A Fuzzy Sliding Controller for Nonlinear Systems

L. K. Wong, *Member, IEEE*, Frank H. F. Leung, *Member, IEEE*, and Peter K. S. Tam, *Member, IEEE*

Abstract—It is well known that sliding-mode control can give good transient performance and system robustness. However, the presence of chattering may introduce problems to the actuators. Many chattering elimination methods use a finite dc gain controller which leads to a finite steady-state error. One method to ensure zero steady-state error is using a proportional plus integral (PI) controller. This paper proposes a fuzzy logic controller which combines a sliding-mode controller (SMC) and a PI controller. The advantages of the SMC and the PI controller can be combined and their disadvantages can be removed. The system stability is proved, although there is one more state variable to be considered in the PI subsystem. An illustrative example shows that good transient and steady-state responses can be obtained by applying the proposed controller.

Index Terms—Combining controllers, fuzzy sliding, Lyapunov, stability.

I. INTRODUCTION

IT IS WELL KNOWN that sliding-mode controllers (SMCs) are powerful in controlling nonlinear systems with bounded unknown disturbances [4]–[6]. They offer good robustness and transient performance, even in large-signal operations. However, since a discontinuous control action is involved, chattering will take place and the steady-state performance will be degraded. One common method to alleviate this drawback is to introduce a boundary layer about the sliding plane [5], [6]. This method can give a chatter-free output response, but a finite steady-state error must exist due to the finite nonswitching gain of the controller in steady state. Some methods embed an SMC in a fuzzy logic controller (FLC) using either the output error and the change of output error as inputs (mainly for second-order systems) [8], [9], or the distance between the states and the sliding plane as input (usually for higher order systems) [10]. Since switching is not present, the dc gain of such controller is finite, and steady-state error may still exist.

There are two methods to achieve zero steady-state error: switching control and integral control. Under these two cases, the dc gain of the controller is infinity. To improve the performance in steady-state, a proportional plus integral (PI) controller can be considered. This paper proposes an FLC that combines an SMC and a PI controller. As the SMC and PI controllers can give good transient and steady-state performance, respectively, the role of the FLC is to schedule them under different operating conditions [3]. System stability will be proved by a newly proposed stability analysis method [1], [2]. This method

requires that every subsystem gives a negative definite $\dot{V}$ for a given Lyapunov function V. Then, under the weighted sum defuzzification method, it can be proved that $\dot{V}$ is negative definite for the whole system, i.e., the Lyapunov stability theorem can be applied. Here, a subsystem is defined as the closed-loop system under the control of either the SMC or the PI controller only. One major difficulty in the stability analysis is that, unlike the systems in [1] and [3], there is one more state variable to be considered in the PI subsystem. Fortunately, it can be proved that by properly designing the input fuzzy sets and the gain of the SMC, this problem can be solved.

The stability analysis method will be detailed in Section II. Then, the stability analysis procedure of the proposed FLC will be developed in Section III. Although the stability proof appears to be complex, the design of the FLC is as easy as following a few formulas. Section IV illustrates the application of the proposed FLC to a nonlinear plant with external disturbances. Finally, a conclusion will be drawn in Section V.

II. STABILITY ANALYSIS METHOD

A. Fuzzy Logic Control System

Consider a single-input nth-order nonlinear system subject to external disturbances. Let the system equation be given by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x})u + \boldsymbol{w} \tag{1}$$

where $\boldsymbol{x}$ is the state vector, $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{b}(\boldsymbol{x})$ are functions describing the dynamics of the plant, u is the control input determined by an FLC, and $\boldsymbol{w} = [w_1, w_2, \ldots, w_n]^T$ is the vector describing the external disturbances. It is assumed that the values of $w_1, w_2, \ldots, w_n$ are unknown but constant, and bounded by a positive constant vector $\boldsymbol{w_b} = [w_{b1}, w_{b2}, \ldots, w_{bn}]^T$ such that $|w_j| < w_{bj}$ for all $j = 1, 2, \ldots, n$. The ith IF–THEN rule in the fuzzy rule base of the FLC is of the following form:

$$Rule\ i: \quad \text{IF } \langle premise\ i \rangle \text{ THEN } u = u_i \tag{2}$$

where $\langle premise\ i \rangle$ is the premise of rule i with respect to a certain general input variable z; $u = u_i$ is the control output of rule i. Let μ_i and r be the degree of membership of fuzzy rule i and the total number of rules, respectively. Under the weighted-sum defuzzification method, the overall output of the FLC is given by

$$u = \frac{\sum_{i=1}^{r} \mu_i u_i}{\sum_{i=1}^{r} \mu_i}. \tag{3}$$

Definition 1: Let $\boldsymbol{z_o} \in \boldsymbol{Z}$ be any input of the FLC. If the fuzzy rule i is fired, that is, the degree of membership μ_i of

fuzzy rule i is nonzero, this fuzzy rule i is called an *active fuzzy rule for z_o*; otherwise, it is called an *inactive fuzzy rule for z_o*. Also, we define a set I_A that contains the rule numbers of the active fuzzy rules for z_o. An *active region* of a fuzzy rule i is defined as the region $Z_r \subset Z$ such that the fuzzy rule i is active for $z_o \in Z_r$.

By excluding inactive fuzzy rules, the output of the FLC can be represented by the following [1]:

$$u = \frac{\sum\limits_{i \in I_A} \mu_i u_i}{\sum\limits_{i \in I_A} \mu_i}. \tag{4}$$

Property 1: For any input $z_o \in Z$, there exist $p, q \in I_A$ such that $u_p \leq u_i \leq u_q$ for all $i \in I_A$.

Remark 1: The values of p and q may change as the value of z_o or u_i changes. Also, if there is only one element in I_A or all the u_i's are the same for all $i \in I_A$, $u_p = u_i = u_q$.

From *Property 1* and (4), we have

$$\Rightarrow u_p \leq u \leq u_q, \text{ equality holds when } u_i = u_p = u_q \tag{5}$$

for $z_o \in Z$ [1]. The significance of (5) is summarized as follows.

Property 2: For any input $z_o \in Z$, the output u of the FLC lies between u_p and u_q if the defuzzification method in (3) is applied.

Remark 2: It is not necessary to find the values of p, q and u_p, u_q in both the stability analysis and the implementation of the FLC. In fact, u_p and u_q are not predefined bounds of the control u. They are the control outputs of fuzzy rules p and q.

Definition 2: A *fuzzy subsystem* associated with fuzzy rule i is a system with a plant of form (1) controlled by only u_i, which is the output of fuzzy rule i of the form (2).

B. Stability Analysis Method

The idea of the proposed stability analysis method is to break down the problem of analyzing the stability of the whole fuzzy logic control system into analyzing the stability of the fuzzy subsystems individually. The complexity of the analysis is drastically decreased as it is easier to check whether every fuzzy subsystem can give a negative-definite $\dot{V}$ for a given Lyapunov function V. However, the condition that all fuzzy subsystems give a negative-definite $\dot{V}$ does not directly imply that the whole fuzzy logic control system gives a negative-definite $\dot{V}$, too. (If the whole system gives a negative-definite $\dot{V}$, the system stability has been proved by the Lyapunov stability theorem.) We propose sufficient conditions that make this implication valid. They are stated in the following theorem.

Theorem 1: Consider a fuzzy logic control system described in Section II-A. If

1) there exists a positive-definite, continuously differentiable, and radially unbounded scalar function $V = x^T P x$, where P is an $n \times n$ constant positive definite matrix,
2) every fuzzy subsystem gives a negative-definite $\dot{V}$ in the active region of the corresponding fuzzy rule, and
3) the defuzzification method of (3) is employed, which leads to *Property 2* such that $u_p \leq u \leq u_q$,

then by the Lyapunov theorem, the equilibrium point at the origin is globally asymptotically stable.

Proof:

$$V = x^T P x \tag{6}$$

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x}. \tag{7}$$

From (1),

$$\dot{V} = (f(x) + b(x)u + w)^T P x + x^T P(f(x) + b(x)u + w)$$
$$= F(x) + B(x)u \tag{8}$$

where

$$F(x) = (f(x) + w)^T P x + x^T P(f(x) + w) \quad \text{and}$$
$$B(x) = b(x)^T P x + x^T P b(x)$$

are scalars.

We need to prove that if every fuzzy subsystem gives a negative-definite $\dot{V}$ [condition 2)] and $u_p \leq u \leq u_q$ [condition 3)], then the overall fuzzy logic control system also gives a negative-definite $\dot{V}$. First, consider $x = 0$, $F(0) = B(0) = 0$,

$$\dot{V}(0) = 0.$$

Second, consider $x \neq 0$, for any input $z_o \in Z$, from condition 2),

$$\dot{V} = F(x) + B(x)u_i < 0 \tag{9}$$

for all $i \in I_A$. Hence, from *Property 1*, there exist $p, q \in I_A$ such that

$$\dot{V} = F(x) + B(x)u_q < 0 \tag{10}$$

$$\dot{V} = F(x) + B(x)u_p < 0. \tag{11}$$

Then, three cases should be considered.

Case 1: If $B(x)$ is positive.
From (10),

$$\dot{V} = F(x) + B(x)u_q < 0.$$

Since $B(x)$ is positive,

$$\dot{V} < 0 \quad \text{for } u \leq u_q. \tag{12}$$

Case 2: If $B(x)$ is negative.
From (11),

$$\dot{V} = F(x) + B(x)u_p < 0.$$

Since $B(x)$ is negative,

$$\dot{V} < 0 \quad \text{for } u \geq u_p. \tag{13}$$

Case 3: If $B(x)$ is zero.
From (9),

$$\dot{V} = F(x) < 0.$$
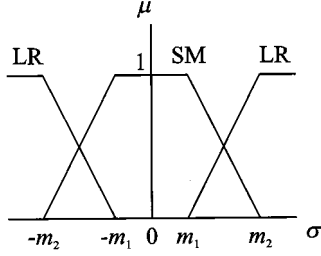$$\Rightarrow \dot{V} < 0 \quad \text{for all } u.$$

Fig. 1.   Membership functions.

From the above three cases, we have, whatever the value of $B(\boldsymbol{x})$,

$$\dot{V} < 0 \quad \text{for } u_p \le u \le u_q. \tag{14}$$

By condition 3), $\dot{V} < 0$. In conclusion, $\dot{V}$ is negative definite. By the Lyapunov theorem, the equilibrium point at the origin is globally asymptotically stable.   QED

Remark 3: A fuzzy subsystem associated with fuzzy rule i gives a negative-definite $\dot{V}$ in its active region which implies that the plant can be stabilized on applying fuzzy rule i individually when $\boldsymbol{z}$ is in the active region.

III. COMBINATION OF SMC AND PI CONTROLLER

An SMC and a PI controller are combined into a single FLC to control a plant in the form of (1). The input variable of the FLC is σ which is defined as

$$\boldsymbol{\sigma} = \boldsymbol{s}\boldsymbol{x} \tag{15}$$

where $\boldsymbol{s} = [s_1, s_2, \cdots, s_n]$ is a constant vector. It should be chosen such that when the system states are in the sliding plane (i.e., $\boldsymbol{\sigma} = 0$), they will slide along the plane to the equilibrium point. To combine an SMC and a PI controller into a single FLC, the fuzzy rules of the FLC are defined as follows.

Rule 1:   IF σ is SM THEN $u = u_1 = (\boldsymbol{s}\boldsymbol{b})^{-1} (-\boldsymbol{s}\boldsymbol{f} - k_i v - k_p \boldsymbol{\sigma})$.

Rule 2:   IF σ is LR THEN $u = u_2 = (\boldsymbol{s}\boldsymbol{b})^{-1} (-\boldsymbol{s}\boldsymbol{f} - k_d \operatorname{sgn}(\boldsymbol{\sigma}))$.

In these rules, SM and LR are membership functions (with parameters m_1 and m_2) as shown in Fig. 1, and k_i, k_p, and k_d are gains to be designed. In Rule 1, we define a state v which will be used when we analyze the system with the PI controller

$$v = \int \boldsymbol{\sigma} \Phi(\boldsymbol{\sigma}) \, dt \tag{16}$$

where $\Phi(\sigma)$ is a function of $\boldsymbol{\sigma}$, as shown in Fig. 2. Furthermore, we define v_r as the reference value of v. It is a constant to cancel out the effect of the unknown disturbance $\boldsymbol{w}$ when the sliding plane is hit. Hence, we have

$$v_r = \frac{\boldsymbol{s}\boldsymbol{w}}{k_i} \tag{17}$$

where k_i is a gain to be determined. In practice, due to the integral action as given by (16), the state v will automatically become v_r under a proper design of the controller when the sliding
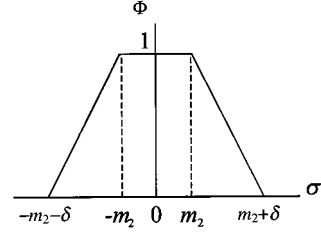
Fig. 2.   Function $\Phi(\sigma)$.

plane is hit. It is not needed to know the value of v_r. However, its maximum bound v_{rb} can be evaluated as follows:

$$v_{rb} = \frac{\max *(\boldsymbol{s}\boldsymbol{w})}{k_i} \tag{18}$$

where $\max *(\boldsymbol{s}\boldsymbol{w}) = |s_1|w_{b1} + |s_2|w_{b2} + \cdots + |s_n|w_{bn}$. Note that v_{rb} is positive. To carry out the stability analysis, we choose an upper bound for v and define an error state e_v as follows:

$$|v| < 10 v_{rb} \tag{19}$$

$$e_v = v_r - v. \tag{20}$$

Then, from (16),

$$\dot{e}_v = -\sigma \Phi(\sigma). \tag{21}$$

On the other hand, from (18) to (20),

$$|e_v| < 11 v_{rb}. \tag{22}$$

To guarantee the system stability using the proposed stability analysis method, we need to find a Lyapunov function V and ensure that every fuzzy subsystem gives a negative-definite $\dot{V}$ in the active region of the corresponding fuzzy rule.

A. PI Subsystem

From Rule 1, (15), and (1), we have

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{s}\boldsymbol{f} + \boldsymbol{s}\boldsymbol{b}(\boldsymbol{s}\boldsymbol{b})^{-1}(-\boldsymbol{s}\boldsymbol{f} - k_i v - k_p \boldsymbol{\sigma}) + \boldsymbol{s}\boldsymbol{w}$$
$$= k_i e_v - k_i v_r - k_p \boldsymbol{\sigma} + \boldsymbol{s}\boldsymbol{w}.$$

Hence, from (17),

$$\dot{\boldsymbol{\sigma}} = k_i e_v - k_p \boldsymbol{\sigma}.$$

Also from (21), since $\Phi(\sigma) = 1$, $\dot{e}_v = -\sigma$, we have

$$\begin{bmatrix} \dot{e}_v \\ \dot{\boldsymbol{\sigma}} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ k_i & -k_p \end{bmatrix} \begin{bmatrix} e_v \\ \boldsymbol{\sigma} \end{bmatrix} = \boldsymbol{A}_1 \begin{bmatrix} e_v \\ \boldsymbol{\sigma} \end{bmatrix}. \tag{23}$$

The closed-loop subsystem behaves like a linear system. If the real part of all eigenvalues of $\boldsymbol{A}_1$ is negative, we can define a symmetric positive-definite matrix $\boldsymbol{Q}_1$ such that a unique symmetric positive-definite matrix $\boldsymbol{P}$ can be found satisfying the following equation [6]:

$$\boldsymbol{A}_1^T \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_1 = -2\boldsymbol{Q}_1. \tag{24}$$

Hence, we can select a Lyapunov function V such that

$$V = \frac{1}{2}[e_v \quad \boldsymbol{\sigma}]\boldsymbol{P} \begin{bmatrix} e_v \\ \boldsymbol{\sigma} \end{bmatrix}$$
$$= \frac{1}{2}[e_v \quad \boldsymbol{\sigma}] \begin{bmatrix} p_1 & p_2 \\ p_2 & p_4 \end{bmatrix} \begin{bmatrix} e_v \\ \boldsymbol{\sigma} \end{bmatrix}. \tag{25}$$

Obviously, from (24) and (25), $\dot{V}$ is negative definite. Moreover, since $\boldsymbol{P}$ is positive definite, p_1 and p_4 must both be positive because the principle minors of $\boldsymbol{P}$ must be positive.

B. SMC Subsystem

From (25), we have

$$\dot{V} = p_1 e_v \dot{e}_v + p_2 \boldsymbol{\sigma} \dot{e}_v + p_2 e_v \dot{\boldsymbol{\sigma}} + p_4 \boldsymbol{\sigma} \dot{\boldsymbol{\sigma}}. \qquad (26)$$

Also from Rule 2, (15), and (1), we have

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{sw} - k_d \, \text{sgn} \, (\boldsymbol{\sigma}). \qquad (27)$$

The active region of this fuzzy rule does not include the origin, so we need to ensure that only $\dot{V} < 0$ for this subsystem. We divide the active region into two subregions. With reference to Figs. 1 and 2, the two subregions are $|\boldsymbol{\sigma}| \geq m_2 + \boldsymbol{\delta}$ and $m_1 < |\boldsymbol{\sigma}| < m_2 + \boldsymbol{\delta}$.

Case 1: $|\boldsymbol{\sigma}| \geq m_2 + \boldsymbol{\delta}$.
From (21), $\dot{e}_v = 0$. Then, (26) can be reduced to

$$\dot{V} = p_2 e_v \dot{\boldsymbol{\sigma}} + p_4 \boldsymbol{\sigma} \dot{\boldsymbol{\sigma}}. \qquad (28)$$

It can be proved that if

$$k_d > k_i v_{rb} \qquad (29)$$
$$m_2 > 11 v_{rb} \qquad (30)$$
$$|p_2| < p_4. \qquad (31)$$

Equation (28) is negative. Hence, $\dot{V}$ is negative definite for $|\boldsymbol{\sigma}| \geq m_2 + \boldsymbol{\delta}$ can be satisfied by properly designing k_d, k_i, k_p, m_2 and $\boldsymbol{Q}_1$. The detailed proof is given in Appendix A.

Case 2: $m_1 < |\boldsymbol{\sigma}| < m_2 + \boldsymbol{\delta}$.
From (21), $\dot{e}_v = -\boldsymbol{\sigma}\boldsymbol{\Phi}(\boldsymbol{\sigma})$. Then, (26) becomes

$$\dot{V} = -p_1 e_v \boldsymbol{\sigma}\boldsymbol{\Phi}(\boldsymbol{\sigma}) - p_2 \boldsymbol{\sigma}^2 \boldsymbol{\Phi}(\boldsymbol{\sigma}) + p_2 e_v \dot{\boldsymbol{\sigma}} + p_4 \boldsymbol{\sigma} \dot{\boldsymbol{\sigma}}. \qquad (32)$$

The conditions for (32) to be negative definite are

$$k_d = \frac{2k_D}{p_4} \qquad (33)$$

and

$$m_1 = 11 v_{rb} |p_2| \left(\frac{k_i v_{rb}}{k_D} + \frac{2}{p_4} \right) \qquad (34)$$

where $k_D = (11 p_1 v_{rb} + |p_2|(m_2 + \boldsymbol{\delta}) + p_4 k_i v_{rb})$. The derivations of (33) and (34) are given in Appendix B.

In conclusion, to ensure that both the PI subsystem and the SMC subsystem give negative-definite $\dot{V}$ with respect to the Lyapunov function of (25), we firstly need to select k_i, k_p, and $\boldsymbol{Q}_1$ to satisfy (31) and select k_d, m_1, and m_2 according to (29), (30), (33), and (34). The values of m_1 and m_2 should satisfy, from Fig. 1, the condition $m_1 < m_2$. The design may be conservative because of this condition and conditions (30) and (31). By *Theorem 1*, the equilibrium point at $[e_v \, \boldsymbol{\sigma}]^T = \boldsymbol{0}$ is globally asymptotically stable. Although only $\boldsymbol{\sigma} \to 0$ has been proved, the condition $\boldsymbol{x} \to 0$ can be ensured by the equation governing $\boldsymbol{\sigma}$ and $\boldsymbol{x}$ ($\boldsymbol{\sigma} = \boldsymbol{sx}$).
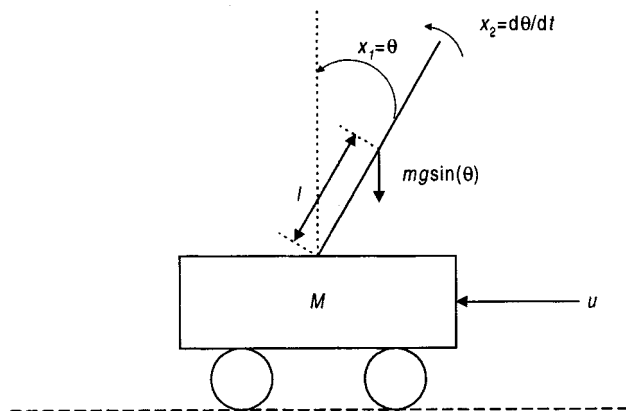

Fig. 3. An inverted pendulum.

IV. ILLUSTRATIVE EXAMPLE

Consider a nonlinear inverted pendulum system (Fig. 3) in the form of (1)

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x})u + \boldsymbol{w}$$

where

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$x_1 = \boldsymbol{\theta}$$
$$x_2 = \dot{\boldsymbol{\theta}}$$
$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_2 \\ \dfrac{9.8 \sin(x_1) - 0.05 x_2^2 \sin(2x_1)}{2/3 - 0.1 \cos^2(x_1)} \end{bmatrix}$$
$$\boldsymbol{b}(\boldsymbol{x}) = \begin{bmatrix} 0 \\ \dfrac{-0.1 \cos(x_1)}{2/3 - 0.1 \cos^2(x_1)} \end{bmatrix}$$
$$\boldsymbol{w} = \begin{bmatrix} 0 \\ w_2 \end{bmatrix}.$$

The magnitude of w_2 is less than 1. The control objective is to regulate $\boldsymbol{x}$ to $\boldsymbol{0}$. Consider the FLC as described in Section III. Select

$$\boldsymbol{s} = \begin{bmatrix} 1 & 0.2 \end{bmatrix}. \qquad (35)$$

The above values are selected because if $\boldsymbol{\sigma} = 0$, we have $\dot{x}_1 + 5x_1 = 0$, which gives a stable system (i.e., the sliding plane $\boldsymbol{\sigma} = 0$ is stable). Also, we have

$$\max *(\boldsymbol{sw}) = 0.2. \qquad (36)$$

Moreover, let $k_i = 2$, $k_p = 5$, and $\boldsymbol{Q}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 200 \end{bmatrix}$; from (24), we have $p_1 = 85.4$, $p_2 = -1$, and $p_4 = 40.2$. Also, from (18) and (36), $v_{rb} = 0.05$. In addition, let $m_2 = 0.6 > 11 v_{rb}$ which satisfies (30). Then, from (33) and (34), $k_d = 2.567$ and $m_1 = 0.0284 (< m_2)$. Choose $\boldsymbol{\delta} = 0.001$ and $w_2 = 0.99$. The transient responses of x_1, x_2 and the control signal are shown in Figs. 4–6, respectively, under the control of the FLC and an SMC (based on Rule 2 alone). The initial state is $\boldsymbol{x}(0) = [0.5236 \; 0]^T$ (0.5236 rad = $30°$). Both systems are stable and have zero steady-state error, even when disturbances are present. From Fig. 4, it can be seen that the response using the FLC is
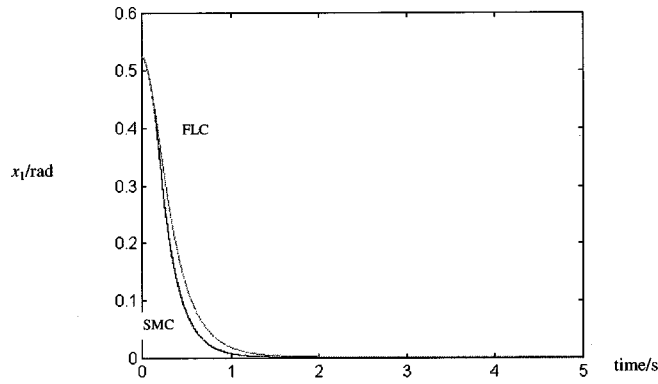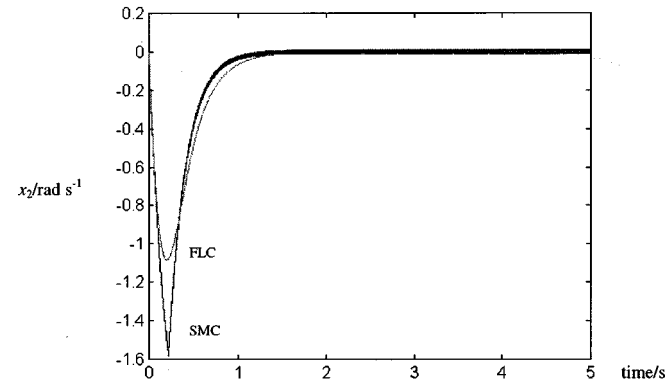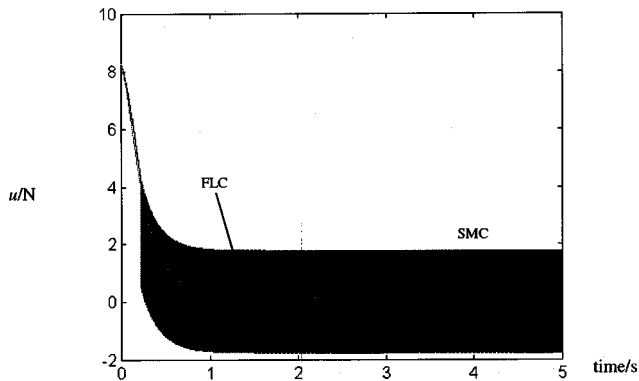
Fig. 4.   Time response of x_1.

Fig. 5.   Time response of x_2.

Fig. 6.   Control signal.

slightly slower than that using the SMC. However, as shown in Figs. 5 and 6, chattering exists when using the SMC alone. The steady-state performance of the system is significantly improved by the proposed FLC.

V. CONCLUSION

An approach to combine an SMC and a PI controller using an FLC has been proposed in this paper. The role of the FLC is to schedule different control action according to the operating conditions. When the states are far from the sliding plane, the FLC is, in fact, an SMC driving the states toward the sliding plane, even under unknown disturbances. The transient performance and system robustness are the same as a pure SMC.

When the states are near the sliding plane, the PI controller takes control and ensures that the states eventually reach the origin under disturbances. Hence, the major drawback of chattering in sliding-mode control is removed. However, the system robustness relies on the robustness of the PI controller only, when the states are near the sliding plane. Although there is one more state variable to be considered in the PI subsystem, the conditions for obtaining a stable closed-loop system are derived. This combined controller is applied to a nonlinear inverted pendulum with disturbances to show its merits. It is found that both good transient response and zero steady-state error can be obtained by applying this controller.

APPENDIX A

It will be proved in this Appendix that (28) will be negative if the following three conditions are satisfied:

$$k_d > k_i v_{rb}$$
$$m_2 > 11 v_{rb}$$
$$|p_2| < p_4.$$

Proof: From (27),

$$\sigma\dot{\sigma} = \sigma sw - \sigma k_d \operatorname{sgn}(\sigma)$$
$$\leq |sw||\sigma| - k_d|\sigma|.$$

From (29) and (18),

$$k_d > k_i v_{rb} = \max *(sw).$$

Hence,

$$\sigma\dot{\sigma} < 0. \tag{A1}$$

Now, $|\sigma| \geq m_2 + \delta > m_2$ for Case 1. From (30) and (22),

$$|\sigma| > |e_v|. \tag{A2}$$

Hence, consider (28),

$$\dot{V} = p_2 e_v \dot{\sigma} + p_4 \sigma\dot{\sigma}$$
$$\leq |p_2 e_v \dot{\sigma}| + p_4 \sigma\dot{\sigma}$$
$$= |p_2||e_v||\dot{\sigma}| + p_4 \sigma\dot{\sigma}$$
$$< |p_2||\sigma\dot{\sigma}| + p_4 \sigma\dot{\sigma} \quad \text{[from (A2)]}$$
$$= |p_2||\sigma\dot{\sigma}| - p_4|\sigma\dot{\sigma}| \quad \text{[from (A1)]}$$
$$< 0 \quad \text{[from (31)].}$$

QED

APPENDIX B

For $m_1 < |\sigma| < m_2 + \delta$, $\dot{e}_v = -\sigma\Phi(\sigma)$. Then, from (26) and (27),

$$\dot{V} = -p_1 e_v \sigma\Phi(\sigma) - p_2\sigma^2\Phi(\sigma) + p_2 e_v \dot{\sigma} + p_4 \sigma\dot{\sigma}$$
$$= -p_1 e_v \sigma\Phi(\sigma) - p_2\sigma^2\Phi(\sigma) + p_2 e_v \dot{\sigma} + p_4\sigma sw - p_4 k_d|\sigma|$$
$$\leq |\sigma|[\max(-p_1 e_v \Phi(\sigma)) + \max(p_2\sigma\Phi(\sigma))$$
$$\quad + \max(p_4 sw)] - p_4 k_d|\sigma| + p_2 e_v \dot{\sigma}$$
$$= |\sigma|[\max(-p_1 e_v) + \max(p_2\sigma)$$
$$\quad + \max(p_4 sw)] - p_4 k_d|\sigma| + p_2 e_v \dot{\sigma} \tag{B1}$$

where $\max(\cdot)$ denotes the maximum value of its argument. Then, let

$$k_D = (11p_1v_{rb} + |p_2|(m_2 + \boldsymbol{\delta}) + p_4k_iv_{rb}) \quad \text{and}$$
$$k_d = \frac{2k_D}{p_4}$$

as in (33). From (22) and (18),

$$11p_1v_{rb} > p_1|e_v|,$$
$$p_4k_iv_{rb} = \max*(p_4\boldsymbol{sw}).$$

Recalling that

$$m_2 + \boldsymbol{\delta} > |\boldsymbol{\sigma}|$$

then

$$k_D = 11p_1v_{rb} + |p_2|(m_2 + \delta) + p_4k_iv_{rb}$$
$$> \max(|p_1e_v|) + \max(|p_2\boldsymbol{\sigma}|) + \max*(p_4\boldsymbol{sw}).$$

Hence, (B1) becomes

$$\dot{V} < k_D|\boldsymbol{\sigma}| - 2k_D|\boldsymbol{\sigma}| + p_2e_v\dot{\boldsymbol{\sigma}}$$
$$= -k_D|\boldsymbol{\sigma}| + p_2e_v\dot{\boldsymbol{\sigma}}$$
$$\leq -k_D|\boldsymbol{\sigma}| + \max(|p_2e_v\dot{\boldsymbol{\sigma}}|).$$

A sufficient condition for $\dot{V} < 0$ is

$$|\boldsymbol{\sigma}| > \frac{\max(|p_2e_v\dot{\boldsymbol{\sigma}}|)}{k_D}. \tag{B2}$$

Since $|\boldsymbol{\sigma}| > m_1$, (B2) can be satisfied by letting

$$m_1 = \frac{\max(|p_2e_v\dot{\boldsymbol{\sigma}}|)}{k_D}$$
$$= 11v_{rb}|p_2|\frac{\max(|\dot{\boldsymbol{\sigma}}|)}{k_D} \quad \text{[from (22)]}$$
$$= 11v_{rb}|p_2|\frac{\max(\boldsymbol{sw} - \frac{2}{p_4}k_D\operatorname{sgn}(\boldsymbol{\sigma}))}{k_D} \quad \text{[from (27)]}$$
$$= 11v_{rb}|p_2|\left(\max\left(\frac{\boldsymbol{sw}}{k_D}\right) + \frac{2}{p_4}\right) \quad (\text{since } p_4 > 0)$$
$$= 11v_{rb}|p_2|\left(\frac{k_iv_{rb}}{k_D} + \frac{2}{p_4}\right) \quad \text{[from (18)]}$$

which gives condition (34). In conclusion, conditions (33) and (34) ensure that $\dot{V} < 0$.

REFERENCES

[1] L. K. Wong, F. H. F. Leung, and P. K. S. Tam, "Lyapunov function based design of fuzzy logic controllers and its application on combining controllers," *IEEE Trans. Ind. Electron.*, vol. 45, pp. 502–509, June 1998.

[2] L. K. Wong, F. H. F. Leung, and P. K. S. Tam, "Lyapunov function based design of heuristic fuzzy logic controllers," in *Proc. 6th IEEE Int. Conf. Fuzzy Systems*, vol. 1, Barcelona, Spain, July 1997, pp. 281–285.

[3] L. K. Wong, F. H. F. Leung, and P. K. S. Tam, "The design of static fuzzy logic controllers with combination of conventional controllers," in *Proc. IEEE ISIE'97*, vol. 3, July 1997, pp. 993–997.

[4] V. I. Utkin, "Variable structure systems with sliding modes," *IEEE Trans. Automat. Contr.*, vol. 22, pp. 212–222, Apr. 1977.

[5] R. A. DeCarlo, S. H. Zak, and G. P. Matthews, "Variable structure control of nonlinear multivariable systems: A tutorial," *Proc. IEEE*, vol. 76, pp. 212–232, Mar. 1988.

[6] J. J. E. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, NJ: Prentice-Hall, 1991.

[7] L. K. Wong, F. H. F. Leung, and P. K. S. Tam, "Combination of sliding mode controller and PI controller using fuzzy logic controller," in *Proc. 1998 IEEE Int. Conf. Fuzzy Systems*, Anchorage, AK, May 1998, pp. 296–301.

[8] C. S. Ting, T. H. S. Li, and F. C. Kung, "An approach to systematic design of the fuzzy control system," *Fuzzy Sets Syst.*, vol. 77, pp. 151–166, 1996.

[9] R. Palm, "Robust control by fuzzy sliding mode," *Automatica*, vol. 30, no. 9, pp. 1429–1437, 1994.

[10] J. S. Glower and J. Munighan, "Designing fuzzy controllers from a variable structures standpoint," *IEEE Trans. Fuzzy Syst.*, vol. 5, pp. 138–144, Feb. 1997.

L. K. Wong (S'96–M'00) received the B.Eng. (Hons.) and Ph.D. degrees from the Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, Hong Kong, in 1994 and 1999, respectively.

He is currently a Research Fellow at The Hong Kong Polytechnic University. His research interests are fuzzy logic control, intelligent control, switching-mode power converters, and intelligent homes.

Frank H. F. Leung (M'92) was born in Hong Kong in 1964. He received the B.Eng. and Ph.D. degrees in electronic engineering from The Hong Kong Polytechnic University, Hong Kong, in 1988 and 1992, respectively.

He took his industry-based training with Lambda Industrial Ltd., Hong Kong, where he worked on PABX and telephone systems. In 1992, he joined The Hong Kong Polytechnic University, where he is currently an Assistant Professor in the Department of Electronic and Information Engineering. He has authored more than 80 published research papers on intelligent control and power electronics. At present, he is actively involved in the Multimedia Intelligent Home and Micro Robot Soccer Tournament projects. He is a reviewer for many international journals and has helped in the organization of many international conferences.

Prof. Leung is a Chartered Engineer in the U.K. and a corporate member of the Institution of Electrical Engineers, U.K. He was a recipient of the Sir Edward Youde Memorial Fellowship in 1989 and 1990 and is listed in *Outstanding People of the 20th Century* and the *International Directory of Distinguished Leadership*.

Peter K. S. Tam (S'74–M'76) received the B.E., M.E., and Ph.D. degrees from the University of Newcastle, Newcastle, Australia, in 1971, 1973, and 1976, respectively, all in electrical engineering.

From 1967 to 1980, he held a number of industrial and academic positions in Australia. In 1980, he joined The Hong Kong Polytechnic University, Hong Kong, as a Senior Lecturer. He is currently an Associate Professor in the Department of Electronic and Information Engineering. He has participated in the organization of a number of symposiums and conferences. His research interests include signal processing, automatic control, fuzzy systems, and neural networks.