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BIQUADRATIC OPTIMIZATION OVER UNIT SPHERES AND SEMIDEFINITE PROGRAMMING RELAXATIONS*

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Abstract. This paper studies the so-called biquadratic optimization over unit spheres $\min_{x\in\mathbb{R}^n, y\in\mathbb{R}^m}\sum_{1\le i,k\le n,\,1\le j,l\le m} b_{ijkl}x_iy_jx_ky_l$, subject to $\|x\|=1$, $\|y\|=1$. We show that this problem is NP-hard, and there is no polynomial time algorithm returning a positive relative approximation bound. Then, we present various approximation methods based on semidefinite programming (SDP) relaxations. Our theoretical results are as follows: For general biquadratic forms, we develop a $\frac{1}{2\max\{m,n\}^2}$-approximation algorithm under a slightly weaker approximation notion; for biquadratic forms that are square-free, we give a relative approximation bound $\frac{1}{nm}$; when $\min\{n,m\}$ is a constant, we present two polynomial time approximation schemes (PTASs) which are based on sum of squares (SOS) relaxation hierarchy and grid sampling of the standard simplex. For practical computational purposes, we propose the first order SOS relaxation, a convex quadratic SDP relaxation, and a simple minimum eigenvalue method and show their error bounds. Some illustrative numerical examples are also included.

Key words. biquadratic optimization, semidefinite programming, approximate solution, sum of squares, polynomial time approximation scheme

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1. Introduction.

Consider the biquadratic polynomial optimization of the form

$$\min_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} \quad b(x,y)=\sum_{1\le i,k\le n,\,1\le j,l\le m} b_{ijkl}x_iy_jx_ky_l \quad \text{subject to} \quad \|x\|=1,\ \|y\|=1, \tag{1.1}$$

where $\|\cdot\|$ denotes the standard 2-norm in Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, where $\mathbb{R}^n$ denotes the space of real n-dimensional column vectors. Without loss of generality, we assume that the coefficients b_{ijkl} satisfy the symmetric property $b_{ijkl}=b_{kjil}=b_{ilkj}$ for $i,k=1,\ldots,n$ and $j,l=1,\ldots,m$. Let $\mathcal{A}:=(b_{ijkl})$. Then $\mathcal{A}$ is a fourth order partially symmetric tensor.

Throughout this paper, $\mathcal{S}^n$ denotes the space of real symmetric $n\times n$ matrices, and ^{T} denotes transpose. $S_{n,m}=\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m : \|x\|=\|y\|=1\}$ denotes the

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unit bisphere. For $x \in \mathbb{R}^n$, x_j denotes the jth component of x. For any matrix A and fourth order tensor $\mathcal{A}$, $\|A\|_F$ and $\|\mathcal{A}\|_F$ denote the Frobenius norms of A and $\mathcal{A}$, respectively, i.e.,

$$\|A\|_F = \left(\mathrm{Tr}(A^T A)\right)^{1/2}, \quad \|\mathcal{A}\|_F = \left(\sum_{1\le i,k\le n, 1\le j,l\le m} b_{ijkl}^2 \right)^{1/2},$$

where $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix. For $A \in \mathcal{S}^n$, $A \succeq 0$ (resp., $A \succ 0$) means that A is positive semidefinite (resp., positive definite). $\mathcal{S}^n_+$ denotes the cone of positive semidefinite matrices in $\mathcal{S}^n$. I stands for the identity matrix in an appropriate dimension.

Problem (1.1) arises from the strong ellipticity condition problem in solid mechanics (for $n = m = 3$) [16, 29, 32, 34, 39] and the entanglement problem in quantum physics. The entanglement problem is to determine whether a quantum state is separable or inseparable (entangled), or to check whether an $mn \times mn$ symmetric matrix $A \succeq 0$ can be decomposed as a convex combination of tensor products of n and m dimensional vectors [6]. It has fundamental importance in quantum science and has attracted much attention since the pioneer work of Einstein, Podolsky, and Rosen [10] and Schrödinger [33]. The entanglement problem was proved to be NP-hard by Gurvits [15].

Biquadratic optimization (1.1) has another application. Suppose that (x^*, y^*) is a global minimizer and $p_{\min}$ is the minimum objective value of (1.1). Let $p_{\max}$ be the maximum objective value of (1.1) under the same sphere constraints and $(\bar{x}, \bar{y})$ be a global maximizer. If $|p_{\min}| \ge |p_{\max}|$, then $p_{\min} \cdot \left(x^*(y^*)^T\right) \otimes \left(x^*(y^*)^T\right)$ is the best rank-one approximation to tensor $\mathcal{A}$. If $|p_{\max}| > |p_{\min}|$, then $p_{\max} \cdot \left(\bar{x}\bar{y}^T\right) \otimes \left(\bar{x}\bar{y}^T\right)$ is the best rank-one approximation to $\mathcal{A}$; see [28, 30] for details. The best rank-one approximation problem has wide applications in signal and image processing, wireless communication systems, data analysis, higher-order statistics, as well as independent component analysis [3, 5, 7, 8, 14, 19, 26, 40].

If $x \in \mathbb{R}^n$ is fixed in (1.1), then we have a quadratic optimization problem

$$\min_{y\in\mathbb{R}^m} y^T B(x) y \ \text{ subject to } \|y\| = 1, \tag{1.2}$$

where $B(x) = \left(\sum_{i,k=1}^n b_{ijkl} x_i x_k\right)_{1\le j,l\le m}$ is an $m \times m$ symmetric matrix. Similarly, if $y \in \mathbb{R}^m$ is fixed, then we have a quadratic optimization problem

$$\min_{x\in\mathbb{R}^n} x^T C(y) x \ \text{ subject to } \|x\| = 1, \tag{1.3}$$

where $C(y) = \left(\sum_{j,l=1}^m b_{ijkl} y_j y_l\right)_{1\le i,k\le n}$ is an $n \times n$ symmetric matrix. Since problem (1.1) is closely related to quadratic optimization, we call it a *biquadratic optimization problem*, or a *biquadratic program*. Using symbol $\mathcal{A}$, we can rewrite $B(x)$ and $C(y)$ as $\mathcal{A}xx^T$ and $yy^T\mathcal{A}$, respectively. Then, it is clear that

$$b(x,y) = (\mathcal{A}xx^T) \bullet (yy^T) = (yy^T\mathcal{A}) \bullet (xx^T),$$

where $X \bullet Y$ stands for the standard matrix inner product, i.e., $X \bullet Y = \mathrm{Tr}(X^T Y)$.

Contributions. In section 2, we show that problem (1.1) is NP-hard. Thus, it is not expected to find a polynomial time algorithm to solve (1.1) for general biquadratic form $b(x, y)$. Actually, we have proved a stronger result: there is no polynomial time algorithm returning a positive relative approximation bound unless P=NP.

In section 3, we propose various approximation methods to solve (1.1) using semidefinite programming (SDP) and analyze their approximation bounds under a slightly weaker approximation notion. For general biquadratic forms $b(x, y)$, we develop a $\frac{1}{2\max\{m,n\}^2}$-approximation algorithm. For $b(x, y)$ that is square-free (contains no quartic term with x_i^2 or y_j^2 for any i and j), we give an SDP relaxation with relative approximation bound $\frac{1}{nm}$. In the case where $\min\{n, m\}$ is a constant, we present two polynomial time approximation schemes (PTASs); one is based on sum of squares (SOS) relaxation hierarchy, and the other is based on grid sampling of the standard simplex originally used by Bomze and de Klerk [1].

In section 4, for practical computational purposes, we propose the first order SOS relaxation, a convex quadratic SDP relaxation, a simple minimum eigenvalue relaxation method, and show their error bounds of the three methods after certain rounding procedures.

Some illustrative numerical examples are included in section 5. We conclude and list a few open problems in the final section.

2. Complexity analysis: Hardness results. Since $b(x, y)$ is a continuous function and the feasible set of (1.1) is compact, the problem (1.1) has a global minimizer (x^*, y^*). When either x or y is fixed, the problem is then reduced to an eigenvalue problem and hence can be solved in polynomial time. However, when x and y are both variables, (1.1) is a nonconvex optimization problem, since its objective is biquadratic and nonconvex. How difficult is it to solve (1.1) globally? In this section, we show that the problem (1.1) is NP-hard to solve. We can even prove a stronger result: there is no polynomial time algorithm returning a positive relative approximation bound unless P=NP.

We first define a quality measure of approximation.

DEFINITION 2.1. *Let $\mathfrak{A}$ be a polynomial time (in n and m) approximation algorithm to solve* (1.1). *We say $\mathfrak{A}$ has a relative approximation bound $\mathcal{C} = C(\mathfrak{A}, b) \in (0, 1]$ if, for any instance of* (1.1), *the algorithm $\mathfrak{A}$ can find an upper bound p for* (1.1) *such that*

$$\begin{cases} C \cdot p \le & p_{\min} & \le p & \quad if\ p_{\min} \ge 0, \\ p_{\min} \le & p & \le C \cdot p_{\min} & \quad if\ p_{\min} < 0, \end{cases} \tag{2.1}$$

where $p_{\min}$ is the minimum value of the instance of (1.1).

In this definition, the closer C is to 1, the better the approximation algorithm would be.

2.1. Hardness of biquadratic optimization. Our main result of this section is the following.

THEOREM 2.2. (i) *The following problem is NP-hard: Given any biquadratic objective function $b(x, y)$ of* (1.1), *find the minimum value $p_{\min}$ of $b(x, y)$ over the bisphere $S_{n,m}$.*

(ii) *Unless P=NP, there does not exist a polynomial time approximation algorithm $\mathfrak{A}$ for* (1.1) *possessing a positive relative approximation bound for every instance of* (1.1).

Proof. (i) We show the NP-hardness when the biquadratic forms are restricted to be square-free and when $n = m$. To see this point, let $G = (V, E)$ be a graph with V being the set of n vertices and E being its edge set. Then define a square-free biquadratic form associated with G as

$$b_G(x, y) := -2 \sum_{(i,j)\in E} x_i x_j y_i y_j.$$

Let $\Delta_n = \{x \in \mathbb{R}^n_+ : x_1 + \cdots + x_n = 1\}$ be the standard simplex. Then we have that

$$\min_{(x,y)\in S_{n,m}} b_G(x, y) = -\max_{\|x\|=1} \sum_{(i,j)\in E} 2x_i^2 x_j^2 = -\max_{x\in\Delta_n} \sum_{(i,j)\in E} 2x_i x_j = -1 + \frac{1}{\alpha(G)},$$

due to a theorem of Motzkin and Straus [23]. Here, $\alpha(G)$ is the stability number of the graph G, i.e., the cardinality of the maximum independent set of G. Therefore, computing the minimum of $b_G(x, y)$ over the bisphere is NP-hard, since it is known to be NP-hard to compute $\alpha(G)$.

(ii) We prove this is impossible even when $n = m$. Given any integer vector a, define biquadratic form

$$b_a(x, y) = (a^T x)^2(a^T y)^2 + \left(1 - \frac{1}{n}\right) \|x\|^2 \cdot \|y\|^2 - 2 \sum_{1\le i<j\le n} x_i x_j y_i y_j. \tag{2.2}$$

In the rest of the proof, we restrict (x, y) to be in $S_{n,m}$. Then we have

$$2 \sum_{1\le i<j\le n} x_i x_j y_i y_j \le \sum_{1\le i<j\le n} x_i^2 x_j^2 + \sum_{1\le i<j\le n} y_i^2 y_j^2 = 1 - \frac{1}{2}\left(\sum_{i=1}^n x_i^4 + \sum_{i=1}^n y_i^4\right) \le 1 - \frac{1}{n}.$$

In the above, all the inequalities become equalities if and only if $x = \pm y$ has the form $\frac{1}{\sqrt{n}}(\pm 1, \ldots, \pm 1)$. Obviously,

$$(a^T x)^2(a^T y)^2 \ge 0,$$

and the inequality becomes an equality if and only if at least one of $a^T x$ and $a^T y$ equals zero. Thus, we can see that $p_{\min} \ge 0$, and the equality holds if and only if the integer vector a can be partitioned into two parts of equal sum, which is known to be NP-hard.

Now we prove (ii) by contradiction. Assume that such an algorithm $\mathfrak{A}$ exists. Then for every integer vector a, we apply the algorithm $\mathfrak{A}$ to the biquadratic form $b_a(x, y)$ defined in (2.2) and would get a bound p and $0 < C = C(\mathfrak{A}, a) \le 1$ such that

$$C \cdot p \le p_{\min} \le p.$$

Then we can see $p_{\min} = 0$ if and only if $p = 0$. This implies that we can decide whether an arbitrary integer vector could be partitioned into two parts of equal sums in polynomial time, which is known to be impossible unless P=NP. ◻

Theorem 2.2 shows that the biquadratic optimization (1.1) is NP-hard, and finding an approximate solution with a positive relative approximation bound is also NP-hard. More precisely, the proof of item (i) of Theorem 2.2 actually indicates a stronger result: Problem (1.1) remains NP-hard when the biquadratic forms are restricted to be square-free and $n = m$. Item (ii) of Theorem 2.2 says that there exists no (problem data dependent or not) positive relative approximation quality bound (the relation (2.1)) for (1.1) unless P=NP.

SDP relaxations are important on approximating quadratic optimization problems and have received much attention recently, e.g., [11, 13, 17, 21, 36], and [38]. A natural question would be, is the standard SDP relaxation of (1.1) polynomially solvable? We give a negative answer to this question next.

2.2. Hardness of bilinear SDP relaxation. We now investigate the standard SDP relaxation for (1.1). It is easy to see that problem (1.1) can be written as

$$(2.3)\qquad \begin{array}{rl} p_{\min} := \displaystyle\min_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} & (\mathcal{A}xx^T)\bullet(yy^T) \\ \text{subject to} & \mathrm{Tr}(xx^T)=1, \\ & \mathrm{Tr}(yy^T)=1, \end{array}$$

which is equivalent to

$$(2.4)\qquad \begin{array}{rl} \displaystyle\min_{X,Y} & (\mathcal{A}X)\bullet Y \\ \text{subject to} & \mathrm{Tr}(X)=1, X\succeq 0, \\ & \mathrm{Tr}(Y)=1, Y\succeq 0, \\ & \mathrm{rank}(X)=1, \mathrm{rank}(Y)=1. \end{array}$$

Here $X\in\mathcal{S}^n$, $Y\in\mathcal{S}^m$, and $\mathcal{A}X$ is an $m\times m$ matrix with

$$(\mathcal{A}X)_{jl}=\sum_{i,k=1}^n b_{ijkl}X_{ik}, \quad j,l=1,2,\ldots,m.$$

Thus, the standard SDP relaxation of (1.1) is a bilinear SDP program:

$$(2.5)\qquad \begin{array}{rl} p_{sdp} := \displaystyle\min_{X,Y} & (\mathcal{A}X)\bullet Y \\ \text{subject to} & \mathrm{Tr}(X)=1, X\succeq 0, \\ & \mathrm{Tr}(Y)=1, Y\succeq 0. \end{array}$$

We denote by p_{sdp} the optimal value of (2.5). It is clear that $p_{sdp}\le p_{\min}$.

We now consider how to generate an optimal solution (x^*, y^*) of the original problem (1.1) from an optimal solution pair (X^*, Y^*) of the bilinear SDP problem (2.5). To this aim, we state a matrix decomposition result first.

LEMMA 2.3 (Sturm and Zhang [35]). *Let $X\in\mathcal{S}^n_+$ be a positive semidefinite matrix of rank r. Let $G\in\mathcal{S}^n$ be such that $G\bullet X\ge 0$. Then, one can always find $x^1,\ldots,x^r\in\mathbb{R}^n$ in polynomial time such that $X=\sum_{i=1}^r x^i(x^i)^T$ and*

$$G\bullet x^i(x^i)^T = G\bullet X/r \quad for \quad i=1,\ldots,r.$$

THEOREM 2.4. *The biquadratic optimization* (1.1) *and bilinear SDP* (2.5) *are equivalent, that is,* (1.1) *and* (2.5) *have the same optimal value, and an optimal solution pair of* (1.1) *can be obtained from an optimal solution pair of* (2.5).

Proof. Let (X^*, Y^*) be an optimal solution matrix pair of (2.5). Without loss of generality, we assume that X^* and Y^* have full ranks n and m, respectively. Then, by Lemma 2.3, one can find the decompositions of X^* and Y^* such that

$$X^*=\sum_{i=1}^n \bar{x}^i(\bar{x}^i)^T, \qquad \|\bar{x}^i\|^2 = I\bullet X^*/n = 1/n \ \ \forall\, i$$

and

$$Y^*=\sum_{j=1}^m \bar{y}^j(\bar{y}^j)^T, \qquad \|\bar{y}^j\|^2 = I\bullet Y^*/m = 1/m \ \ \forall\, j.$$

There must exist an index, say, 1, such that $(\mathcal{A}X^*) \bullet \bar{y}^1(\bar{y}^1)^T \leq p_{sdp}/m$, since

$$p_{sdp} = (\mathcal{A}X^*) \bullet Y^* = (\mathcal{A}X^*) \bullet \left(\sum_{j=1}^{m} \bar{y}^j (\bar{y}^j)^T \right).$$

Let $y^* = \sqrt{m}\bar{y}^1$. Then we must have

$$(\mathcal{A}X^*) \bullet y^*(y^*)^T \leq p_{sdp}, \qquad |y^*\|^2 = 1.$$

Continue this process on X^*. There must be an index, say, 1, such that

$$(\mathcal{A}\bar{x}^1(\bar{x}^1)^T) \bullet y^*(y^*)^T \leq p_{sdp}/n.$$

Let $x^* = \sqrt{n}\bar{x}^1$; we must have

$$(\mathcal{A}x^*(x^*)^T) \bullet y^*(y^*)^T \leq p_{sdp}, \qquad \|x^*\|^2 = 1, \|y^*\|^2 = 1.$$

That is, (x^*, y^*) is a feasible solution pair for the original problem (1.1) so that

$$p_{\min} \leq (\mathcal{A}x^*(x^*)^T) \bullet y^*(y^*)^T \leq p_{sdp} \leq p_{\min},$$

which implies that $p_{\min} = p_{sdp} = (\mathcal{A}x^*(x^*)^T) \bullet y^*(y^*)^T$. We complete the proof. ∎

Theorem 2.4 shows that we can obtain a solution of (1.1) in polynomial time from a solution of (2.5). Therefore, (2.5) must be still hard to solve.

COROLLARY 2.5. *It is NP-hard to solve the bilinear SDP relaxation* (2.5).

Proof. Theorem 2.4 shows that the biquadratic optimization (1.1) and its bilinear SDP relaxation (2.5) have the same optimal value. From Theorem 2.2, we know (1.1) is NP-hard, which immediately implies the relaxation (2.5) is also NP-hard. ∎

Our result is in contrast to the bilinear optimization over two vector simplexes:

$$\min_{u \in \mathbb{R}^n, v \in \mathbb{R}^m} \quad u^T A v \quad \text{subject to} \quad \sum_{i=1}^{n} u_i = 1, \quad \sum_{j=1}^{m} v_j = 1, \quad u \geq 0, \quad v \geq 0.$$

The above problem is solvable in polynomial time by simply choosing the minimum element in the matrix A.

3. Approximation quality bounds. Although approximating (1.1) is NP-hard, it does not exclude the approximatability when the biquadratic form $b(x, y)$ in (1.1) has special structures. In this section, we give various approximation results when $b(x, y)$ is general or has special features using SDP relaxation methods. To present our results, we begin with another quality measure of approximation.

DEFINITION 3.1. *Let $1 > \epsilon \geq 0$, and let $\mathfrak{A}$ be an approximation algorithm for* (1.1). *We say $\mathfrak{A}$ is a $(1-\epsilon)$-approximation algorithm for* (1.1) *if for any instance of* (1.1) *the algorithm $\mathfrak{A}$ returns a feasible pair $(\bar{x}, \bar{y})$ to* (1.1) *such that*

$$b(\bar{x}, \bar{y}) - p_{\min} \leq \epsilon(p_{\max} - p_{\min}).$$

Recall that $p_{\min}$ (resp., $p_{\max}$) is the minimum (resp., maximum) value of the objective in (1.1). *We say* (1.1) *has a PTAS if for every $1 > \epsilon > 0$ there exists a $(1-\epsilon)$-approximation algorithm.*

One can see that Definition 3.1 is weaker than Definition 2.1. If $p_{\max} = 0$, then the two definitions coincide each other with $C = 1 - \epsilon$.

We will consider the general biquadratic form $b(x, y)$ first and develop a $\frac{1}{2\max\{m,n\}^2}$-approximation algorithm for (1.1) under Definition 3.1. When $b(x, y)$ has only squared terms in x or y, we show that (1.1) can be solved in polynomial time. When $b(x, y)$ is square-free, we show that (1.1) has a polynomial-time approximation algorithm with a relative bound $\frac{1}{nm}$ under Definition 2.1. When $\min\{n, m\}$ is a constant, we present two PTASs for solving (1.1).

3.1. SDP approximation bounds based on ellipsoids. As we descried earlier, the standard SDP relaxation of (1.1) is the bilinear program (2.5). Theorem 2.4 actually indicates that this relaxation is tight, namely, given any (X, Y) feasible for (2.5), one can in polynomial time find feasible solution pairs $(x', y'), (x'', y'')$ of (1.1) such that

$$b(x', y') \le (\mathcal{A}X) \bullet Y \le b(x'', y'').$$

The bilinear SDP program (2.5) can be rewritten as

$$\begin{aligned} p_{\min} := \min_{X,Y} \quad & (\mathcal{A}X) \bullet Y + \frac{1}{n}(\mathcal{A}I_n) \bullet Y + \frac{1}{m}(\mathcal{A}X) \bullet I_m + \frac{1}{mn}(\mathcal{A}I_n) \bullet I_m \\ \text{subject to} \quad & \mathrm{Tr}(X) = 0, X + \frac{1}{n}I_n \succeq 0, \\ & \mathrm{Tr}(Y) = 0, Y + \frac{1}{m}I_m \succeq 0 \end{aligned} \tag{3.1}$$

after some linear transformations $X := X - \frac{1}{n}I_n$ and $Y := Y - \frac{1}{m}I_m$.

The objective function in (3.1) contains linear and constant terms, which are all zeros when the biquadratic form $b(x, y)$ is square-free. The constant term $\bar{p} := \frac{1}{mn}(\mathcal{A}I_n) \bullet I_m$ is the objective value of (2.5) for the feasible pair $(\frac{1}{n}I_n, \frac{1}{m}I_m)$. Thus, we know

$$p_{\min} \le \bar{p} \le p_{\max}.$$

We denote

$$\phi(X, Y) = (\mathcal{A}X) \bullet Y + \frac{1}{n}(\mathcal{A}I_n) \bullet Y + \frac{1}{m}(\mathcal{A}X) \bullet I_m.$$

Note that the following relation holds for matrices in $\mathcal{S}^n$:

$$\left\{ X : \begin{array}{c} \mathrm{Tr}(X) = 0 \\ \|X\|_F \le \frac{1}{n} \end{array} \right\} \subseteq \left\{ X : \begin{array}{c} \mathrm{Tr}(X) = 0 \\ X \succeq -\frac{1}{n}I_n \end{array} \right\} \subseteq \left\{ X : \begin{array}{c} \mathrm{Tr}(X) = 0 \\ \|X\|_F \le \sqrt{1 - \frac{1}{n}} \end{array} \right\}. \tag{3.2}$$

For any scalars $\lambda > 0$ and $\mu > 0$, denote $\Omega(\lambda, \mu)$ for the optimization problem:

$$\begin{aligned} p(\lambda, \mu) := \min_{X,Y} \quad & \phi(X, Y) \\ \text{subject to} \quad & \mathrm{Tr}(X) = \mathrm{Tr}(Y) = 0, \\ & \|X\|_F \le \lambda, \ \|Y\|_F \le \mu. \end{aligned} \tag{3.3}$$

This is a nonhomogeneous quadratic program over two ellipsoidal constraints. It can be viewed as using an ellipsoidal set to approximate the affine conic feasible set of (2.5), which was first used in Ye [37] and by Fu, Luo, and Ye [13] for polyhedral constrained nonconvex quadratic optimization, and more recently by Luo and Zhang

[22] for homogeneous quartic polynomial optimization. Note again the relationship between the optimal values

$$p(1,1) \le p_{\min} - \bar{p} \le p\left(\frac{1}{n}, \frac{1}{m}\right) \le p\left(\frac{1}{\max\{m,n\}}, \frac{1}{\max\{m,n\}}\right).$$

For any optimal pair (X^*, Y^*) of (3.3), the linear sum $\frac{1}{n}(\mathcal{A}I_n) \bullet Y^* + \frac{1}{m}(\mathcal{A}X^*) \bullet I_m$ must be nonpositive; otherwise, we can replace (X^*, Y^*) by $(-X^*, -Y^*)$ to get a smaller objective value. Hence, we have the relation

$$p(1,1) \le p\left(\frac{1}{n}, \frac{1}{m}\right) \le p\left(\frac{1}{\max\{m,n\}}, \frac{1}{\max\{m,n\}}\right) \le \frac{1}{\max\{m,n\}^2} p(1,1).$$

Thus, if one can compute a feasible pair $(\bar{X}, \bar{Y})$ for $\Omega\left(\frac{1}{n}, \frac{1}{m}\right)$ such that $\phi(\bar{X}, \bar{Y}) \le \alpha p\left(\frac{1}{n}, \frac{1}{m}\right)$, then

$$\phi(\bar{X}, \bar{Y}) \le \frac{\alpha}{\max\{m,n\}^2} p(1,1) \le \frac{\alpha}{\max\{m,n\}^2}(p_{\min} - \bar{p}).$$

Taking $X = \bar{X} + \frac{1}{n} I_n$ and $Y = \bar{Y} + \frac{1}{m} I_m$, we have

$$(\mathcal{A}X) \bullet Y - \bar{p} \le \frac{\alpha}{\max\{m,n\}^2}(p_{\min} - \bar{p}).$$

From the proof of Theorem 2.4, one can, in polynomial time, compute a solution (x', y') feasible to (1.1) such that

$$b(x', y') - \bar{p} \le (\mathcal{A}X) \bullet Y - \bar{p} \le \frac{\alpha}{\max\{m,n\}^2}(p_{\min} - \bar{p}).$$

This, together with $\bar{p} \le p_{\max}$, imply $b(x', y') - p_{\max} \le \frac{\alpha}{\max\{m,n\}^2}(p_{\min} - p_{\max})$ and

$$b(x', y') - p_{\min} \le \left(1 - \frac{\alpha}{\max\{m,n\}^2}\right)(p_{\max} - p_{\min}).$$

In other words, we should be able to establish a $\frac{1}{2\max\{m,n\}^2}$-approximation algorithm for (1.1) if we can approximate $\Omega\left(\frac{1}{n}, \frac{1}{m}\right)$ with a relative approximation bound $\alpha = \frac{1}{2}$.

THEOREM 3.2. *There is a polynomial time algorithm that returns a solution* (x', y') *for the biquadratic optimization* (1.1) *such that*

$$b(x', y') - p_{\min} \le \left(1 - \frac{1}{2\max\{m,n\}^2}\right)(p_{\max} - p_{\min}).$$

Proof. From the above discussion, we know it suffices to show that $\Omega(\frac{1}{n}, \frac{1}{m})$ allows a polynomial-time solution of relative approximation bound $\alpha = \frac{1}{2}$. To see this, note that (3.3) can be equivalently formulated as a quadratic optimization problem

$$\begin{array}{llc} & \min\limits_{z \in \mathbb{R}^N} & q(z) := z^T Q z + 2c^T z \\ (3.4) & \text{subject to} & z^T A_1 z \le 1, \\ & & z^T A_2 z \le 1, \end{array}$$

where $A_1, A_2 \succeq 0$ and $A_1 + A_2 \succ 0$, $N = \frac{1}{2}[n(n+1) + m(m+1)] - 2$, and Q is symmetric. Denote its minimal value by $q_{\min}$. Then $q_{\min} \le 0$, as $z = 0$ is a feasible solution of (3.4). The standard SDP relaxation for the above problem is

$$\begin{array}{ll} \min\limits_{W,z} & Q \bullet W + 2c^T z \\ \text{subject to} & A_1 \bullet W \le 1,\ A_2 \bullet W \le 1, \\ & \begin{pmatrix} 1 & z^T \\ z & W \end{pmatrix} \succeq 0. \end{array}$$

This SDP has three constraints, so that an optimal $\begin{pmatrix} 1 & (z^*)^T \\ z^* & W^* \end{pmatrix}$ can be computed in polynomial time such that its rank equals two (e.g., see [38]). Hence the Schur complement $W^* - z^*(z^*)^T$ must be rank one and one can write

$$W^* = z^*(z^*)^T + w^*(w^*)^T$$

for some $w^* \in \mathbb{R}^N$. Let us choose w^* such that $c^T w^* \le 0$ (otherwise, we choose $-w^*$ as w^*). Note that both z^* and w^* are feasible for (3.4) because both A_1 and A_2 are positive semidefinite. Then we have

$$q(z^*) = Q \bullet z^*(z^*)^T + 2c^T z^*, \qquad q(w^*) = Q \bullet w^*(w^*)^T + 2c^T w^*.$$

Adding these two, together with $c^T w^* \le 0$, we have

$$q(z^*) + q(w^*) = Q \bullet \left(z^*(z^*)^T + w^*(w^*)^T\right) + 2c^T(z^* + w^*) \le Q \bullet W^* + 2c^T z^* = q_{\min},$$

which implies

$$\min\{q(z^*), q(w^*)\} \le \frac{1}{2} q_{\min}.$$

Thus, either z^* or w^* is a solution with relative approximation bound $\alpha = \frac{1}{2}$ for (3.4). □

Theorem 3.2 establishes an approximation bound for general biquadratic form $b(x,y)$ under Definition 3.1. When $b(x,y)$ has special features, better results are possible.

THEOREM 3.3. *For the biquadratic optimization* (1.1)*, we have*

(i) *If $b(x,y)$ in* (1.1) *is square-free, then the SDP relaxation $\Omega\left(\frac{1}{n}, \frac{1}{m}\right)$ can be solved in polynomial time and*

$$p_{\min} \le p\left(\frac{1}{n}, \frac{1}{m}\right) \le \frac{1}{nm} p_{\min}.$$

(ii) *If $b(x,y)$ has only squared terms in x or has only squared terms in y, then biquadratic optimization* (1.1) *can be solved in polynomial time.*

Proof. (i) When $b(x,y)$ is square-free, $\bar{p} = 0$ and $\phi(X,Y)$ is homogeneous and quadratic, so that $p(\frac{1}{n}, \frac{1}{m}) = \frac{1}{nm} p(1,1)$. Then $p(1,1) \le p_{\min} \le p(\frac{1}{n}, \frac{1}{m})$ immediately implies the inequalities in (i). On the other hand, when $b(x,y)$ is square-free, problem (3.3) is polynomial time solvable, since by eliminating equality constraints it can be reduced to minimizing a homogeneous quadratic objective over two homogeneous quadratic inequality constraints. The latter problem can be solved by using the S-lemma; see Ye and Zhang [38].

(ii) Now we consider the special case where $b(x,y)$ in (1.1) has only squared terms in x or has only squared terms in y. Assume that this is the latter case. Then (1.1) has the form

$$\min_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} \sum_{1\le i,k\le n,\, 1\le j\le m} b_{ijkj}x_ix_ky_j^2 \text{ subject to } \quad \|x\|^2=1,\ \|y\|^2=1$$

$$= \min_{x\in\mathbb{R}^n,\, \|x\|^2=1} \min_{1\le j\le m} \sum_{1\le i,k\le n} b_{ijkj}x_ix_k = \min_{1\le j\le m} \min_{x\in\mathbb{R}^n,\, \|x\|^2=1} \sum_{1\le i,k\le n} b_{ijkj}x_ix_k$$

$$= \min_{1\le j\le m} \lambda_{\min}(B_j),$$

where for $j=1,\dots,m$, $\lambda_{\min}(B_j)$ is the smallest eigenvalue of the symmetric $n\times n$ matrix $B_j=(b_{ijkj})_{1\le i,k\le n}$. Since one can find the smallest eigenvalue of a symmetric $n\times n$ matrix in polynomial time, this case can be solved in polynomial time. Similarly, one can solve the case where $b_{ijkl}=0$ whenever $i\neq k$ in polynomial time. $\square$

Remark. In the case of square-free biquadratic forms, Theorem 3.3 gives a better bound for the optimal value than Theorem 3.2. On the other hand, Theorem 3.2 provides information about the quality of a feasible solution (x',y') for a general biquadratic form.

3.2. A partial PTAS for (1.1) based on SOS. Let $B(x)$ be the symmetric matrix in (1.2). Then the original biquadratic optimization (1.1) can be equivalently formulated as

$$\begin{array}{ll} p_{\min} := \max & \gamma \\ \text{subject to} & B(x)-\gamma(x^Tx)I_m \succeq 0\,\forall x\in\mathbb{R}^n. \end{array} \tag{3.5}$$

A sequence of SDP relaxations based on SOS can be applied to solve problem (3.5). SOS methods have received much attention recently in solving nonconvex polynomial optimization problems [9, 18, 20, 27, 24]. Usually a hierarchy of SDP relaxations based on SOS can be applied to obtain a sequence of lower bounds converge to the optimal value of polynomial optimization problems, where a general convergence rate was established by Nie and Schweighofer [25].

Let $N\ge 0$ be an integer. Consider the following Nth order SOS relaxation:

$$\begin{array}{ll} p_N := \max & \gamma \\ \text{subject to} & (x^Tx)^N\big(B(x)-\gamma(x^Tx)I_m\big) \text{ is SOS }. \end{array} \tag{3.6}$$

For a symmetric matrix polynomial $F(x)$, we say $F(x)$ is SOS if there exists some matrix polynomial $G(x)$ such that $F(x)=G(x)^TG(x)$. Obviously, for any integer N, p_N is a lower bound of $p_{\min}$. When $N=0$, the dual of the relaxation (3.6) is the problem (4.2) of the next section. The convergence result is as follows.

THEOREM 3.4. *For any* $N\ge \frac{3n}{\log 2}-\frac{1}{2}n-2$, *it holds that*

$$0\le \frac{p_{\min}-p_N}{p_{\max}-p_{\min}} \le \frac{6n}{(2N+n+4)\log 2-6n},$$

where $p_{\max}$ *is the maximum of* $b(x,y)$ *over the bisphere* $S_{n,m}$.

SOS methods have been applied to minimize forms (homogeneous scalar polynomials) over unit spheres. Faybusovich [11] proved a quality bound like in Theorem 3.4

for minimizing general even forms over unit spheres, using a result of Reznick [31] on degree bounds of representing positive definite forms by using SOS. To prove Theorem 3.4, we need to generalize that result of degree bounds to positive definite matrix forms (homogeneous matrix polynomials). That is the following lemma.

LEMMA 3.5. *Let $F(x)$ be a homogeneous symmetric matrix polynomial of degree $2d$ such that $F(x) \succ 0$ for any $x \neq 0$. Let*

$$c(F) = \max_{\|\xi\|=1} \frac{\max_{\|x\|=1} \xi^T F(x)\xi}{\min_{\|x\|=1} \xi^T F(x)\xi}.$$

Then for any integer N such that

$$N \geq \frac{nd(2d-1)}{(2\log 2)} c(F) - \frac{n+2d}{2},$$

the matrix polynomial $(\sum_i x_i^2)^N F(x)$ is SOS.

Proof. We generalize the proof in section 7 of Reznick [31] for scalar forms to matrix forms. Write $F(x) = \sum_i F_i f_i(x)$, where F_i are matrices and $f_k(x)$ are scalar homogeneous polynomials. Let $G(x) = x_1^2 + \cdots + x_n^2$. For any polynomial $p(x)$, the differential operator $p(\partial)$ is defined by replacing each x_j by $\frac{\partial}{\partial x_j}$, e.g., $G(\partial) = \Delta$ is the Laplacian operator. The matrix differential operator $F(\partial)$ is defined to be $\sum_k F_k f_k(\partial)$. For every polynomial h of degree $2d$, it holds that

$$h(\partial)G^N = \Phi_N(h)G^{N-2d}, \quad \text{where} \quad \Phi_N(h) = \sum_{k\geq 0} \frac{(N)_{d-k}}{2^{2k-d}d!} \Delta^k(h)G^k.$$

Here, $(N)_t = N(N-1)\cdots(N-(t-1))$. The above two identities imply that

$$h(\partial)G^N = h(\partial)\left(\sum_{k=1}^{N} \lambda_k(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2N}\right),$$
$$\Phi_N(h)G^{N-2d} = (2N)_d \sum_{k=1}^{N} \lambda_k h(\alpha_{k1}, \ldots, \alpha_{kn})(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2N-2d}.$$

If we choose $h = \Phi_N^{-1}(f_i)$, then we have

$$f_i(x)G^{N-2d} = (2N)_d \sum_{k=1}^{N} \lambda_k \Phi_N^{-1}(f_i)(\alpha_{k1}, \ldots, \alpha_{kn})(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2N-2d}.$$

Therefore, it holds that

$$F(x)G^{N-2d} = (2N)_d \sum_{k=1}^{N} \lambda_k \sum_i F_i \Phi_N^{-1}(f_i)(\alpha_{k1}, \ldots, \alpha_{kn})(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2N-2d}. \tag{3.7}$$

For any polynomial p, $\Phi_N^{-1}(p)$ has the formula

$$\Phi_N^{-1}(p) = \frac{1}{(N)_d 2^d}\left(p - \frac{\Delta(p)G}{2(n+2N-2)} + \frac{\Delta^2(p)G^2}{8(n+2N-2)(n+2N-4)} - \cdots\right).$$

Hence,

$$\sum_i F_i \Phi_N^{-1}(f_i) = \frac{1}{(N)_d 2^d}\left(F(x) - \frac{\Delta(F)G}{2(n+2N-2)} + \frac{\Delta^2(F)G^2}{8(n+2N-2)(n+2N-4)} - \cdots\right).$$

Obviously, it holds that

$$\lim_{N\to\infty}(N)_d 2^d \sum_i F_i \Phi_N^{-1}(f_i(x)) = F(x).$$

When $F(x) \succ 0$, we can choose N big enough such that $\sum_i F_i \Phi_N^{-1}(f_i(x)) \succ 0$.

For any vector ξ with $\|\xi\| = 1$, it holds that

$$\xi^T\left(\sum_i F_i \Phi_N^{-1}(f_i)\right)\xi = \frac{1}{(N)_d 2^d}\left(\xi^T F\xi - \frac{\Delta(\xi^T F\xi)G}{2(n+2N-2)} + \frac{\Delta^2(\xi^T F\xi)G^2}{8(n+2N-2)(n+2N-4)} - \cdots\right).$$

By the theorem in section 7 in [31], when

$$N \geq \frac{nd(2d-1)}{(2\log 2)}\frac{\max_{\|x\|=1}\xi^T F(x)\xi}{\min_{\|x\|=1}\xi^T F(x)\xi} - \frac{n+2d}{2},$$

$\xi^T\left(\sum_i F_i \Phi_N^{-1}(f_i)\right)\xi$ is positive. Choose a uniform N for all $\|\xi\| = 1$. For

$$N \geq \frac{nd(2d-1)}{(2\log 2)}c(F) - \frac{n+2d}{2},$$

we have $\sum_i F_i \Phi_N^{-1}(f_i(x)) \succ 0$. So $(\sum_i x_i^2)^N F(x)$ is SOS by (3.7). ∎

Proof of Theorem 3.4. Note that we have the inequality

$$p_{\min} I_m \preceq B(x) \preceq p_{\max} I_m \quad \forall\, x \in \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

Let $\gamma < p_{\min}$. Then it holds that

$$(p_{\min} - \gamma)I_m \preceq B(x) - \gamma(x^T x)I_m \preceq (p_{\max} - \gamma)I_m \quad \forall x \in \{x \in \mathbb{R}^n : \|x\| = 1\},$$

and hence

$$c(B(x) - \gamma(x^T x)I_m) \leq \frac{p_{\max} - \gamma}{p_{\min} - \gamma}.$$

Now fix one $N > \frac{3n}{\log 2} - \frac{1}{2}n - 2$, and choose

$$\gamma_N = p_{\min} - \frac{6n(p_{\max} - p_{\min})}{(2N+n+4)\log 2 - 6n}.$$

Then we can verify that

$$N = \frac{3n}{(\log 2)}\frac{p_{\max} - \gamma_N}{p_{\min} - \gamma_N} - \frac{n+4}{2}.$$

By Lemma 3.5, we know $(x^Tx)^N(B(x)-\gamma_N(x^Tx)I_m)$ is SOS. By the definition of p_N, we know p_N satisfies the inequality claimed by Theorem 3.4. $\square$

Let $C(y)$ be the symmetric quadratic matrix defined in (1.3). Then the equivalent formulation (1.3) of (1.1) can be formulated as

$$(3.8)\qquad \begin{array}{rl} p_{\min} := \max & \gamma \\ \text{subject to} & C(y)-\gamma(y^Ty)I_n \succeq 0\,\forall y\in\mathbb{R}^m. \end{array}$$

Similarly, a sequence of convergent SDP relaxations using SOS can be applied to solve the problem (3.8), as we have done for (3.5). Let $N \geq 0$ be an integer. The Nth order SOS relaxation for (3.8) is

$$(3.9)\qquad \begin{array}{rl} \tilde{p}_N := \max & \gamma \\ \text{subject to} & (y^Ty)^N\big(C(y)-\gamma(y^Ty)I_n\big) \text{ is SOS .} \end{array}$$

Obviously, for any integer N, $\tilde{p}_N$ is a lower bound of $p_{\min}$. When $N=0$, the dual of the relaxation (3.9) is also the same as (4.2). A similar convergence result is as follows.

THEOREM 3.6. *For any $N \geq \frac{3m}{\log 2}-\frac{1}{2}m-2$, it holds that*

$$0 \leq \frac{p_{\min}-\tilde{p}_N}{p_{\max}-p_{\min}} \leq \frac{6m}{(2N+m+4)\log 2-6m},$$

where $p_{\max}$ is the maximum of $b(x,y)$ over the bisphere $S_{n,m}$.

Note that when $\min\{n,m\}$ and N are fixed, both SOS relaxations (3.6) and (3.9) can be solved in polynomial time. Thus Theorems 3.4 and 3.6 imply the following corollary.

COROLLARY 3.7. *If $\min\{n,m\}$ is fixed, then for every $\epsilon > 0$ we can find a lower bound p_{low} for the optimal value $p_{\min}$ of (1.1) in polynomial time such that*

$$0 \leq p_{\min}-p_{low} \leq \epsilon(p_{\max}-p_{\min}).$$

Proof. Let $K=\min\{n,m\}$ be a fixed constant. Then choose an integer N such that

$$N \geq \frac{3K}{\log 2}-\frac{1}{2}K-2, \qquad \frac{6K}{(2N+K+4)\log 2-6K} \leq \epsilon.$$

If $m=K$, we apply SOS relaxation (3.9), which can be solved in polynomial time, and set $p_{low}=\tilde{p}_N$. If $n=K$, we apply SOS relaxation (3.6), which can also be solved in polynomial time, and set $p_{low}=p_N$. In either case of $m=K$ or $n=K$, from Theorems 3.6 or 3.4, we know p_{low} is a lower bound for the minimum value $p_{\min}$, and it satisfies the relation we want. $\square$

3.3. Another partial PTAS for (1.1) based on grid sampling on simplex. Now consider the biquadratic optimization of the special form

$$(3.10)\qquad \begin{array}{rl} p_{\min} := \min\limits_{x\in\mathbb{R}^n,y\in\mathbb{R}^m} & \sum\limits_{1\leq i,k\leq n,\,1\leq j,l\leq m} b_{ijkl}x_iy_jx_ky_l \\ \text{subject to} & \|x\|=1, \quad \|y\|=1, \\ & y\geq 0. \end{array}$$

The difference between (3.10) and the original biquadratic optimization (1.1) is that (3.10) requires $y \geq 0$. In this case, one can choose $y\in\mathbb{R}^m_+$ to be from grid points

$\{0, \sqrt{\frac{1}{d}}, \ldots, \sqrt{\frac{d-1}{d}}, 1\}$ such that $y_1^2 + \cdots + y_m^2 = 1$ for some given integer d. They represent uniform grid points on the partial sphere $\{y \in \mathbb{R}^m_+ : \|y\| = 1\}$. The total number of such feasible grid points is $\binom{m+d-1}{d}$, which is polynomial in m for any fixed integer $d \geq 1$.

For each feasible grid point $\hat{y}$, one can solve the minimum eigenvalue problem

$$p_{\hat{y}} := \min_{x \in \mathbb{R}^n} \sum_{1 \leq i,k \leq n} \sum_{1 \leq j,l \leq m} b_{ijkl} x_i \hat{y}_j x_k \hat{y}_l \quad \text{subject to} \quad \|x\| = 1.$$

The above problem can be solved in polynomial time for each fixed $\hat{y}$. Then, one can choose $\hat{y}$ among these grid points such that $p_{\hat{y}}$ is the smallest, which gives a $(1 - \frac{1}{d})$-approximation solution to (3.10) (see Bomze and de Klerk [1]). Thus we have the following.

THEOREM 3.8. *There is a PTAS for solving problem* (3.10).

Similarly, if in problem (3.10) the constraint $y \in \mathbb{R}^m_+$ is replaced by $x \in \mathbb{R}^n_+$, then a similar PTAS exists. So, for the original biquadratic optimization (1.1), if we know in advance the sign of optimal vector x^* or y^*, the above PTAS can be slightly modified to solve (1.1). For instance, when all the coefficients of the biquadratic form are nonpositive, the optimal x^* and y^* must be nonnegative, and hence a PTAS exists.

Note that the number of sign patterns for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are at most 2^n and 2^m, respectively. If $\min\{n, m\}$ is fixed, then we can yield a PTAS for (1.1) by solving subproblems of the form (3.10) at most $2^{\min\{n,m\}}$ times. Hence, this presents a PTAS for solving (1.1) whenever $\min\{n, m\}$ is fixed.

COROLLARY 3.9. *If* $\min\{m, n\}$ *is fixed, there exists a PTAS based on the grid sampling on simplex for solving* (1.1).

4. Some practical semidefinite relaxations. Section 2 proved the NP-hardness of the biquadratic optimization (1.1), while section 3 presented several approximation results. In this section, we present further semidefinite relaxations that might be more practical and can be effectively implemented. They are based on the first order SOS relaxation and a convex quadratic SDP relaxation.

4.1. First order SOS relaxation and the minimum eigenvalue method. Note that the bilinear SDP (2.5) can be equivalently formulated as

$$(4.1) \qquad \begin{aligned} p_{\min} := \min \quad & \sum_{1 \leq i,k \leq n,\, 1 \leq j,l \leq m} b_{ijkl} X_{ik} Y_{jl} \\ \text{subject to} \quad & \mathrm{Tr}(X \otimes Y) = 1, \\ & X \otimes Y \succeq 0. \end{aligned}$$

Here $\otimes$ denotes the standard Kronecker product. In (4.1), define $m \times m$ matrices $B^{(i,k)} = (b_{ijkl})_{1 \leq j,l \leq m}$. Then we can further relax the above bilinear SDP (4.1) as the linear SDP

$$(4.2) \qquad \begin{aligned} p_{sos} := \min_{Z} \quad & \sum_{1 \leq i,k \leq n} B^{(i,k)} \bullet Z^{(i,k)} \\ \text{subject to} \quad & \sum_{i=1}^{n} \mathrm{Tr}(Z^{(i,i)}) = 1, \\ & Z^{(i,k)} = Z^{(k,i)}, \quad (Z^{(i,k)})^T = Z^{(i,k)}, \quad \forall\, (i,k), \\ & Z := \left(Z^{(i,j)}\right)_{1 \leq i,j \leq n} \succeq 0. \end{aligned}$$

Obviously, the optimal value p_{sos} of (4.2) is a lower bound for the minimum value $p_{\min}$ of (1.1). The dual of the SDP relaxation (4.2) can be shown to have the form

$$
\begin{array}{lll}
 & \max\limits_{\gamma, W} & \gamma \\
(4.3) & \text{subject to} & B = W + \gamma I_{nm}, \\
 & & W^{(i,k)} = W^{(k,i)}, \quad (W^{(i,k)})^T = W^{(i,k)}, \quad \forall\,(i,k), \\
 & & W := \left(W^{(i,j)}\right)_{1\le i,j\le n} \succeq 0,
\end{array}
$$

where the matrix B is defined as $B = \left(B^{(i,j)}\right)_{1\le i,j\le n}$.

THEOREM 4.1. *The semidefinite relaxation* (4.2) *has the following properties:*

(i) *For any feasible* γ *in* (4.3), *the difference* $b(x,y) - \gamma x^T x \cdot y^T y$ *is an SOS, i.e., there exist matrices* $A_1, \dots, A_K \in \mathbb{R}^{n\times m} (K \le nm)$ *such that*

$$b(x,y) - \gamma \cdot x^T x \cdot y^T y = \sum_{k=1}^{K} (x^T A_k y)^2.$$

In particular, the difference $b(x,y) - p_{sos} \cdot x^T x \cdot y^T y$ *is an SOS.*

(ii) *It holds that* $\lambda_{\min}(B) \le p_{sos} \le p_{\min}$.

(iii) *If* $\min\{n,m\} = 2$, *then* $p_{\min} = p_{sos}$.

Proof. (i) Let (γ, W) be a feasible pair for (4.3). Then we have the relation

$$(x\otimes y)^T B (x\otimes y) = (x\otimes y)^T W (x\otimes y) + \gamma \|(x\otimes y)\|^2.$$

Hence we get the polynomial identity

$$b(x,y) - \gamma \cdot x^T x \cdot y^T y = (x\otimes y)^T W (x\otimes y).$$

Since $W \succeq 0$, there exists a matrix $L \in \mathbb{R}^{nm\times K}$ such that $W = LL^T$. Here K is the rank of W. For every $k = 1, \dots, K$, let A_k be a matrix such that the vectorization of A_k equals the kth column of L. Thus the first part of (i) is proved.

Since the feasible set of (4.2) has nonempty interior, the optimal value of the dual (4.3) is attainable and must equal p_{sos}. Hence there exists some W^* such that (p_{sos}, W^*) is feasible for (4.3). So the second part of (i) of Theorem 4.1 can be implied by the first part of (i) of Theorem 4.1.

(ii) The second inequality is obvious. In SDP relaxation (4.2), if we do not require any off-diagonal block of Z to be symmetric, then it can be further relaxed to

$$\min \quad B \bullet Z \quad \text{subject to} \quad \mathrm{Tr}(Z) = 1, \quad Z \succeq 0. \tag{4.4}$$

The optimal value above is exactly $\lambda_{\min}(B)$. Then we can see $\lambda_{\min}(B) \le p_{sos}$.

(iii) By definition of $p_{\min}$, we know $b(x,y) - p_{\min} \cdot x^T x \cdot y^T y$ is a nonnegative biquadratic form. When $n = 2$ or $m = 2$, Calderón [2] showed that every nonnegative biquadratic form $b(x,y)$ must be an SOS. So $p_{\min} \le p_{sos}$ follows from the definition of p_{sos}, and so by (ii) of Theorem 4.1, $p_{\min} = p_{sos}$. ∎

From (i) of Theorem 4.1, we can see that the dual problem (4.3) is actually the first one ($N = 0$) in the hierarchy defined in (3.6). Hence $p_{sos} = p_0$. Once the SDP relaxation (4.2) is solved, we obtain a lower bound p_{sos} and an optimal matrix $Z^* \succeq 0$. When Z^* has rank one, the block-symmetric structures of Z^* imply that there are some vectors $x^* \in \mathbb{R}^n, y^* \in \mathbb{R}^m$ such that $Z^* = (x^*\otimes y^*)(x^*\otimes y^*)^T$, and hence (x^*, y^*) is one global optimizer for (1.1). Now we consider the general case where

$$Z^* = \lambda_1 z^1 (z^1)^T + \cdots + \lambda_r z^r (z^r)^T$$

for orthonormal vectors $z^1, \ldots, z^r$ and scalars $\lambda_1 \geq 0, \ldots, \lambda_r \geq 0$ with $\lambda_1 + \cdots + \lambda_r = 1$. For each z^i, we pack it back into an $m \times n$ matrix $U_i = \mathrm{mat}(z^i)$ by columns, i.e., the m elements in the jth column of U_i consist of $z^i_{(j-1)m+1}, \ldots, z^i_{jm}$ of z^i. Then find the singular value decomposition (SVD)

$$U_i = \sigma_{i,1} u^{i,1} (v^{i,1})^T + \cdots + \sigma_{i,k_i} u^{i,k_i} (v^{i,k_i})^T.$$

From the set of all pairs $(v^{i,p}, u^{j,q})$ obtained above, choose one pair (x^*, y^*) such that

$$b(x^*, y^*) = \min_{\substack{1 \leq i,j \leq r \\ 1 \leq p \leq k_i,\, 1 \leq q \leq k_j}} b(v^{i,p}, u^{j,q}).$$

The performance of the above pair selection process is as follows.

THEOREM 4.2. *Let $\lambda_{\max}(B)$ be the largest eigenvalue of the symmetric matrix B in* (4.2). *Then, the first order SOS relaxation method, together with the pair selection process described above, can produce a feasible pair (x^*, y^*) to* (1.1) *such that*

$$\lambda_{\max}(B) - b(x^*, y^*) \geq \frac{1}{\min\{n, m\}} (\lambda_{\max}(B) - p_{\min}).$$

Proof. From the rank one decomposition of optimal matrix Z^*, there exists one z^i, say, z^1, such that

$$(z^1)^T B z^1 \leq p_{sos} \quad \text{and} \quad \|z^1\|^2 = 1. \tag{4.5}$$

Note that $z^1 (z^1)^T$ may not have the desired block-symmetry anymore. We can pack z^1 back into an $m \times n$ matrix $U_1 = \mathrm{mat}(z^1)$ by columns. The rank of U_1 is k_1 ($\leq \min\{m, n\}$). $\|z^1\| = 1$ implies $\sigma_{1,1}^2 + \cdots + \sigma_{1,k_1}^2 = 1$. Hence, from (4.5), we have

$$\begin{aligned}
&\lambda_{\max}(B) - p_{sos} \\
&\leq \mathrm{vec}(U_1)^T \left(\lambda_{\max}(B) I_{mn} - B\right) \mathrm{vec}(U_1) \\
&= \left(\sum_{j=1}^{k_1} \sigma_{1,j} \mathrm{vec}\left(u^{1,j} (v^{1,j})^T\right) \right)^T (\lambda_{\max}(B) I_{mn} - B) \left(\sum_{j=1}^{k_1} \sigma_{1,j} \mathrm{vec}\left(u^{1,j} (v^{1,j})^T\right) \right) \\
&\leq k_1 \cdot \left(\sum_{j=1}^{k_1} \sigma_{1,j}^2 \mathrm{vec}\left(u^{1,j} (v^{1,j})^T\right)^T (\lambda_{\max}(B) I_{mn} - B)\, \mathrm{vec}\left(u^{1,j} (v^{1,j})^T\right) \right) \\
&= k_1 \cdot \left(\lambda_{\max}(B) - \sum_{j=1}^{k_1} \sigma_{1,j}^2 b\left(v^{1,j}, u^{1,j}\right) \right),
\end{aligned}$$

where the first inequality comes from (4.5) and the second inequality comes from $\lambda_{\max}(B) I_{nm} - B \succeq 0$. From $\sum_{j=1}^{k_1} \sigma_{1,j}^2 = 1$, we must have one j, say, $j = 1$ such that

$$\lambda_{\max}(B) - b\left(v^{1,1}, u^{1,1}\right) \geq \frac{1}{k_1} (\lambda_{\max}(B) - p_{sos}) \geq \frac{1}{\min\{m, n\}} (\lambda_{\max}(B) - p_{sos}),$$

that is, $(v^{1,1}, u^{1,1})$ is an approximate solution to the original problem (1.1) such that

$$\lambda_{\max}(B) - b\left(v^{1,1}, u^{1,1}\right) \geq \frac{\lambda_{\max}(B) - p_{sos}}{\min\{m, n\}} \geq \frac{\lambda_{\max}(B) - p_{\min}}{\min\{m, n\}},$$

where the second inequality comes from $p_{sos} \le p_{\min}$. From the selection of the pair (x^*, y^*), we immediately have the claim of the theorem. ∎

Note that the approximation result here depends only on $\min\{m, n\}$, which is probably why the first order SOS relaxation (4.2) is more effective than other SDP relaxation methods like (4.7) below in practice.

One may solve the linear SDP (4.2) without the block-symmetry constraints, that is, solve (4.4) instead by computing a minimum-eigenvalue eigenvector of B and proceed with the SVD rounding. Then a similar analysis gives the approximation result:

$$\lambda_{\max}(B) - b\,(x^*, y^*) \ge \frac{1}{\min\{m,n\}}(\lambda_{\max}(B) - \lambda_{\min}(B)).$$

4.2. A convex quadratic SDP relaxation. In this subsection, we present another method for estimating the optimal value $p_{\min}$ of (1.1). This method generates a lower bound of $p_{\min}$ from a solution pair $(\bar{X}, \bar{Y})$ of a convex SDP relaxation of (1.1). At the same time, we also obtain an approximate solution of (1.1).

Note that the biquadratic optimization (1.1) is equivalent to

$$\tag{4.6} \begin{array}{ll} \min\limits_{X,Y} & (\mathcal{A}X) \bullet Y + \alpha\,\{X \bullet X + Y \bullet Y\} \\ \text{subject to} & \mathrm{Tr}(X) = 1, X \succeq 0, \\ & \mathrm{Tr}(Y) = 1, Y \succeq 0, \\ & \mathrm{rank}(X) = 1, \mathrm{rank}(Y) = 1 \end{array}$$

for any constant $\alpha > 0$. Thus, we consider the standard bilinear SDP relaxation

$$\tag{4.7} \begin{array}{ll} p_{csdp}(\alpha) := \min\limits_{X,Y} & (\mathcal{A}X) \bullet Y + \alpha\,\{X \bullet X + Y \bullet Y\} \\ \text{subject to} & \mathrm{Tr}(X) = 1, X \succeq 0, \\ & \mathrm{Tr}(Y) = 1, Y \succeq 0, \end{array}$$

where $\alpha > 0$ is large enough such that (4.7) is convex. Denote by $\hat{b}(X, Y)$ the objective function in (4.7). In fact, $\hat{b}(X, Y)$ can be written as

$$\hat{b}(X,Y) = \left(\mathrm{vec}(X)^T, \mathrm{vec}(Y)^T\right)(F(\mathcal{A}) + \alpha I)\begin{pmatrix} \mathrm{vec}(X) \\ \mathrm{vec}(Y) \end{pmatrix},$$

where the operator "vec" and $F(\mathcal{A})$ are defined as

$$\mathrm{vec}(X) = \left(X_{11}, \sqrt{2}X_{12}, \ldots, \sqrt{2}X_{1n}, X_{22}, \sqrt{2}X_{23}, \ldots, \sqrt{2}X_{n-1,n}, X_{nn}\right)^T,$$

$$F(\mathcal{A}) = \frac{1}{2}\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}.$$

Here, A is a $\frac{1}{2}n(n+1) \times \frac{1}{2}m(m+1)$ matrix such that $(\mathcal{A}X) \bullet Y = \mathrm{vec}(X)^T A \mathrm{vec}(Y)$. It is well known that $\hat{b}(X, Y)$ is convex if and only if $F(\mathcal{A}) + \alpha I \succeq 0$, which is equivalent to that $4\alpha^2 I - A^T A \succeq 0$. Therefore, we may choose $\alpha \ge \frac{1}{2}\|A\|_2$ to guarantee the convexity of (4.7), where $\|A\|_2 = (\lambda_{\max}(A^T A))^{1/2}$.

Note that the convex quadratic SDP (4.7) is equivalent to the standard linear SDP

$$
\begin{array}{ll}
\min\limits_{X,Y,W} & \begin{pmatrix} 0 & 0 \\ 0 & F(\mathcal{A})+\alpha I \end{pmatrix} \bullet W \\
\text{subject to} & \mathrm{Tr}(X)=1, \qquad \mathrm{Tr}(Y)=1, \\
& W := \begin{pmatrix} 1 & \mathrm{vec}(X)^T \ \mathrm{vec}(Y)^T \\ \mathrm{vec}(X) & \\ \mathrm{vec}(Y) & Z \end{pmatrix} \succeq 0, \\
& X \succeq 0, \qquad Y \succeq 0.
\end{array} \tag{4.8}
$$

We mention that (4.8) is relatively easier to solve than (4.2) because the numbers of equality constraints in (4.8) and (4.2) are $\mathcal{O}(n^2+m^2)$ and $\mathcal{O}(n^2m^2)$, respectively. This is also observed in the numerical results.

Once the convex quadratic SDP (4.7) is solved, we can extract an approximate solution pair $(\bar{x},\bar{y})$ of (1.1) as follows. Let $(\bar{X},\bar{Y})$ be an optimal solution pair of (4.7) with α. By eigenvalue decomposition, one knows that

$$
\bar{X} = \bar{\lambda}_1 \bar{x}^1 \left(\bar{x}^1\right)^T + \cdots + \bar{\lambda}_r \bar{x}^r \left(\bar{x}^r\right)^T, \qquad \bar{Y} = \bar{\mu}_1 \bar{y}^1 \left(\bar{y}^1\right)^T + \cdots + \bar{\mu}_s \bar{y}^s \left(\bar{y}^s\right)^T.
$$

Here, $\bar{x}^1,\ldots,\bar{x}^r$ and $\bar{y}^1,\ldots,\bar{y}^s$ are the orthonormal eigenvectors of $\bar{X}$ and $\bar{Y}$ with respect to positive eigenvalues $\bar{\lambda}_1 \geq \cdots \geq \bar{\lambda}_r > 0$ and $\bar{\mu}_1 \geq \cdots \geq \bar{\mu}_s > 0$, respectively. Let $(\bar{x},\bar{y})$ be a vector pair satisfying

$$
b\left(\bar{x},\bar{y}\right) = \min\left\{ b\left(\bar{x}^i,\bar{y}^j\right) \ : \ 1 \leq i \leq r, 1 \leq j \leq s \right\}.
$$

For any $\alpha \geq \frac{1}{2}\|A\|_2$ and $(\bar{x},\bar{y})$ generated above, $b(\bar{x},\bar{y})$ is an upper bound for $p_{\min}$. A lower bound for (1.1) is readily given by $p_{csdp} := p_{csdp}(\alpha) - 2\alpha$, since (4.7) is an SDP relaxation of (4.6) which is equivalent to the original problem (1.1), but its optimal value is larger than that of (1.1) by 2α.

The quality of the convex SDP relaxation (4.7) and the extraction process described above is given below.

THEOREM 4.3. *The approximate solution* $(\bar{x},\bar{y})$ *of problem* (1.1), *generated as above from the optimal solution of the convex quadratic SDP relaxation* (4.7), *satisfies*

$$
b(\bar{x},\bar{y}) - p_{\min} \quad \leq \quad \alpha\left(2 - \frac{1}{n} - \frac{1}{m}\right), \tag{4.9}
$$

where α *is a number satisfying* $\alpha \geq \frac{1}{2}\|A\|_2$.

Proof. Since $(\bar{X},\bar{Y})$ is an optimal solution of (4.7), there exist $\bar{\zeta},\bar{\eta} \in \mathbb{R}$ such that the following system holds:

$$
\left\{
\begin{array}{l}
\mathcal{A}\bar{X} + 2\alpha\bar{Y} - \bar{\zeta}I \succeq 0, \\
\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I \succeq 0, \\
(\mathcal{A}\bar{X} + 2\alpha\bar{Y} - \bar{\zeta}I) \bullet \bar{Y} = 0, \\
(\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I) \bullet \bar{X} = 0.
\end{array}
\right. \tag{4.10}
$$

Since $\mathrm{Tr}(\bar{X}) = 1$ and $\mathrm{Tr}(\bar{Y}) = 1$, from the third and the fourth equations of (4.10), we have

$$
\bar{\zeta} = (\mathcal{A}\bar{X}) \bullet \bar{Y} + 2\alpha\bar{Y} \bullet \bar{Y} \quad \text{and} \quad \bar{\eta} = (\bar{Y}\mathcal{A}) \bullet \bar{X} + 2\alpha\bar{X} \bullet \bar{X},
$$

which imply that

$$(\bar{\zeta} + \bar{\eta})/2 = p_{csdp}(\alpha). \tag{4.11}$$

Moreover, it is readily seen that

$$(\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I) \bullet \bar{x}^1(\bar{x}^1)^T = 0, \qquad (\mathcal{A}\bar{X} + 2\alpha\bar{Y} - \bar{\zeta}I) \bullet \bar{y}^1(\bar{y}^1)^T = 0.$$

By this, we have

$$\begin{aligned} \sum_{j=1}^{s} \bar{\mu}_j \left(\bar{y}^j(\bar{y}^j)^T\mathcal{A}\right) \bullet \bar{x}^1(\bar{x}^1)^T &= \bar{\eta} - 2\alpha\bar{\lambda}_1, \\ \sum_{i=1}^{r} \bar{\lambda}_i \left(\mathcal{A}\bar{x}^i(\bar{x}^i)^T\right) \bullet \bar{y}^1(\bar{y}^1)^T &= \bar{\zeta} - 2\alpha\bar{\mu}_1. \end{aligned} \tag{4.12}$$

From the definition of $(\bar{x}, \bar{y})$, it is clear that

$$\begin{aligned} b\left(\bar{x}, \bar{y}\right) &\le b\left(\bar{x}^i, \bar{y}^1\right) = \left(\mathcal{A}\bar{x}^i(\bar{x}^i)^T\right) \bullet \bar{y}^1(\bar{y}^1)^T, \\ b\left(\bar{x}, \bar{y}\right) &\le b\left(\bar{x}^1, \bar{y}^j\right) = \left(\bar{y}^j(\bar{y}^j)^T\mathcal{A}\right) \bullet \bar{x}^1(\bar{x}^1)^T, \end{aligned}$$

which imply, together with (4.12), that

$$b\left(\bar{x}, \bar{y}\right) \le \bar{\zeta} - 2\alpha\bar{\mu}_1 \quad \text{and} \quad b\left(\bar{x}, \bar{y}\right) \le \bar{\eta} - 2\alpha\bar{\lambda}_1, \tag{4.13}$$

since $\sum_{i=1}^r \bar{\lambda}_i = 1$ and $\sum_{j=1}^s \bar{\mu}_j = 1$. By (4.11) and (4.13), we have

$$b\left(\bar{x}, \bar{y}\right) \le p_{csdp}(\alpha) - \alpha(\bar{\lambda}_1 + \bar{\mu}_1), \tag{4.14}$$

which implies, together with $p_{\min} \le b(\bar{x}, \bar{y})$ and $p_{csdp}(\alpha) - 2\alpha \le p_{\min}$, that

$$b\left(\bar{x}, \bar{y}\right) - p_{\min} \le \alpha\left(2 - \bar{\lambda}_1 - \bar{\mu}_1\right).$$

By this and the fact that $\bar{\lambda}_1 \ge 1/s \ge 1/n$ and $\bar{\mu}_1 \ge 1/r \ge 1/m$, we obtain the desired result and complete the proof. ∎

We should point out, for the convex quadratic SDP (4.7) to approximate the biquadratic optimization (1.1) efficiently, the constant $\alpha > 0$ in (4.7) cannot be too large. This will be shown in Theorem 4.4. In general, the lower bound obtained for (1.1) by solving (4.7) is better when α is chosen close to $\frac{1}{2}\|A\|_2$.

THEOREM 4.4. *Assume that $b(x, y) \ge 0$ for every (x, y), i.e., $\mathcal{A}X \in \mathcal{S}_+^m$ whenever $X \in \mathcal{S}_+^n$ and $Y\mathcal{A} \in \mathcal{S}_+^n$ whenever $Y \in \mathcal{S}_+^m$. If $(\bar{X}, \bar{Y})$ is an optimal solution of (4.7) with*

$$\alpha > \frac{1}{2}\max\{n-1, m-1\}\|\mathcal{A}\|_F, \tag{4.15}$$

then we have

$$\operatorname{rank}(\bar{X}) = n \quad \textit{and} \quad \operatorname{rank}(\bar{Y}) = m. \tag{4.16}$$

Proof. Since $(\bar{X}, \bar{Y})$ is an optimal solution of (4.7), there exist $\bar{\zeta}, \bar{\eta} \in \mathbb{R}$ such that

$$\begin{cases} \mathcal{A}\bar{X} + 2\alpha\bar{Y} - \bar{\zeta}I \succeq 0, \\ \bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I \succeq 0, \\ (\mathcal{A}\bar{X} + 2\alpha\bar{Y} - \bar{\zeta}I) \bullet \bar{Y} = 0, \\ (\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I) \bullet \bar{X} = 0. \end{cases} \tag{4.17}$$

Let $\text{rank}(\bar{X}) = r$ and $\text{rank}(\bar{Y}) = s$. It is clear that $r \geq 1$ and $s \geq 1$ because $\text{Tr}(\bar{X}) = 1$ and $\text{Tr}(\bar{Y}) = 1$, respectively. Moreover, since $\text{Tr}(\bar{X}) = 1$, by Lemma 2.3, there exist $\bar{x}^i \in \mathbb{R}^n$ $(i = 1, \ldots, r)$ such that

$$\bar{X} = \sum_{i=1}^{r} \bar{x}^i (\bar{x}^i)^T, \qquad I \bullet \bar{x}^i (\bar{x}^i)^T = 1/r \quad \text{for} \quad i = 1, \ldots, r.$$

Consequently,

$$(\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I) \bullet \bar{x}^i (\bar{x}^i)^T = 0 \quad \text{for} \quad i = 1, \ldots, r. \tag{4.18}$$

On the other hand, from the second expression in (4.17), we have that $\text{Tr}(\bar{Y}\mathcal{A}) + 2\alpha\text{Tr}(\bar{X}) - \bar{\eta}\text{Tr}(I) \geq 0$, which implies

$$\bar{\eta}n \leq 2\alpha + \|\mathcal{A}\|_F, \tag{4.19}$$

since $\text{Tr}(\bar{Y}\mathcal{A}) \leq \|\mathcal{A}\|_F \|\bar{Y}\|_F \leq \|\mathcal{A}\|_F$. Moreover, we have that for every k,

$$\begin{aligned}
&\left(\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I\right) \bullet \bar{x}^k (\bar{x}^k)^T \\
&\geq \left(2\alpha \sum_{i=1}^{r} \bar{x}^i (\bar{x}^i)^T - \bar{\eta}I\right) \bullet \bar{x}^k (\bar{x}^k)^T \\
&\geq \left(2\alpha \bar{x}^k (\bar{x}^k)^T - \bar{\eta}I\right) \bullet \bar{x}^k (\bar{x}^k)^T \\
&= \frac{1}{r}\left(2\alpha\frac{1}{r} - \bar{\eta}\right),
\end{aligned} \tag{4.20}$$

where the first inequality comes from the assumption that $Y\mathcal{A}$ is positive semidefinite for any $Y \in \mathcal{S}_+^m$ and the second inequality comes from the fact that $xx^T \bullet \tilde{x}\tilde{x}^T \geq 0$ for any $x, \tilde{x} \in \mathbb{R}^n$.

Now we prove the conclusion for $\bar{X}$ by contradiction. Suppose that $\text{rank}(\bar{X}) = r < n$. Then, it is readily seen that $\dfrac{1}{r} \geq \dfrac{1}{n-1}$, which implies, together with (4.19), that

$$2\alpha\frac{1}{r} - \bar{\eta} \geq 2\alpha\frac{1}{n-1} - \frac{2\alpha}{n} - \frac{\|\mathcal{A}\|_F}{n} = \frac{1}{n}\left(2\alpha\frac{1}{n-1} - \|\mathcal{A}\|_F\right) > 0, \tag{4.21}$$

where the final inequality comes from (4.15). Equation (4.21) shows, together with (4.20), that for any $i = 1, \ldots, r$,

$$\left(\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I\right) \bullet \bar{x}^i (\bar{x}^i)^T > 0,$$

which contradicts (4.18). Therefore, it holds that $\text{rank}(\bar{X}) = n$. The conclusion for $\bar{Y}$ can be proved similarly. ∎

5. Illustrative numerical results. This section reports some numerical results on the computational performances of the first order SOS relaxation (4.2), the convex SDP relaxation (4.7), and the minimum eigenvalue method (4.4). For the first order SOS method, we solve the SDP (4.2) to find a lower bound p_{sos} and an optimal solution Z^*, and then we apply the SVD rounding procedure described in front of Theorem 4.2 to produce an approximate solution pair (x^*, y^*) of (1.1). For the convex quadratic SDP method, we choose $\alpha = \frac{1}{2}\|A\|_2$ and solve the SDP (4.8) to obtain an optimal solution matrix pair $(\bar{X}, \bar{Y})$. Then, we follow the rounding procedure described in front of Theorem 4.3 to get an approximate solution pair $(\bar{x}, \bar{y})$ of (1.1) and a lower bound $p_{csdp} := p_{csdp}(\alpha) - 2\alpha$. For the minimum eigenvalue method, we first compute the minimal eigenvalue $\lambda_{\min}(B)$ and the corresponding eigenvector $\hat{z}$ by solving (4.4). Then we apply the same SVD rounding procedure on the matrix $\hat{U} = \mathrm{mat}(\hat{z})$ to obtain an approximate solution $(\hat{x}, \hat{y})$ of (1.1).

All the numerical computations here were done by using an Intel Core 2 Duo 2.4GHz computer with 2GB of RAM, and all the SDP problems were solved by the SDP software SDPA-M (version 6.2.0) [12].

Example 5.1. Consider the biquadratic optimization

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^3, y \in \mathbb{R}^3} & x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + 2(x_1^2 y_2^2 + x_2^2 y_3^2 + x_3^2 y_1^2) \\ & -2x_1 x_2 y_1 y_2 - 2x_1 x_3 y_1 y_3 - 2x_2 x_3 y_2 y_3 \\ \text{subject to} & \|x\|^2 = 1, \ \|y\|^2 = 1. \end{array}$$

First, we use the first order SOS relaxation (4.2) to find a lower bound of $p_{\min}$ and then extract an approximate solution for it. It can be shown [4] that $p_{\min} = 0$, and the objective biquadratic form is not SOS. From the given fourth order tensor $\mathcal{A}$, it can be verified that the coefficient matrix B in (4.2) has $\lambda_{\max}(B) = 2.118$ and $\lambda_{\min}(B) = -0.118$. By solving (4.2), we get $p_{sos} = -0.0972$. It is clear that $\lambda_{\min}(B) < p_{sos} < p_{\min}$. Now, we extract an approximate solution of the original problem from Z^* by applying the SVD rounding procedure, and get $x^* = (-1, 0, 0)^T$ and $y^* = (0, 0, -1)^T$. Note that $b(x^*, y^*) = 0$ attains the exact minimum objective value.

Second, we use the convex quadratic SDP relaxation (4.6) to solve the problem. Choose $\alpha = \frac{1}{2}\|A\|_2 = 1.5$. It is not difficult to obtain the optimal value $p_{csdp}(\alpha) = 2$ of (4.6) and an optimal matrix pair

$$\bar{X} = \bar{Y} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

Hence, we obtain a lower bound -1 for the minimum $p_{\min}$. Moreover, after rounding we obtain an approximation solution pair $\bar{x} = (-1, 0, 0)^T$ and $\bar{y} = (0, 0, 1)^T$, which also attains the exact minimum objective value.

Furthermore, based upon the $\lambda_{\min}(B)$ and its eigenvector $\hat{z}$, we extract the exact solutions $\hat{x} = (-1, 0, 0)^T$ and $\hat{y} = (0, 0, -1)^T$.

Example 5.2. Consider the biquadratic optimization

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^6, y \in \mathbb{R}^6} & \sum\limits_{i=1}^{5} x_i x_{i+1} y_i y_{i+1} \\ \text{subject to} & \|x\|^2 = 1, \ \|y\|^2 = 1. \end{array}$$

First, we use the first order SDP relaxation (4.2) to solve the problem. It can be verified that $\lambda_{\min}(B) = -0.4505$ and $\lambda_{\max}(B) = 0.4505$. We obtain $p_{sos} = -0.25$ and a corresponding optimal solution Z^*. Then, by applying the SVD rounding procedure, we extract an approximate solution from Z^*:

$$x^* = (0, 0, 0, 0, -0.7066, -0.7076)^T, \qquad y^* = (0, 0, 0, 0.0001, -0.7077, 0.7065)^T$$

such that $b(x^*, y^*) = -0.2500$.

Second, we use convex quadratic SDP relaxation to solve the problem. Choosing $\alpha = \frac{1}{2}\|A\|_2 = 1/4$, we obtain a lower bound $p_{csdp} = -0.4167$ and optimal matrices

$$\bar{X} = \bar{Y} = \begin{pmatrix} 0.1667 & -0.0016 & 0 & 0 & 0 & 0 \\ -0.0016 & 0.1667 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1667 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1667 \end{pmatrix}.$$

Hence, we obtain a lower bound -0.4167 for the minimum $p_{\min}$. From the rounding procedure, we obtain an approximate solution with objective value -0.2500 as follows:

$$\bar{x} = (-0.7071, -0.7071, 0, 0, 0, 0)^T, \qquad \bar{y} = (0.7071, -0.7071, 0, 0, 0, 0)^T.$$

Third, from the eigenvector $\hat{z}$ corresponding to $\lambda_{\min}(B)$, we extract an approximate solution

$$\hat{x} = (0, 0, 1, 0, 0, 0)^T, \qquad \hat{y} = (0, 0, 1, 0, 0, 0)^T$$

such that $b(\hat{x}, \hat{y}) = 0$, which does not attain the minimum objective value.

Example 5.3. Consider the biquadratic optimization

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^9, y\in\mathbb{R}^{12}} & \sum\limits_{1\le i,k\le 9,\, 1\le j,l\le 12} x_i y_j x_k y_l \\ \text{subject to} & \|x\|^2 = 1,\ \|y\|^2 = 1. \end{array}$$

It can be verified that $\lambda_{\min}(B) = 0$ and $\lambda_{\max}(B) = 108$. By solving the first order SOS relaxation (4.2), we obtain $p_{sos} = 0$ and extract a pair (x^*, y^*) with objective value 0:

$$\begin{aligned} x^* &= (0.5144, 0.1874, 0.6634, 0.4207, 0.1573, \\ &\quad -0.1194, 0.1873, -0.0378, 0.0877)^T, \\ y^* &= (-0.2207, 0.1225, -0.4439, 0.3975, 0.3158, -0.0189, \\ &\quad -0.5694, 0.0357, 0.3487, -0.1163, 0.0055, 0.1434)^T. \end{aligned}$$

For the convex SDP method, choosing $\alpha = \frac{1}{2}\|A\|_2 = 54$ and solving the SDP (4.8), we get a lower bound -97.0909 and extract a pair $(\bar{x}, \bar{y})$ with objective value 0:

$$\begin{aligned} \bar{x} &= (-0.8445, -0.0163, 0.1397, 0.1302, 0.0452, \\ &\quad -0.0028, 0.4257, 0.0198, -0.2576)^T, \\ \bar{y} &= (0.0464, 0.0617, 0.4152, -0.0305, -0.1712, 0.0110, \\ &\quad 0.0655, -0.0409, 0.0778, 0.5078, -0.6675, -0.2754)^T. \end{aligned}$$

For the minimum eigenvalue method, we also extract an approximate solution with objective value 0:

$$\begin{aligned} \hat{x} &= (0.0972, 0.2778, -0.3277, -0.1575, -0.6329 \\ &\quad -0.1641, -0.0369, 0.5925, 0.0366)^T, \\ \hat{y} &= (-0.3893, 0.2134, 0.1229, 0.0646, 0.3544, -0.3069 \\ &\quad -0.0483, 0.0915, -0.3173, 0.6016, -0.2807, -0.1088)^T. \end{aligned}$$

TABLE 1
Computational results for random examples.

Dim	First order SOS (4.2)			Convex quadratic SDP (4.7)			Minimum eig. M. (4.4)		
	Low.B.	$b(x^*, y^*)$	Cpu	Low.B	$b(\bar{x}, \bar{y})$	Cpu	Low.B	$b(\hat{x}, \hat{y})$	Cpu
(6, 7)	−260.31	−260.31	0.39	−417.54	−257.50	0.12	−311.88	−251.17	0.01
(5, 8)	−119.14	−119.14	0.21	−246.91	−45.71	0.09	−146.28	−116.42	0.01
(7, 8)	−268.10	−268.10	1.03	−337.78	−263.78	0.28	−309.76	−262.41	0.03
(7, 9)	−565.19	−565.19	1.79	−587.20	−564.91	0.43	−605.64	−563.64	0.04
(8, 9)	−526.71	−526.71	3.45	−593.00	−525.31	0.57	−591.49	−521.98	0.04
(9, 9)	−609.39	−609.39	6.45	−783.48	−602.14	0.84	−695.60	−597.92	0.06
(10, 10)	−752.19	−752.19	18.81	−1003.22	−739.71	1.59	−880.63	−738.41	0.10
(11, 11)	−362.66	−362.66	54.46	−980.97	−342.77	2.84	−444.13	−334.49	0.17
(12, 12)	−499.41	−499.41	142.18	−982.55	−483.11	4.29	−623.55	−474.42	0.31
(13, 13)	–	–	–	−491.36	−7.06	5.15	−35.48	−14.12	0.48
(14, 14)	–	–	–	−509.56	−66.72	7.59	−82.96	−73.14	0.67
(20, 20)	–	–	–	−1360.31	−220.52	75.23	−250.54	−231.30	5.43
(50, 50)	–	–	–	–	–	–	−9.34	−4.76	2.14
(100, 100)	–	–	–	–	–	–	−9.28	−8.88	28.54
(150, 150)	–	–	–	–	–	–	−13.26	−11.30	190.45
(200, 300)	–	–	–	–	–	–	−8.17	−6.46	1256.63
(300, 300)	–	–	–	–	–	–	−8.41	−6.32	1678.65
(300, 600)	–	–	–	–	–	–	−8.20	−7.06	17826.25

Finally we test some dense and sparse random examples for relatively larger dimension (n, m). The coefficients of the biquadratic form $b(x, y)$ in (1.1) are generated randomly by normal distribution. For (n, m) with $(6, 7) - (20, 20)$, the coefficients of the biquadratic form $b(x, y)$ are dense, while for (n, m) beyond 50, they are sparse. Again, the first order SOS relaxation (4.2), the convex quadratic SDP relaxation (4.7), and the minimum eigenvalue method (4.4) are applied to solving these randomly generated biquadratic optimization problems. The computational results are summarized in Table 1, where "Dim" stands for the dimension pair (n, m), "Low.B." denotes the computed lower bound p_{sos}, p_{csdp}, or $\lambda_{\min}(B)$, and "Cpu" the consumed CPU time in seconds.

From Table 1, we see that the first order SOS relaxation (4.2) provides a better lower bound than both the convex quadratic SDP relaxation (4.7) and the minimum eigenvalue method (4.4), while the latter two consume less CPU time, especially for large-scale problems. This is because (4.2) has $\mathcal{O}(m^2 n^2)$ equality constraints, (4.7) has only $\mathcal{O}(m^2 + n^2)$ equality constraints, and (4.4) is just a problem of finding the minimum eigenvalue and the corresponding eigenvector of B. For $(n, m) = (13, 13)$, $(14, 14)$, and $(20, 20)$, we obtain a lower bound and an approximate solution $(\bar{x}, \bar{y})$ from solving (4.7). For $(n, m) = (50, 50)$ and beyond, we are only able to obtain the eigenvector $\hat{z}$ corresponding to $\lambda_{\min}(B)$ and an approximate solution $(\hat{x}, \hat{y})$ from solving (4.4), due to the memory limit when solving the SDP problems. It seems that there is a trade-off on choosing among the relaxation methods: the first order SOS relaxation (4.2), the convex quadratic SDP relaxation (4.7), and the minimum eigenvalue method (4.4).

6. Conclusion and open problems. This paper discusses minimizing biquadratic forms over unit spheres. We proved that this problem is NP-hard. Subsequently, based on SDP relaxation, we developed several approximation algorithms with guaranteed approximation bounds. When $\min\{m, n\}$ is a constant, we established a PTAS for solving (1.1). We also proposed three practical computational methods: the first order SOS relaxation, the minimum eigenvalue method, and the convex quadratic SDP relaxation. Preliminary computational results indicate that they are all promising. In particular, it seems that the minimum eigenvalue method with the SVD rounding procedure is the most time efficient while still generating good quality solutions.

Theorem 4.1(iii) shows that when $\min\{m, n\} = 2$, (1.1) is polynomial time solvable. When $\min\{m, n\}$ is a constant bigger than 2, is (1.1) still polynomial time solvable? Is there a PTAS for solving (1.1) for general biquadratic form $b(x, y)$? Does (1.1) have a PTAS when $b(x, y)$ is restricted to be square-free? In Theorem 3.2, can we improve the approximation bound to $\mathcal{O}(\frac{1}{mn})$? To the best knowledge of the authors, all such questions are open.

One natural generalization of biquadratic optimization (1.1) is

$$\begin{array}{ll} \min & b(x,y) \\ \text{subject to} & x^T A_i x \leq 1,\ i = 1, \ldots, m_1, \\ & y^T B_j y \leq 1,\ j = 1, \ldots, m_2. \end{array} \tag{6.1}$$

Here $b(x, y)$ is still a biquadratic form, and A_i, B_j are constant symmetric matrices. We can see that (1.1) is a special case of (6.1). Hence problem (6.1) is also NP-hard. Are our approximation results in section 3 applicable to approximating (6.1)? Again, this is an open question.

REFERENCES

[1] I.M. Bomze and E. de Klerk, *Solving standard quadratic optimization problems via linear, semidefinite and copositive programming*, J. Global Optim., 24 (2002), pp. 163–185.

[2] A.P. Calderón, *A note on bi-quadratic forms*, Linear Algebra Appl., 7 (1973), pp. 175–177.

[3] J.F. Cardoso, *High-order contrasts for independent component analysis*, Neural Comput., 11 (1999), pp. 157–192.

[4] M.-D. Choi, *Positive semidefinite biquadratic forms*, Linear Algebra Appl., 12 (1975), pp. 95–100.

[5] P. Comon, *Independent component analysis, a new concept?*, Signal Process., 36 (1994), pp. 287–314.

[6] G. Dahl, J.M. Leinaas, J. Myrheim, and E. Ovrum, *A tensor product matrix approximation problem in quantum physics*, Linear Algebra Appl., 420 (2007), pp. 711–725.

[7] L. De Lathauwer, P. Comon, B. De Moor, and J. Vandewalle, *Higher-order power method—application in independent component analysis*, in Proceedings of the International Symposium on Nonlinear Theory and its Applications (NOLTA'95), Las Vegas, NV, 1995, IEICE, pp. 91–96.

[8] L. De Lathauwer, B. De Moor, and J. Vandewalle, *On the best rank-1 and rank-$(R_1, R_2, \ldots, R_N)$ approximation of higher-order tensors*, SIAM J. Matrix Anal. Appl., 21 (2000), pp. 1324–1342.

[9] J. Demmel, J. Nie, and V. Powers, *Representations of positive polynomials on non-compact semialgebraic sets via KKT ideals*, J. Pure Appl. Algebra, 209 (2007), pp. 189–200.

[10] A. Einstein, B. Podolsky, and N. Rosen, *Can quantum-mechanical description of physical reality be considered complete?*, Phys. Rev., 47 (1935), pp. 777–780.

[11] L. Faybusovich, *Global optimization of homogeneous polynomials on the simplex and on the sphere*, in Frontiers in Global Optimization, C. Floudas and P. Pardalos, eds., Kluwer Academic Publishers, Norwell, MA, 2003.

[12] K. Fujisawa, Y. Futakata, M. Kojima, S. Matsuyama, S. Nakamura, K. Nakata, and M. Yamashita, *SDPA-M (SemiDefinite Programming Algorithm in MATLAB)*, http://homepage.mac.com/klabtitech/sdpa-homepage/download.html.

[13] M. Fu, Z.-Q. Luo, and Y. Ye, *Approximation algorithms for quadratic programming*, J. Comb. Optim., 2 (1998), pp. 29–50.

[14] V.S. Grigorascu and P.A. Regalia, *Tensor displacement structures and polyspectral matching*, in Fast Reliable Algorithms for Structured Matrices, T. Kailath and A.H. Sayed, eds., SIAM, Philadelphia, 1999.

[15] L. Gurvits, *Classical deterministic complexity of Edmonds' problem and quantum entanglement*, in Proceedings of the Thirty-Fifth ACM Symposium on Theory of Computing 10-19, New York, 2003.

[16] D. Han, H.H. Dai, and L. Qi, *Conditions for strong ellipticity of anisotropic elastic materials*, J. Elasticity, 97 (2009), pp. 1–13.

[17] S. He, Z-Q. Luo, J. Nie, and S. Zhang, *Semidefinite relaxation bounds for indefinite homogeneous quadratic optimization*, SIAM J. Optim., 19 (2008), pp. 503–523.

[18] D. Jibetean and M. Laurent, *Semidefinite approximations for global unconstrained polynomial optimization*, SIAM J. Optim., 16 (2005), pp. 490–514.

[19] E. Kofidis and P.A. Regalia, *On the best rank-1 approximation of higher-order supersymmetric tensors*, SIAM J. Matrix Anal. Appl., 23 (2002), pp. 863–884.

[20] J. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim., 11 (2001), pp. 796–817.

[21] Z.-Q. Luo, N.D. Sidiropoulos, P. Tseng, and S. Zhang, *Approximation bounds for quadratic optimization with homogeneous quadratic constraints*, SIAM J. Optim., 18 (2007), pp. 1–28.

[22] Z.-Q. Luo and S.Z. Zhang, *A Semidefinite Relaxation Scheme for Multivariate Quartic Polynomial Optimization with Quadratic Constraints*, manuscript, Department of SEEM, Chinese University of Hong Kong, Ma Liu Shui, Sha Tin, New Territories, Hong Kong, 2009.

[23] T.S. Motzkin and E.G. Straus, *Maxima for graphs and a new proof of a theorem of Túran*, Canad. J. Math., 17 (1965), pp. 533–540.

[24] J. Nie, J. Demmel, and B. Sturmfels, *Minimizing polynomials via sum of squares over the gradient ideal*, Math. Program., 106 (2006), pp. 587–606.

[25] J. Nie and M. Schweighofer, *On the complexity of putinar's positivstellensatz*, J. Complexity, 23 (2007), pp. 135–150.

[26] C.L. Nikias and A.P. Petropulu, *Higher-Order Spectra Analysis, A Nonlinear Signal Processing Framework*, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[27] P. Parrilo, *Semidefinite programming relaxations for semialgebraic problems*, Math. Program. Ser. B, 96 (2003), pp. 293–320.

[28] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput., 40 (2005), pp. 1302–1324.

[29] L. Qi, H.H. Dai, and D. Han, *Conditions for Strong Ellipticity and M-eigenvalues*, Front. Math. China 4, 2009, pp. 349–364.

[30] L. Qi, F. Wang, and Y. Wang, *Z-Eigenvalue methods for a global optimization polynomial optimization problem*, Math. Program., 118 (2009), pp. 301–306.

[31] B. Reznick, *Some concrete aspects of Hilbert's 17th problem*, in Contemp. Math. 253, American Mathematical Society, Providence, RI, 2000, pp. 251–272,

[32] P. Rosakis, *Ellipticity and deformations with discontinuous deformation gradients in finite elastostatics*, Arch. Ration. Mech. Anal., 109 (1990), pp. 1–37.

[33] E. Schrödinger, *Die gegenwärtige situation in der quantenmechanik*, Naturwissenschaften, 23 (1935), pp. 807–812, 823–828, 844–849.

[34] H.C. Simpson and S.J. Spector, *On copositive matrices and strong ellipticity for isotropic elastic materials*, Arch. Ration. Mech. Anal., 84 (1983), pp. 55–68.

[35] J.F. Sturm and S.Z. Zhang, *On cones of nonnegative quadratic functions*, Math. Oper. Res., 28 (2003), pp. 246–267.

[36] P. Tseng, *Further results on approximating nonconvex quadratic optimization by semidefinite programming relaxation*, SIAM J. Optim., 14 (2003), pp. 268–283.

[37] Y. Ye, *On affine scaling algorithms for nonconvex quadratic programming*, Math. Program., 56 (1992), pp. 285-300.

[38] Y. Ye and S.Z. Zhang, *New results on quadratic minimization*, SIAM J. Optim., 14 (2003), pp. 245–267.

[39] Y. Wang and M. Aron, *A reformulation of the strong ellipticity conditions for unconstrained hyperelastic media*, J. Elasticity, 44 (1996), pp. 89–96.

[40] T. Zhang and G.H. Golub, *Rank-one approximation of higher-order tensors*, SIAM J. Matrix Anal. Appl., 23 (2001), pp. 534–550.