LMI-Based Stability and Performance Conditions for Continuous-Time Nonlinear Systems in Takagi–Sugeno's Form

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***Abstract*—This correspondence presents the stability analysis and performance design of the continuous-time fuzzy-model-based control systems. The idea of the nonparallel-distributed-compensation (non-PDC) control laws is extended to the continuous-time fuzzy-model-based control systems. A nonlinear controller with non-PDC control laws is proposed to stabilize the continuous-time nonlinear systems in Takagi–Sugeno's form. To produce the stability-analysis result, a parameter-dependent Lyapunov function (PDLF) is employed. However, two difficulties are usually encountered: 1) the time-derivative terms produced by the PDLF will complicate the stability analysis and 2) the stability conditions are not in the form of linear-matrix inequalities (LMIs) that aid the design of feedback gains. To tackle the first difficulty, the time-derivative terms are represented by some weighted-sum terms in some existing approaches, which will increase the number of stability conditions significantly. In view of the second difficulty, some positive-definitive terms are added in order to cast the stability conditions into LMIs. In this correspondence, the favorable properties of the membership functions and nonlinear control laws, which allow the introduction of some free matrices, are employed to alleviate the two difficulties while retaining the favorable properties of PDLF-based approach. LMI-based stability conditions are derived to ensure the system stability. Furthermore, based on a common scalar performance index, LMI-based performance conditions are derived to guarantee the system performance. Simulation examples are given to illustrate the effectiveness of the proposed approach.**

***Index Terms*—Fuzzy control, parameter-dependent Lyapunov function (PDLF), performance realization, stability analysis.**

I. INTRODUCTION

Fuzzy-model-based control approach offers a systematic and effective framework to investigate the system stability. In general, the stability analysis is carried out based on the T–S fuzzy models [1], [2], which represent the system dynamics of the nonlinear plants. In the last two decade, fruitful stability-analysis results [3]–[16] have been obtained to guarantee the system stability of the continuous-time or discrete-time fuzzy-model-based control systems. Basic linear-matrix-inequality (LMI)-based stability conditions were developed in the study in [3] and [4] using Lyapunov approach for the fuzzy-model-based control systems. In [4], an efficient design technique, namely, parallel-distributed compensation (PDC) technique, was proposed to design the fuzzy controllers and relax the stability conditions. Further relaxed stability conditions were obtained in the study in [5]–[13] based on the PDC-design technique. In [3]–[13], the stability analysis of fuzzy-model-based control systems was investigated based on a parameter-independent Lyapunov function. The stability analysis was extended to parameter-dependent Lyapunov functions (PDLFs) for continuous-time [13] and discrete-time systems [15], [16]. Furthermore, in [15] and [16], a non-PDC nonlinear controller was proposed to stabilize the discrete-time nonlinear systems represented by the T–S fuzzy models. It was shown that the non-PDC control laws derived by the PDLF-based approach could further relax the stability conditions.

On using the continuous-time PDC approach with PDLF, two difficulties are faced during the stability analysis: 1) unlike the discrete-time case, the continuous-time design using PDLF will generate time-derivative information of the membership functions, increasing the difficulty of the stability analysis and 2) the resultant stability conditions cannot be simply expressed in LMI forms. To deal with the problem in 1), the time-derivative information is represented by some weighted functions in the study in [13]. However, the number of stability condition is increased by the multiplication property of the fuzzy-model-based approach. To deal with the problem in 2), some positive-definitive terms were employed to formulate the stability conditions into LMI forms by using the Schur complement technique. However, conservativeness is introduced to the stability analysis result by the additional positive-definitive terms added. Furthermore, the dimension of the matrices in the stability conditions will be increased by the Schur complement technique. In [14], under a particular premise structure of the fuzzy model, the time-derivative information can be eliminated. However, it cannot be applied to general fuzzy models.

In this correspondence, the non-PDC design approach using the PDLF proposed in [15], and the study in [16] is extended to the continuous-time nonlinear systems. To deal with the problem in 1), the property of the membership functions, which allows introducing some free matrices, is employed during the stability analysis. Unlike the weighted-sum representation of the time-derivative information in the study in [13] that increases the order of the multiplication, our approach converts the time-derivative information into additive terms only. The difficulty in the problem in 1) can thus be reduced. To deal with the problem in 2), the non-PDC control laws are employed. As some of the nonlinear terms can be compensated by the non-PDC control laws during the system analysis, the order of the multiplication can be further reduced and, more importantly, the stability conditions can be expressed in LMI forms without introducing extra positive-definite terms. The LMI-based stability conditions are derived using the PDLF-based approach to guarantee the stability of the fuzzy-model-based systems. In order to realize the system performance, a commonly used scalar performance index is employed to quantitatively measure the system performance. Based on this performance index, some LMI-based performance conditions are derived to guarantee the system performance.

This correspondence is organized as follows. In Section II, the fuzzy model and the non-PDC nonlinear controller are presented. In Section III, the LMI-based stability and performance conditions are derived by using the Lyapunov stability theory. In Section IV, simulation examples are presented to illustrate the effectiveness of the proposed approach. A conclusion will be drawn in Section V.

II. FUZZY MODEL AND NON-PDC NONLINEAR CONTROLLER

A multivariable fuzzy-model-based control system comprising a nonlinear plant represented by a fuzzy model and a non-PDC nonlinear controller connected in a closed loop is considered.

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A. Fuzzy Model

Let p be the number of fuzzy rules describing the nonlinear plant. The ith rule is of the following format:

$$\text{Rule } i: \text{IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND} \ldots \text{AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i \quad \text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \tag{1}$$

where M_α^i is a fuzzy term of rule i corresponding to the known function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2 \ldots, \Psi$; $i = 1, 2, \ldots, p$; Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices, respectively; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector; and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The system dynamics are described by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)) \tag{2}$$

where

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, \quad w_i(\mathbf{x}(t)) \in [0 \quad 1] \text{ for all } i \tag{3}$$

and (4), shown at the bottom of the page, is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{M_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, $\alpha = 1, 2, \ldots, \Psi$, is the grade of membership corresponding to the fuzzy term of M_α^i.

B. Non-PDC Nonlinear Controller

A non-PDC nonlinear controller for the nonlinear plant represented by the fuzzy model of (2) is proposed as follows:

$$\mathbf{u}(t) = \sum_{j=1}^{p} w_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{\Gamma}(\mathbf{x}(t))^{-1}\mathbf{x}(t) + \sum_{j=1}^{p} \dot{w}_j(\mathbf{x}(t))\bar{\mathbf{G}}_j\mathbf{\Gamma}(\mathbf{x}(t))^{-1}\mathbf{x}(t) \tag{5}$$

where $\mathbf{G}_j \in \Re^{m\times n}$, and $\overline{\mathbf{G}}_j \in \Re^{m\times n}$ are constant feedback gains to be designed; $\mathbf{\Gamma}(\mathbf{x}(t)) = \mathbf{\Gamma}(\mathbf{x}(t))^{\mathrm{T}} = (\sum_{k=1}^{p} w_k(\mathbf{x}(t))\mathbf{P}_k)$; $\mathbf{P}_k = \mathbf{P}_k^{\mathrm{T}} \in \Re^{n\times n} > 0$, $k = 1, 2, \ldots, p$.

Remark 1: Referring to (5), when the fuzzy controller contains the feedback gain of $\overline{\mathbf{G}}_j$, the time derivative of membership function, $\dot{w}_j(\mathbf{x}(t)) = (dw_j(\mathbf{x}(t))/d\mathbf{x}(t))\dot{\mathbf{x}}(t) = (dw_j(\mathbf{x}(t))/d\mathbf{x}(t))(\sum_{i=1}^{p} w_i(\mathbf{x}(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)))$, has to be known. Hence, $\dot{w}_j(\mathbf{x}(t))$ should not depend on $\mathbf{u}(t)$; otherwise, the fuzzy controller of (5) cannot be implemented easily.

Remark 2: As $\mathbf{P}_k$, $k = 1, 2, \ldots, p$ is a positive-definite matrix that implies nonsingularity, the inverse of $\mathbf{P}_k$ must exist. Referring to (3), i.e., $\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1$, $w_i(\mathbf{x}(t)) \in [0 \quad 1]$ for all i, $\sum_{k=1}^{p} w_k(\mathbf{x}(t))\mathbf{P}_k$ is a linear combination of positive-definitive matrices $\mathbf{P}_k$. As the sum of positive-definite matrices is still a positive-definite matrix; hence, $\sum_{k=1}^{p} w_k(\mathbf{x}(t))\mathbf{P}_k$ is a positive-definite matrix, which is nonsingular, and $\mathbf{\Gamma}(\mathbf{x}(t))^{-1} = (\sum_{k=1}^{p} w_k(\mathbf{x}(t))\mathbf{P}_k)^{-1}$ exists.

III. Stability Analysis and Performance Design

In this section, the system stability and performance design of the fuzzy-model-based control system are presented. In the following analysis, $w_i(\mathbf{x}(t))$, $\dot{w}_j(\mathbf{x}(t))$, and $\mathbf{\Gamma}(\mathbf{x}(t))$ are denoted by w_i, $\dot{w}_j$, and $\mathbf{\Gamma}$, respectively, for simplicity, and the properties that $\sum_{i=1}^{p} w_i = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j = 1$ and $\sum_{i=1}^{p} \dot{w}_i = 0$ are used. From (2) and (5), the fuzzy-model-based control system is defined as follows:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{p} w_i \left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\left(\sum_{j=1}^{p} w_j\mathbf{G}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t) + \sum_{j=1}^{p} \dot{w}_j\overline{\mathbf{G}}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t)\right)\right) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\mathbf{\Gamma}^{-1}\right)\mathbf{x}(t) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\dot{w}_j\mathbf{B}_i\overline{\mathbf{G}}_j\mathbf{\Gamma}^{-1}\mathbf{x}(t).\end{aligned} \tag{6}$$

The system stability and performance of the fuzzy-model-based control system are guaranteed by the LMI-based stability and performance conditions in the following theorem.

Theorem 1: The fuzzy model-based control system of (6), which is formed by the nonlinear system in the form of (2) and the non-PDC nonlinear controller of (5), is guaranteed to be asymptotically stable if there exist nonzero positive scalars $0 < \rho < 1$, $\eta > 0$ and $\sigma > 1$, such that $w_k(\mathbf{x}(t)) + \rho\dot{w}_k(\mathbf{x}(t)) > 0$ for all k and $\mathbf{x}(t)$, and matrices $\mathbf{F} \in \Re^{(3n+m)\times(3n+m)}$, $\mathbf{G}_j \in \Re^{m\times n}$, $\overline{\mathbf{G}}_j \in \Re^{m\times n}$, $\mathbf{J}_1 = \mathbf{J}_1^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{J}_2 \in \Re^{n\times m}$, $\mathbf{J}_3 = \mathbf{J}_3^{\mathrm{T}} \in \Re^{m\times m}$, $\mathbf{P}_k = \mathbf{P}_k^{\mathrm{T}}$, $\mathbf{R}_{ijk} = \mathbf{R}_{jik}^{\mathrm{T}} \in \Re^{n\times n}$, $\overline{\mathbf{R}}_{ij} = \overline{\mathbf{R}}_{ij}^{\mathrm{T}} \in \Re^{n\times n}$, and $\mathbf{\Lambda}_i = \mathbf{\Lambda}_i^{\mathrm{T}} \in \Re^{n\times n}$, such that the following LMI-based stability and performance conditions are satisfied.

A. LMI-Based Stability Conditions

$$\mathbf{P}_k > 0, \qquad k = 1, 2, \ldots, p$$

$$\mathbf{R}_{ijk} + \mathbf{R}_{ijk}^{\mathrm{T}} \geq 0, \qquad j, k = 1, 2, \ldots, p; \quad i < j$$

$$\overline{\mathbf{R}}_{ij} + \overline{\mathbf{R}}_{ij}^{\mathrm{T}} \leq 0, \qquad j = 1, 2, \ldots, p; \quad i < j$$

$$\mathbf{S}_k = \begin{bmatrix} \mathbf{Q}_{11k} & \mathbf{S}_{12k} & \cdots & \mathbf{S}_{1pk} \\ \mathbf{S}_{21k} & \mathbf{Q}_{22k} & \cdots & \mathbf{S}_{2pk} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{S}_{p1k} & \mathbf{S}_{p2k} & \cdots & \mathbf{Q}_{ppk} \end{bmatrix} < 0, \qquad k = 1, 2, \ldots, p$$

$$\overline{\mathbf{S}} = \begin{bmatrix} \mathbf{Q}_{11} & \overline{\mathbf{S}}_{12} & \cdots & \overline{\mathbf{S}}_{1p} \\ \overline{\mathbf{S}}_{21} & \mathbf{Q}_{22} & \cdots & \overline{\mathbf{S}}_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{\mathbf{S}}_{p1} & \overline{\mathbf{S}}_{p2} & \cdots & \mathbf{Q}_{pp} \end{bmatrix} > 0$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{M_1^i}(f_1(\mathbf{x}(t))) \times \mu_{M_2^i}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{M_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\left(\mu_{M_1^k}(f_1(\mathbf{x}(t))) \times \mu_{M_2^k}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{M_\Psi^k}(f_\Psi(\mathbf{x}(t)))\right)} \tag{4}$$

where

$$\mathbf{S}_{ijk} = \frac{\mathbf{Q}_{ijk} + \mathbf{Q}_{jik}}{2} + \mathbf{R}_{ijk}, \qquad j,k = 1,2,\ldots,p; \quad i<j$$

$$\overline{\mathbf{S}}_{ij} = \frac{\overline{\mathbf{Q}}_{ij} + \overline{\mathbf{Q}}_{ji}}{2} + \overline{\mathbf{R}}_{ij}, \qquad j = 1,2,\ldots,p; \quad i<j$$

$$\mathbf{Q}_{ijk} = \mathbf{P}_j\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{P}_j + \mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{G}_j + \frac{1}{\rho}\left(\mathbf{\Lambda}_i + \overline{\mathbf{G}}_k^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\overline{\mathbf{G}}_k - \mathbf{P}_k\right), \qquad i,j,k = 1,2,\ldots,p$$

$$\overline{\mathbf{Q}}_{ij} = \mathbf{\Lambda}_i + \overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\overline{\mathbf{G}}_j - \mathbf{P}_j, \qquad i,j = 1,2,\ldots,p.$$

B. LMI-Based Performance Conditions

$$\mathbf{T}_{ij} + \mathbf{F} < 0, \qquad i,j = 1,2,\ldots,p$$
$$\overline{\mathbf{T}}_i + \mathbf{F} > 0, \qquad i = 1,2,\ldots,p$$

where the expression for $\mathbf{T}_{ij}$ is shown at the bottom of the page and where

$$\overline{\mathbf{T}}_i = \begin{bmatrix} \frac{\eta}{\sigma}\mathbf{P}_i & \mathbf{0} & -\frac{1}{\sigma}\mathbf{P}_i & \mathbf{0} \\ \mathbf{0} & \frac{\eta}{\sigma}\mathbf{P}_i & \mathbf{0} & \frac{1}{\rho}\overline{\mathbf{G}}_i^{\mathrm{T}} \\ -\frac{1}{\sigma}\mathbf{P}_i & \mathbf{0} & \frac{1}{\sigma}\mathbf{K}_1 & \frac{1}{\sigma}\mathbf{K}_2 \\ \mathbf{0} & \frac{1}{\rho}\overline{\mathbf{G}}_i & \frac{1}{\sigma}\mathbf{K}_2^{\mathrm{T}} & \frac{1}{\sigma}\mathbf{K}_3 \end{bmatrix}, \qquad i,j = 1,2,\ldots,p.$$

It should be noted that the values of $\mathbf{J}_1$, $\mathbf{J}_2$, and $\mathbf{J}_3$ have to be determined, such that

$$\begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_2^{\mathrm{T}} & \mathbf{K}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix}^{-1} > 0$$

prior to applying Theorem 1. The proof of Theorem 1 is given in the following sections.

C. Stability Analysis

The stability of the fuzzy-model-based control system of (6) is investigated. Considering the following Lyapunov function candidate:

$$V(t) = \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{x}(t). \tag{7}$$

From (6) and (7), we have

$$\begin{aligned} \dot{V}(t) &= \dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\dot{\mathbf{x}}(t) - \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\dot{\mathbf{\Gamma}}\mathbf{\Gamma}^{-1}\mathbf{x}(t) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{x}(t)^{\mathrm{T}}\Big(\mathbf{A}_i^{\mathrm{T}}\mathbf{\Gamma}^{-1} + \mathbf{\Gamma}^{-1}\mathbf{A}_i + \mathbf{\Gamma}^{-1}\big(\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{G}_j\big)\mathbf{\Gamma}^{-1}\Big)\mathbf{x}(t) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i \dot{w}_j \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\left(\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\overline{\mathbf{G}}_j\right)\mathbf{\Gamma}^{-1}\mathbf{x}(t) \\ &\quad - \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\dot{\mathbf{\Gamma}}\mathbf{\Gamma}^{-1}\mathbf{x}(t) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\big(\mathbf{\Gamma}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{\Gamma} + \mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{G}_j\big)\mathbf{\Gamma}^{-1}\mathbf{x}(t) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i \dot{w}_j \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\left(\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\overline{\mathbf{G}}_j\right)\mathbf{\Gamma}^{-1}\mathbf{x}(t) \\ &\quad - \mathbf{x}(t)^{\mathrm{T}}\mathbf{\Gamma}^{-1}\dot{\mathbf{\Gamma}}\mathbf{\Gamma}^{-1}\mathbf{x}(t). \end{aligned} \tag{8}$$

Let $\overline{\mathbf{x}}(t) = \mathbf{\Gamma}^{-1}\mathbf{x}(t)$, and put $\mathbf{\Gamma} = \sum_{j=1}^{p} w_j\mathbf{P}_j$, $\dot{\mathbf{\Gamma}} = \sum_{j=1}^{p} \dot{w}_j\mathbf{P}_j = (\sum_{i=1}^{p} w_i)\sum_{j=1}^{p}\dot{w}_j\mathbf{P}_j = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\dot{w}_j\mathbf{P}_j$ and $\sum_{j=1}^{p}\dot{w}_j(\sum_{i=1}^{p} w_i\mathbf{\Lambda}_i) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\dot{w}_j\mathbf{\Lambda}_i = \mathbf{0}$, where $\mathbf{\Lambda}_i = \mathbf{\Lambda}_i^{\mathrm{T}} \in \Re^{n\times n}$, $i = 1,2,\ldots,p$ are arbitrary matrices. We then have

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \overline{\mathbf{x}}(t)^{\mathrm{T}}\left(\mathbf{\Gamma}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{\Gamma} + \mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{G}_j\right)\overline{\mathbf{x}}(t) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\dot{w}_j\overline{\mathbf{x}}^{\mathrm{T}}\left(\mathbf{\Lambda}_i + \overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\overline{\mathbf{G}}_j\right)\overline{\mathbf{x}}(t) \\ &\quad - \sum_{i=1}^{p}\sum_{j=1}^{p} w_i\dot{w}_j\overline{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}_j\overline{\mathbf{x}}(t) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\overline{\mathbf{x}}(t)^{\mathrm{T}} \\ &\quad \times\left(\mathbf{P}_j\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{P}_j + \mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{G}_j\right)\overline{\mathbf{x}}(t) \\ &\quad + \frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i(w_j + \rho\dot{w}_j - w_j)\overline{\mathbf{x}}^{\mathrm{T}} \\ &\quad \times\left(\mathbf{\Lambda}_i + \overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\overline{\mathbf{G}}_j - \mathbf{P}_j\right)\overline{\mathbf{x}}(t) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\overline{\mathbf{x}}(t)^{\mathrm{T}} \\ &\quad \times\left(\mathbf{P}_j\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{P}_j + \mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{G}_j\right)\overline{\mathbf{x}}(t) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(w_j + \rho\dot{w}_j)\overline{\mathbf{x}}^{\mathrm{T}} \\ &\quad \times\frac{1}{\rho}\left(\mathbf{\Lambda}_i + \overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\overline{\mathbf{G}}_j - \mathbf{P}_j\right)\overline{\mathbf{x}}(t) \\ &\quad - \frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\overline{\mathbf{x}}^{\mathrm{T}}\left(\mathbf{\Lambda}_i + \overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\overline{\mathbf{G}}_j - \mathbf{P}_j\right)\overline{\mathbf{x}}(t) \end{aligned} \tag{9}$$

$$\mathbf{T}_{ij} = \begin{bmatrix} -\eta\left(1-\frac{1}{\sigma}\right)\mathbf{P}_i & \mathbf{0} & \left(1-\frac{1}{\sigma}\right)\mathbf{P}_i & \mathbf{0} \\ \mathbf{0} & -\eta\left(1-\frac{1}{\sigma}\right)\mathbf{P}_i & \mathbf{0} & \mathbf{G}_i^{\mathrm{T}} + \frac{1}{\rho}\overline{\mathbf{G}}_j^{\mathrm{T}} \\ \left(1-\frac{1}{\sigma}\right)\mathbf{P}_i & \mathbf{0} & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_1 & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_2 \\ \mathbf{0} & \mathbf{G}_i + \frac{1}{\rho}\overline{\mathbf{G}}_j & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_2^{\mathrm{T}} & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_3 \end{bmatrix}$$

where ρ is a nonzero positive scalar. From the above equations, it can be seen that the free matrices $\mathbf{\Lambda}_i$ are introduced and can take arbitrary values to produce less conservative stability-analysis result. Considering the property that $\sum_{j=1}^{p} \dot{w}_j = 0$, which leads to $\sum_{j=1}^{p}(w_j+\rho\dot{w}_j)=\sum_{j=1}^{p} w_j+\rho\sum_{j=1}^{p}\dot{w}_j=1$, from (9), we have

$$\begin{aligned}\dot{V}(t)=&\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i w_j(w_k+\rho\dot{w}_k)\overline{\mathbf{x}}(t)^{\mathrm{T}}\\&\times\Bigg(\mathbf{P}_j\mathbf{A}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{P}_j+\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j\\&\quad+\frac{1}{\rho}\left(\mathbf{\Lambda}_i+\overline{\mathbf{G}}_k^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_k-\mathbf{P}_k\right)\Bigg)\overline{\mathbf{x}}(t)\\&-\frac{1}{\rho}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\overline{\mathbf{x}}^{\mathrm{T}}\left(\mathbf{\Lambda}_i+\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_j-\mathbf{P}_j\right)\overline{\mathbf{x}}(t)\\=&\sum_{i=1}^{p}\sum_{k=1}^{p} w_i^2(w_k+\rho\dot{w}_k)\overline{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{Q}_{iik}\overline{\mathbf{x}}(t)\\&+\sum_{j=1}^{p}\sum_{i<j}\sum_{k=1}^{p} w_i w_j(w_k+\rho\dot{w}_k)\overline{\mathbf{x}}(t)^{\mathrm{T}}(\mathbf{Q}_{ijk}+\mathbf{Q}_{jik})\overline{\mathbf{x}}(t)\\&-\frac{1}{\rho}\sum_{i=1}^{p} w_i^2\overline{\mathbf{x}}^{\mathrm{T}}\overline{\mathbf{Q}}_{ii}\overline{\mathbf{x}}(t)-\frac{1}{\rho}\sum_{j=1}^{p}\sum_{i<j} w_i w_j\overline{\mathbf{x}}^{\mathrm{T}}(\overline{\mathbf{Q}}_{ij}+\overline{\mathbf{Q}}_{ji})\overline{\mathbf{x}}(t)\end{aligned}\tag{10}$$

where $\mathbf{Q}_{ijk}=\mathbf{P}_j\mathbf{A}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{P}_j+\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j+1/\rho(\mathbf{\Lambda}_i+\overline{\mathbf{G}}_k^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_k-\mathbf{P}_k)$, i, j, $k=1,2,\ldots,p$; $\overline{\mathbf{Q}}_{ij}=\mathbf{\Lambda}_i+\overline{\mathbf{G}}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\overline{\mathbf{G}}_j-\mathbf{P}_j$, i, $j=1,2,\ldots,p$. Let $\mathbf{R}_{ijk}+\mathbf{R}_{ijk}^{\mathrm{T}}\geq 0$, j, $k=1,2,\ldots,p$; $i<j$ and $\overline{\mathbf{R}}_{ij}+\overline{\mathbf{R}}_{ij}^{\mathrm{T}}\leq 0$, $j=1,2,\ldots,p$; $i<j$, where $\mathbf{R}_{ijk}=\mathbf{R}_{jik}^{\mathrm{T}}\in\Re^{n\times n}$ and $\mathbf{R}_{ij}=\overline{\mathbf{R}}_{ji}^{\mathrm{T}}\in\Re^{n\times n}$. From (10), we have

$$\begin{aligned}\dot{V}(t)\leq&\sum_{i=1}^{p}\sum_{k=1}^{p} w_i^2(w_k+\rho\dot{w}_k)\overline{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{Q}_{iik}\overline{\mathbf{x}}(t)\\&+\sum_{j=1}^{p}\sum_{i<j}\sum_{k=1}^{p} w_i w_j(w_k+\rho\dot{w}_k)\overline{\mathbf{x}}(t)^{\mathrm{T}}\\&\times(\mathbf{Q}_{ijk}+\mathbf{Q}_{jik}+\mathbf{R}_{ijk}+\mathbf{R}_{ijk})\overline{\mathbf{x}}(t)-\frac{1}{\rho}\sum_{i=1}^{p} w_i^2\overline{\mathbf{x}}^{\mathrm{T}}\overline{\mathbf{Q}}_{ii}\overline{\mathbf{x}}(t)\\&-\frac{1}{\rho}\sum_{j=1}^{p}\sum_{i<j} w_i w_j\overline{\mathbf{x}}^{\mathrm{T}}\left(\overline{\mathbf{Q}}_{ij}+\overline{\mathbf{Q}}_{ji}+\overline{\mathbf{R}}_{ij}+\overline{\mathbf{R}}_{ij}^{\mathrm{T}}\right)\overline{\mathbf{x}}(t)\\=&\sum_{k=1}^{p}(w_k+\rho\dot{w}_k)\begin{bmatrix}w_1\overline{\mathbf{x}}(t)\\w_p\overline{\mathbf{x}}(t)\\\vdots\\w_p\overline{\mathbf{x}}(t)\end{bmatrix}^{\mathrm{T}}\mathbf{S}_k\begin{bmatrix}w_1\overline{\mathbf{x}}(t)\\w_p\overline{\mathbf{x}}(t)\\\vdots\\w_p\overline{\mathbf{x}}(t)\end{bmatrix}\\&-\frac{1}{\rho}\begin{bmatrix}w_1\overline{\mathbf{x}}(t)\\w_p\overline{\mathbf{x}}(t)\\\vdots\\w_p\overline{\mathbf{x}}(t)\end{bmatrix}^{\mathrm{T}}\overline{\mathbf{S}}\begin{bmatrix}w_1\overline{\mathbf{x}}(t)\\w_p\overline{\mathbf{x}}(t)\\\vdots\\w_p\overline{\mathbf{x}}(t)\end{bmatrix}\end{aligned}\tag{11}$$

where

$$\mathbf{S}_k=\begin{bmatrix}\mathbf{Q}_{11k}&\mathbf{S}_{12k}&\cdots&\mathbf{S}_{1pk}\\\mathbf{S}_{21k}&\mathbf{Q}_{22k}&\cdots&\mathbf{S}_{2pk}\\\vdots&\vdots&\vdots&\vdots\\\mathbf{S}_{p1k}&\mathbf{S}_{p2k}&\cdots&\mathbf{Q}_{ppk}\end{bmatrix},\qquad k=1,2,\ldots,p$$

$$\mathbf{S}_{ijk}=\frac{\mathbf{Q}_{ijk}+\mathbf{Q}_{jik}}{2}+\mathbf{R}_{ijk},\quad j,k=1,2,\ldots,p;\qquad i<j$$

$$\overline{\mathbf{S}}=\begin{bmatrix}\overline{\mathbf{Q}}_{11}&\overline{\mathbf{S}}_{12}&\cdots&\overline{\mathbf{S}}_{1p}\\\overline{\mathbf{S}}_{21}&\mathbf{Q}_{22}&\cdots&\overline{\mathbf{S}}_{2p}\\\vdots&\vdots&\vdots&\vdots\\\overline{\mathbf{S}}_{p1}&\overline{\mathbf{S}}_{p2}&\cdots&\mathbf{Q}_{pp}\end{bmatrix}$$

$$\overline{\mathbf{S}}_{ij}=\frac{\overline{\mathbf{Q}}_{ij}+\overline{\mathbf{Q}}_{ji}}{2}+\overline{\mathbf{R}}_{ij},\qquad j=1,2,\ldots,p;\quad i<j.$$

Let the value of ρ be designed such that $w_k+\rho\dot{w}_k>0$ for all k and $\mathbf{x}(t)$, it can be seen from (10) that the time derivatives $\dot{w}_k$ are dealt with by these conditions. The time derivatives are turned to additive terms rather than going for the weighted-sum representation as in the study in [13], which results in increasing the number of stability conditions drastically.

It can be seen from (11) that if $\mathbf{S}_k<0$, $k=1,2,\ldots,p$, and $\overline{\mathbf{S}}>0$, we obtain $\dot{V}(t)\leq 0$ (equality holds when $\overline{\mathbf{x}}(t)=\mathbf{x}(t)=\mathbf{0}$), which implies the asymptotic stability of the fuzzy-mode-based control system of (6). This proves the LMI-based stability conditions in Theorem 1.

Remark 3: It can be seen from the stability analysis above that the analysis is valid when the time derivatives of the membership functions exists; otherwise, the PDLF-based analysis cannot be applied. This is the drawback of the PDLF-based-analysis approach.

D. Performance Consideration of Non-PDC Fuzzy-Model-Based Control Systems

In this section, LMI-based performance conditions are derived to guarantee the system performance of the fuzzy-model-based control system. The system performance is quantitatively measured by the following performance index, which is commonly used in optimal control [17]:

$$J=\int_0^{\infty}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{u}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{J}_1&\mathbf{J}_2\\\mathbf{J}_2^{\mathrm{T}}&\mathbf{J}_3\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{u}(t)\end{bmatrix}dt\tag{12}$$

where $\mathbf{J}_1=\mathbf{J}_1^{\mathrm{T}}\in\Re^{n\times n}>0$, $\mathbf{J}_2\in\Re^{n\times m}$, $\mathbf{J}_3=\mathbf{J}_3^{\mathrm{T}}\in\Re^{m\times m}>0$, and $\begin{bmatrix}\mathbf{J}_1&\mathbf{J}_2\\\mathbf{J}_2^{\mathrm{T}}&\mathbf{J}_3\end{bmatrix}=\begin{bmatrix}\mathbf{K}_1&\mathbf{K}_2\\\mathbf{K}_2^{\mathrm{T}}&\mathbf{K}_3\end{bmatrix}^{-1}\in\Re^{(n+m)\times(n+m)}>0$. From (5) and (12), we have

$$\begin{aligned}J=&\int_0^{\infty}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\sum_{i=1}^{p} w_i\mathbf{\Gamma}^{-1}\mathbf{G}_i^{\mathrm{T}}+\sum_{i=1}^{p}\dot{w}_i\mathbf{\Gamma}^{-1}\overline{\mathbf{G}}_i^{\mathrm{T}}\end{bmatrix}\\&\times\begin{bmatrix}\mathbf{J}_1&\mathbf{J}_2\\\mathbf{J}_2^{\mathrm{T}}&\mathbf{J}_3\end{bmatrix}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\sum_{j=1}^{p} w_j\mathbf{G}_j\mathbf{\Gamma}^{-1}+\sum_{j=1}^{p}\dot{w}_j\overline{\mathbf{G}}_j\mathbf{\Gamma}^{-1}\end{bmatrix}\\&\times\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t)\end{bmatrix}dt.\end{aligned}\tag{13}$$

Let

$$J<\eta\int_0^{\infty}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{\Gamma}^{-1}&\mathbf{0}\\\mathbf{0}&\mathbf{\Gamma}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t)\end{bmatrix}dt\tag{14}$$

where η is a nonzero positive scalar. The objective of (14) is to attenuate the scalar performance index J to a prescribed level governed by the value of η. From (13) and (14), we have

$$\int_0^\infty \begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t)\end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \sum_{i=1}^{p} w_i \mathbf{\Gamma}^{-1^{\mathrm{T}}} \mathbf{G}_i^{\mathrm{T}} + \sum_{i=1}^{p} \dot{w}_i \mathbf{\Gamma}^{-1^{\mathrm{T}}} \overline{\mathbf{G}}_i^{\mathrm{T}}\end{bmatrix} \begin{bmatrix}\mathbf{J}_1 & \mathbf{J}_2\\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3\end{bmatrix} \times \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \sum_{j=1}^{p} w_j \mathbf{G}_j \mathbf{\Gamma}^{-1} + \sum_{j=1}^{p} \dot{w}_j \overline{\mathbf{G}}_j \mathbf{\Gamma}^{-1}\end{bmatrix} - \eta \begin{bmatrix}\mathbf{\Gamma}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-1}\end{bmatrix} \right) \begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t)\end{bmatrix} dt < 0. \tag{15}$$

From (15), we have

$$\int_0^\infty \begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t)\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\mathbf{\Gamma}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-1}\end{bmatrix} \mathbf{W} \begin{bmatrix}\mathbf{\Gamma}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-1}\end{bmatrix} \begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t)\end{bmatrix} dt < 0 \tag{16}$$

where

$$\mathbf{W} = \begin{bmatrix}\mathbf{\Gamma} & \mathbf{0}\\ \mathbf{0} & \sum_{i=1}^{p} w_i \mathbf{G}_i^{\mathrm{T}} + \sum_{i=1}^{p} \dot{w}_i \overline{\mathbf{G}}_i^{\mathrm{T}}\end{bmatrix} \begin{bmatrix}\mathbf{J}_1 & \mathbf{J}_2\\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3\end{bmatrix} \times \begin{bmatrix}\mathbf{\Gamma} & \mathbf{0}\\ \mathbf{0} & \sum_{j=1}^{p} w_j \mathbf{G}_j + \sum_{j=1}^{p} \dot{w}_j \overline{\mathbf{G}}_j\end{bmatrix} - \eta \begin{bmatrix}\mathbf{\Gamma} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}\end{bmatrix}. \tag{17}$$

It can be seen that the inequality of (16) holds when $\mathbf{W} < 0$. By using the property of $\sum_{i=1}^{p} \dot{w}_i = 0$, we have $\sum_{i=1}^{p} \dot{w}_i \mathbf{F} = \mathbf{0}$, where $\mathbf{F} \in \Re^{(3n+m)\times(3n+m)}$ is an arbitrary matrix that is used to facilitate the analysis. From (17), by Schur complement, $\mathbf{W} < 0$ is equivalent to $\overline{\mathbf{W}}$ and $\mathbf{T}_{ij}$, shown at the bottom of the page, and

$$\overline{\mathbf{T}}_i = \begin{bmatrix} \frac{\eta}{\sigma}\mathbf{P}_i & \mathbf{0} & -\frac{1}{\sigma}\mathbf{P}_i & \mathbf{0}\\ \mathbf{0} & \frac{\eta}{\sigma}\mathbf{P}_i & \mathbf{0} & \frac{1}{\rho}\overline{\mathbf{G}}_i^{\mathrm{T}}\\ -\frac{1}{\sigma}\mathbf{P}_i & \mathbf{0} & \frac{1}{\sigma}\mathbf{K}_1 & \frac{1}{\sigma}\mathbf{K}_2\\ \mathbf{0} & \frac{1}{\rho}\overline{\mathbf{G}}_i & \frac{1}{\sigma}\mathbf{K}_2^{\mathrm{T}} & \frac{1}{\sigma}\mathbf{K}_3 \end{bmatrix}, \qquad \sigma > 1.$$

From (18) and the fact that $w_i + \rho\dot{w}_i > 0$ for all i and $\mathbf{x}(t)$, it can be seen that $\overline{\mathbf{W}} < 0$ if $\mathbf{T}_{ij} + \mathbf{F} < 0, i, j = 1, 2, \ldots, p$ and $\overline{\mathbf{T}}_i + \mathbf{F} > 0, i = 1, 2, \ldots, p$, which are the performance conditions. It can be seen from above equation that the free matrix $\mathbf{F}$ and scalar σ are introduced to ease the satisfaction of the performance conditions. This proves the LMI-based performance conditions in Theorem 1.

IV. Simulation Examples

Two simulation examples are given in this section to illustrate the effectiveness of the proposed non-PDC approach. The first numerical example is given to show that the proposed non-PDC nonlinear controller offers stronger stabilization ability than those of the traditional ones. The second simulation example is on the stabilization of an inverted pendulum on a cart. The proposed non-PDC nonlinear controller is designed based on the LMI-based stability and performance conditions. It will be shown that the proposed non-PDC nonlinear controller performs better than the traditional PDC ones.

A. Numerical Example

A numerical example is given to demonstrate the effectiveness of the LMI-based stability and performance conditions. Consider a fuzzy model with the following rules [13]:

Rule i : IF $x_1(t)$ is M_1^i

$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t), \qquad i = 1, 2 \tag{19}$$

where $\mathbf{A}_1 = \begin{bmatrix}-5 & -4\\ 10 & -2\end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix}-2 & -4\\ -a & -2\end{bmatrix}$, $\mathbf{B}_1 = \begin{bmatrix}0\\ 10\end{bmatrix}$, $\mathbf{B}_2 = \begin{bmatrix}0\\ b\end{bmatrix}$, $-100 \le a \le 20$, and $1 \le b \le 51$. The membership functions are defined as $w_1(x_1(t)) = \mu_{M_1^1}(x_1(t)) = 1 + \sin x_1(t)/2$ and

$$\overline{\mathbf{W}} = \begin{bmatrix} -\eta\sum_{i=1}^{p} w_i\mathbf{P}_i & \mathbf{0} & \sum_{i=1}^{p} w_i\mathbf{P}_i & \mathbf{0}\\ \mathbf{0} & -\eta\sum_{i=1}^{p} w_i\mathbf{P}_i & \mathbf{0} & \sum_{i=1}^{p} w_i\mathbf{G}_i^{\mathrm{T}} + \sum_{i=1}^{p} \frac{1}{\rho}(w_i + \rho\dot{w}_i - w_i)\overline{\mathbf{G}}_i^{\mathrm{T}}\\ \sum_{i=1}^{p} w_i\mathbf{P}_i & \mathbf{0} & -\mathbf{K}_1 & -\mathbf{K}_2\\ \mathbf{0} & \sum_{i=1}^{p} w_i\mathbf{G}_i + \sum_{i=1}^{p} \frac{1}{\rho}(w_i + \rho\dot{w}_i - w_i)\overline{\mathbf{G}}_i & -\mathbf{K}_2^{\mathrm{T}} & -\mathbf{K}_3 \end{bmatrix} + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(w_j + \rho\dot{w}_j - w_j)\mathbf{F}$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(w_j + \rho\dot{w}_j)(\mathbf{T}_{ij} + \mathbf{F}) - \sum_{i=1}^{p} w_i(\overline{\mathbf{T}}_i + \mathbf{F}) < 0 \tag{18}$$

$$\mathbf{T}_{ij} = \begin{bmatrix} -\eta\left(1-\frac{1}{\sigma}\right)\mathbf{P}_i & \mathbf{0} & \left(1-\frac{1}{\sigma}\right)\mathbf{P}_i & \mathbf{0}\\ \mathbf{0} & -\eta\left(1-\frac{1}{\sigma}\right)\mathbf{P}_i & \mathbf{0} & \mathbf{G}_i^{\mathrm{T}} + \frac{1}{\rho}\overline{\mathbf{G}}_j^{\mathrm{T}}\\ \left(1-\frac{1}{\sigma}\right)\mathbf{P}_i & \mathbf{0} & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_1 & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_2\\ \mathbf{0} & \mathbf{G}_i + \frac{1}{\rho}\overline{\mathbf{G}}_j & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_2^{\mathrm{T}} & -\left(1-\frac{1}{\sigma}\right)\mathbf{K}_3 \end{bmatrix}$$

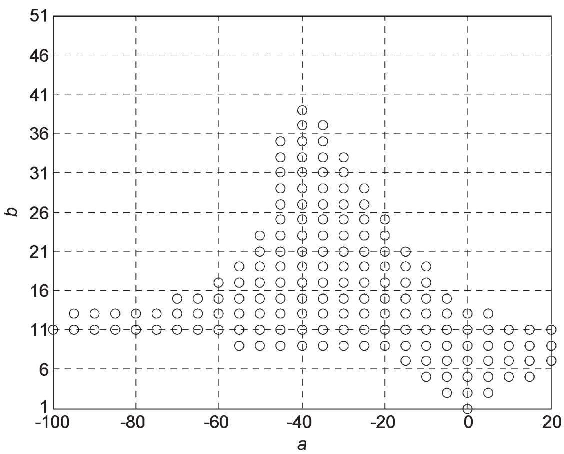

Fig. 1. Stability region based on the published stability conditions in the study in [9]–[12].

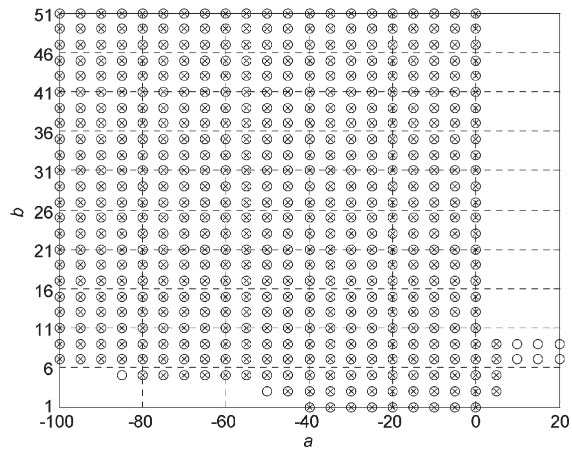

Fig. 2. Stability regions based on the proposed stability conditions in Theorem 1 for $\rho = 0.007$. Cross ($\times$): with $\overline{\mathbf{G}}_i = \mathbf{0}$. Circle ($\circ$): with $\overline{\mathbf{G}}_i$ determined by Theorem 1.

$w_2(x_1(t)) = \mu_{M_1^2}(x_1(t)) = (1 - \sin x_1(t)/2)$. The system dynamics are described as

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} w_i(x_1(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)) \tag{20}$$

where $\mathbf{x}(t) = [x_1(t) \quad x_2(t)]^{\mathrm{T}}$. It is assumed that $x_1(t) \in [-(4\pi/9) \quad (4\pi/9)]$ and $x_2(t) \in [-(4\pi/9) \quad (4\pi/9)]$. A nonlinear controller in the following form is used:

$$u(t) = \sum_{j=1}^{2} w_j(x_1(t))\mathbf{G}_j \left(\sum_{k=1}^{2} w_k(x_1(t))\mathbf{P}_k\right)^{-1} \mathbf{x}(t) + \sum_{j=1}^{2} \dot{w}_j(x_1(t))\overline{\mathbf{G}}_j \left(\sum_{k=1}^{2} w_k(x_1(t))\mathbf{P}_k\right)^{-1} \mathbf{x}(t). \tag{21}$$

The time derivatives of the membership functions are obtained as $\dot{w}_1(x_1(t)) = (\cos(x_1(t))\dot{x}_1(t)/2)$ and $\dot{w}_2(x_1(t)) = -(\cos(x_1(t))\dot{x}_1(t)/2)$. From (20), we have

$$\begin{aligned}
\dot{x}_1(t) &= [1 \quad 0] \times \sum_{i=1}^{2} w_i(x_1(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)) \\
&= \sum_{i=1}^{2} w_i(x_1(t))[1 \quad 0]\mathbf{A}_i\mathbf{x}(t) \\
&= \frac{1}{2}((1 + \sin(x_1(t)))[-5 \quad -4] \\
&\qquad + (1 - \sin(x_1(t)))[-2 \quad -4]) \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\
&= \frac{1}{2}(-7x_1(t) - 8x_2(t) - 3x_1(t)\sin(x_1(t))).
\end{aligned}$$

Hence, we have $\dot{w}_1(\mathbf{x}(t)) = -(\cos(x_1(t))/4)(7x_1(t) + 8x_2(t) + 3x_1(t)\sin(x_1(t)))$. By choosing $\rho = 0.007$, it can be shown that the conditions of $w_i(x_1(t)) + \rho\dot{w}_i(x_1(t)) > 0$ for all i and $x_1(t)$ (in the operating domain) are satisfied. In this example, the values of the feedback gains $\mathbf{G}_i$ are designed such that the eigenvalues of $\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i$ are located at -1 and -3, respectively. As the main concern in this example is on the system stability, the performance conditions in Theorem 1 are not used in this example. Fig. 1 shows the stability region given by the stability conditions in the study in [9]–[12]. It should be noted that the stability conditions in the study in [9]–[12] are more relaxed compared with those in [3]–[8]. Fig. 2 shows the stability regions for $\rho = 0.007$ with $\overline{\mathbf{G}}_i = \mathbf{0}$ and $\overline{\mathbf{G}}_i$ determined by Theorem 1. It can be seen from Figs. 1 and 2 that the proposed stability conditions offer larger stability regions than those of the published ones. Moreover, the feedback gains of $\overline{\mathbf{G}}_i$ enhance the stabilization ability of the proposed non-PDC nonlinear controller, which is reflected by a larger stability region.

B. Inverted Pendulum on a Cart

A simulation example on stabilizing an inverted pendulum on a cart [18] is given to demonstrate the effectiveness of the LMI-based stability and performance conditions in Theorem 1.

Step 1) The dynamic equations of the inverted pendulum on a cart [18] are given by (22)–(25), shown at the bottom of the next page, where $x_1(t)$ and $x_2(t)$ denote the angular displacement (in radians) and the angular velocity (in radians per second) of the pendulum from the vertical, respectively, $x_3(t)$ and $x_4(t)$ denote the displacement (in meters) and the velocity (in meters per second) of the cart, respectively, $g = 9.8\ \mathrm{m/s^2}$ is the acceleration due to gravity, $m = 0.22$ kg is the mass of the pendulum, $M = 1.3282$ kg is the mass of the cart, $l = 0.304$ m is the length from the center of mass of the pendulum to the shaft axis, $J_o = ml^2/3\ \mathrm{kgm^2}$ is the moment of inertia of the pendulum around the center of mass, $F_0 = 22.915$ N/m/s and $F_1 = 0.007056$ N/rad/s are the friction factors of the cart and the pendulum, respectively, and $u(t)$ is the force (N) applied to the cart. The nonlinear plant can be represented by a fuzzy model with two fuzzy rules [18]. The ith rule is given by

$$\begin{aligned}
&\text{Rule } i: \text{IF } x_1(t) \text{ is } M_1^i \\
&\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t) \qquad \text{for } i = 1, 2.
\end{aligned} \tag{26}$$

The system dynamics are described by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} w_i(x_1(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)) \tag{27}$$

where we have the expression shown at the bottom of the next page. The membership functions [18] are defined as $w_1(x_1(t)) = \mu_{M_1^1}(x_1(t)) = (1 - (1/1 + e^{-7(x_1(t) - \pi/6)}))$

TABLE I
$\mathbf{G}_i$ AND $\mathbf{P}_i$ AND THE MINIMUM AND MAXIMUM MAGNITUDES OF CONTROL SIGNALS

Fuzzy Controller	$\mathbf{G}_i$ and $\mathbf{P}_i$, $i = 1, 2$	Min. $u(t)$ (N)	Max. $u(t)$ (N)
Traditional fuzzy controller with PDC design approach	$\mathbf{F}_1 = [901.0367 \quad 60.4927 \quad 9.5440 \quad 65.8864]$ $\mathbf{F}_2 = [1216.4236 \quad 72.2941 \quad 11.1648 \quad 72.7950]$	–32.8411	1265.5923
Proposed non-PDC nonlinear controller	$\mathbf{G}_1 = [3.6925\times10^2 \quad 5.7065\times10^3 \quad -2.0022\times10^4 \quad 2.8115\times10^4]$ $\mathbf{G}_2 = [5.0773\times10^2 \quad 5.3774\times10^3 \quad -2.0006\times10^4 \quad 2.8219\times10^4]$ $\mathbf{P}_1 = \begin{bmatrix} 1.3016\times10^1 & -1.1300\times10^2 & -1.0710\times10^1 & -2.3747\times10^1 \\ -1.1300\times10^2 & 1.8965\times10^3 & 8.8627\times10^1 & -7.8427\times10^2 \\ -1.0710\times10^1 & 8.8627\times10^1 & 2.0498\times10^3 & -8.3085\times10^2 \\ -2.3747\times10^1 & -7.8427\times10^2 & -8.3085\times10^2 & 1.7090\times10^3 \end{bmatrix}$ $\mathbf{P}_2 = \begin{bmatrix} 1.2484\times10^1 & -1.1308\times10^2 & -1.0498\times10^1 & -2.3654\times10^1 \\ -1.1308\times10^2 & 1.8967\times10^3 & 8.9000\times10^1 & 2.0502\times10^3 \\ -1.0498\times10^1 & 8.9000\times10^1 & 2.0502\times10^3 & -8.3009\times10^2 \\ -2.3654\times10^1 & -7.8327\times10^2 & -8.3009\times10^2 & 1.7102\times10^3 \end{bmatrix}$	–41.8354	2713.4397

$(1/1 + e^{-7(x_1(t)+\pi/6)})$ and $w_2(x_1(t)) = \mu_{M_1^2}(x_1(t)) = 1 - \mu_{M_1^1}(x_1(t))$. The operating domain of the nonlinear plant are assumed to be $x_1(t) \in [-\pi/3 \quad \pi/3]$ and $\dot{x}_1(t) = x_2(t) \in [-20 \quad 20]$. The time derivatives of the membership functions are obtained as $\dot{w}_1(\mathbf{x}(t)) = (7x_2(t)/h_1(x_1(t)))((h_3(x_1(t))/h_1(x_1(t))) - (h_3(x_1(t)/h_1(x_1(t)) \times h_2(x_1(t))) - (h_4(x_1(t))/h_2(x_1(t))^2))$ and $\dot{w}_2(\mathbf{x}(t)) = -\dot{w}_1(\mathbf{x}(t))$, where $h_1(x_1(t)) = 1 + e^{-7(x_1(t)+\pi/6)}$, $h_2(x_1(t)) = 1 + e^{-7(x_1(t)-\pi/6)}$, $h_3(x_1(t)) = e^{-7(x_1(t)+\pi/6)}$, and $h_4(x_1(t)) = e^{-7(x_1(t)-\pi/6)}$.

Step 2) The following non-PDC nonlinear controller is proposed to control the nonlinear plant to achieve $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$:

$$u(t) = \sum_{j=1}^{2} w_j(x_1(t))\, \mathbf{G}_j \left(\sum_{k=1}^{2} w_k(x_1(t))\, \mathbf{P}_k \right)^{-1} \mathbf{x}(t). \quad (28)$$

Step 3) The LMI-based stability conditions in Theorem 1 (without the LMI-based performance conditions) are employed to help design a stable non-PDC controller for the inverted

$$\dot{x}_1(t) = x_2(t) \quad (22)$$

$$\dot{x}_2(t) = \frac{(-F_1(M+m)x_2(t) - m^2l^2x_2(t)^2 \sin x_1(t)\cos x_1(t) + F_0 mlx_4(t)\cos x_1(t) + (M+m)mgl\sin x_1(t) - ml\cos x_1(t)u(t))}{(M+m)(J_o+ml^2) - m^2l^2(\cos x_1(t))^2} \quad (23)$$

$$\dot{x}_3(t) = x_4(t) \quad (24)$$

$$\dot{x}_4(t) = \frac{(F_1 mlx_2(t)\cos x_1(t) + (J+ml^2)mlx_2(t)^2 \sin x_1(t) - F_0(J_o+ml^2)x_4(t) - m^2gl^2 \sin x_1(t)\cos x_1(t) + (J_o+ml^2)u(t))}{(M+m)(J_o+ml^2) - m^2l^2(\cos x_1(t))^2} \quad (25)$$

$$\mathbf{x}(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)]^{\mathrm{T}}$$

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (M+m)mgl/a_1 & -F_1(M+m)/a_1 & 0 & F_0 ml/a_1 \\ 0 & 0 & 1 & 0 \\ -m^2gl^2/a_1 & F_1 ml/a_1 & 0 & -F_0(J_o+ml^2)/a_1 \end{bmatrix}$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ -ml/a_1 \\ 0 \\ (J_o+ml^2)/a_1 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{3\sqrt{3}}{2\pi}(M+m)mgl/a_2 & -F_1(M+m)/a_2 & 0 & F_0 ml\cos(\pi/3)/a_2 \\ 0 & 0 & 1 & 0 \\ -\frac{3\sqrt{3}}{2\pi}m^2gl^2\cos(\pi/3)/a_2 & F_1 ml\cos(\pi/3)/a_2 & 0 & -F_0(J_o+ml^2)/a_1 \end{bmatrix}$$

$$\mathbf{B}_2 = \begin{bmatrix} 0 \\ -ml\cos(\pi/3)/a_2 \\ 0 \\ (J_o+ml^2)/a_2 \end{bmatrix}$$

$$a_1 = (M+m)(J+ml^2) - m^2l^2$$

$$a_2 = (M+m)(J_o+ml^2) - m^2l^2\cos(\pi/3)^2$$

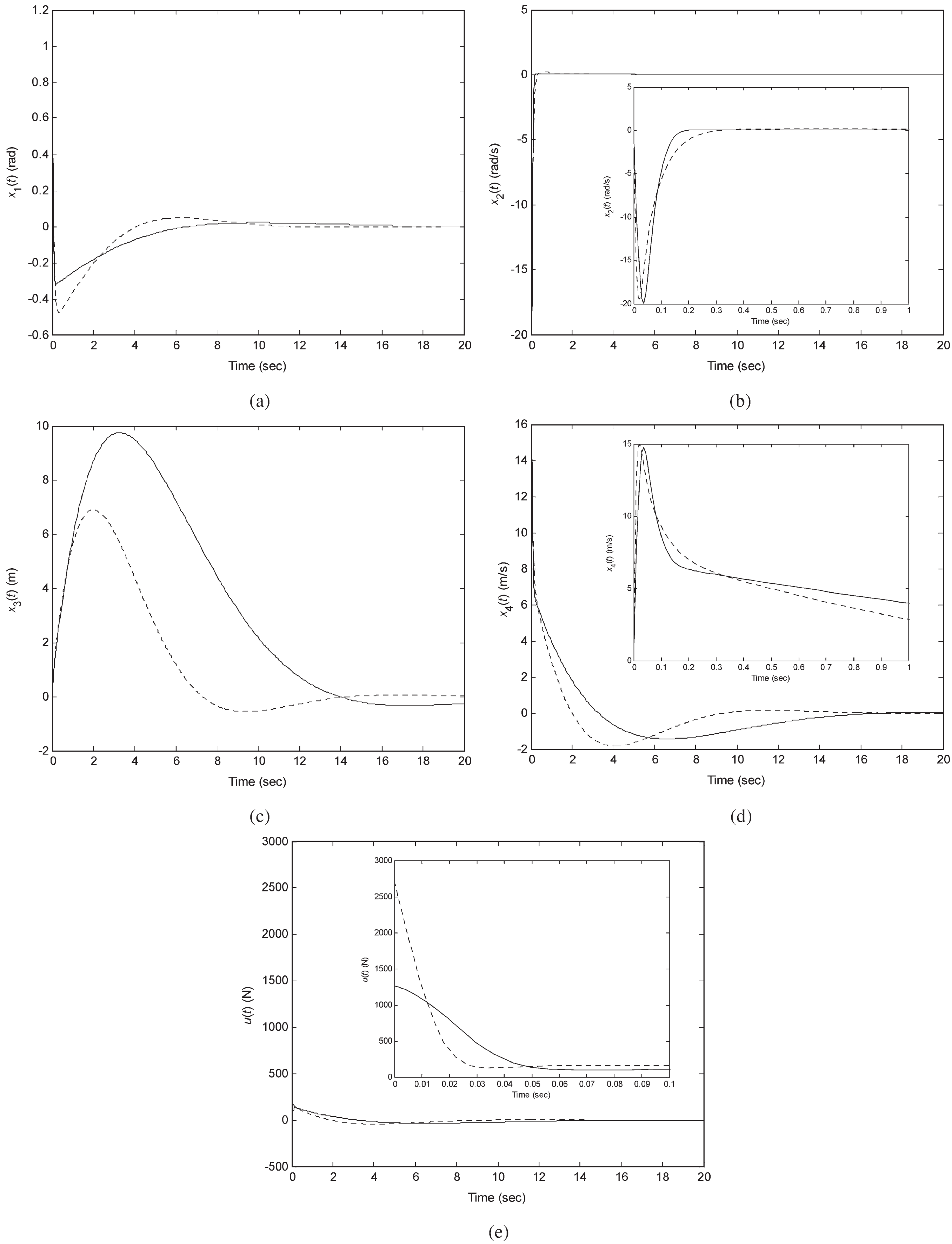

Fig. 3. System state responses and control signals for the nonlinear plant with (dotted lines) non-PDC nonlinear controller and (solid lines) traditional fuzzy controller. (a) $x_1(t)$. (b) $x_2(t)$. (c) $x_3(t)$. (d) $x_4(t)$. (e) $u(t)$.

TABLE II
PERFORMANCE INDEX PARAMETERS, $\mathbf{G}_i$ AND $\mathbf{P}_i$, AND THE MINIMUM AND MAXIMUM MAGNITUDES OF CONTROL SIGNALS

Performance Index Parameters	$\mathbf{G}_i$ and $\mathbf{P}_i$, $i = 1, 2$	Min. $u(t)$ (N)	Max. $u(t)$ (N)
$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $J_3 = 1$	$\mathbf{G}_1 = [-7.7229\times10^{-4} \ \ 1.2873\times10^{-2} \ \ -2.1307\times10^{-2} \ \ -1.4536\times10^{-3}]$ $\mathbf{G}_2 = [-2.0009\times10^{-4} \ \ 5.0092\times10^{-3} \ \ -2.1654\times10^{-2} \ \ -5.7682\times10^{-4}]$ $\mathbf{P}_1 = \begin{bmatrix} 1.6651\times10^{-4} & -3.1571\times10^{-3} & -8.4247\times10^{-5} & 1.1973\times10^{-3} \\ -3.1571\times10^{-3} & 6.0085\times10^{-2} & 1.6064\times10^{-3} & -2.2937\times10^{-2} \\ -8.4247\times10^{-5} & 1.6064\times10^{-3} & 7.7416\times10^{-3} & -1.5692\times10^{-4} \\ 1.1973\times10^{-3} & -2.2937\times10^{-2} & -1.5692\times10^{-4} & 8.9253\times10^{-3} \end{bmatrix}$ $\mathbf{P}_2 = \begin{bmatrix} 1.5952\times10^{-4} & -3.0377\times10^{-3} & -8.1160\times10^{-5} & 1.1523\times10^{-3} \\ -3.0377\times10^{-3} & 5.8051\times10^{-2} & 1.5526\times10^{-3} & -2.2171\times10^{-2} \\ -8.1160\times10^{-5} & 1.5526\times10^{-3} & 7.7416\times10^{-3} & -1.5487\times10^{-4} \\ 1.1523\times10^{-3} & -2.2171\times10^{-2} & -1.5487\times10^{-4} & 8.6415\times10^{-3} \end{bmatrix}$	−3.2246	645.1568
$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $J_3 = 1$	$\mathbf{G}_1 = [-7.7594\times10^{-4} \ \ 1.2952\times10^{-2} \ \ -6.6270\times10^{-3} \ \ -1.4953\times10^{-3}]$ $\mathbf{G}_2 = [-1.9675\times10^{-4} \ \ 5.0430\times10^{-3} \ \ -7.0085\times10^{-3} \ \ -6.0673\times10^{-4}]$ $\mathbf{P}_1 = \begin{bmatrix} 1.6450\times10^{-4} & -3.1187\times10^{-3} & -8.0898\times10^{-5} & 1.1816\times10^{-3} \\ -3.1187\times10^{-3} & 5.9356\times10^{-2} & 1.5409\times10^{-3} & -2.2642\times10^{-2} \\ -8.0898\times10^{-5} & 1.5409\times10^{-3} & 7.7418\times10^{-3} & -9.0241\times10^{-4} \\ 1.1816\times10^{-3} & -2.2642\times10^{-2} & -9.0241\times10^{-4} & 8.8084\times10^{-3} \end{bmatrix}$ $\mathbf{P}_2 = \begin{bmatrix} 1.5725\times10^{-4} & -2.9945\times10^{-3} & -7.7714\times10^{-5} & 1.1349\times10^{-3} \\ -2.9945\times10^{-3} & 5.7238\times10^{-2} & 1.4853\times10^{-3} & -2.1845\times10^{-2} \\ -7.7714\times10^{-5} & 1.4853\times10^{-3} & 7.7404\times10^{-3} & -8.8127\times10^{-4} \\ 1.1349\times10^{-3} & -2.1845\times10^{-2} & -8.8127\times10^{-4} & 8.5106\times10^{-3} \end{bmatrix}$	−8.5199	627.7489
$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $J_3 = 0.01$	$\mathbf{G}_1 = [-1.1025\times10^{-2} \ \ 6.4730\times10^{-2} \ \ -2.5556\times10^{-1} \ \ 2.9586\times10^{-1}]$ $\mathbf{G}_2 = [7.5620\times10^{-3} \ \ -5.9964\times10^{-2} \ \ -2.6784\times10^{-1} \ \ 1.7489\times10^{-1}]$ $\mathbf{P}_1 = \begin{bmatrix} 2.5751\times10^{-4} & -3.8371\times10^{-3} & 2.3636\times10^{-5} & 7.2333\times10^{-4} \\ -3.8371\times10^{-3} & 7.0025\times10^{-2} & -4.3935\times10^{-4} & -2.4385\times10^{-2} \\ 2.3636\times10^{-5} & -4.3935\times10^{-4} & 7.8882\times10^{-2} & -1.1692\times10^{-2} \\ 7.2333\times10^{-4} & -2.4385\times10^{-2} & -1.1692\times10^{-2} & 2.3599\times10^{-2} \end{bmatrix}$ $\mathbf{P}_2 = \begin{bmatrix} 2.5482\times10^{-4} & -3.8455\times10^{-3} & 2.0876\times10^{-5} & 7.2434\times10^{-3} \\ -3.8455\times10^{-3} & 6.9930\times10^{-2} & -4.9431\times10^{-4} & -2.4243\times10^{-2} \\ 2.0876\times10^{-5} & -4.9431\times10^{-4} & 7.8882\times10^{-2} & -1.1654\times10^{-2} \\ 7.2434\times10^{-3} & -2.4243\times10^{-2} & -1.1654\times10^{-2} & 2.3961\times10^{-2} \end{bmatrix}$	−22.4385	610.3116
$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $J_3 = 0.01$	$\mathbf{G}_1 = [-5.2677\times10^{-3} \ \ 8.5923\times10^{-2} \ \ -8.2069\times10^{-2} \ \ 1.2244\times10^{-1}]$ $\mathbf{G}_2 = [1.0244\times10^{-2} \ \ -7.5488\times10^{-2} \ \ -9.5951\times10^{-2} \ \ 9.4212\times10^{-2}]$ $\mathbf{P}_1 = \begin{bmatrix} 2.4406\times10^{-4} & -3.8109\times10^{-3} & -7.0835\times10^{-6} & 9.6208\times10^{-4} \\ -3.8109\times10^{-3} & 7.0779\times10^{-2} & -1.8510\times10^{-4} & -2.4876\times10^{-2} \\ -7.0835\times10^{-6} & -1.8510\times10^{-4} & 8.5912\times10^{-3} & -3.6465\times10^{-3} \\ 9.6208\times10^{-4} & -2.4876\times10^{-2} & -3.6465\times10^{-3} & 1.6134\times10^{-2} \end{bmatrix}$ $\mathbf{P}_2 = \begin{bmatrix} 2.4083\times10^{-4} & -3.8315\times10^{-3} & -7.0186\times10^{-6} & 9.6915\times10^{-4} \\ -3.8315\times10^{-3} & 7.0509\times10^{-2} & -2.0716\times10^{-4} & -2.4594\times10^{-2} \\ -7.0186\times10^{-6} & -2.0716\times10^{-4} & 8.5917\times10^{-3} & -3.6329\times10^{-3} \\ 9.6915\times10^{-4} & -2.4594\times10^{-2} & -3.6329\times10^{-3} & 1.6096\times10^{-2} \end{bmatrix}$	−38.4704	891.1842

pendulum on a cart. We choose $\rho = 0.007$ and $\sigma = 1/\rho$, it can be seen that the conditions of $w_i(x_1(t)) + \rho\dot{w}_i(x_1(t)) > 0$ for all i and $x_1(t)$ (in the operating domain) are satisfied. Based on Theorem 1 and with the help of the MATLAB LMI toolbox, the feedback gains of $\mathbf{G}_i$ and $\mathbf{P}_i$ are obtained and listed in Table I. For comparison purpose, the following traditional PDC fuzzy controller [8]–[12] is employed to control the nonlinear plant:

$$u(t) = \sum_{j=1}^{2} w_j\left(x_1(t)\right)\mathbf{F}_j\mathbf{x}(t) \tag{29}$$

where $\mathbf{F}_j \in \Re^{m\times n}$, $j = 1, 2$ are constant feedback gains. The feedback gains of $\mathbf{F}_j$ are determined by the stability conditions in the study in [9] and listed in Table I. Fig. 3 shows the system responses and control signals of nonlinear plant with the non-PDC nonlinear and the traditional PDC fuzzy controller under the initial system state conditions of $\mathbf{x}(0) = [\pi/3 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$. It can be seen that the proposed non-PDC fuzzy controller offer better system state responses in terms of lower overshoot/undershoot magnitude and faster settling time. However, the proposed non-PDC nonlinear controller produces higher magnitudes of the control signal.

For comparison purpose and to illustrate the effectiveness of the LMI-based performance conditions, they are employed to help design non-PDC nonlinear controllers for the nonlinear plant. Based on Theorem 1, with $\eta = 0.1$ and $\sigma = 1/\rho$, the feedback gains of $\mathbf{G}_i$ and

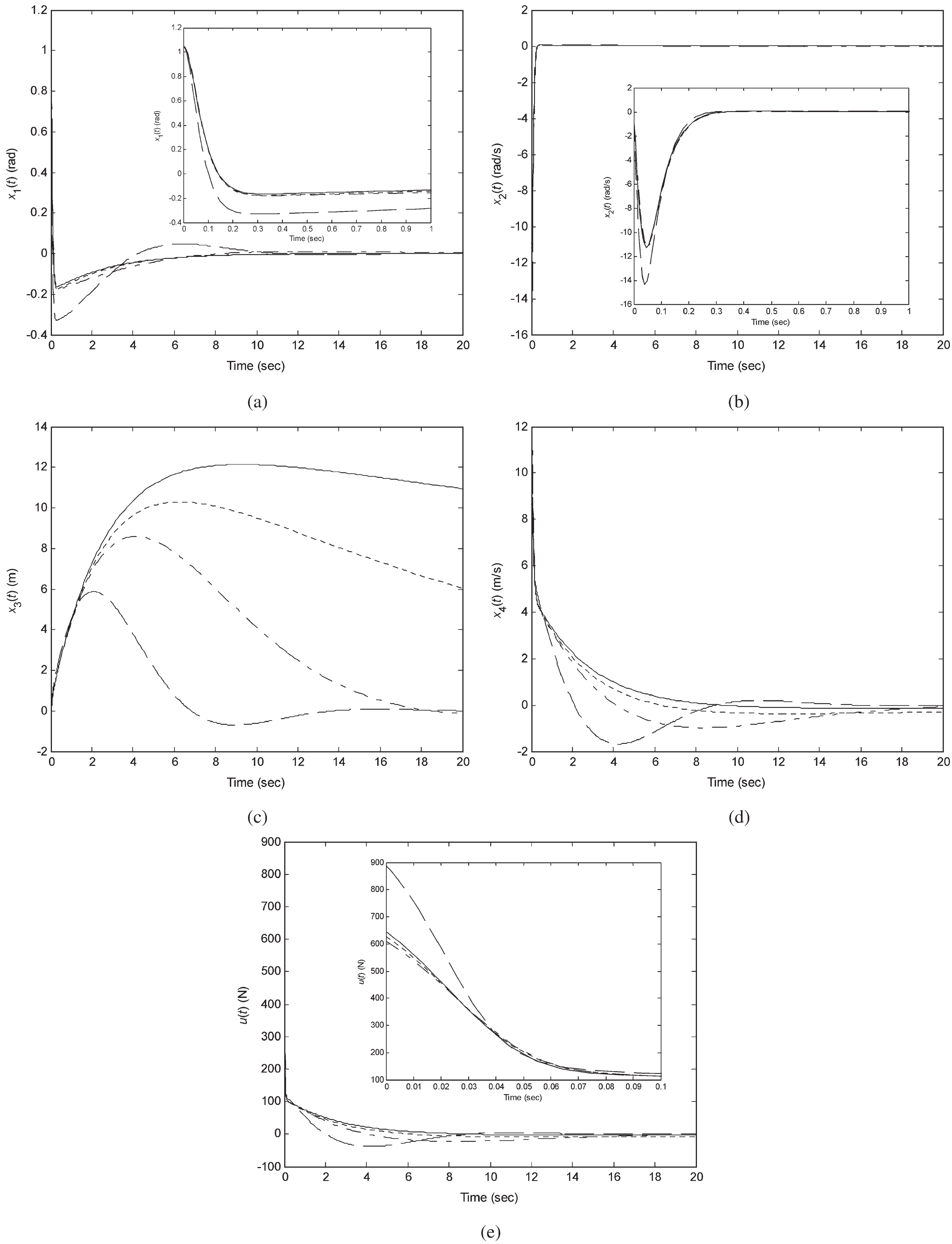

Fig. 4. System state responses and control signals for the nonlinear plant with (solid lines) non-PDC nonlinear controller one, (dotted lines) controller two, (dash–dot) controller three, and (dashed lines) controller four. (a) $x_1(t)$. (b) $x_2(t)$. (c) $x_3(t)$. (d) $x_4(t)$. (e) $u(t)$.

$\mathbf{P}_i, i = 1, 2$ are obtained under different weighting matrices of $\mathbf{J}_1, \mathbf{J}_2$, and $\mathbf{J}_3$. The values of the feedback gains of the non-PDC nonlinear controllers one to four are tabulated in Table II. Referring to Table II, it can be seen that different weighting matrices place different weights on $x_3(t)$ and $u(t)$ to specify the system performance. Fig. 4 shows the system state responses and control signals of the nonlinear plant with the non-PDC nonlinear controllers under different sets of feedback gains. It can be seen that all non-PDC nonlinear controllers are able to stabilize the nonlinear system. Taking the non-PDC nonlinear controller 1 with $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{J}_3 = 1$ as a reference, when we change $\mathbf{J}_1$ to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, which places heavy weight on $x_3(t)$, it can be seen that the transient response of $x_3(t)$ of the nonlinear plant with non-PDC nonlinear controller two is improved. When we change $\mathbf{J}_3$ to 0.01 for non-PDC nonlinear controller three to relax the constraint on the control energy in the performance index, it can be seen that the system responses can be improved. For non-PDC nonlinear controller four, we change $\mathbf{J}_1$ to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ which places heavy weight on $x_3(t)$, it can be seen that the system responses can be further improved at the cost of larger magnitudes of the control signal. In this example, it can be seen that the LMI-based stability and performance conditions provide a systematic and effective tool for the design of stable and well-performed non-PDC nonlinear controller.

V. Conclusion

A nonlinear controller has been proposed to control nonlinear plants represented by fuzzy models based on the PDLF approach. The difficulties given by the PDLF approach are reduced while less conservative stability-analysis results are produced by considering the favorable properties of the membership functions and the proposed nonlinear controller. LMI-based stability conditions have been derived based on the PDLF approach to guarantee the system stability. Based on the commonly used scalar performance index, LMI-based performance conditions have been derived to guarantee the system performance. Simulation examples have been given to illustrate the effectiveness of the proposed approach.

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