Optimal and Stable Fuzzy Controllers for Nonlinear Systems Based on an Improved Genetic Algorithm

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***Abstract*—This paper addresses the optimization and stabilization problems of nonlinear systems subject to parameter uncertainties. The methodology is based on a fuzzy logic approach and an improved genetic algorithm (GA). The TSK fuzzy plant model is employed to describe the dynamics of the uncertain nonlinear plant. A fuzzy controller is then obtained to close the feedback loop. The stability conditions are derived. The feedback gains of the fuzzy controller and the solution for meeting the stability conditions are determined using the improved GA. In order to obtain the optimal fuzzy controller, the membership functions are further tuned by minimizing a defined fitness function using the improved GA. An application example on stabilizing a two-link robot arm will be given.**

***Index Terms*—Fuzzy control, nonlinear systems, optimality and stability.**

I. Introduction

FUZZY CONTROL is one of the useful control techniques for uncertain and ill-defined nonlinear systems. Control actions of a fuzzy controller are described by some linguistic rules. This property makes the control algorithm easy to understand. Heuristic fuzzy controllers incorporate the experience or knowledge into rules, which are fine tuned based on trial and error. In order to have a systematic tuning procedure, a fuzzy controller implemented by a neural network was proposed in [6]. A genetic algorithm (GA) is a powerful searching algorithm [5]. It has been applied to fuzzy systems to generate the membership functions and/or the rule sets [13], [14]. These methodologies make the design simple; however, they do not guarantee the system stability and robustness.

In order to investigate the system stability, the TSK fuzzy plant model approach was proposed [1], [18]. A nonlinear system is modeled as a weighted sum of some simple subsystems. It gives a fixed structure that facilitates the system analysis to some nonlinear systems. There are two ways to obtain the fuzzy plant model: 1) by performing system identification based on the input–output data of the plant [1] and 2) by derivation from the mathematical model of the nonlinear plant. Stability of a fuzzy system formed by a fuzzy plant model and a fuzzy controller has been investigated recently. Different stability conditions have been obtained by employing Lyapunov stability theory [3], [8], [19] and other methods [11], [15], [16]. Most of the proposed fuzzy control laws are functions of the fuzzy plant model's membership functions, which must be known. It implies that the parameters of the nonlinear plant must be known, or be constant when the identification method is used to derive the fuzzy plant model. Practically, the parameters of many nonlinear plants will change during the operation, e.g., the load of a power system, the number of passengers on board a train. In these cases, the robustness property of the fuzzy controller is an important concern. Robustness analysis of fuzzy control systems can be found [7], [9]–[12], [15], [16]. In most of these works, only a stability and robustness testing condition is provided. The system performance and the determination of the control parameters (e.g., gains and membership functions) are seldom discussed.

In order to systematically obtain a fuzzy controller that guarantees the system stability [19], robustness, optimality and good performance, a fuzzy controller derived from an improved GA [17] is proposed. The contributions of this paper are twofold: 1) stability conditions for fuzzy control systems subject to parameter uncertainties are derived and 2) based on the derived stability conditions, the parameters of the fuzzy controller are obtained using an improved GA [17]. The membership functions of the fuzzy controller are also obtained automatically using the improved GA to achieve the optimal system performance. Both stability and performance are of concern.

This paper is organized as follows. The fuzzy plant model and fuzzy controller are presented in Section II. The fuzzy control system subject to parameter uncertainties are analyzed in Section III. The stability conditions will be derived. The problems of solving the derived stability conditions, obtaining the feedback gains of the fuzzy controller, and optimizing the system performance using the improved GA are presented in Section IV. An application example on stabilizing a two-link robot arm using the proposed fuzzy controller will be presented. A conclusion will be drawn in Section V.

II. TSK Fuzzy Plant Model With Parameter Uncertainties and Fuzzy Controller

We consider a nonlinear plant subject to parameter uncertainties connected with a fuzzy controller in closed loop. In order to obtain the fuzzy controller, a TSK fuzzy plant model is employed to describe the dynamics of the nonlinear plant subject to parameter uncertainties.

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A. TSK Fuzzy Plant Model With Parameter Uncertainties

Let p be the number of fuzzy rules describing the uncertain nonlinear plant. The ith rule is of the following format:

$$\text{Rule } i: \text{IF} f_1(\mathbf{x}(t)) \text{ is } \mathrm{M}_1^i \text{ and} \ldots \text{and } f_\Psi(\mathbf{x}(t)) \text{ is } \mathrm{M}_\Psi^i \text{ THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \quad (1)$$

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$ containing the parameter uncertainties of the nonlinear plant, $\alpha = 1, 2, \ldots, \Psi$, $i = 1, 2, \ldots, p$, Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices, respectively; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The inferred system is given by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)) \quad (2)$$

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, w_i(\mathbf{x}(t)) \in [0,\ 1] \text{ for all } i \quad (3)$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\left(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\right)} \quad (4)$$

is a nonlinear function of $x(t)$ and $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$ is the membership function corresponding to M_α^i. The value of $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$ is unknown as $f_\alpha(\mathbf{x}(t))$ is related to parameter uncertainties of the nonlinear plant. A fuzzy controller will be obtained based on the TSK fuzzy plant model of (2).

B. Fuzzy Controller

A fuzzy controller with c fuzzy rules is to be designed for the plant. The jth rule of the fuzzy controller is of the following format:

$$\text{Rule } j: \text{IF } g_1(\mathbf{x}(t)) \text{ is } \mathrm{N}_1^j \text{ and} \ldots \text{and } g_\Omega(\mathbf{x}(t)) \text{ in } \mathrm{N}_\Omega^j \text{ THEN } \mathbf{u}(t) = \mathbf{G}_j\mathbf{x}(t) + \mathbf{r} \quad (5)$$

where N_β^j is a fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta = 1, 2, \ldots, \Omega$, $j = 1, 2, \ldots, c$, Ω is a positive integer; $\mathbf{G}_j \in \Re^{m\times n}$ is the feedback gain of rule j to be designed, $\mathbf{r} \in \Re^{n\times 1}$ is the reference input vector. The inferred output of the fuzzy controller is given by

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t))\left(\mathbf{G}_j\mathbf{x}(t) + \mathbf{r}\right) \quad (6)$$

$$\sum_{j=1}^{c} m_j(\mathbf{x}(t)) = 1, m_j(\mathbf{x}(t)) \in [0,\ 1] \text{ for all } j \quad (7)$$

$$m_j(\mathbf{x}(t)) = \frac{\mu_{\mathrm{N}_1^j}(g_1(\mathbf{x}(t))) \times \mu_{\mathrm{N}_2^j}(g_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{N}_\Omega^j}(g_\Omega(\mathbf{x}(t)))}{\sum_{k=1}^{c}\left(\mu_{\mathrm{N}_1^k}(g_1(\mathbf{x}(t))) \times \mu_{\mathrm{N}_2^k}(g_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{N}_\Omega^k}(g_\Omega(\mathbf{x}(t)))\right)} \quad (8)$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{N}_\beta^j}(g_\beta(\mathbf{x}(t)))$ is the membership function corresponding to N_β^j to be designed.

C. Fuzzy Control System

In order to carry out the analysis, the closed-loop fuzzy system should be obtained. From (2) and (6), the fuzzy control system is given by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))\left(\mathbf{H}_{ij}\mathbf{x}(t) + \mathbf{B}_i\mathbf{r}\right) \quad (9)$$

$$\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j. \quad (10)$$

III. Stability Analysis

In the following, the stability of the fuzzy control system of (9) subject to parameter uncertainties will be analyzed [15], [16]. Consider the Taylor series

$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \dot{\mathbf{x}}(t)\Delta t + \mathbf{o}(\Delta t) \quad (11)$$

where $\mathbf{o}(\Delta t) = -\mathbf{x}(t) - \dot{\mathbf{x}}(t)\Delta t + \mathbf{x}(t+\Delta t)$ is the error term and $\Delta t > 0$

$$\lim_{\Delta t\to 0^+} \frac{\|\mathbf{o}(\Delta t)\|}{\Delta t} = 0. \quad (12)$$

From (9) and (11), writing $w_i(\mathbf{x}(\mathrm{t}))$ as w_i and $m_j(\mathbf{x}(\mathrm{t}))$ as m_j, and multiplying a transformation matrix $\mathbf{T} \in \Re^{n\times n}$ of rank n to both sides, we have

$$\begin{aligned}\mathbf{Tx}(t+\Delta t) &= \mathbf{Tx}(t) + \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{T}\left(\mathbf{H}_{ij}\mathbf{x}(t) + \mathbf{B}_i\mathbf{r}\right)\Delta t + \mathbf{To}(\Delta t)\\ &= \left(\mathbf{I} + \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}\Delta t\right)\mathbf{Tx}(t)\\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{T}\mathbf{B}_i\mathbf{r}\Delta t + \mathbf{To}(\Delta t).\end{aligned}$$

The reason for introducing $\mathbf{T}$ will be given at the end of this section. Taking the norm on both sides of the above equation,

$$\begin{aligned}\|\mathbf{Tx}(t+\Delta t)\| \le& \left\|\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\mathbf{I} + \mathbf{T}\mathbf{H}_{ij}\mathbf{T}^{-1}\Delta t\right)\right\| \|\mathbf{Tx}(t)\|\\ &+ \left\|\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{T}\mathbf{B}_i\mathbf{r}\Delta t\right\| + \|\mathbf{To}(\Delta t)\| \quad (13)\end{aligned}$$

where $\|\cdot\|$ denotes the l_2 norm for vectors and l_2 induced norm for matrices. From (13), we derive (14), shown at the bottom of the page. From (12) and (14),

$$\begin{aligned}\frac{\mathrm{d}\,\|\mathbf{Tx}(t)\|}{\mathrm{d}t} &\le \lim_{\Delta t\to 0^+}\frac{\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\left\|\mathbf{I}+\mathbf{TH}_{ij}\mathbf{T}^{-1}\Delta t\right\|-1\right)}{\Delta t}\|\mathbf{Tx}(t)\| + \left\|\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{TB}_i\mathbf{r}\right\| \\ &\le \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mu\left[\mathbf{TH}_{ij}\mathbf{T}^{-1}\right]\|\mathbf{Tx}(t)\| + \left\|\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{TB}_i\mathbf{r}\right\| \end{aligned} \tag{15}$$

where

$$\begin{aligned}\mu\left[\mathbf{TH}_{ij}\mathbf{T}^{-1}\right] &= \lim_{\Delta t\to 0^+}\frac{\left\|\mathbf{I}+\mathbf{TH}_{ij}\mathbf{T}^{-1}\Delta t\right\|-1}{\Delta t} \\ &= \lambda_{\max}\left(\frac{\mathbf{TH}_{ij}\mathbf{T}^{-1}+(\mathbf{TH}_{ij}\mathbf{T}^{-1})^*}{2}\right)\end{aligned} \tag{16}$$

is the corresponding matrix measure [4] of the induced matrix norm of $\left\|\mathbf{TH}_{ij}\mathbf{T}^{-1}\right\|$ (or the logarithmic derivative of $\left\|\mathbf{TH}_{ij}\mathbf{T}^{-1}\right\|$); $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue, and * denotes the conjugate transpose. From (15),

$$\frac{\mathrm{d}\,\|\mathbf{Tx}(t)\|}{\mathrm{d}t} \le \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mu\left[\mathbf{TH}_{ij}\mathbf{T}^{-1}\right]\|\mathbf{Tx}(t)\| + \left\|\sum_{i=1}^{p} w_i \mathbf{TB}_i\mathbf{r}\right\|. \tag{17}$$

Let $\mu\left[\mathbf{TH}_{ij}\mathbf{T}^{-1}\right]$ satisfy the following inequalities:

$$\mu\left[\mathbf{TH}_{ij}\mathbf{T}^{-1}\right] \le -\varepsilon \text{ for all } i \text{ and } j \tag{18}$$

where ε is a designed nonzero positive constant, and it can be proved that (17) implies a stable system of (9). Before conducting the proof, consider the following inequality obtained from (17) and (18):

$$\begin{aligned}\frac{\mathrm{d}\,\|\mathbf{Tx}(t)\|}{\mathrm{d}t} &\le -\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \varepsilon\|\mathbf{Tx}(t)\| + \sum_{i=1}^{p} w_j\|\mathbf{TB}_i\mathbf{r}\| \\ &= -\varepsilon\|\mathbf{Tx}(t)\| + \sum_{i=1}^{p} w_i\|\mathbf{TB}_i\mathbf{r}\| \\ &\Rightarrow \left(\frac{\mathrm{d}\,\|\mathbf{Tx}(t)\|}{\mathrm{d}t} + \varepsilon\|\mathbf{Tx}(t)\|\right)e^{\varepsilon(t-t_o)} \\ &\le \sum_{i=1}^{p} w_i\|\mathbf{TB}_i\mathbf{r}\|e^{\varepsilon(t-t_o)} \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{Tx}(t)\|e^{\varepsilon(t-t_o)}\right) \le \sum_{i=1}^{p} w_i\|\mathbf{TB}_i\mathbf{r}\|e^{\varepsilon(t-t_o)}\end{aligned} \tag{19}$$

where $t_o < t$ is an arbitrary initial time. Based on (19), there are two cases to investigate the system behavior: $\mathbf{r} = \mathbf{0}$ and $\mathbf{r} \neq \mathbf{0}$. For the former case, it can be shown that if the condition of (18) is satisfied, the closed-loop system of (9) is exponentially stable, and $\|\mathbf{x}(t)\| \to 0$ as $t \to \infty$.

Proof: For $\mathbf{r} = \mathbf{0}$, from (19),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{Tx}(t)\|e^{\varepsilon(t-t_o)} \le 0 &\Rightarrow \|\mathbf{Tx}(t)\|e^{\varepsilon(t-t_o)} \le \|\mathbf{Tx}(t_o)\| \\ &\Rightarrow \|\mathbf{Tx}(t)\| \le \|\mathbf{Tx}(t_o)\|e^{-\varepsilon(t-t_o)}.\end{aligned} \tag{20}$$

$$\begin{aligned}&\|\mathbf{Tx}(t+\Delta t)\| \\ &\le \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left\|\mathbf{I}+\mathbf{TH}_{ij}\mathbf{T}^{-1}\Delta t\right\|\|\mathbf{Tx}(t)\| + \left\|\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\mathbf{TB}_i\mathbf{r}\Delta t\right\| + \|\mathbf{To}(\Delta t)\| \\ &\Rightarrow \|\mathbf{Tx}(t+\Delta t)\| - \|\mathbf{Tx}(t)\| \\ &\le \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\left\|\mathbf{I}+\mathbf{TH}_{ij}\mathbf{T}^{-1}\Delta t\right\|-1\right)\|\mathbf{Tx}(t)\| + \left\|\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\mathbf{TB}_i\mathbf{r}\Delta t\right\| + \|\mathbf{To}(\Delta t)\| \\ &\Rightarrow \lim_{\Delta t\to 0^+}\frac{\|\mathbf{Tx}(t+\Delta t)\| - \|\mathbf{Tx}(t)\|}{\Delta t} \\ &\le \lim_{\Delta t\to 0^+}\frac{\left[\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\left\|\mathbf{I}+\mathbf{TH}_{ij}\mathbf{T}^{-1}\Delta t\right\|-1\right)\|\mathbf{Tx}(t)\| + \left\|\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\mathbf{TB}_i\mathbf{r}\Delta t\right\| + \|\mathbf{To}(\Delta t)\|\right]}{\Delta t}\end{aligned} \tag{14}$$

Since ε is a positive value, $\|\mathbf{Tx}(t)\| \to 0$ as $t \to \infty$, and

$$\begin{aligned}\sigma_{\min}(\mathbf{T}^{\mathrm{T}}\mathbf{T})\,\|\mathbf{x}(t)\|^2 &\le \|\mathbf{Tx}(t)\|^2 \\ &= \mathbf{x}(t)^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{Tx}(t) \le \sigma_{\max}(\mathbf{T}^{\mathrm{T}}\mathbf{T})\,\|\mathbf{x}(t)\|^2 \end{aligned} \tag{21}$$

where $\sigma_{\max}(\mathbf{T}^{\mathrm{T}}\mathbf{T})$ and $\sigma_{\min}(\mathbf{T}^{\mathrm{T}}\mathbf{T})$ denote the maximum and minimum singular values of $\mathbf{T}^{\mathrm{T}}\mathbf{T}$, respectively. As $\mathbf{T}^{\mathrm{T}}\mathbf{T}$ is symmetric positive definite ($\mathbf{T}$ has rank n), from (21), $\|\mathbf{Tx}(t)\| \to 0$ only when $\|\mathbf{x}(t)\| \to 0$. **QED**

For the latter case of $\mathbf{r} \neq \mathbf{0}$, the closed-loop system of (9) is input-to-state stable, i.e., the system states are bounded if the condition of (18) is satisfied and $\mathbf{r}$ is bounded.

Proof: For $\mathbf{r} \neq \mathbf{0}$, from (19),

$$\|\mathbf{Tx}(t)\|e^{\varepsilon(t-t_o)} \le \|\mathbf{Tx}(t_o)\| + \int_{t_o}^{t}\sum_{i=1}^{p} w_i\,\|\mathbf{TB}_i\mathbf{r}\|e^{\varepsilon(\tau-t_o)}\mathrm{d}\tau$$

$$\Rightarrow \|\mathbf{Tx}(t)\|\,e^{\varepsilon(t-t_o)} \le \|\mathbf{Tx}(t_o)\| + \|\mathbf{TB}_i\mathbf{r}\|_{\max}\int_{t_o}^{t} e^{\varepsilon(\tau-t_o)}\mathrm{d}\tau$$

where $\max_i \|\mathbf{TB}_i\mathbf{r}\|_{\max} \ge \|\mathbf{TB}_i\mathbf{r}\|$. Then,

$$\begin{aligned}\|\mathbf{Tx}(t)|\,e^{\varepsilon(t-t_o)} &\le \|\mathbf{Tx}(t_o)\| + \frac{\|\mathbf{TB}_i\mathbf{r}\|_{\max}}{\varepsilon}(e^{\varepsilon(t-t_o)} - 1) \\ \Rightarrow \|\mathbf{Tx}(t)\| &\le \|\mathbf{Tx}(t_o)\|\,e^{-\varepsilon(t-t_o)} \\ &\quad + \frac{\|\mathbf{TB}_i\mathbf{r}\|_{\max}}{\varepsilon}(1-e^{-\varepsilon(t-t_o)}). \end{aligned} \tag{22}$$

Since the right-hand side of (22) is finite if $\mathbf{r}$ is bounded, the system states are also bounded. **QED**

The stability conditions of the closed-loop fuzzy system can be summarized by the following lemma:

Lemma 1: The fuzzy control system, subject to parameter uncertainties, as given by (9) *is exponentially stable for* $\mathbf{r} = \mathbf{0}$ *or input-to-state stable for* $\mathbf{r} \neq \mathbf{0}$ *if* $\mathbf{TH}_{ij}\mathbf{T}^{-1}$ *is designed such that*

$$\mu\left[\mathbf{TH}_{ij}\mathbf{T}^{-1}\right] \le -\varepsilon \text{ for all } i \text{ and } j$$

where ε *is a nonzero positive constant scalar.*

It should be noted that with the use of a suitable transformation matrix $\mathbf{T}$, any Hurwitz matrix having a positive or zero matrix measure can be transformed into another matrix having a negative matrix measure [see (18)]. The stability conditions derived can then be applied. The problem left is how to find such a matrix $\mathbf{T}$ for a given system. This will be discussed later. From the above derivation and Lemma 1, we also see the system stability is not affected by the membership functions of the fuzzy controller. Therefore, the membership functions of the fuzzy controller can be determined using a GA to obtain the optimal system performance.

IV. Solving the Stability Conditions, Obtaining the Feedback Gains, and Optimizing the System Performance

In this section, the problems of solving the stability conditions derived in the previous section, obtaining the feedback gains of the fuzzy controller, and optimizing the system performance will be tackled using the improved GA [17] proposed by the same authors.

A. Solving the Stability Conditions and Obtaining the Feedback Gains

From Lemma 1, the uncertain fuzzy control system is stable if the following conditions hold:

$$\mu\left[\mathbf{T}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\mathbf{T}^{-1}\right] \le -\varepsilon, i=1,2,\ldots,p;\ j = 1,2,\ldots,c. \tag{23}$$

We therefore have to find $\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn}\end{bmatrix}$ and $\mathbf{G}_j = \begin{bmatrix} G_{11}^j & G_{11}^j & \cdots & G_{1n}^j \\ G_{21}^j & G_{22}^j & \cdots & G_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1}^j & G_{m2}^j & \cdots & G_{mn}^j\end{bmatrix}$ such that the conditions of (23) are satisfied. Define a fitness function as follows:

$$\text{fitness} = \sum_{i=1}^{p}\sum_{j=1}^{c} n_{ij}\mu\left[\mathbf{T}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\mathbf{T}^{-1}\right] \tag{24}$$

where $n_{ij} \ge 0$, $i = 1,2,\ldots,p$, $j = 1,2,\ldots,c$, are constant scalar. The problems of finding $\mathbf{T}$ and $\mathbf{G}_j$ can be formulated into a minimization problem. The aim is to minimize the fitness function of (24) with $\mathbf{T}$ and $\mathbf{G}_j$ using the improved GA. As $\mathbf{T}$ and $\mathbf{G}_j$ are the variables of the fitness function of (24), they will be used to form the genes of the chromosomes. The finding of the solution to this minimization problem, however, does not imply that the conditions of (23) are satisfied. Hence, different n_{ij}, $i = 1,2,\ldots,p$, $j = 1,2,\ldots,c$, may need to be tried to weight the conditions of (23) in order to change the significance of different terms on the right hand side of (24). For instance, one of the terms in (24) is very positive, which returns a very large fitness value. Under this case, the conditions of (23) are not satisfied. A large value of n_{ij} corresponding to that term can be used to increase the tuning effect to that term in the fitness function. This may help the GA process to find a solution that satisfies the conditions of (23) during the minimizing process.

B. Optimizing the System Performance

After $\mathbf{T}$ and $\mathbf{G}_j$ have been determined, what follows is to determine the membership functions of the fuzzy controller using the improved GA such that the performance of the uncertain fuzzy control system is optimal subject to a defined performance index. The dynamics of the uncertain fuzzy control system is restated as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t), \mathbf{z}_j)\left(\mathbf{H}_{ij}\mathbf{x}(t) + \mathbf{B}_i\mathbf{r}\right) \tag{25}$$

where $\mathbf{z}_j$ is the parameter vector governing the membership functions of rule j of the fuzzy controller, e.g., the values of the means and the standard deviations of various Gaussian membership functions. A fitness function (performance index) is defined as follows:

$$\text{fitness} = \int \mathbf{x}(t)^{\mathrm{T}}\mathbf{W}_x\mathbf{x}(t) + \mathbf{u}(t)^{\mathrm{T}}\mathbf{R}_u\mathbf{u}(t)\mathrm{d}t \tag{26}$$

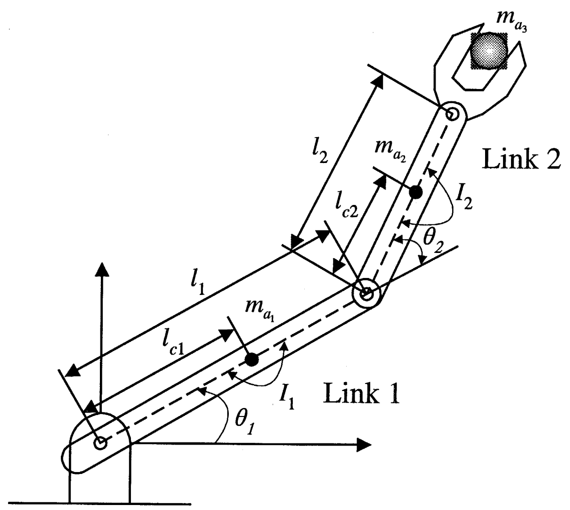

Fig. 1. Two-link robot arm.

where $\mathbf{W}_x \in \Re^{n\times n}$ and $\mathbf{R}_u \in \Re^{m\times m}$ are constant semipositive or positive-definite matrices. This fitness function is the performance index used in the conventional optimal control [2]. The optimization problem formulated here will be handled by the improved GA. $\mathbf{z}_j, j = 1, 2, \ldots, c$, will be used to form the genes of the chromosomes for the GA process. As the fuzzy control system is an uncertain system, the nominal system parameters will be used for determining the membership functions of the fuzzy plant model. Hence, all $w_i(\mathbf{x}(t))$ in (25) are known. The procedure to obtain the fuzzy controller using the improved GA can be summarized into the following steps.

1) Obtain the mathematical model of the nonlinear plant subject to parameter uncertainties. Convert the mathematical model into the fuzzy plant model of **(2)**.
2) Determine the number of rules for the fuzzy controller. Solve $\mathbf{T}$ and $\mathbf{G}_j$ with the fitness function defined in **(24)** and $n_{ij} = 1, i = 1, 2, \ldots, p, j = 1, 2, \ldots, c$ using the improved GA. If $\mathbf{T}$ and $\mathbf{G}_j$ cannot be found, adjust n_{ij} accordingly.
3) Determine the membership functions of the fuzzy controller. Obtain the parameters of the membership functions using the improved GA to optimize the system performance with respect to the performance index of **(26)**.

V. Application Example

An application example will be given in this section. A multiple-input multiple-output (MIMO) two-link robot arm [12] shown in Fig. 1 is taken as the nonlinear plant. Referring to Fig. 1, m_{a_1} is the center of mass of link 1, m_{a_2} is the center of mass of link 2, m_{a_3} is the mass of the load; l_1 is the length of link 1, l_2 is the length of link 2; l_{c_1} is the length from the joint of link 1 to its center of mass, l_{c_2} is the length from the joint of link 2 to its center of mass; I_1 is the lengthwise centroidal inertia of link 1, I_2 is the lengthwise centroidal inertia of link 2; θ_1 and θ_2 are the angles of the joints as shown in Fig. 1. A fuzzy controller will be obtained to stabilize the two-link robot arm by following the procedure in the previous section.

1) The system dynamics of the two-link robot arm is governed by the following dynamical equations:

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \Delta\mathbf{A}(\mathbf{x}(t)))\,\mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{E} \qquad (27)$$

where

$$\mathbf{x}(t) = [\,x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)\,]^{\mathrm{T}} = [\,\theta_1(t) \quad \dot{\theta}_1(t) \quad \theta_2(t) \quad \dot{\theta}_2(t)\,]^{\mathrm{T}}$$

$x_1(t) \in [\,x_{1_{\min}} \quad x_{1_{\max}}\,] = [-\pi \quad \pi]$, $x_2(t) \in [\,x_{2_{\min}} \quad x_{2_{\max}}\,] = [-3 \quad 3]$, $x_3(t) \in [\,x_{3_{\min}} \quad x_{3_{\max}}\,] = [-\pi \quad \pi]$, $x_4(t) \in [\,x_{4_{\min}} \quad x_{4_{\max}}\,] = [-3 \quad 3]$, $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and

$$\Delta\mathbf{A}(\mathbf{x}(t)) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & f_2(\mathbf{x}(t))hx_2(t) & 0 & f_1(\mathbf{x}(t))h\,(2x_2(t) + x_4(t)) \\ 0 & 0 & 0 & 0 \\ 0 & -f_3(\mathbf{x}(t))hx_2(t) & 0 & -f_2(\mathbf{x}(t))h\,(2x_2(t) + x_4(t)) \end{bmatrix}$$

$$\mathbf{B}(\mathbf{x}(t)) = \begin{bmatrix} 0 & 0 \\ f_1(\mathbf{x}(t)) & -f_2(\mathbf{x}(t)) \\ 0 & 0 \\ -f_2(\mathbf{x}(t)) & f_3(\mathbf{x}(t)) \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 0 \\ -f_1(\mathbf{x}(t))g_1 + f_2(\mathbf{x}(t))g_2 \\ 0 \\ f_2(\mathbf{x}(t))g_1 - f_3(\mathbf{x}(t))g_2 \end{bmatrix}$$

and

$$f_1(\mathbf{x}(t)) = M_{22}/(M_{11}M_{22} - M_{12}M_{21})$$
$$f_2(\mathbf{x}(t)) = M_{12}/(M_{11}M_{22} - M_{12}M_{21})$$
$$f_3(\mathbf{x}(t)) = M_{11}/(M_{11}M_{22} - M_{12}M_{21})$$

$$M_{11} = I_1 + I_2 + m_{a_1}l_{c_1}^2 + m_{a_2}\left(l_1^2 + l_{c_2}^2 + 2l_1l_{c_2}\cos(x_3(t))\right) + m_{a_3}\left(l_1^2 + l_2^2 + 2l_1l_2\cos(x_3(t))\right),$$
$$M_{12} = M_{21} = I_1 + m_{a_2}\left(l_{c_2}^2 + l_1l_{c_2}\cos(x_3(t))\right) + m_{a_3}\left(l_2^2 + l_1l_2\cos(x_3(t))\right)$$

$$M_{22} = I_2 + m_{a_2}l_{c_2}^2 + m_{a_3}l_2^2,\ h = m_{a_2}l_1l_{c_2}\sin(x_3(t)),$$

$$g_1 = m_{a_1}l_{c_1}g\cos(x_1(t)) + m_{a_2}g(l_{c_2}\cos(x_1(t) + x_3(t)) + l_1\cos(x_1(t)))$$

$g_2 = m_{a_2}l_{c_2}g\cos(x_1(t) + x_3(t))$; $m_{a_1} = 10$ kg, $m_{a_2} = 10$ kg, $m_{a_3} \in [\,0\text{ kg} \quad 3\text{ kg}\,]$, $I_1 = 5\,\text{kg}\cdot\text{m}^2$, $I_2 = 3.5\,\text{kg}\cdot\text{m}^2$, $l_1 = 1$ m, $l_2 = 0.5$ m, $l_{c_1} = 0.8$ m, $l_{c_2} = 0.2$ m $g = 9.8\ \text{ms}^{-2}\mathbf{u}(t) =$

$\begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^{\mathrm{T}}$ is the control inputs. It should be noted that $M_{11}M_{22} - M_{12}M_{21} > 0$ [12] and the parameter uncertainties of m_3 are included in $f_1(\mathbf{x}(t))$, $f_2(\mathbf{x}(t))$, and $f_3(\mathbf{x}(t))$. Let

$$\mathbf{u}(t) = \mathbf{u}_1(t) + \mathbf{R}(\mathbf{x}(t)) \tag{28}$$

where

$$\mathbf{R} = \begin{bmatrix} -hx_4(t)\,(2x_2(t) + x_4(t)) + g_1 \\ hx_2(t)^2 + g_2 \end{bmatrix} \tag{29}$$

such that $\Delta\mathbf{A}\mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t))\mathbf{R}(\mathbf{x}(t)) + \mathbf{E} = \mathbf{0}$. From (28) and (27), we have

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t))\mathbf{u}_1(t). \tag{30}$$

The nonlinear plant of (30) can be represented by an 8-rule TSK fuzzy plant model. The eight rules are shown as follows.

$$\begin{aligned} &\text{Rule } i: \text{ IF } f_1(\mathbf{x}(t)) \text{ is } \mathrm{M}_1^i \text{ AND } f_2(\mathbf{x}(t)) \\ &\quad \text{is } \mathrm{M}_2^i \text{ AND } f_3(\mathbf{x}(t)) \text{ is } \mathrm{M}_3^i \\ &\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}_1(t), \qquad i = 1, 2, \ldots, 8 \end{aligned} \tag{31}$$

where

$$\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{A}_3 = \mathbf{A}_4 = \mathbf{A}_5 = \mathbf{A}_6 = \mathbf{A}_7 = \mathbf{A}_8 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 & 0 \\ f_{1_{\min}} & -f_{2_{\min}} \\ 0 & 0 \\ -f_{2_{\min}} & f_{3_{\min}} \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 & 0 \\ f_{1_{\min}} & -f_{2_{\min}} \\ 0 & 0 \\ -f_{2_{\min}} & f_{3_{\max}} \end{bmatrix}$$

$$\mathbf{B}_3 = \begin{bmatrix} 0 & 0 \\ f_{1_{\min}} & -f_{2_{\max}} \\ 0 & 0 \\ -f_{2_{\max}} & f_{3_{\min}} \end{bmatrix}, \quad \mathbf{B}_4 = \begin{bmatrix} 0 & 0 \\ f_{1_{\min}} & -f_{2_{\max}} \\ 0 & 0 \\ -f_{2_{\max}} & f_{3_{\max}} \end{bmatrix}$$

$$\mathbf{B}_5 = \begin{bmatrix} 0 & 0 \\ f_{1_{\max}} & -f_{2_{\min}} \\ 0 & 0 \\ -f_{2_{\min}} & f_{3_{\min}} \end{bmatrix}, \quad \mathbf{B}_6 = \begin{bmatrix} 0 & 0 \\ f_{1_{\max}} & -f_{2_{\min}} \\ 0 & 0 \\ -f_{2_{\min}} & f_{3_{\max}} \end{bmatrix}$$

$$\mathbf{B}_7 = \begin{bmatrix} 0 & 0 \\ f_{1_{\max}} & -f_{2_{\max}} \\ 0 & 0 \\ -f_{2_{\max}} & f_{3_{\min}} \end{bmatrix}, \quad \mathbf{B}_8 = \begin{bmatrix} 0 & 0 \\ f_{1_{\max}} & -f_{2_{\max}} \\ 0 & 0 \\ -f_{2_{\max}} & f_{3_{\max}} \end{bmatrix}$$

$f_{1_{\min}} = 0.0794$, $f_{1_{\max}} = 0.1050$, $f_{2_{\min}} = -0.0306$, $f_{2_{\max}} = 0.2237$, $f_{3_{\min}} = 0.7752$, $f_{3_{\max}} = 2.0490$. The membership functions of M_j^i, $i = 1, 2, \ldots, 8$, $j = 1, 2, 3$, which are shown in Fig. 2, are defined as follows:

$$\begin{aligned} \mu_{\mathrm{M}_1^1}(f_1(\mathbf{x}(t)) &= \mu_{\mathrm{M}_1^2}(f_1(\mathbf{x}(t)) = \mu_{\mathrm{M}_1^3}(f_1(\mathbf{x}(t)) \\ &= \mu_{\mathrm{M}_1^4}(f_1(\mathbf{x}(t)) = \frac{-f_1(\mathbf{x}(t)) + f_{1_{\max}}}{f_{1_{\max}} - f_{1_{\min}}} \end{aligned} \tag{32}$$

$$\begin{aligned} \mu_{\mathrm{M}_1^5}(f_1(\mathbf{x}(t)) &= \mu_{\mathrm{M}_1^6}(f_1(\mathbf{x}(t)) = \mu_{\mathrm{M}_1^7}(f_1(\mathbf{x}(t)) \\ &= \mu_{\mathrm{M}_1^8}(f_1(\mathbf{x}(t)) = \frac{f_1(\mathbf{x}(t)) - f_{1_{\min}}}{f_{1_{\max}} - f_{1_{\min}}} \end{aligned} \tag{33}$$

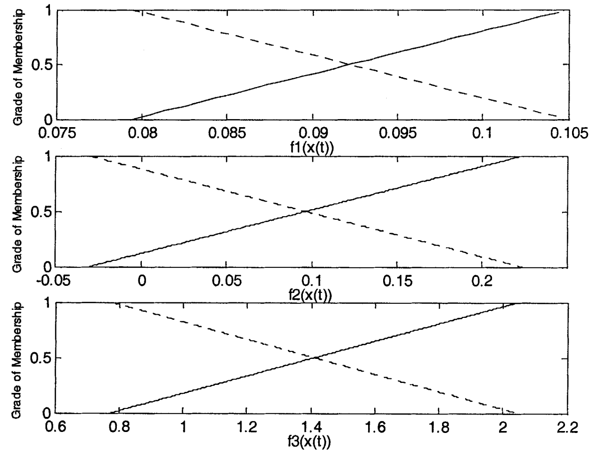

Fig. 2. Membership functions of the fuzzy plant model of the two-link robot arm.

$$\begin{aligned} \mu_{\mathrm{M}_2^1}(f_2(\mathbf{x}(t)) &= \mu_{\mathrm{M}_2^2}(f_2(\mathbf{x}(t)) = \mu_{\mathrm{M}_2^5}(f_2(\mathbf{x}(t)) \\ &= \mu_{\mathrm{M}_2^6}(f_2(\mathbf{x}(t)) = \frac{-f_2(\mathbf{x}(t)) + f_{2_{\max}}}{f_{2_{\max}} - f_{2_{\min}}} \end{aligned} \tag{34}$$

$$\begin{aligned} \mu_{\mathrm{M}_2^3}(f_2(\mathbf{x}(t)) &= \mu_{\mathrm{M}_2^4}(f_2(\mathbf{x}(t)) = \mu_{\mathrm{M}_2^7}(f_2(\mathbf{x}(t)) \\ &= \mu_{\mathrm{M}_2^8}(f_2(\mathbf{x}(t)) = \frac{f_2(\mathbf{x}(t)) - f_{2_{\min}}}{f_{2_{\max}} - f_{2_{\min}}} \end{aligned} \tag{35}$$

$$\begin{aligned} \mu_{\mathrm{M}_3^1}(f_3(\mathbf{x}(t)) &= \mu_{\mathrm{M}_3^3}(f_3(\mathbf{x}(t)) = \mu_{\mathrm{M}_3^5}(f_3(\mathbf{x}(t)) \\ &= \mu_{\mathrm{M}_3^7}(f_3(\mathbf{x}(t)) = \frac{-f_3(\mathbf{x}(t)) + f_{3_{\max}}}{f_{3_{\max}} - f_{3_{\min}}} \end{aligned} \tag{36}$$

$$\begin{aligned} \mu_{\mathrm{M}_3^2}(f_3(\mathbf{x}(t)) &= \mu_{\mathrm{M}_3^4}(f_3(\mathbf{x}(t)) = \mu_{\mathrm{M}_3^6}(f_3(\mathbf{x}(t)) \\ &= \mu_{\mathrm{M}_3^8}(f_3(\mathbf{x}(t)) = \frac{f_3(\mathbf{x}(t)) - f_{3_{\min}}}{f_{3_{\max}} - f_{3_{\min}}}. \end{aligned} \tag{37}$$

In order to regulate $x_1(t)$ and $x_3(t)$ of the two-link robot arm, a fuzzy controller with integral control will be employed. Therefore, the TSK fuzzy plant model of (31) has to be augmented to one with the following rules.

$$\begin{aligned} &\text{Rule } i: \text{ IF } f_1(\mathbf{x}(t)) \text{ is } \mathrm{M}_1^i \text{ AND } f_2(\mathbf{x}(t)) \\ &\quad \text{is } \mathrm{M}_2^i \text{ AND } f_3(\mathbf{x}(t)) \text{ is } \mathrm{M}_3^i \\ &\text{THEN } \dot{\bar{\mathbf{x}}}(t) = \bar{\mathbf{A}}_i\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}_i\mathbf{u}_1(t), \qquad i = 1, 2, \ldots, 8 \end{aligned} \tag{38}$$

where $\bar{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) & x_5(t) & x_6(t) \end{bmatrix}^{\mathrm{T}}$, $x_5(t) = x_5(t_o) + \int_{t_o}^{\infty} (\theta_{1_{\mathrm{ref}}} - x_1(t))\mathrm{d}t$, and $x_6(t) = x_6(t_o) + \int_{t_o}^{\infty} (\theta_{2_{\mathrm{ref}}} - x_3(t))\mathrm{d}t$; $x_5(t_o)$ and $x_6(t_o)$ are the initial values of $x_5(t)$ and $x_6(t)$ at an arbitrary time t_o, respectively. $\theta_{1_{\mathrm{ref}}}$ and $\theta_{2_{\mathrm{ref}}}$ are the reference values of $x_1(t)$ and $x_3(t)$, respectively,

$$\bar{\mathbf{A}}_1 = \bar{\mathbf{A}}_2 = \bar{\mathbf{A}}_3 = \bar{\mathbf{A}}_4 = \bar{\mathbf{A}}_5 = \bar{\mathbf{A}}_6 = \bar{\mathbf{A}}_7 = \bar{\mathbf{A}}_8 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

TABLE I
STABILITY ANALYSIS RESULTS

i,j	$\mu[\mathbf{T}(\bar{\mathbf{A}}_i+\bar{\mathbf{B}}_i\mathbf{G}_j)\mathbf{T}^{-1}]$	i,j	$\mu[\mathbf{T}(\bar{\mathbf{A}}_i+\bar{\mathbf{B}}_i\mathbf{G}_j)\mathbf{T}^{-1}]$
1, 1	−0.8125	5, 1	−0.8123
1, 2	−0.8135	5, 2	−0.8133
1, 3	−0.8074	5, 3	−0.7648
1, 4	−0.8141	5, 4	−0.7904
2, 1	−0.8001	6, 1	−0.8182
2, 2	−0.8132	6, 2	−0.8130
2, 3	−0.2816	6, 3	−0.3463
2, 4	−0.4134	6, 4	−0.4648
3, 1	−0.8070	7, 1	−0.8109
3, 2	−0.8054	7, 2	−0.8128
3, 3	−0.2116	7, 3	−0.7807
3, 4	−0.0477	7, 4	−0.7762
4, 1	−0.8010	8, 1	−0.8016
4, 2	−0.8142	8, 2	−0.8141
4, 3	−0.7338	8, 3	−0.7484
4, 4	−0.7688	8, 4	−0.7706

$$\bar{\mathbf{B}}_1=\begin{bmatrix}0&0\\f_{1_{\min}}&-f_{2_{\min}}\\0&0\\-f_{2_{\min}}&f_{3_{\min}}\\0&0\\0&0\end{bmatrix},\quad \bar{\mathbf{B}}_2=\begin{bmatrix}0&0\\f_{1_{\min}}&-f_{2_{\min}}\\0&0\\-f_{2_{\min}}&f_{3_{\max}}\\0&0\\0&0\end{bmatrix}$$

$$\bar{\mathbf{B}}_3=\begin{bmatrix}0&0\\f_{1_{\min}}&-f_{2_{\max}}\\0&0\\-f_{2_{\max}}&f_{3_{\min}}\\0&0\\0&0\end{bmatrix},\quad \bar{\mathbf{B}}_4=\begin{bmatrix}0&0\\f_{1_{\min}}&-f_{2_{\max}}\\0&0\\-f_{2_{\max}}&f_{3_{\max}}\\0&0\\0&0\end{bmatrix}$$

$$\bar{\mathbf{B}}_5=\begin{bmatrix}0&0\\f_{1_{\max}}&-f_{2_{\min}}\\0&0\\-f_{2_{\min}}&f_{3_{\min}}\\0&0\\0&0\end{bmatrix},\quad \bar{\mathbf{B}}_6=\begin{bmatrix}0&0\\f_{1_{\max}}&-f_{2_{\min}}\\0&0\\-f_{2_{\min}}&f_{3_{\max}}\\0&0\\0&0\end{bmatrix}$$

$$\bar{\mathbf{B}}_7=\begin{bmatrix}0&0\\f_{1_{\max}}&-f_{2_{\max}}\\0&0\\-f_{2_{\max}}&f_{3_{\min}}\\0&0\\0&0\end{bmatrix},\quad \bar{\mathbf{B}}_8=\begin{bmatrix}0&0\\f_{1_{\max}}&-f_{2_{\max}}\\0&0\\-f_{2_{\max}}&f_{3_{\max}}\\0&0\\0&0\end{bmatrix}.$$

A fuzzy controller will be found based on the TSK fuzzy plant model of (38).

2) A four-rule fuzzy controller will be found. The rules of the fuzzy controllers are as follows.

$$\text{Rule } i: \text{ IF } x_1(t) \text{ is } \mathrm{N}_1^j \text{ AND } x_3(t) \text{ is } \mathrm{N}_2^j \text{ THEN } \mathbf{u}_1(t) = \mathbf{G}_j\bar{\mathbf{x}}(t), \qquad j=1,2,3,4. \tag{39}$$

The output of the fuzzy controller is given by

$$\mathbf{u}_1(t)=\sum_{j=1}^{4}m_j(\mathbf{x}(t))\mathbf{G}_j\bar{\mathbf{x}}(t) \tag{40}$$

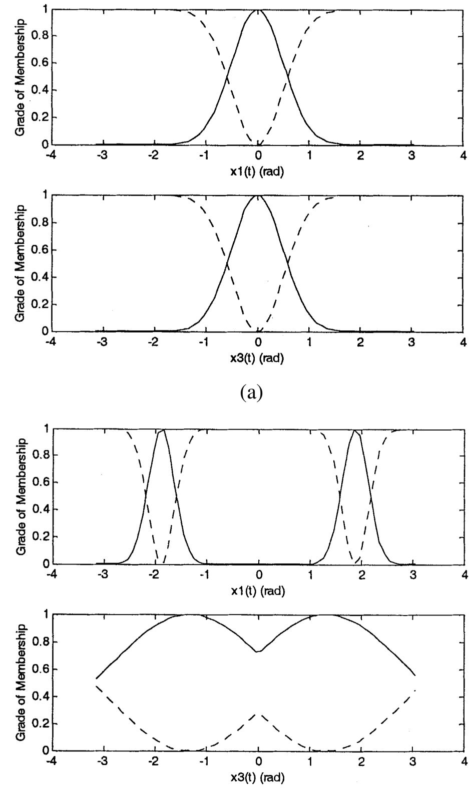

Fig. 3. Membership functions of the fuzzy controller before and after tuning. (a) Before tuning. (b) After tuning.

where $m_j(\mathbf{x}(t))$ is defined in (8) with $\mu_{\mathrm{N}_1^1}(x_1(t)) = \mu_{\mathrm{N}_1^2}(x_1(t)) = e^{-(|x_1(t)|-\bar{m}_1)^2/2\bar{\sigma}_1^2}$; $\mu_{\mathrm{N}_1^3}(x_1(t)) = \mu_{\mathrm{N}_1^4}(x_1(t)) = 1 - e^{-(|x_1(t)|-\bar{m}_1)^2/2\bar{\sigma}_1^2}$; $\mu_{\mathrm{N}_2^1}(x_3(t)) = \mu_{\mathrm{N}_2^3}(x_3(t)) = e^{-(|x_3(t)|-\bar{m}_2)^2/2\bar{\sigma}_2^2}$; $\mu_{\mathrm{N}_2^2}(x_3(t)) = \mu_{\mathrm{N}_2^4}(x_3(t)) = 1 - e^{-(|x_3(t)|-\bar{m}_2)^2/2\bar{\sigma}_2^2}$. $\bar{m}_1$, $\bar{m}_2$, $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are the parameters governing the Gaussian membership functions. These four parameters will be tuned by the improved GA to optimize the system performance. From (28) and (40), the final fuzzy controller is defined as

$$\mathbf{u}(t)=\sum_{j=1}^{4}m_j(\mathbf{x}(t))\mathbf{G}_j\bar{\mathbf{x}}(t)+\mathbf{R}(\mathbf{x}(t)). \tag{41}$$

To obtain the $\mathbf{T}$ and $\mathbf{G}_j$, we choose the minimum and maximum values of each element of $\mathbf{T}$ to be −1.5 and 1.5, respectively; the minimum and maximum values of each element of $\mathbf{G}_j$, $j=1$, 2, 3, 4, are chosen to be −1500 and 1500, respectively. A fitness function is defined as

$$\text{fitness}=\sum_{i=1}^{8}\sum_{j=1}^{4}n_{ij}\mu\left[\mathbf{T}\left(\bar{\mathbf{A}}_i+\bar{\mathbf{B}}_i\mathbf{G}_j\right)\mathbf{T}^{-1}\right] \tag{42}$$

where $n_{ij}=1, i=1,2,3,4,5,6,7,8; j=1,2,3,4$. $\mathbf{T}$ and $\mathbf{G}_j$, $j=1$, 2, 3, 4, are obtained automatically using the improved GA under the consideration of the system stability conditions of (23). The parameters of the improved GA [17], namely the weight w and the probability of acceptance p_a are chosen to be 0.5 and 0.1 respectively. The population size and number of iterations are chosen to be 10 and 5000 respectively. The initial values of $\mathbf{T}$ and $\mathbf{G}_j$, $j = 1, 2, 3, 4$, are randomly generated for the GA process. After applying the improved GA process, we obtain the equations shown at the bottom of the page. The stability analysis result is tabulated in Table I. It can be seen that the values of $\mu[\mathbf{T}\left(\bar{\mathbf{A}}_i + \bar{\mathbf{B}}_i\mathbf{G}_j\right)\mathbf{T}^{-1}]$, $i = 1, 2, 3, 4, 5, 6, 7, 8$; $j = 1, 2, 3, 4$, are all negative. According to Lemma 1, the fuzzy control system is guaranteed to be stable.

3) The optimal performance of the fuzzy control system will be obtained by tuning the membership functions of the fuzzy controller. The tunable parameters of the membership functions are $\bar{m}_1 \in [-\pi \quad \pi]$, $\bar{\sigma}_1 \in [0 \quad 5]$, $\bar{m}_2 \in [-\pi \quad \pi]$ and $\bar{\sigma}_2 \in [0 \quad 5]$. The parameters of w and p_a of the improved GA [17] are chosen to be 0.5 and 0.1 respectively. The population size and number of iterations are chosen to be 10 and 500, respectively. The initial values are $\bar{m}_1 = 0$, $\bar{\sigma}_1 = 0.5$, $\bar{m}_2 = 0$ and $\bar{\sigma}_2 = 0.5$. The fitness function is chosen as follows:

$$\text{fitness} = \int_0^{20} \bar{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{W}_x\bar{\mathbf{x}}(t) + \mathbf{u}(t)^{\mathrm{T}}\mathbf{R}_u\mathbf{u}(t)\mathrm{d}t \tag{43}$$

where

$$\mathbf{W}_x = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R}_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

After the improved GA process, $\bar{m}_1 = 1.8802$, $\bar{\sigma}_1 = 0.2519$, $\bar{m}_2 = 1.3212$ and $\bar{\sigma}_2 = 1.6046$. The fitness values before and after the GA process are 111 876.0098 and 104 424.5023, respectively. It is about 6.6605% improvement. Fig. 3 shows the membership functions of the fuzzy controller before and after the GA tuning. Fig. 4 shows the responses of the system states, with $m_{a_3} = 0$ kg, $\theta_{1_{\mathrm{ref}}} = \pi/2$ and $\theta_{2_{\mathrm{ref}}} = \pi/2$ for $0 \le t < 10$ and $m_{a_3} = 3$ kg, $\theta_{1_{\mathrm{ref}}} = -\pi/2$ and $\theta_{2_{\mathrm{ref}}} = -\pi/2$ for $t \ge 10$, before (dotted line) and after (solid line) the GA process has tuned the membership functions under the initial condition of $\bar{\mathbf{x}}(0) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$. Fig. 5 shows the control signals of the fuzzy controller before and after the tuning process. It can be seen from the simulation results that the responses after optimization are better. In this paper, the derived stability conditions form the basis for designing a stable fuzzy controller. From the application example, it can be observed that finding of the feedback gains of the fuzzy controller is mainly for the system stability, which will not be affected by the fuzzy controller's membership functions. To improve the system performance, we can further tune the membership functions.

For comparison purposes, a linear state feedback controller designed based on the linearized model of (27) is employed. The linearized model of (27) around the origin with integral control and $m_{a_3} = 0$ is as follows:

$$\dot{\bar{\mathbf{x}}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \bar{\mathbf{x}}(t) + \begin{bmatrix} 0 & 0 \\ \bar{f}_1 & -\bar{f}_2 \\ 0 & 0 \\ \bar{f}_2 & \bar{f}_3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \bar{\mathbf{u}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \theta_{1_{\mathrm{ref}}} \\ \theta_{2_{\mathrm{ref}}} \end{bmatrix} \tag{44}$$

where

$$\begin{aligned} \bar{f}_1 &= \bar{M}_{22}/(\bar{M}_{11}\bar{M}_{22} - \bar{M}_{12}\bar{M}_{21}) \\ \bar{f}_2 &= \bar{M}_{12}/(\bar{M}_{11}\bar{M}_{22} - \bar{M}_{12}\bar{M}_{21}) \\ \bar{f}_3(\mathbf{x}(t)) &= \bar{M}_{11}/(\bar{M}_{11}\bar{M}_{22} - \bar{M}_{12}\bar{M}_{21}) \end{aligned}$$

and

$$\begin{aligned} \bar{M}_{11} &= I_1 + I_2 + m_{a_1}l_{c_1}^2 + m_{a_2}\left(l_1^2 + l_{c_2}^2 + 2l_1l_{c_2}\right) \\ \bar{M}_{12} &= \bar{M}_{21} = I_1 + m_{a_2}\left(l_{c_2}^2 + l_1l_{c_2}\right) \\ M_{22} &= I_2 + m_{a_2}l_{c_2}^2 + m_{a_3}l_2^2. \end{aligned}$$

A linear state feedback controller is designed based on the linear quadratic regulator (LQR) technique [2] by minimizing

$$\mathbf{T} = \begin{bmatrix} -0.3535 & 0.4190 & 0.8451 & 0.4185 & -0.4937 & -0.9036 \\ -0.7364 & -0.8006 & -0.1412 & 0.2738 & 0.3417 & -0.2373 \\ -1.4941 & -0.0884 & -1.0955 & -0.1150 & 1.1466 & 0.8027 \\ -1.4976 & -0.5144 & 1.0884 & -0.0660 & 1.4986 & -0.2179 \\ -0.5464 & -0.2925 & -0.4113 & -0.4133 & -0.5956 & -0.2079 \\ 0.3071 & -0.0250 & 0.1705 & 0.0163 & -1.3510 & 0.8340 \end{bmatrix}$$

$$\mathbf{G}_1 = \begin{bmatrix} -1086.6560 & -831.5020 & 0 & 0 & 824.9274 & 0 \\ 0 & 0 & -1057.7074 & -899.7480 & 0 & 837.0547 \end{bmatrix}$$

$$\mathbf{G}_2 = \begin{bmatrix} -995.7281 & -735.9732 & 0 & 0 & 746.4286 & 0 \\ 0 & 0 & -948.5759 & -770.1179 & 0 & 736.6712 \end{bmatrix}$$

$$\mathbf{G}_3 = \begin{bmatrix} -302.1633 & -199.9928 & 0 & 0 & 250.0390 & 0 \\ 0 & 0 & -250.0412 & -174.9477 & 0 & 183.9495 \end{bmatrix}$$

$$\mathbf{G}_4 = \begin{bmatrix} -274.2215 & -179.9977 & 0 & 0 & 225.0594 & 0 \\ 0 & 0 & -225.0425 & -157.4589 & 0 & 164.4405 \end{bmatrix}$$

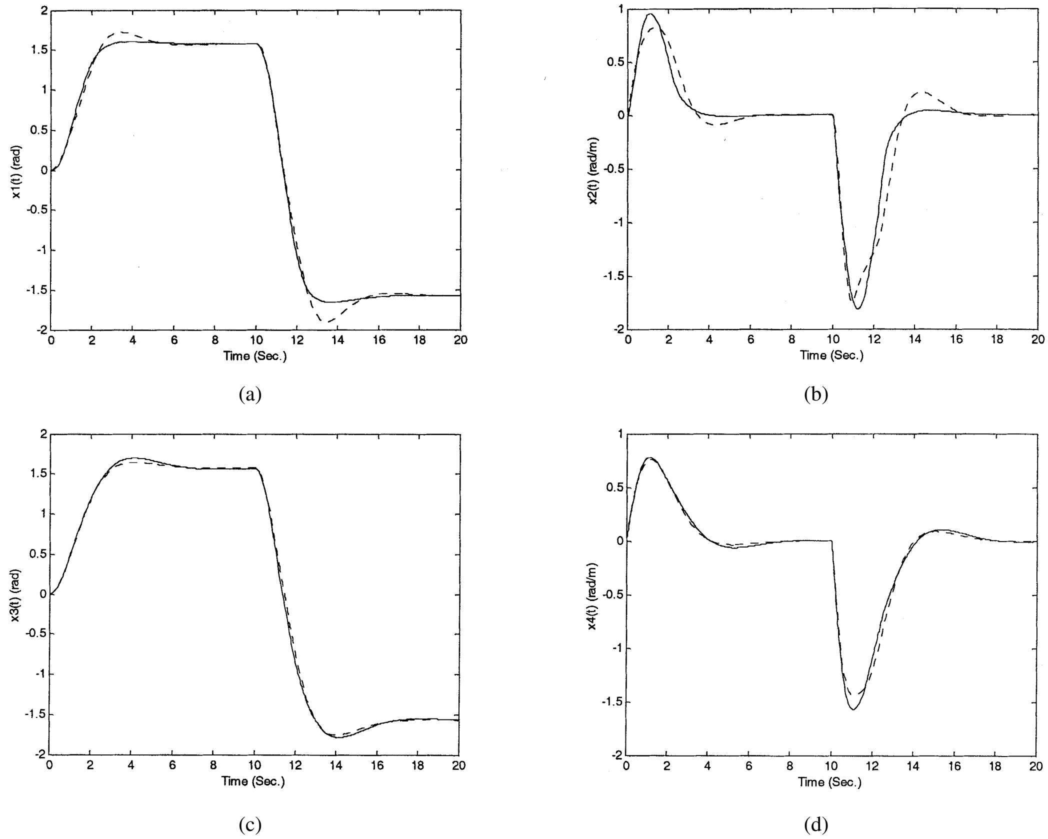

Fig. 4. Responses of $\mathbf{x}(t)$ of the two-link robot arm with the fuzzy controller before (dotted lines) and after (solid lines) tuning. (a) $x_1(t)$. (b) $x_2(t)$. (c) $x_3(t)$. (d) $x_4(t)$.

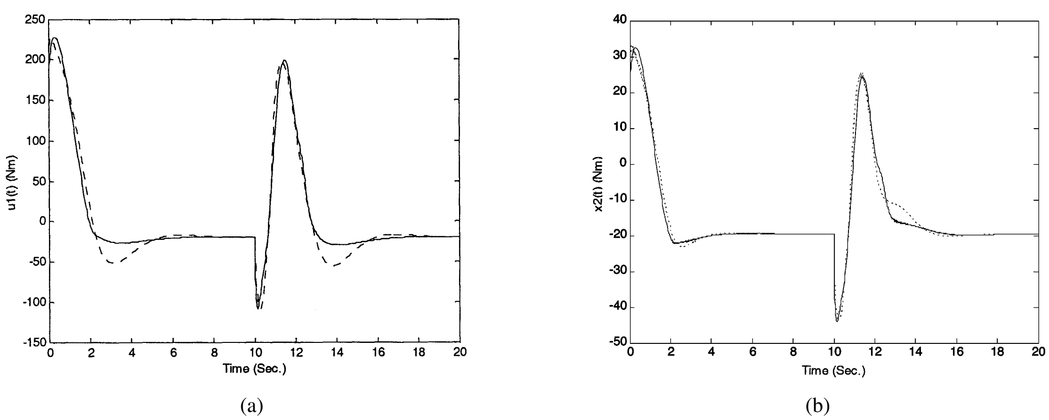

Fig. 5. Control signals of the fuzzy controller before (dotted lines) and after (solid lines) tuning. (a) $u_1(t)$. (b) $u_2(t)$.

the performance index of (43) with the same $\mathbf{W}_x$ and $\mathbf{R}_u$. The linear state feedback control law is given by (45), shown at the bottom of the page. The output responses of and the control signals for the two-link robot arm controlled by the linear state feedback controller under the initial conditions $\bar{\mathbf{x}}(0) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$ are shown in Figs. 6 and 7, respectively. It can be seen that the linear state feedback controller cannot successfully control the system.

VI. Conclusion

Fuzzy control of nonlinear systems subject to parameter uncertainties has been presented. Stability conditions have been

$$\bar{\mathbf{u}}(t) = \begin{bmatrix} -6.9212 & -19.1361 & -0.6074 & -3.4898 & 1.0000 & 0.0000 \\ -0.6074 & -3.4898 & -4.5694 & -5.6240 & 0.0000 & 1.0000 \end{bmatrix} \bar{\mathbf{x}}(t) \tag{45}$$

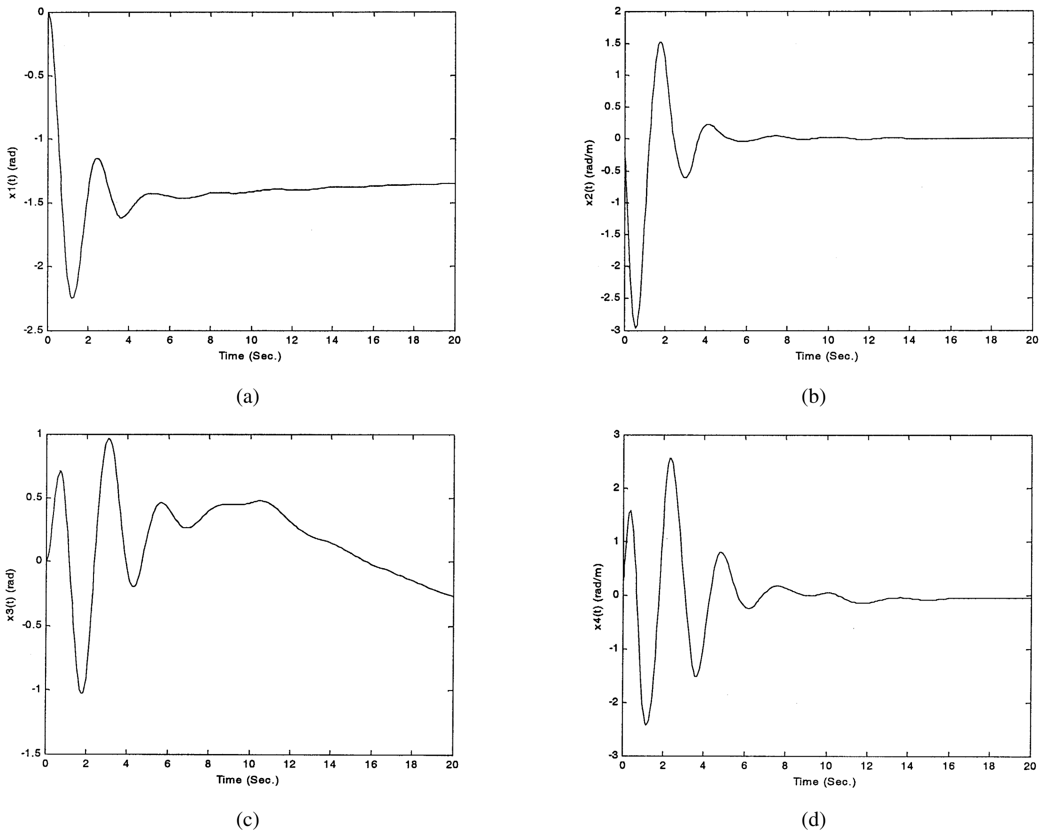

Fig. 6. Responses of $\mathbf{x}(t)$ of the two-link robot arm with the linear state feedback controller. (a) $x_1(t)$. (b) $x_2(t)$. (c) $x_3(t)$. (d) $x_4(t)$.

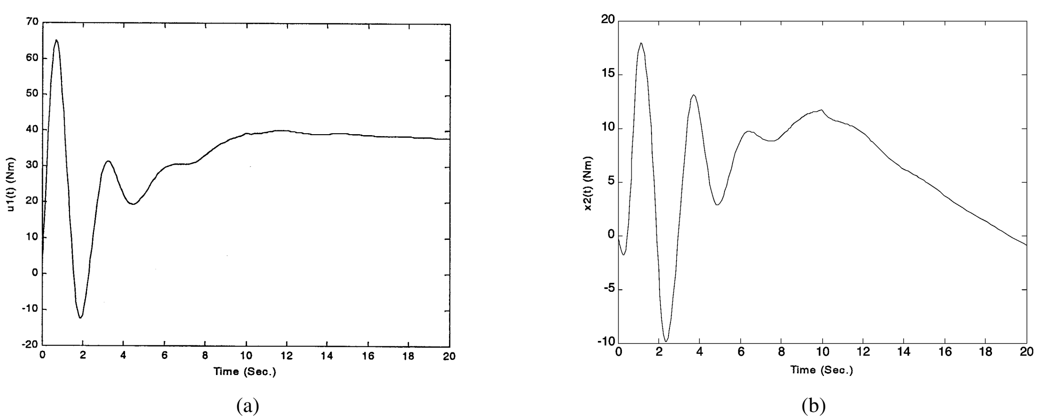

Fig. 7. Control signals of the linear state feedback controller. (a) $u_1(t)$. (b) $u_2(t)$.

derived for this class of uncertain fuzzy control system. An improved GA has been employed to help find the solution to the stability conditions and determine the feedback gains of the fuzzy controller. Moreover, the membership functions of the fuzzy controller can be tuned automatically for optimal system performance. An application example on stabilizing a two-link robot arm has been presented to illustrate the merits of the proposed fuzzy controller.

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