Anomalous transport in lattice and continuum percolating systems

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Anomalous diffusion is studied on a certain class of percolation models in which the diffusion properties arise both from the underlying network and from the singular distribution of transfer rates. Scaling arguments parallel to those of Kogut and Straley lead to nonuniversal modifications of the dynamical exponents. The scaling results are demonstrated in a Sierpiński honeycomb and in the effective-medium theory and explicit values for the exponents are obtained. The diffusion-coefficient exponent and the spectral dimension are found to be $2/(1-a)$ and $2(1-a)/(2-a)$, respectively, $(0<a<1)$ in the effective-medium theory, with values being significantly different from the corresponding ones in ordinary percolation. As a concrete example, the frequency-dependent conductivity has been calculated for a three-dimensional cubic network and several interesting crossover behaviors are obtained.

I. INTRODUCTION

In a recent paper, Halperin, Feng, and Sen[1] considered a class of "random-void" continuum percolation models, where spherical holes are randomly placed in a uniform medium. A mapping (Fig. 1) of this model onto a discrete random network[2] shows that the resulting network contains a large number of narrow channels, which are weak links and can be realized as a distribution consisting of a large probability of small but finite conductances. As for the usual discrete-lattice percolation models, the conductance for each bond is either zero or finite with probability $1-p$ and p, respectively. It has been found that the exponent governing the behavior of network conductivity near the percolation threshold can be quite different from the corresponding one in the discrete-lattice models.[1,3,4] On the other hand, the static exponents for geometrical properties of percolation, such as the correlation length exponent ν, the order parameter exponent β, and the fractal dimension of the infinite percolating cluster $\bar{d}$, have been confirmed by simulations to be the same for these models as for usual percolation.[2]

Halperin, Feng, and Sen[1] explicitly calculated the exponents for conductivity, elasticity, and fluid permeability for these continuum models and concluded that the values of the exponents are significantly larger than those of the corresponding ones in the discrete-lattice percolation models. These studies lead us to consider dynamical properties such as diffusion and lattice vibrations on these random-void models. The exponent governing diffusion and lattice phonons on percolation clusters is known as the spectral dimension $\bar{\bar{d}}$, first proposed by Alexander and Orbach,[5] which is also closely related to the diffusion coefficient exponent θ (Refs. 6 and 7) by the relation $\bar{\bar{d}}=2\bar{d}/(2+\theta)$. Anomalous diffusion occurs at short times or small length scales for diffusion on percolating clusters;[7] it also occurs when the transfer-rate distribution is statistically self-similar.[8,9] While the singular [class (c)] distribution of transfer rates is known to give rise to anomalous diffusion on a one-dimensional chain,[10,11] the physical origin of this distribution can be thought of as resulting from a mapping of the random-void model onto a discrete random network.[1]

In this paper, we study a continuous-time random walk[10] on a percolating network whose transfer-rate distribution has the following form

$$P(W)=(1-p)\delta(W)+ph_a(W)\ ,$$

where

$$h_a(W)=(1-a)W^{-a},\quad 0<W<1\ ,$$
$$=0 \text{ otherwise}\ , \tag{1.1}$$

$p\,(0<p<1)$ being the percolation probability and $0<a<1$.

The introduction of a singular distribution of finite transfer rates is arbitrary but it is shown to have relevance

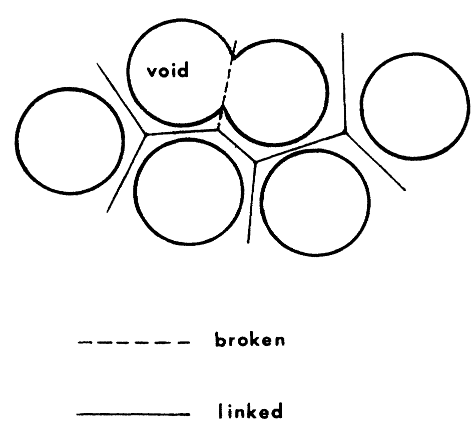

FIG. 1. Mapping of a random-void continuum percolation model onto a discrete-lattice model.

to the continuum percolation models. We shall calculate the frequency-dependent conductivity to extract spectral properties on this model. The paper is organized as follows. In Sec. II, we study the model on an exact fractal lattice. The static conductivity is calculated as of Gefen *et al.*[12] A singular distribution of conductance is introduced so as to calculate the nonuniversal exponent for conductivity. In Sec. III, we give a brief account of the results of frequency-dependent conductivity from calculations of the effective-medium theory. A general scaling form for the results of conductivity is suggested to interpolate the effective-medium theory. In Sec. IV, numerical calculations of the effective conductivity are shown. The results are in good accord with scaling predictions.

II. SIERPINSKI HONEYCOMB

The static limit at the percolation threshold can be easily studied by examining a Sierpinski honeycomb, which is obtained by a dual transformation of the Sierpinski gasket.[13] As illustrated in two dimensions (2D), a basic tetrahedral unit is started out. Three basic units form a larger unit which in turn serves as a building block, again three of which form an even larger unit. The procedure is repeated to infinity. As shown in Fig. 2, the assignment of conductance for each link is $g=1$, R, R^2, R^3, etc., $(0<R<1)$ at the lowest, next, third, fourth, etc., level, respectively. If one specifies $f(g)\sim g^{-a}$ for the distribution of conductance, one finds

$$a=1-\ln 3/|\ln R| \; . \tag{2.1}$$

The condition for nonuniversal contribution to anomalous diffusion is when $0<R<R_c=\frac{1}{3}$, or $0<a<1$.

Using the decimation method,[12] we increase the length scale by a factor of 2 and examine the change of conductance. The Sierpinski honeycomb is believed to serve as a model of percolation backbone at the percolation threshold. Monte Carlo simulations and experimental results suggest that percolation backbone at p_c is finitely and

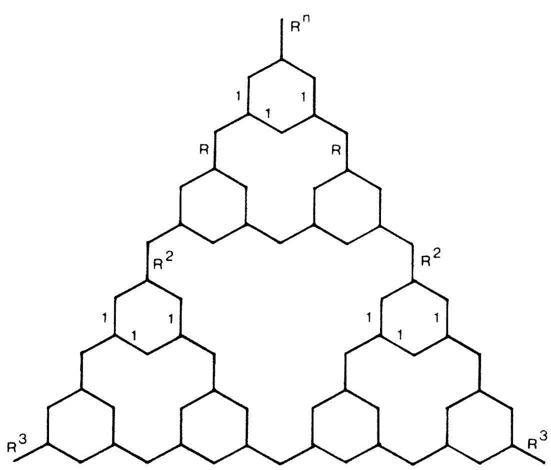

FIG. 2. Construction of a Sierpinski honeycomb in two dimensions. The assignment of conductance for each link is discussed in text.

highly ramified but not quasi one-dimensional.[12] Figure 1 suggests an interesting relation between the honeycomb and the random-void model. Let us write the conductance $g\sim(p-p_c)^{\zeta}$. Since $g=\sigma\xi^{d-2}$, we find that $t=(d-2)\nu+\zeta$. The percolation correlation length ξ behaves as $(p-p_c)^{-\nu}$, we find $g\sim\xi^{-\zeta/\nu}$. If we increase the length scale by a factor of 2, the number of units will increase by a factor of 3, and we obtain the fractal dimension $\bar{d}=\ln 3/\ln 2$. In the case $R=1$, we recover the usual fractal model.[12] By the star-triangle transformation, we increase the length by a factor of 2 and change the conductance g by a factor of $\frac{3}{5}$, we thus find that $\zeta/\nu=\ln(5/3)/\ln 2$. These considerations are readily generalized to a d-dimensional honeycomb:

$$\bar{d}=d-\beta'/\nu=\ln(d+1)/\ln 2 \; , \tag{2.2}$$

$$\zeta/\nu=\ln[(d+3)/(d+1)]/\ln 2 \; , \tag{2.3}$$

and the spectral dimension[5]

$$\bar{\bar{d}}=2\ln(d+1)/\ln(d+3) \; . \tag{2.4}$$

For R being different from unity, the star-triangle transformation gives us a renormalization (RG) equation:

$$R'=R(3+2R)/(3+2R^2) \quad (2\text{D}) \; . \tag{2.5}$$

An asymptotic scaling form can be obtained for $R\ll 1$ (or $a\rightarrow 1^-$). A change of g by a factor R results, and one finds

$$\zeta'/\nu=|\ln R|/\ln 2=\bar{d}/(1-a) \; .$$

For R not small, numerical iterations of the RG equation can be performed. We suggest

$$t'/\nu=t/\nu+a\bar{d}/(1-a) \; . \tag{2.6}$$

Since $\theta=(t-\beta)/\nu$, one finds

$$\theta'=\theta+(t'-t)/\nu=\theta+a\bar{d}/(1-a) \; . \tag{2.7}$$

The random walk dimensionality is known to be $d_w=2$ for normal diffusion, now becomes $d_w=2+\theta$. One finds the change

$$d_w'-d_w=\theta'-\theta=a\bar{d}/(1-a) \; , \tag{2.8}$$

which we shall call the fractal dimensionality of the distribution of transfer rates, in the same spirit of Ref. 8. These results support our previous statement that the anomalous diffusion arises both from the underlying network and from the distribution of transfer rates. The spectral dimension has a corresponding change

$$\bar{\bar{d}}\,'^{-1}=\bar{\bar{d}}^{\,-1}+a/2(1-a) \; . \tag{2.9}$$

For $R>R_c=1/(d+1)$, the parameter a becomes negative, and the model crosses over to usual results.

Kogut and Straley[3] who considered a similar model for a random resistor network, suggested $t'=t+a/(1-a)$, from the double considerations of the effective-medium theory and the Cayley tree. This is in accord with our results because in both cases $\bar{d}\nu=1$. This distribution-induced change in exponents is nonuniversal and depends explicitly on the parameter a describing the singular distribution.

III. EFFECTIVE-MEDIUM THEORY AND THE FREQUENCY-DEPENDENT CONDUCTIVITY

As an implementation of our results Eqs. (2.6), (2.7), and (2.9), we shall calculate the exponents for a d-dimensional hypercubic network using the effective-medium approximation (EMA) (Refs. 14 and 15). We solve the model described by Eq. (1.1) for a continuous-time random walk in EMA. The random percolating network is replaced by a homogeneous effective network, in which the random transfer rates W are replaced by a single homogeneous value $\overline{W}$, which is to be determined by self-consistency. The EMA transfer rate $\overline{W}$ satisfies the following equation (see the Appendix):

$$(p-f)/f=(1-a)K(a)[\overline{W}(1-f)/f]^{(1-a)}\ , \qquad (3.1)$$

where

$$f=(1-sP_0)/d\ , \qquad (3.2)$$

and $P_0(s)$ is the zero-site probability as a function of the Laplace transform variable s:

$$P_0(s)=\frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int_0^{2\pi}dq_1\cdots dq_d\frac{1}{s+2\overline{W}\sum_{i=1}^{d}(1-\cos q_i)}\ , \qquad (3.3)$$

and $K(a)$ is a real number:

$$K(a)=\int_0^\infty\frac{\chi^{-a}d\chi}{1+\chi}=\pi/\sin(\pi a),\quad 0<a<1\ . \qquad (3.4)$$

One can solve for $\overline{W}$ from Eqs. (2.1)–(2.3) as a function of $(p-p_c)$ and s, then the frequency-dependent conductivity is obtained by an analytic continuation of s into $i\omega$ (Ref. 15):

$$\sigma(\omega)=\overline{W}(s=i\omega)\ . \qquad (3.5)$$

In the static limit $\omega=0$, $f=1/d=p_c$, one finds immediately

$$\sigma\sim(p-p_c)^{1/(1-a)}\ , \qquad (3.6)$$

which is consistent with $t'=1/(1-a)$ (Ref. 3). At $p=p_c$, one can approximate f by $p_c-Bs/\overline{W}$ for $d>2$, where B is a constant dependent on dimension. Thus one finds

$$\sigma(\omega)\sim\omega^{1/(2-a)}\ . \qquad (3.7a)$$

And yet for $1<d<2$, $f=p_c-B'(s/\overline{W})^{d/2}$, where B' is a constant dependent on dimension, one finds

$$\sigma(\omega)\sim\omega^{d/[d+2(1-a)]}\ . \qquad (3.7b)$$

The general results from EMA can be summaried as follows:

$$t'=1/(1-a)\quad\text{for } d>1\ , \qquad (3.8)$$

$$\theta'=2/(1-a)\quad\text{for } d>2\ , \qquad (3.9)$$

and

$$\bar{\bar{d}}\,'=2(1-a)/(2-a)\quad\text{for } d>2\ , \qquad (3.10)$$

in contrast with the corresponding values $t=1$, $\theta=2$, and $\bar{\bar{d}}=1$ in the usual percolation model.

These results are in accord with the corresponding scaling predictions. The spectral dimension $\bar{\bar{d}}$ also describes the density of states in the fraction[5] region. Our result shows that $\bar{\bar{d}}\,'$ becomes smaller for the model suggesting a modification of the Alexander-Orbach[5] result for the density of phonon states in the continuum percolation model. At the same time, we calculate the frequency-dependent conductivity at the percolation threshold for small frequencies. We find

$$\sigma(\omega)\sim\begin{cases}\omega^{1/(2-a)} & \text{for } d>2\ , & (3.11a)\\ \omega^{d/[d+2(1-a)]} & \text{for } 1<d<2\ . & (3.11b)\end{cases}$$

These results are significantly different from the one-dimensional results[10,15] $\sigma(\omega)\sim\omega^{a/(2-a)}$, indicating that the infinite percolating cluster at the percolation threshold is not quasi one-dimensional.[12] In addition, we have also calculated numerically the conductivity as a function of frequency and $(p-p_c)$ for a three-dimensional cubic percolating network. To this end, we note that an interesting crossover behavior in the conductivity is observed from normal to anomalous diffusion, arising both from the underlying percolation network and from the singular distribution of transfer rates.[8] The frequency-dependent conductivity obeys a scaling form[14]

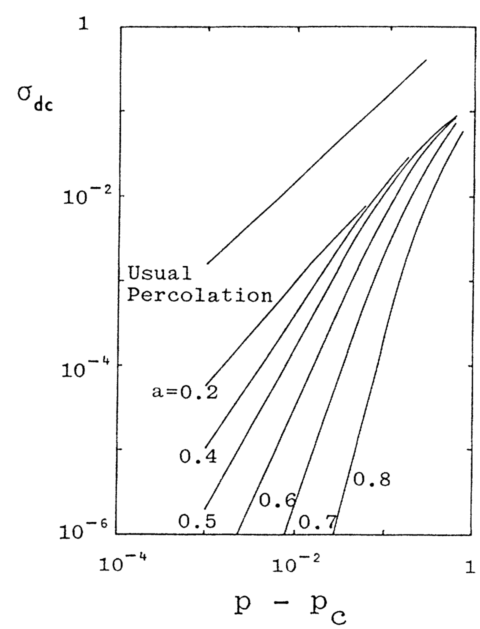

FIG. 3. The dc conductivity plotted as a function of the percolation probability for several singular distributions parametrized by a. A linear relationship obtained in the small $(p-p_c)$ limit suggests scaling behavior, with exponent increases as a increases, in accord with analytic results. The results for usual percolation are plotted for comparison.

$$\sigma(\omega)=(p-p_c)^{(t-\beta)}\varphi(\omega/(p-p_c)^{(2+\theta)\nu}) . \quad (3.12)$$

This relation suggests a crossover frequency from normal to anomalous diffusion

$$\omega_{c0}\sim(p-p_c)^{(2+\theta)\nu} . \quad (3.13)$$

The numerical calculations are presented in the next section.

IV. NUMERICAL CALCULATIONS

For a three-dimensional cubic network, the *dc* conductivity is calculated as a function of $(p-p_c)$ for several values of a. As shown in Fig. 3, the general results indicate that σ_{dc} is a monotonic increasing function of $(p-p_c)$. For small $(p-p_c)$, σ_{dc} shows scaling behavior $\sigma_{dc}\sim(p-p_c)^{t'}$, with t' being very different from the result of usual percolation, which is also calculated for comparison. This is in quantitative accord with the analytic result $t'=t+a/(1-a)$ as suggested by Kogut and Straley.[3] As an example, we consider the case $a=0.8$, where we find that $t'=1/(1-a)=5$, which is exactly the slope of the $a=0.8$ curve in Fig. 3. For larger $(p-p_c)$ the results begin to deviate from usual percolation. A discontinuity between the $a\to0$ result and the usual percolation result indicates that there is indeed a singularity.[16] The results for $a\to1$ become steeper, indicating that singular distribution alters the exponent in a nontrivial way. In Fig. 4, the real (σ_R) and imaginary (σ_I) parts of the conductivity are calculated as a function of frequency for several values of $(p-p_c)$ and for fixed values of a. One can see that for small a, the results for $\sigma(\omega)$ are not very different from those of usual percolation.[14] However, as p approaches p_c, scaling behavior $\sigma(\omega)\sim\omega^{1/(2-a)}$ is ob-

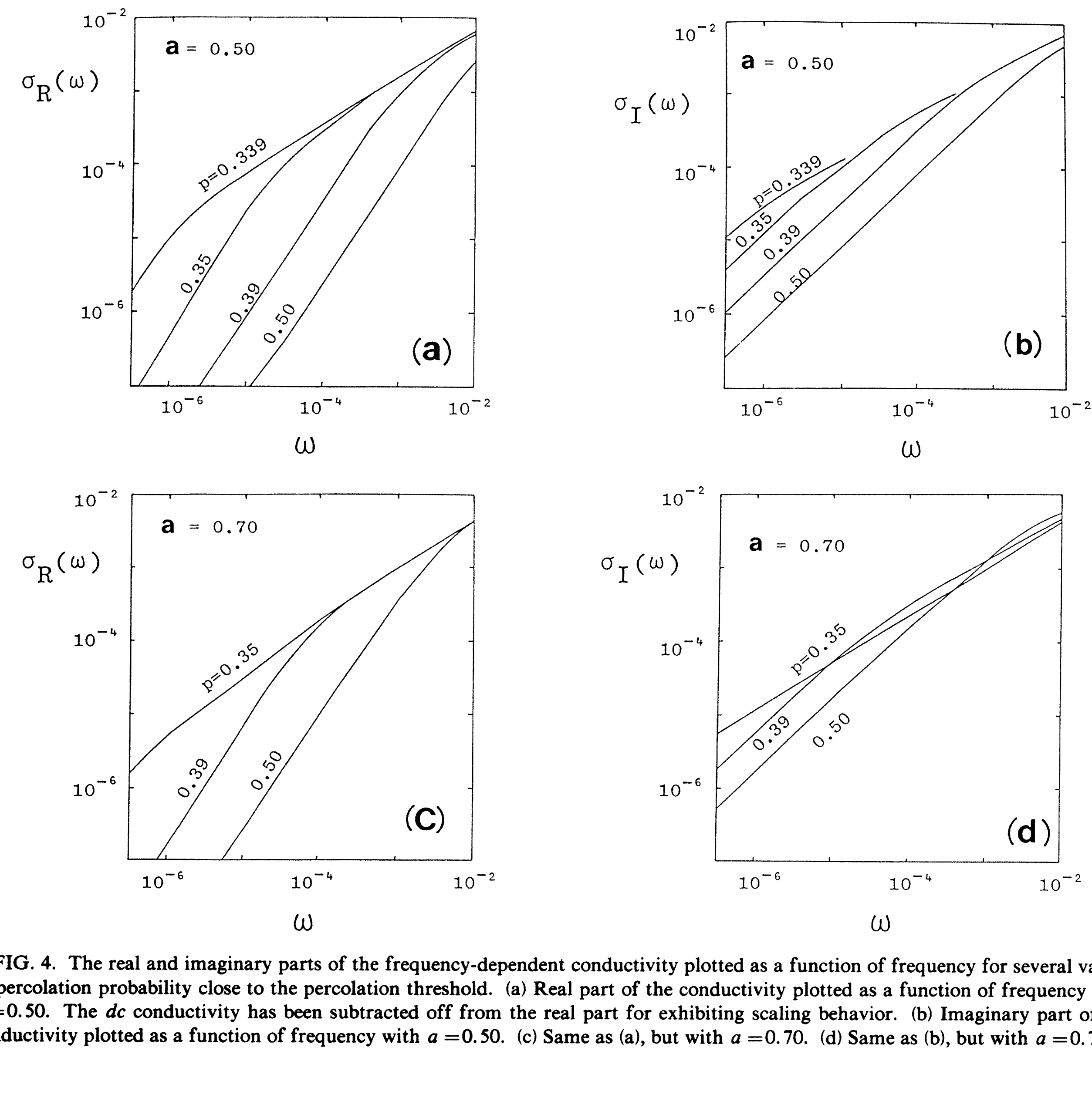

FIG. 4. The real and imaginary parts of the frequency-dependent conductivity plotted as a function of frequency for several values of percolation probability close to the percolation threshold. (a) Real part of the conductivity plotted as a function of frequency with $a=0.50$. The *dc* conductivity has been subtracted off from the real part for exhibiting scaling behavior. (b) Imaginary part of the conductivity plotted as a function of frequency with $a=0.50$. (c) Same as (a), but with $a=0.70$. (d) Same as (b), but with $a=0.70$.

tained, in good accord with the analytic expression for finite frequency expansion at $p=p_c$. For large $(p-p_c)$, one recovers the normal diffusion behavior, namely, $\sigma_R \sim \omega^{3/2}$ and $\sigma_I \sim \omega$; the singular distribution is irrelevant.

In conclusion, we have studied anomalous diffusion on a class of percolation model in which the diffusion properties arise both from the underlying network and from the singular distribution of transfer rates. Decimation methods on the Sierpinski honeycomb as well as effective medium theory provide a powerful tool of extracting important results from the model. Explicit calculations for the dynamic exponents strongly suggest that the distribution of transfer rates alters the transport properties near the percolation threshold in a nonuniversal way. Finite-frequency conductivity were also calculated in EMA and results thus obtained are found in accord with the scaling arguments. These EMA results should be compared to direct numerical simulations of Eq. (1.1) in a future publication.

Note added in proof. A recent paper of J. Machta, R. A. Guyer, and S. M. Moore [Phys. Rev. B (to be published)] gives a result for the conductivity exponent which cannot be obtained from our Eq. (2.6). Their results are based on a numerical simulation on hierarchical lattices which mimic percolation clusters.

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APPENDIX: EFFECTIVE-MEDIUM APPROXIMATION

Here we wish to solve $\overline{W}$ from the following self-consistency equations:[14,15]

$$\langle (W-\overline{W})/[1+Q(W-\overline{W})]\rangle = 0 , \tag{A1}$$

where $Q=(1-sP_0)/d\overline{W}$ is the EMA impedance and $P_0(s)$ is given by Eq. (3.3). The brackets denote an average over the distribution of transfer rates given by Eq. (1.1). Upon an explicit evaluation of the average, one finds

$$(1-p)\overline{W}/(1-Q\overline{W}) = p\langle (W-\overline{W})/[1+Q(W-\overline{W})]\rangle a , \tag{A2}$$

where the brackets denote an average over $h_a(W)$ only. Let $Q=f/\overline{W}$ and $h=\overline{W}(1-f)/f$, and thus $f=(1-sP_0)/d$. Upon a change of variable $W=h\zeta$, Eq. (A2) becomes

$$(p-f)/f = (1-a)K(a)[\overline{W}(1-f)/f]^{(1-a)} , \tag{A3}$$

which is exactly Eq. (3.1).

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