHARP MINIMA FOR SEMI-INFINITE OPTIMIZATION PROBLEMS WITH APPLICATIONS*

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Abstract. We study local weak sharp minima and sharp minima for smooth semi-infinite optimization problems SIP. We provide several dual and primal characterizations for a point to be a sharp minimum or a weak sharp minimum of SIP. As applications, we present several sufficient and necessary conditions of calmness for infinitely many smooth inequalities. In particular, we improve some calmness results in [R. Henrion and J. Outrata, *Math. Program.*, 104 (2005), pp. 437–464].

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1. Introduction.

The notion of a sharp minimum, namely, a strong isolated minimum or a strong unique local minimum, of real-valued functions, introduced in [24], plays an important role in the convergence analysis of numerical algorithms in mathematical programming problems (see [4, 12, 22, 30]). As such, it has received extensive attention and investigation. As a generalization of sharp minima, weak sharp minima for real-valued functions were introduced and studied in [5]. Extensive study of weak sharp minima for real-valued convex functions has been done in the literature (cf. [2, 3, 28, 31, 33]). It has been found that the weak sharp minimum is closely related to the error bound in convex programming (cf. [32]), a notion that has received much attention and has produced a vast number of publications (see [16, 17, 23, 31, 32]).

The calmness is an important type of Lipschitz-like property for multifunctions, which play a key role in many issues of mathematical programming such as sensitivity analysis, error bounds, and optimality conditions. Thus, the study of the calmness has recently received increasing attention in the mathematical programming literature (see [8, 9, 10, 15]).

In this paper, we will study local weak sharp minima for the following semi-infinite optimization problem:

$$\text{(SIP)} \qquad \min f(x) \ \text{ subject to } \phi(x,y) \le 0 \text{ for all } y \in Y,$$

where $f : X \to R$ is a smooth function, X is an Euclidean space, Y is an infinite index set, and $\phi : X \times Y \to R$ is a function such that the function $x \mapsto \phi(x,y)$ is smooth for each index $y \in Y$. It is known that (SIP) has many important and interesting applications in engineering design, control of robots, mechanical stress of

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materials, and social sciences; see the survey paper [11] and the books [6, 21, 25]. In the past three decades, (SIP) and its broad range of applications have been an active study area in mathematical programming (see [1, 7, 13, 14, 20, 27, 29] and references therein).

Let Z denote the set of all feasible points for (SIP); that is,

$$Z := \{x \in X : \ \phi(x, y) \leq 0 \text{ for all } y \in Y\}.$$

We say that $\bar{x} \in X$ is a local sharp minimum of (SIP) if $\bar{x} \in Z$ and there exist $\eta, \delta \in (0, +\infty)$ such that

$$\eta\|x - \bar{x}\| \leq f(x) - f(\bar{x}) + \sup_{y\in Y}[\phi(x, y)]_+ \text{ for all } x \in B(\bar{x}, \delta), \tag{1.1}$$

where $B(\bar{x}, \delta)$ denotes the open ball with center $\bar{x}$ and radius δ.

We say that $\bar{x}$ is a local weak sharp minimum of (SIP) if $\bar{x} \in Z$ and there exist $\eta, \delta \in (0, +\infty)$ such that

$$\eta d(x, L_f(\bar{x}) \cap Z) \leq f(x) - f(\bar{x}) + \sup_{y\in Y}[\phi(x, y)]_+ \text{ for all } x \in B(\bar{x}, \delta), \tag{1.2}$$

where $L_f(\bar{x}) := \{x \in X : \ f(x) = f(\bar{x})\}$ and $d(x, L_f(\bar{x}) \cap Z) := \inf\{\|x - u\| : \ u \in L_f(\bar{x}) \cap Z\}$.

Recall a known optimality condition of (SIP) (cf. [11, 13, 34]) that if $\bar{x}$ is a local minimum of (SIP) and a constraint qualification is satisfied at $\bar{x}$, then there exist $t_i \geq 0$ and $y_i \in I_0(\bar{x})$, $i = 1, \dots, p$, such that

$$0 = f'(\bar{x}) + \sum_{i=1}^{p} t_i \phi'_x(\bar{x}, y_i), \tag{1.3}$$

where $I_0(\bar{x})$ denotes the index set of active inequality constraints at $\bar{x}$. Furthermore, under a convexity assumption, the optimality condition (1.3) also becomes sufficient.

When Y is a compact topological space and $\phi(x, y)$ and $\phi'_x(x, y)$ satisfy some continuity conditions, we will prove that $\bar{x}$ is a local weak sharp minimum of (SIP) if and only if there exist $\eta, \delta \in (0, +\infty)$ such that for all $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$

$$\tilde{N}(L_f(\bar{x}) \cap Z, u) \cap \eta B_{X^*} \subset f'(u) + [0, 1]\text{co}\{\phi'_x(u, y) : \ y \in I_0(\bar{x})\}, \tag{1.4}$$

where X^* denotes the dual space of X, B_{X^*} denotes the unit ball of X^*, and $\tilde{N}(A, u)$ is any one of the Fréchet, limiting, or Clarke normal cones of A at u; in particular, $\bar{x}$ is a local sharp minimum of (SIP) if and only if

$$0 \in \text{int}(f'(\bar{x}) + [0, 1]\text{co}\{\phi'_x(\bar{x}, y) : \ y \in I_0(\bar{x})\}). \tag{1.5}$$

It is interesting to compare (1.5) and (1.4) with (1.3). These are referred to as dual characterizations. We also obtain a set of primal ones for a local weak sharp minimum of (SIP). Moreover, we obtain mixed characterizations for a local (weak) sharp minimum.

Motivated by Henrion and Outrata [10], we consider the calmness of multifunctions defined by infinitely many smooth inequalities. As applications of several characterizations of weak sharp minima mentioned above, we provide several equivalent conditions for the calmness; in particular, we improve one of the main results in [10].

The outline of the paper is as follows. In section 2, some preliminaries on notions of variational analysis are given. In section 3, several characterizations for a local weak sharp minimum and a local sharp minimum of (SIP) are obtained. In section 4, some equivalent conditions for the calmness of the system of infinitely many smooth inequalities are provided.

2. Preliminaries. Let X be an Euclidean space and $\psi : X \to R \cup \{+\infty\}$ a proper lower semicontinuous function. For $x \in \mathrm{dom}(\psi) := \{x \in X : \phi(x) < +\infty\}$, let $\hat{\partial}\psi(x)$ denote the Fréchet subdifferential of ψ at x; that is,

$$\hat{\partial}\psi(x) := \left\{ x^* \in X^* : \liminf_{u \xrightarrow{\psi} x} \frac{\psi(u) - \psi(x) - \langle x^*, u - x\rangle}{\|u - x\|} \geq 0 \right\},$$

where $u \xrightarrow{\psi} x$ means $u \to x$ and $\psi(u) \to \psi(x)$. The limiting subdifferential of ψ at x is denoted by $\partial\psi(x)$ and is defined by

$$\partial\psi(x) := \limsup_{u \xrightarrow{\psi} x} \hat{\partial}\psi(u);$$

that is, $x^* \in \partial\psi(x)$ if and only if there exist sequences $x_k \xrightarrow{\psi} x$ and $x_k^* \to x^*$ with $x_k^* \in \hat{\partial}\psi(x_k)$.

The following proposition is well known (cf. [18, Theorem 2.33]) and is useful for us.

PROPOSITION 2.1. *Let $\psi_1, \psi_2 : X \to R \cup \{+\infty\}$ be proper lower semicontinuous functions and $x \in \mathrm{dom}(\psi_1) \cap \mathrm{dom}(\psi_2)$. Suppose that ψ_1 is locally Lipschitz at x. Then*

$$\partial(\psi_1 + \psi_2)(x) \subset \partial\psi_1(x) + \partial\psi_2(x).$$

For a closed subset A of X and $a \in A$, let $\hat{N}(A, a)$ and $N(A, a)$ denote the Fréchet normal cone and the limiting normal cone of A at a, respectively; that is,

$$\hat{N}(A, a) = \hat{\partial}\delta_A(a) \ \text{ and } \ N(A, a) = \partial\delta_A(a),$$

where δ_A denotes the indicator function of A. Thus, $x^* \in \hat{N}(A, a)$ if and only if $\limsup_{x \xrightarrow{A} a} \frac{\langle x^*, x-a\rangle}{\|x-a\|} \leq 0$, where $x \xrightarrow{A} a$ means $x \in A$ and $x \to a$, and $x^* \in N(A, a)$ if and only if there exist $x_k \xrightarrow{A} a$ and $x_k^* \to x^*$ such that $x_k^* \in \hat{N}(A, x_k)$ for all $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all natural numbers.

Let $T(A, a)$ denote the tangent cone of A at a; that is,

$$T(A, a) := \{h \in X : \exists t_k \to 0^+ \text{ and } h_k \to h \text{ such that } a + t_k h_k \in A \text{ for all } k \in \mathbb{N}\}.$$

It is known (cf. [26, Theorem 6.28]) that

$$\hat{N}(A, a) = \{x^* \in X^* : \langle x^*, h\rangle \leq 0 \text{ for all } h \in T(A, a)\}. \tag{2.1}$$

Let $T_c(A, a)$ denote the Clarke tangent cone; that is, $v \in T_c(A, a)$ if and only if, for each sequence $\{a_k\}$ in A converging to a and each sequence $\{t_k\}$ in $(0, \infty)$ decreasing to 0, there exists a sequence $\{v_k\}$ in X converging to v such that $a_k + t_k v_k \in A$ for all $k \in \mathbb{N}$. Let $N_c(A, a)$ denote the Clarke normal cone of A at a and be defined by

$$N_c(A, a) := \{x^* \in X^* : \langle x^*, v\rangle \leq 0 \quad \text{for all } v \in T_c(A, a)\}. \tag{2.2}$$

It is well known (cf. [26, Proposition 6.5] and [18, Theorem 3.57]) that

$$\hat{N}(A,a) \subset N(A,a) \subset N_c(A,a) \text{ and } N_c(A,a) = \overline{\text{co}}N(A,a). \tag{2.3}$$

Histories of the subdifferentials and the normal cones can be found in [18, 19, 26].

For any $x \in X$, let $P_A(x)$ denote the projection of x on A; that is,

$$P_A(x) := \{a \in A : \|x - a\| = d(x, A)\}.$$

We will need the following known result (cf. [26, Example 6.16]).

LEMMA 2.1. *Let A be a closed subset of X and $x \in X$. Then*

$$x - a \in \hat{N}(A,a) \text{ for any } a \in P_A(x). \tag{2.4}$$

3. Weak sharp minima for smooth semi-infinite optimization problems. Throughout the remainder of this paper, let X be an Euclidean space of dimension m and Y a compact topological space (e.g., a bounded closed subset of an Euclidean space). Let $f : X \to R$ and $\phi : X \times Y \to R$ be as in section 1. We always assume that the following properties hold:

(P1) The function $x \mapsto \phi(x,y)$ is smooth for each $y \in Y$, and the function $y \mapsto \phi(x,y)$ is continuous for each $x \in X$.

(P2) The functions $(x,y) \mapsto \phi(x,y)$ and $(x,y) \mapsto \phi'_x(x,y)$ are continuous on $X \times Y$, where $\phi'_x(x,y)$ denotes the derivative of the function $x \mapsto \phi(x,y)$.

In the literature on semi-infinite optimization, assumptions (P1) and (P2) have been extensively used.

Since Y is compact and (P1) holds, it is easy to verify that $\bar{x} \in Z$ is a local sharp minimum and a local weak sharp minimum of (SIP) if and only if there exist $\eta, \delta \in (0, +\infty)$ such that

$$\eta\|x - \bar{x}\| \le f(x) - f(\bar{x}) + \max_{y \in Y}[\phi(x,y)]_+ \text{ for all } x \in B(\bar{x}, \delta) \tag{3.1}$$

and

$$\eta d(x, L_f(\bar{x}) \cap Z) \le f(x) - f(\bar{x}) + \max_{y \in Y}[\phi(x,y)]_+ \text{ for all } x \in B(\bar{x}, \delta), \tag{3.2}$$

respectively.

It follows from (3.2) that every local weak sharp minimum of (SIP) is a local solution of (SIP). Clearly, $\bar{x}$ is a local sharp minimum of (SIP) if and only if $\bar{x}$ is a local weak sharp minimum of (SIP) and

$$L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta) = \{\bar{x}\} \text{ for some } \delta > 0.$$

For convenience, let

$$\Phi(x) := \max\{\phi(x,y) : y \in Y\} \text{ and } I(x) := \{y \in Y : \phi(x,y) = \Phi(x)\}.$$

From (P1) and the compactness of Y, it is clear that $I(x) \neq \emptyset$ for all $x \in X$. For each $x \in Z$, let $I_0(x)$ denote the index set of active inequality constraints at x; that is,

$$I_0(x) := \{y \in Y : \phi(x,y) = 0\}.$$

We will provide characterizations for $\bar{x}$ to be a local weak sharp minimum or a local sharp minimum of (SIP). We need the following lemma.

LEMMA 3.1. *Let $\bar{x} \in X$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that for any $x \in B(\bar{x}, \delta)$ and $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$*

$$\langle f'(u), x - u\rangle \leq f(x) - f(\bar{x}) + \varepsilon\|x - u\|$$

and

$$\langle \phi'_x(u, y), x - u\rangle \leq \phi(x, y) + \varepsilon\|x - u\| \quad \textit{for all } y \in I_0(u).$$

Proof. Since $(x, y) \mapsto \phi'_x(x, y)$ is continuous, for any $y \in Y$ there exist open neighborhoods U_y and V_y of $\bar{x}$ and y, respectively, such that

$$\|\phi'_x(x_1, v_1) - \phi'_x(x_2, v_2)\| < \varepsilon \quad \text{for all } x_1, x_2 \in U_y \text{ and for all } v_1, v_2 \in V_y.$$

Since Y is compact, there exist $y_1, \ldots, y_k \in Y$ such that $Y = \bigcup_{i=1}^k V_{y_i}$. Let $U := \bigcap_{i=1}^n U_{y_i}$, and take $\delta > 0$ such that $B(\bar{x}, \delta) \subset U$. It is easy to verify that

$$\|\phi'_x(x_1, y) - \phi'_x(x_2, y)\| < \varepsilon \quad \text{for all } x_1, x_2 \in B(\bar{x}, \delta) \text{ and for all } y \in Y. \tag{3.3}$$

Since f is continuously differentiable, we assume without loss of generality that

$$\|f'(x_1) - f'(x_2)\| < \varepsilon \quad \text{for all } x_1, x_2 \in B(\bar{x}, \delta) \tag{3.4}$$

(considering smaller δ if necessary). Let $x \in B(\bar{x}, \delta)$, $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$, and $y \in I_0(u)$. By the mean value theorem, there exist $\theta_1, \theta_2 \in (u,\ x) := \{tu + (1 - t)x : 0 < t < 1\}$ such that

$$f(x) - f(\bar{x}) = f(x) - f(u) = \langle f'(\theta_1), x - u\rangle$$

and

$$\phi(x, y) = \phi(x, y) - \phi(u, y) = \langle \phi'_x(\theta_2, y), x - u\rangle.$$

It follows from (3.4) and (3.3) that

$$\begin{aligned}\langle f'(u), x - u\rangle &= \langle f'(u) - f'(\theta_1), x - u\rangle + \langle f'(\theta_1), x - u\rangle \\ &\leq f(x) - f(\bar{x}) + \varepsilon\|x - u\|\end{aligned}$$

and

$$\langle \phi'_x(u, y), x - u\rangle \leq \phi(x, y) + \varepsilon\|x - u\|.$$

The proof is completed. ∎

LEMMA 3.2. *Let $\bar{x} \in Z$ and $u \in L_f(\bar{x}) \cap Z$. Then*

$$\langle f'(u), h\rangle = 0 \text{ and } \langle \phi'_x(u, y), h\rangle \leq 0, \quad \textit{for all } h \in T(L_f(\bar{x}) \cap Z, u) \textit{ and for all } y \in I_0(u).$$

Proof. Let $h \in T(L_f(\bar{x}) \cap Z, u)$ and $y \in I_0(u)$. Then there exist $t_k \to 0^+$ and $h_k \to h$ such that $u + t_k h_k \in L_f(\bar{x}) \cap Z$ for all $k \in \mathbb{N}$. Hence

$$f(u + t_k h_k) = f(u) = f(\bar{x}) \text{ and } \phi(u + t_k h_k, y) \leq 0 \quad \text{for all } k \in \mathbb{N}.$$

Since f is continuously differentiable,

$$f(u + t_k h_k) - f(u) = \langle f'(u), t_k h_k\rangle + o(t_k).$$

It follows that $\langle f'(u), h_k\rangle + \frac{o(t_k)}{t_k} = 0$. This implies that $\langle f'(u), h\rangle = 0$. Let ε be an arbitrary positive number. Then Lemma 3.1 implies that

$$\langle \phi'(u,y), t_k h_k\rangle \le \phi(u + t_k h_k, y) + \varepsilon\|t_k h_k\| \le \varepsilon\|t_k h_k\|$$

for all k large enough, and so $\langle \phi'(u,y), h\rangle \le \varepsilon\|h\|$. Since ε is arbitrary, it follows that $\langle \phi'(u,y), h\rangle \le 0$. This completes the proof. □

In the next theorems we first provide some dual characterizations and then some primal characterizations for a feasible point of (SIP) to be a local weak sharp minimum. As usual, let co$A$ denote the convex hull of A. For convenience, we adopt the conventions that if $u \in L_f(\bar{x}) \cap Z$, with $I_0(u) = \emptyset$, then

$$[0,\ 1]\text{co}\{\phi'_x(u,y):\ y \in I_0(u)\} := \{0\}$$

and

$$\max_{y\in I_0(u)} [\langle \phi'_x(u,y), h\rangle]_+ := 0 \ \text{ for all } h \in X.$$

THEOREM 3.1. *Let $\bar{x}$ be a feasible point of* (SIP) *(i.e., $\bar{x} \in Z$). Then the following statements are equivalent:*

(i) *$\bar{x}$ is a local weak sharp minimum of* (SIP).

(ii) *There exist $\eta, \delta \in (0,\ +\infty)$ such that*

$$(3.5)\quad \hat{N}(L_f(\bar{x}) \cap Z, u) \cap \eta B_{X^*} \subset f'(u) + [0,\ 1]\text{co}\{\phi'_x(u,y):\ y \in I_0(u)\}$$

for all $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$.

(iii) *There exist $\eta, \delta \in (0,\ +\infty)$ such that*

$$(3.6)\quad N(L_f(\bar{x}) \cap Z, u) \cap \eta B_{X^*} \subset f'(u) + [0,\ 1]\text{co}\{\phi'_x(u,y):\ y \in I_0(u)\}$$

for all $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$.

(iv) *There exist $\eta, \delta \in (0,\ +\infty)$ such that*

$$(3.7)\quad N_c(L_f(\bar{x}) \cap Z, u) \cap \eta B_{X^*} \subset f'(u) + [0,\ 1]\text{co}\{\phi'_x(u,y):\ y \in I_0(u)\}$$

for all $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$.

Proof. (i)⇒(ii). Suppose that there exist $\eta, \delta \in (0,\ +\infty)$ such that (3.2) holds. Let $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \frac{\delta}{2})$ and $u^* \in \hat{N}(L_f(\bar{x}) \cap Z, u) \cap B_{X^*}$. Let $\varepsilon > 0$, and take $r \in (0,\ \frac{\delta}{2})$ such that

$$\langle u^*, v - u\rangle \le \varepsilon\|v - u\| \quad \text{for all } v \in L_f(\bar{x}) \cap Z \cap B(u, r).$$

Let $x \in B(u, \frac{r}{2}) \subset B(\bar{x}, \delta)$. Then there exists $v \in L_f(\bar{x}) \cap Z$ such that $\|x - v\| = d(x, L_f(\bar{x}) \cap Z)$. Hence,

$$\|v - u\| \le \|v - x\| + \|x - u\| \le 2\|x - u\| < r.$$

Therefore,

$$\begin{aligned}\langle u^*, x - u\rangle &= \langle u^*, x - v\rangle + \langle u^*, v - u\rangle \\ &\le \|x - v\| + \varepsilon\|v - u\| \\ &\le (1+\varepsilon)\|x - v\| + \varepsilon\|x - u\| \\ &= (1+\varepsilon)d(x, L_f(\bar{x}) \cap Z) + \varepsilon\|x - u\|.\end{aligned}$$

It follows from (3.2) and $B(u, \frac{r}{2}) \subset B(\bar{x}, \delta)$ that

$$\eta\langle u^*, x-u\rangle \leq (1+\varepsilon)(f(x)-f(\bar{x})+[\Phi(x)]_+)+\eta\varepsilon\|x-u\| \quad \text{for all } x \in B\left(u, \frac{r}{2}\right).$$

Noting that $f(u)=f(\bar{x})$ and $[\Phi(u)]_+=0$, it follows that u is a local minimum of the function

$$x \mapsto -\eta\langle u^*, x-u\rangle+(1+\varepsilon)(f(x)-f(\bar{x})+[\Phi(x)]_+)+\eta\varepsilon\|x-u\|.$$

This and Proposition 2.1 imply that

$$\eta u^* \in (1+\varepsilon)(f'(u)+\partial[\Phi(\cdot)]_+(u))+\eta\varepsilon B_{X^*}.$$

Letting $\varepsilon \to 0$, one has

$$\eta u^* \in f'(u)+\partial[\Phi(\cdot)]_+(u)=f'(u)+\text{co}(\{0\}\cup\partial\Phi(u))=f'(u)+[0,\ 1]\partial\Phi(u).$$

Noting (by [26, Theorem 10.31]) that

$$\partial\Phi(u)=\begin{cases}\{0\} & I_0(u)=\emptyset,\\ \text{co}\{\phi'_x(u,y):\ y\in I_0(u)\} & I_0(u)\neq\emptyset,\end{cases}$$

it follows that (3.5) holds.

(ii)⇒(iii). Suppose that there exist $\eta, \delta \in (0,\ +\infty)$ such that (3.5) holds for all $u \in L_f(\bar{x})\cap Z\cap B(\bar{x},\delta)$. Let $u \in L_f(\bar{x})\cap Z\cap B(\bar{x},\delta)$ and $u^* \in N(L_f(\bar{z})\cap Z, u)\cap\eta B_{X^*}$. Take a sequence $\{u_k\}$ in $L_f(\bar{x})\cap Z$ and a sequence $\{u_k^*\}$ in X^* such that $u_k \to u$, $u_k^* \to u^*$, and $u_k^* \in \hat{N}(L_f(\bar{x})\cap Z, u_k)$ for all $k \in \mathbb{N}$. Without loss of generality, we assume that $u_k \in B(\bar{x},\delta)$ and $u_k^* \in \eta B_{X^*}$ for each $k \in \mathbb{N}$. By (3.5), one has

$$u_k^* \in f'(u_k)+[0,\ 1]\text{co}\{\phi'_x(u_k,y):\ y\in I_0(u_k)\} \quad \text{for all } k\in\mathbb{N}.$$

We divide into two cases: 1) $I_0(u_k)=\emptyset$ for infinitely many k and 2) $I_0(u_k)\neq\emptyset$ for infinitely many k.

Case 1. Without loss of generality we assume that $I_0(u_k)=\emptyset$ for all $k\in\mathbb{N}$ (passing to a subsequence if necessary). Thus, $u_k^*=f'(u_k)$ for all $k\in\mathbb{N}$. It follows that $u^*=f'(u)$. Hence (3.6) holds.

Case 2. We can assume that $I_0(u_k)\neq\emptyset$ for all $k\in\mathbb{N}$. Noting that X is of dimension m, it follows from the Caratheodory theorem (cf. [26, Theorem 2.29]) that there exist $t_{1k},\ldots,t_{m+1k}\in[0,\ 1]$ and $y_{1k},\ldots,y_{m+1k}\in I_0(u_k)$ such that

$$\sum_{i=1}^{m+1} t_{ik}\leq 1 \ \text{ and } \ u_k^*=f'(u_k)+\sum_{i=1}^{m+1}t_{ik}\phi'_x(u_k,y_{ik}) \quad \text{for all } k\in\mathbb{N}.$$

Without loss of generality, we assume that

$$t_{ik}\to t_i \ \text{ and } \ y_{ik}\to y_i\in I_0(u) \ \text{ as } k\to\infty,\ i=1,\ldots,m+1$$

(passing to subsequences if necessary). Thus,

$$\sum_{i=1}^{m+1} t_i\leq 1 \quad \text{and} \quad u^*=f'(u)+\sum_{i=1}^{m+1}t_i\phi'_x(u,y_i).$$

This shows that (3.6) holds for all $u\in L_f(\bar{x})\cap Z\cap B(\bar{x},\delta)$.

(iii)⇒(iv). Suppose that there exist $\eta, \delta \in (0,\ +\infty)$ such that (3.6) holds for all $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$. It follows that

$$N(L_f(\bar{x}) \cap Z, u) \subset R_+ f'(u) + R_+ \mathrm{co}\{\phi'_x(u, y) :\ y \in I_0(u)\} \quad \text{for all } u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta).$$

On the other hand, by (2.1) and Lemma 3.2 one has

$$R_+ f'(u) + R_+ \mathrm{co}\{\phi'_x(u, y) :\ y \in I_0(u)\} \subset \hat{N}(L_f(\bar{x}) \cap Z, u) \quad \text{for all } u \in L_f(\bar{x}) \cap Z.$$

It follows that

$$\hat{N}(L_f(\bar{x}) \cap Z, u) = N(L_f(\bar{x}) \cap Z, u) \quad \text{for all } u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta).$$

By (2.1) and (2.3), one has

$$N(L_f(\bar{x}) \cap Z, u) = N_c(L_f(\bar{x}) \cap Z, u) \quad \text{for all } u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta).$$

Therefore, (iv) holds.

(iv)⇒(i). Suppose that there exist $\eta, \delta \in (0,\ +\infty)$ such that (3.7) holds for all $x \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$. Let $x \in B(\bar{x}, \frac{\delta}{2}) \setminus L_f(\bar{x}) \cap Z$, and take $u \in P_{L_f(\bar{x}) \cap Z}(x)$. Then $u \in B(\bar{x}, \delta)$, and it follows from Lemma 2.1 and (2.3) that $\frac{x-u}{\|x-u\|} \in N_c(L_f(\bar{x}) \cap Z, u)$. We claim that $I_0(u) \neq \emptyset$. Suppose to the contrary that $I_0(u) = \emptyset$. Then, by the definition, $[0,\ 1]\mathrm{co}\{\phi'_x(u, y) :\ u \in I_0(u)\} = \{0\}$. This and (3.7) imply that the intersection $N_c(L_f(\bar{x}) \cap Z, u) \cap \eta B_{X^*}$ is the singleton $\{f'(u)\}$, contradicting the fact that it contains 0 and $\frac{\eta(x-u)}{\|x-u\|}$. Hence $I_0(u) \neq \emptyset$. By (3.7), there exist $t_1, \ldots, t_q \in [0,\ +\infty)$ and $y_1, \ldots, y_q \in I_0(u)$ such that

$$\sum_{i=1}^{q} t_i \leq 1 \ \text{ and } \ \frac{\eta(x-u)}{\|x-u\|} = f'(u) + \sum_{i=1}^{q} t_i \phi'_x(u, y_i).$$

Hence

$$\eta\|x-u\| = \langle f'(u), x-u \rangle + \sum_{i=1}^{q} t_i \langle \phi'_x(u, y_i), x-u \rangle.$$

Let $\varepsilon \in (0,\ \frac{\eta}{2})$. By Lemma 3.1, without loss of generality we assume that

$$\langle f'(u), x-u \rangle \leq f(x) - f(\bar{x}) + \varepsilon\|x-u\|$$

and

$$\langle \phi'_x(u, y), x-u \rangle \leq \phi(x, y) + \varepsilon\|x-u\| \quad \text{for all } y \in I_0(u)$$

(considering smaller δ if necessary). Therefore,

$$\begin{aligned} \eta\|x-u\| &\leq f(x) - f(\bar{x}) + \sum_{i=1}^{q} t_i \phi(x, y_i) + 2\varepsilon\|x-u\| \\ &\leq f(x) - f(\bar{x}) + \max_{y \in Y}[\phi(x, y)]_+ + 2\varepsilon\|x-u\|. \end{aligned}$$

It follows that

$$(\eta - 2\varepsilon)\|x-u\| \leq f(x) - f(\bar{x}) + \max_{y \in Y}[\phi(x, y)]_+;$$

that is,

$$(\eta - 2\varepsilon) d(x, L_f(\bar{x}) \cap Z) \leq f(x) - f(\bar{x}) + \max_{y \in Y}[\phi(x, y)]_+.$$

Since $f(x) = f(\bar{x})$ and $\max_{y\in Y}[\phi(x,y)]_+ = 0$ if $x \in L_f(\bar{x}) \cap Z$, the last inequality holds trivially if $x \in L_f(\bar{x}) \cap Z$. This shows that (i) holds. The proof is completed. ∎

Remark. In view of the proof of Theorem 3.1, one can see that the implication (i)⇒(ii) of Theorem 3.1 holds even when X is a Banach space of infinite dimension.

THEOREM 3.2. *Let $\bar{x} \in Z$. Then the following statements are equivalent:*

(i) *$\bar{x}$ is a local weak sharp minimum of* (SIP).

(ii) *There exist $\eta, \delta \in (0, +\infty)$ such that*

$$\eta d(h, T(L_f(\bar{x}) \cap Z, u)) \le \langle f'(u), h\rangle + \max_{y\in I_0(u)}[\langle \phi'_x(u,y), h\rangle]_+ \tag{3.8}$$

for all $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$ and $h \in X$.

(iii) *There exist $\eta, \delta \in (0, +\infty)$ such that*

$$\eta d(h, T_c(L_f(\bar{x}) \cap Z, u)) \le \langle f'(u), h\rangle + \max_{y\in I_0(u)}[\langle \phi'_x(u,y), h\rangle]_+ \tag{3.9}$$

for all $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$ and $h \in X$.

(iv) *There exist $\eta, \delta \in (0, +\infty)$ such that*

$$\eta\|x-u\| \le \langle f'(u), x-u\rangle + \max_{y\in I_0(u)}[\langle \phi'_x(u,y), x-u\rangle]_+ \tag{3.10}$$

for any $x \in B(\bar{x}, \delta)$ and $u \in P_{L_f(\bar{x})\cap Z}(x)$.

Proof. (i)⇒(iii). Suppose that (i) holds. Then by Theorem 3.1 there exist $\eta, \delta \in (0, +\infty)$ such that (3.7) holds for all $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$. Let $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$ and $h \in X$. By Lemma 3.2, (3.9) holds if $h \in T_c(L_f(\bar{x}) \cap Z, u)$. Now we assume that $h \notin T_c(L_f(\bar{x}) \cap Z, u)$. Take $h_0 \in P_{T_c(L_f(\bar{x})\cap Z,u)}(h)$. Then by Lemma 2.1 and (2.3) one has

$$h - h_0 \in N_c(T_c(L_f(\bar{x}) \cap Z, u), h_0).$$

Since $T_c(L_f(\bar{x}) \cap Z, u)$ is a convex cone,

$$\langle h - h_0, z - h_0\rangle \le 0 \quad \text{for all } z \in T_c(L_f(\bar{x}) \cap Z, u).$$

Hence

$$\langle h - h_0, h_0\rangle = 0 \ \text{ and } \ \langle h - h_0, z\rangle \le 0 \quad \text{for all } z \in T_c(L_f(\bar{x}) \cap Z, u).$$

This and (2.2) imply that $\frac{\eta(h-h_0)}{\|h-h_0\|} \in N_c(L_{f(\bar{x})} \cap Z, u)$. It follows from (3.7) that $I_0(u) \neq \emptyset$, and there exist $t_1, \dots, t_q \in [0, +\infty)$ and $y_1, \dots, y_q \in I_0(u)$ such that

$$\sum_{i=1}^{q} t_i \le 1 \ \text{ and } \ \frac{\eta(h-h_0)}{\|h-h_0\|} = f'(u) + \sum_{i=1}^{q} t_i \phi'_x(u, y_i).$$

Hence

$$\begin{aligned}
\eta d(h, T_c(L_f(\bar{x}) \cap Z, u)) &= \left\langle \frac{\eta(h-h_0)}{\|h-h_0\|}, h - h_0 \right\rangle \\
&= \left\langle \frac{\eta(h-h_0)}{\|h-h_0\|}, h \right\rangle \\
&= \langle f'(u), h\rangle + \sum_{i=1}^{q} t_i \langle \phi'_x(u, y_i), h\rangle \\
&\le \langle f'(u), h\rangle + \max_{y\in I_0(u)}[\langle \phi'_x(u,y), h\rangle]_+.
\end{aligned}$$

Therefore, (3.9) holds. This shows that (iii) holds.

Since $T_c(L_f(\bar{x}) \cap Z, u) \subset T(L_f(\bar{x}) \cap Z, u)$ for any $u \in L_f(\bar{x}) \cap Z$,

$$d(h, T(L_f(\bar{x}) \cap Z, u)) \le d(h, T_c(L_f(\bar{x}) \cap Z, u)) \quad \text{for all } h \in X \text{and for all } u \in L_f(\bar{x}) \cap Z.$$

Hence (iii)⇒(ii) holds trivially.

Suppose that (ii) holds. Take $\eta, \delta \in (0, +\infty)$ such that (3.8) holds for all $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta)$ and $h \in X$. Let $x \in B(\bar{x}, \frac{\delta}{2}) \setminus L_f(\bar{x}) \cap Z$, and take $u \in P_{L_f(\bar{x}) \cap Z}(x)$. By Lemma 2.1, one has $\frac{x-u}{\|x-u\|} \in \hat{N}(L_f(\bar{x}) \cap Z, u)$. Hence $\langle \frac{x-u}{\|x-u\|}, z \rangle \le 0$ for any $z \in T(L_f(\bar{x}) \cap Z, u)$. This implies that

$$\|x-u\| \le \left\langle \frac{x-u}{\|x-u\|}, x-u-z \right\rangle \le \|x-u-z\| \quad \text{for all } z \in T(L_f(\bar{x}) \cap Z, u).$$

Hence $\|x-u\| = d(x-u, T(L_f(\bar{x}) \cap Z, u))$. Noting that $u \in B(\bar{x}, \delta)$, it follows from (3.8) that (3.10) holds. This shows that the implication (ii)⇒(iv) holds.

Suppose that (iv) holds. Take $\eta, \delta \in (0, +\infty)$ such that (3.10) holds for any $x \in B(\bar{x}, \delta)$ and $u \in P_{L_f(\bar{x}) \cap Z}(x)$. Let $x \in B(\bar{x}, \frac{\delta}{2}) \setminus L_f(\bar{x}) \cap Z$, and take $u \in P_{L_f(\bar{x}) \cap Z}(x)$. Then $u \in B(\bar{x}, \delta)$. Hence (3.10) holds for such x and u. Let $\varepsilon \in (0, \frac{\eta}{2})$. By Lemma 3.1, without loss of generality we assume that

$$\langle f'(u), x-u \rangle \le f(x) - f(\bar{x}) + \varepsilon \|x-u\|$$

and

$$\langle \phi'_x(u, y), x-u \rangle \le \phi(x, y) + \varepsilon \|x-u\| \quad \text{for all } y \in I_0(u)$$

(taking smaller δ if necessary). Hence

$$[\langle \phi'_x(u, y), x-u \rangle]_+ \le [\phi(x, y)]_+ + \varepsilon \|x-u\| \quad \text{for all } y \in I_0(u).$$

It follows from (3.10) that

$$\eta \|x-u\| \le f(x) - f(\bar{x}) + \max_{y \in Y} [\phi(x, y)]_+ + 2\varepsilon \|x-u\|.$$

Therefore,

$$(\eta - 2\varepsilon) d(x, L_f(\bar{x}) \cap Z) = (\eta - 2\varepsilon) \|x-u\| \le f(x) - f(\bar{x}) + \max_{y \in Y} [\phi(x, y)]_+.$$

This shows that (i) holds. The proof is completed. ∎

Now we provide a mixed characterization for $\bar{x}$ to be a weak sharp minimum of (SIP), which is inspired from [10, Theorem 4].

PROPOSITION 3.1. *Let $\bar{x} \in Z$. Then $\bar{x}$ is a local weak sharp minimum of* (SIP) *if and only if the following conditions are satisfied:*

(i) $T(L_f(\bar{x}) \cap Z, \bar{x}) = \{h \in X : \langle f'(\bar{x}), h \rangle + \max_{y \in I_0(\bar{x})} [\langle \phi'_x(\bar{x}, y), h \rangle]_+ \le 0\}$.

(ii) *There exist $\eta_0, \delta \in (0, +\infty)$ such that for any $u \in L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta) \setminus \{\bar{x}\}$*

$$\hat{N}(L_f(\bar{x}) \cap Z, u) \cap \eta_0 B_{X^*} \subset f'(u) + [0, 1] \text{co}\{\phi'_x(u, y) : y \in I_0(u)\}.$$

Proof. By Lemma 3.2, one has

$$T(L_f(\bar{x}) \cap Z, \bar{x}) \subset \{h \in X : \langle f'(\bar{x}), h \rangle + \max_{y \in I_0(\bar{x})} [\langle \phi'_x(\bar{x}, y), h \rangle]_+ \le 0 \quad \text{for all } y \in I_0(\bar{x})\}.$$

It follows from (ii) of Theorem 3.2 and (ii) of Theorem 3.1 that the necessity part holds.

To prove the sufficiency part, suppose that (i) and (ii) hold. We claim that there exists $\eta_1 > 0$ such that

$$\eta_1\|h\| \leq \langle f'(\bar{x}), h\rangle + \max_{y\in I_0(\bar{x})}[\langle \phi'_x(\bar{x}, y), h\rangle]_+ \quad \text{for all } h \in \hat{N}(L_f(\bar{x}) \cap Z, \bar{x}). \tag{3.11}$$

Suppose to the contrary that there exists a sequence $\{h_k\}$ in $\hat{N}(L_f(\bar{x}) \cap Z, \bar{x})$ such that

$$\|h_k\| = 1 \text{ and } \langle f'(\bar{x}), h_k\rangle + \max_{y\in I_0(\bar{x})}[\langle \phi'_x(\bar{x}, y), h_k\rangle]_+ < \frac{1}{k} \quad \text{for all } k \in \mathbb{N}.$$

Without loss of generality we assume that $h_k \to h_0$. Then

$$h_0 \in \hat{N}(L_f(\bar{x}) \cap Z, \bar{x}) \text{ and } \langle f'(\bar{x}), h_0\rangle + \max_{y\in I_0(\bar{x})}[\langle \phi'_x(\bar{x}, y), h_0\rangle]_+ \leq 0.$$

It follows from (i) that $h_0 \in T(L_f(\bar{x}) \cap Z, \bar{x})$, contradicting $\|h_0\| = 1$ and (2.1). This shows that (3.11) holds. Let $x \in B(\bar{x}, \frac{\delta}{2}) \setminus L_f(\bar{x}) \cap Z$ and $u \in P_{L_f(\bar{x})\cap Z}(x)$. Then $u \in B(\bar{x}, \delta) \setminus \{x\}$ and $\frac{x-u}{\|x-u\|} \in \hat{N}(L_f(\bar{x}) \cap Z, u)$. In the case when $u = \bar{x}$, by (3.11) one has

$$\eta_1\|x - \bar{x}\| \leq \langle f'(\bar{x}), x - \bar{x}\rangle + \max_{y\in I_0(\bar{x})}[\langle \phi'_x(\bar{x}, y), x - \bar{x}\rangle]_+. \tag{3.12}$$

In the case when $u \neq \bar{x}$, by (ii) there exist $t_i \in [0, +\infty)$ and $y_i \in I_0(u)$, $i = 1, \ldots, q$, such that

$$\sum_{i=1}^{q} t_i \leq 1 \text{ and } \frac{\eta_0(x - u)}{\|x - u\|} = f'(u) + \sum_{i=1}^{q} t_i\phi'(u, y_i).$$

It follows that

$$\begin{aligned}\eta_0\|x - u\| &= \langle f'(u), x - u\rangle + \sum_{i=1}^{q} t_i\langle \phi'_x(u, y_i), x - u\rangle \\ &\leq \langle f'(u), x - u\rangle + \max_{y\in I_0(u)}[\langle \phi'_x(u, y_i), x - u\rangle].\end{aligned}$$

This and (3.12) imply that (iv) of Theorem 3.2 holds with $\eta = \min\{\eta_0, \eta_1\}$. It follows from Theorem 3.2 that the sufficiency part holds. The proof is completed. □

Remark. Letting

$$\psi(x) := f(x) + \max_{y\in Y}[\phi(x, y)]_+ \text{ for all } x \in X,$$

it is clear that if $\bar{x}$ is a local weak sharp minimum of (SIP), then $\bar{x}$ is a local weak sharp minimum of ψ: There exist $\eta, \delta \in (0, +\infty)$ such that

$$\eta d(x, L_\psi(\bar{x})) \leq \psi(x) - \psi(\bar{x}) \text{ for all } x \in B(\bar{x}, \delta).$$

The converse implication may not be true. Indeed, let $X = R$, $Y = \{y_0\}$, $f(x) = -x^2$, and $\phi(x, y_0) = x^2$ for all $x \in R$. Then $Z = \{0\}$, and $\bar{x} = 0$ is not a local weak sharp minimum of (SIP). But, noting that $\psi(x) = f(x) + \max_{y\in Y}[\phi(x, y)]_+ = 0$ for all $x \in X$, 0 is a weak sharp minimum of ψ.

When ψ is a convex function, in terms of the normal and tangent cones of the solution set as well as the subdifferential and the directional derivative of ψ, some characterizations for the weak sharp minimum of ψ have been established (cf. [2, 33]). To the best of our knowledge, in the nonconvex case no one considers corresponding characterizations.

Finally, we provide characterizations for $\bar{x} \in Z$ to be a local sharp minimum of (SIP).

THEOREM 3.3. *Let $\bar{x} \in Z$. Then the following statements are equivalent:*

(i) *$\bar{x}$ is a local sharp minimum of* (SIP).

(ii) *There exists $\eta > 0$ such that*

$$\eta B_{X^*} \subset f'(\bar{x}) + [0,\ 1]\text{co}\{\phi'_x(\bar{x}, y) :\ y \in I_0(\bar{x})\}.$$

(iii) *There exists $\eta > 0$ such that*

$$\eta\|h\| \leq \langle f'(\bar{x}), h\rangle + \max_{y \in I_0(\bar{x})}[\langle \phi'_x(\bar{x}, y), h\rangle]_+ \quad \textit{for all } h \in X.$$

(iv) $\{h \in X :\ \langle f'(\bar{x}), h\rangle + \max_{y\in I_0(\bar{x})}[\langle \phi'_x(\bar{x}, y), h\rangle]_+ \leq 0\} = \{0\}$.

Proof. (i)⇒(ii) is immediate from Theorem 3.1 and $N(\{\bar{x}\}, \bar{x}) = X^*$. (ii)⇒(iii) and (iii)⇒(iv) are trivial.

It remains to prove (iv)⇒(i). Suppose that (iv) holds. Noting that $T(\{\bar{x}\}, \bar{x}) = \{0\}$, by Proposition 3.1 we need only show that $L_f(\bar{x}) \cap Z \cap B(\bar{x}, \delta) = \{\bar{x}\}$ for some $\delta > 0$. Suppose to the contrary that there exists a sequence $\{x_k\}$ in $L_f(\bar{x}) \cap Z \setminus \{\bar{x}\}$ such that $x_k \to \bar{x}$. Without loss of generality we assume that $\frac{x_k-\bar{x}}{\|x_k-\bar{x}\|} \to h$ (passing to a subsequence if necessary). Thus, $h \in T(L_f(\bar{x}) \cap Z, \bar{x})$. It follows from Lemma 3.2 that

$$\langle f'(\bar{x}), h\rangle + \max_{y\in I_0(\bar{x})}[\langle \phi'_x(\bar{x}, y), h\rangle]_+ \leq 0,$$

contradicting (iv) and $\|h\| = 1$. The proof is completed. □

4. Calmness for infinitely many smooth inequalities. Recently Henrion and Outrata [10] studied the calmness of infinitely many smooth inequalities. Let $C(Y)$ denote the Banach space of all continuous functions on Y equipped with the maximum norm, and consider the multifunction $M : C(Y) \rightrightarrows X$ defined by

$$M(g) := \{x \in X :\ \phi(x, y) \leq -g(y) \quad \text{for all } y \in Y\} \quad \text{for all } g \in C(Y), \tag{4.1}$$

where X, Y, and $\phi(x, y)$ are as in section 3. For $\bar{g} \in C(Y)$ and $\bar{x} \in M(\bar{g})$, recall that M is calm at $(\bar{g}, \bar{x})$ if there exist $L,\ \delta \in (0,\ +\infty)$ such that

$$d(x, M(\bar{g})) \leq L\|g - \bar{g}\| \quad \text{for all } g \in B(\bar{g}, \delta) \text{ and for all } x \in B(\bar{x}, \delta) \cap M(g).$$

We say that M is strongly calm at $(\bar{g}, \bar{x})$ if there exist $L,\ \delta \in (0,\ +\infty)$ such that

$$\|x - \bar{x}\| \leq L\|g - \bar{g}\| \quad \text{for all } g \in B(\bar{g}, \delta) \text{ and for all } x \in B(\bar{x}, \delta) \cap M(g).$$

It is clear that M is strongly calm at $(\bar{g}, \bar{x})$ if and only if M is calm at $(\bar{g}, \bar{x})$ and $M(\bar{g}) \cap B(\bar{x}, \delta) = \{\bar{x}\}$ for some $\delta > 0$. Let $\Lambda := \{g \in C(Y) :\ g(y) \leq 0 \quad \text{for all } y \in Y\}$ and $\bar{x} \in M(0)$. It is known (cf. [10]) that M is calm at $(0, \bar{x})$ if and only if there exist $L,\ \delta \in (0,\ +\infty)$ such that

$$d(x, M(0)) \leq Ld(\phi(x, \cdot), \Lambda) \quad \text{for all } x \in B(\bar{x}, \delta).$$

Noting that

$$M(0) = Z \text{ and } d(\phi(x,\cdot),\Lambda) = \max_{y\in Y}[\phi(x,y)]_+,$$

it follows that M is calm at $(0,\bar{x})$ if and only if there exist $\eta,\ \delta \in (0,\ +\infty)$ such that

$$\eta d(x,Z) \le \max_{y\in Y}[\phi(x,y)]_+ \quad \text{for all } x \in B(\bar{x},\delta). \tag{4.2}$$

Setting $f(x) = 0$ for all $x \in X$ in (SIP), one sees that (4.2) means that $\bar{x}$ is a local weak sharp minimum of (SIP). Thus, by Theorems 3.1 and 3.2 and Proposition 3.1 we have the following characterizations for M to be calm at $(0,\bar{x})$.

THEOREM 4.1. *Let M be as in* (4.1) *and $\bar{x} \in M(0)$. Then the following statements are equivalent:*

(i) *M is calm at $(0,\bar{x})$.*

(ii) *There exist $\tau,\delta \in (0,\ +\infty)$ such that*

$$\hat{N}(M(0),u)\cap B_{X^*} \subset [0,\ \tau]\mathrm{co}\{\phi'_x(u,y):\ y \in I_0(u)\} \quad \textit{for all } u \in M(0)\cap B(\bar{x},\delta).$$

(iii) *There exist $\tau,\delta \in (0,\ +\infty)$ such that*

$$N(M(0),u)\cap B_{X^*} \subset [0,\ \tau]\mathrm{co}\{\phi'_x(u,y):\ y \in I_0(u)\} \quad \textit{for all } u \in M(0)\cap B(\bar{x},\delta).$$

(iv) *There exist $\tau,\delta \in (0,\ +\infty)$ such that*

$$N_c(M(0),u)\cap B_{X^*} \subset [0,\ \tau]\mathrm{co}\{\phi'_x(u,y):\ y \in I_0(u)\} \quad \textit{for all } u \in M(0)\cap B(\bar{x},\delta).$$

(v) *There exist $\tau,\delta \in (0,\ +\infty)$ such that*

$$d(h,T(M(0),u)) \le \tau \max_{y\in I_0(u)}[\langle\phi'_x(u,y),h\rangle]_+$$

for all $u \in M(0)\cap B(\bar{x},\delta)$ and $h \in X$.

(vi) *There exist $\tau,\delta \in (0,\ +\infty)$ such that*

$$d(h,T_c(M(0),u)) \le \tau \max_{y\in I_0(u)}[\langle\phi'_x(u,y),h\rangle]_+$$

for all $u \in M(0)\cap B(\bar{x},\delta)$ and $h \in X$.

(vii) *There exist $\tau,\delta \in (0,\ +\infty)$ such that*

$$\|x-u\| \le \tau \max_{y\in I_0(u)}[\langle\phi'_x(u,y),x-u\rangle]_+$$

for any $x \in B(\bar{x},\delta)$ and $u \in P_{M(0)}(x)$.

(viii) *$T(M(0),\bar{x}) = \{h \in X:\ \langle\phi'_x(\bar{x},y),h\rangle \le 0 \quad$ for all $y \in I_0(\bar{x})\}$, and there exist $\tau,\delta \in (0,\ +\infty)$ such that for any $u \in M(0)\cap B(\bar{x},\delta)\setminus\{\bar{x}\}$*

$$\hat{N}(M(0),u)\cap B_{X^*} \subset [0,\ \tau]\mathrm{co}\{\phi'_x(u,y):\ y \in I_0(u)\}.$$

In the remainder of this section, we assume that Y is a compact subset of R^n. Following Henrion and Outrata [10], let

$$\mathcal{J} := \{S \in \mathcal{K}(Y):\ \exists x_i \stackrel{\mathrm{bd}M(0)\setminus\{\bar{x}\}}{\longrightarrow} \bar{x} \text{ such that } d_H(S,I_0(x_i)) \to 0\},$$

where $\mathcal{K}(Y)$ denotes the family of all compact subsets of Y and d_H denotes the Hausdorff distance between compact sets.

COROLLARY 4.1. *Let M be as in* (4.1) *and $\bar{x} \in M(0)$. Suppose that the following conditions are satisfied:*

1. $T(M(0),\bar{x}) = \{h \in X : \langle \phi'_x(\bar{x},y),h\rangle \le 0 \quad \text{for all } y \in I_0(\bar{x})\}$.
2. *There exists* $\rho > 0$ *such that*

$$d(0,\mathrm{co}\{\phi'_x(\bar{x},y) : y \in S\}) > \rho \quad \text{for all } S \in \mathcal{J}.$$

Then M *is calm at* $(0,\bar{x})$.

Proof. We claim that there exists $\delta > 0$ such that

$$d(0,\mathrm{co}\{\phi'_x(x,y) : y \in I_0(x)\}) > \rho \quad \text{for all } x \in \mathrm{bd}(M(0)) \cap B(\bar{x},\delta) \setminus \{\bar{x}\}. \tag{4.3}$$

If this is not the case, then there exists a sequence $\{x_i\}$ in $\mathrm{bd}(M(0)) \setminus \{\bar{x}\}$ such that

$$x_i \to \bar{x} \text{ and } d(0,\mathrm{co}\{\phi'_x(x_i,y) : y \in I_0(x_i)\}) \le \rho \quad \text{for all } i \in \mathbb{N}.$$

Noting that X is of dimension m, it follows from the Caratheodory theorem that for each $i \in \mathbb{N}$ there exist $t_{ji} \ge 0$ and $y_{ji} \in I_0(x_i)$, $j = 1,\dots,m+1$, such that

$$\sum_{j=1}^{m+1} t_{ji} = 1 \text{ and } \left\|\sum_{j=1}^{m+1} t_{ji}\phi'_x(x_i,y_{ji})\right\| \le \rho.$$

By (P2) and the compactness of Y, without loss of generality we assume that

$$t_{ji} \to t_j \ge 0 \text{ and } y_{ji} \to y_j \in I_0(\bar{x}),\ j = 1,\dots,m+1.$$

Hence,

$$\sum_{j=1}^{m+1} t_j = 1 \text{ and } \left\|\sum_{j=1}^{m+1} t_j\phi'_x(\bar{x},y_j)\right\| \le \rho. \tag{4.4}$$

Since the space of compact subsets of X endowed with the Hausdorff distance is itself compact, without loss of generality we assume that there exists $S_0 \in \mathcal{J}$ such that $d_H(I_0(x_i),S_0) \to 0$. It is clear that $y_j \in S_0$, $j = 1,\dots,m+1$. This and (4.4) imply that $d(0,\mathrm{co}\{\phi'_x(\bar{x},y) : y \in S_0)\}) \le \rho$, contradicting condition 2. Hence there exists $\delta > 0$ such that (4.3) holds. Recalling (cf. [26, Definition 7.25 and Theorem 10.31]) that $\Phi(x) = \max_{y\in Y}\phi(x,y)$ is regular and $\partial\Phi(x) = \mathrm{co}\{\phi'(x,y) : y \in I(x)\}$, it follows from (4.3) and [26, Proposition 10.3] that

$$N(M(0),x) = R_+\mathrm{co}\{\phi'_x(x,y) : y \in I_0(x)\} \quad \text{for all } x \in \mathrm{bd}(M(0)) \cap B(\bar{x},\delta) \setminus \{\bar{x}\}.$$

Let $x \in \mathrm{bd}(M(0)) \cap B(\bar{x},\delta) \setminus \{\bar{x}\}$ and $x^* \in N(M(0),x) \cap B_{X^*}$. Then there exist $t \in [0, +\infty)$ and $u^* \in \mathrm{co}\{\phi'_x(x,y) : y \in I_0(x)\}$ such that $x^* = tu^*$. This and (4.3) imply that $t < \frac{1}{\rho}$. Hence,

$$N(M(0),x) \cap B_{X^*} \subset \left[0,\ \frac{1}{\rho}\right]\mathrm{co}\{\phi'_x(x,y) : y \in I_0(x)\}.$$

It is clear that

$$N(M(0),x) \cap B_{X^*} = \{0\} \subset \left[0,\ \frac{1}{\rho}\right]\mathrm{co}\{\phi'_x(\bar{x},y) : y \in I_0(x)\} \quad \text{for all } x \in \mathrm{int}(M(0)).$$

Hence, (viii) of Theorem 4.1 holds, and so M is calm at $(0,\bar{x})$. The proof is completed. ∎

Remark 4.1. Corollary 4.1 is a slight improvement of [10, Theorem 4], which, in addition to all assumptions on Corollary 4.1, requires that $(x, y) \mapsto \phi(x, y)$ is continuously differentiable and $(x, y) \mapsto \phi'(x, y)$ is locally Lipschitz. Noting that

$$\mathcal{J} = \{S \subset Y : \exists x_i \overset{\mathrm{bd}M(0)\setminus\{\bar{x}\}}{\longrightarrow} \bar{x} \text{ such that } I_0(x_i) = S \text{ for all } i \in \mathbb{N}\}$$

when Y is a finite set, Corollary 4.1 recaptures [10, Theorem 3].

The following example shows that implication (viii)⇒(i) of Theorem 4.1 properly improves [10, Theorems 4 and 3]. Let $X = R^2$, $Y = \{0, 1\}$, $\phi((s, t), 0) = 0$, and $\phi((s, t), 1) = s - t$ for all $(s, t) \in R^2$. Then $M(0) = \{(s, t) \in R^2 : \ s \leq t\}$ and $I_0(x) = Y$ for any $x \in \mathrm{bd}(M(0))$. Let $\bar{x} = (0, 0)$. Then $\mathcal{J} = \{Y\}$. Noting that

$$\mathrm{co}\{\phi'((s, t), y) : \ y \in Y\} = \{(u, -u) : \ 0 \leq u \leq 1\} \quad \text{for all } (s, t) \in R^2,$$

it follows that $d(0, \mathrm{co}\{\phi'_x(\bar{x}, y) : \ y \in S\}) = 0$ for all $S \in \mathcal{J}$. Thus, Corollary 4.1 and so [10, Theorems 4 and 3] are not applicable. On the other hand, noting that $\mathrm{bd}(M(0)) = \{(s, s) : \ s \in R\}$,

$$T(M(0), (s, s)) = M(0) \ \text{ and } \ N(M(0), (s, s)) = \{(t, -t) : \ t \geq 0\} \quad \text{for all } s \in R,$$

one can see that (viii) of Theorem 4.1 holds. Hence, applying implication (viii)⇒(i) of Theorem 4.1, one obtains that M is calm at $(0, \bar{x})$.

We conclude with characterizations for M to be strongly calm at $(0, \bar{x})$.

THEOREM 4.2. *Let M be as in* (4.1) *and $\bar{x} \in M(0)$. Then the following statements are equivalent:*

(i) *M is strongly calm at $(0, \bar{x})$.*

(ii) *There exists $\tau \in (0, +\infty)$ such that*

$$B_{X^*} \subset [0, \ \tau]\mathrm{co}\{\phi'_x(\bar{x}, y) : \ y \in I_0(\bar{x})\}.$$

(iii) $X^* = R_+ \mathrm{co}\{\phi'_x(\bar{x}, y) : \ y \in I_0(\bar{x})\}$.

(iv) *There exists $\tau \in (0, +\infty)$ such that*

$$\|h\| \leq \tau \max_{y \in I_0(\bar{x})} [\langle \phi'_x(\bar{x}, y), h\rangle]_+ \quad \text{for all } h \in X.$$

(v) $\{h \in X : \ \langle \phi'_x(\bar{x}, y), h\rangle \leq 0 \quad$ *for all* $y \in I_0(\bar{x})\} = \{0\}$.

Proof. Noting that M is strongly calm at $(0, \bar{x})$ if and only if $\bar{x}$ is a local sharp minimum of (SIP) with $f \equiv 0$, (i)⇔(ii)⇔(iv)⇔(v) are immediate from Theorem 3.3. It is clear that (ii)⇒(iii)⇒(v) hold. The proof is completed. ∎

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