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A SCHWARZ LEMMA FOR COMPLETE RIEMANNIAN MANIFOLDS

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We prove a Schwarz Lemma for conformal mappings between two complete Riemannian manifolds when the domain manifold has Ricci curvature bounded below in terms of its distance function. This gives a partial result to a conjecture of Chua.

1. INTRODUCTION

In recent paper [2], Chua has given an interesting generalisation of the classical Ahlfors-Schwarz Lemma to conformal mappings from the hyperbolic space into a Riemannian manifold. His result is the following:

THEOREM. [2] *Let (M^n, g) be an n-dimensional Riemannian manifold and (H^n, g_0) be the n-dimensional hyperbolic space of constant sectional curvature -1. Let $f\colon (H^n, g_0) \to (M^n, g)$ be a conformal map. If $\operatorname{scal}(M^n) \leqslant -n(n-1)$ where $\operatorname{scal}(M^n)$ denotes the scalar curvature of M^n, then f is distance decreasing, that is, $f^*g \leqslant g_0$.*

In the same paper [2], Chua proposed the following interesting generalisation.

CONJECTURE. [2] *Let (M_0^n, g_0) and (M^n, g) be two complete n-dimensional Riemannian manifolds. Let $f\colon (M_0^n, g_0) \to (M^n, g)$ be a conformal map. If there exist positive constants c and k such that $f^*(\operatorname{scal}(M^n)) \leqslant k^2 \operatorname{scal}(M_0^n) \leqslant -c^2 < 0$, then $k^2(f^*g) \leqslant g_0$.*

Our purpose in this note is to prove the following partial result on Chua's Conjecture.

MAIN THEOREM. *Let (M_0^n, g_0) and (M^n, g) be two complete n-dimensional Riemannian manifolds. Let $f\colon (M_0^n, g_0) \to (M^n, g)$ be a conformal map. Assume that $\operatorname{Ric}(M_0^n) \geqslant -a^2\left(1 + y^2 \log^2(y+2)\right)$, where $\operatorname{Ric}(M_0^n)$ denotes the Ricci curvature of M_0^n, a denotes a constant and y is the distance function from a fixed origin $x_0 \in M_0^n$. If there exist positive constants c and k such that $f^*(\operatorname{scal}(M^n)) \leqslant k^2 \operatorname{scal}(M_0^n) \leqslant -c^2 < 0$, then $k^2(f^*g) \leqslant g_0$.*

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2. Proof of the main theorem

Since f is conformal, there exists a smooth function ρ on M_0^n such that

$$f^*g = e^{2\rho}g_0.$$

(See [**3**, p.53].) Following [**3**, p.62] we put $u = e^{-\rho}$.

Since $u \geqslant 0$, therefore $r := \inf u$ exists.

The assumption on $\mathrm{Ric}\,(M_0^n)$ allows us to apply the Generalised Maximum Principle of Chen and Xin [**1**, Theorem 2.2, p.359] to the function $-u$ to obtain the following:

For each positive integer m, there exists a point $x_m \in M_0^n$ such that

$$0 \leqslant r \leqslant u(x_m) < r + \frac{1}{m},$$

$$|\nabla u(x_m)|^2 < \frac{1}{m},$$

and

$$\Delta u(x_m) > -\frac{1}{m}.$$

Here ∇u and Δu denote the gradient and the Laplacian of u respectively.

Let $K_0 = \mathrm{scal}\,(M_0^n)$ and $K = f^*(\mathrm{scal}\,(M^n))$. Then by [**3**, p.62, (3.12)] we have

$$K = u^2K_0 + 2(n-1)u\Delta u - n(n-1)\,|\nabla u|^2.$$

Therefore

$$k^2K_0 \geqslant u^2K_0 + 2(n-1)u\Delta u - n(n-1)\,|\nabla u|^2.$$

(since $K \leqslant k^2K_0$) and hence

$$\begin{aligned}\left(k^2 - u(x_m)^2\right)K_0(x_m) &\geqslant (n-1)\left[2u(x_m)\Delta u(x_m) - n\,|\nabla u(x_m)|^2\right]\\ &\geqslant (n-1)\left[2\left(r+\frac{1}{m}\right)\left(-\frac{1}{m}\right) - \frac{n}{m}\right]\\ &= -\frac{n-1}{m}\left(2r + \frac{2}{m} + n\right).\end{aligned} \tag{2.1}$$

Our Main Theorem will be proven if we can show that $r \geqslant k$. Suppose on the contrary that we have $r < k$. Then for m large enough, we have

$$u(x_m) < r + \frac{1}{m} < k.$$

Since $K \leqslant k^2K_0 \leqslant -c^2$, it follows from (2.1) that for m large enough, we have

$$\left(k^2 - u(x_m)^2\right)(-c^2) \geqslant -\frac{n-1}{m}\left(2r + \frac{2}{m} + n\right)k^2.$$

Letting $m \to \infty$, we obtain

$$(k^2 - r^2)(-c^2) \geqslant 0$$

and this contradicts the assumption that $r < k$. Therefore we must have $r \geqslant k$. $\square$

References

[1] Q. Chen and Y.L. Xin, 'A generalized maximum principle and its applications in geometry', *Amer. J. Math.* **114** (1992), 355–366.

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