Fuzzy Combination of Fuzzy and Switching State-Feedback Controllers for Nonlinear Systems Subject to Parameter Uncertainties

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***Abstract*—This paper presents a fuzzy controller, which involves a fuzzy combination of local fuzzy and global switching state-feedback controllers, for nonlinear systems subject to parameter uncertainties with known bounds. The nonlinear system is represented by a fuzzy combined Takagi–Sugeno–Kang model, which is a fuzzy combination of the global and local fuzzy plant models. By combining the local fuzzy and global switching state-feedback controllers using fuzzy logic techniques, the advantages of both controllers can be retained and the undesirable chattering effect introduced by the global switching state-feedback controller can be eliminated. The steady-state error introduced by the global switching state-feedback controller when a saturation function is used can also be removed. Stability conditions, which are related to the system matrices of the local and global closed-loop systems, are derived to guarantee the closed-loop system stability. An application example will be given to demonstrate the merits of the proposed approach.**

***Index Terms*—Fuzzy combination, fuzzy-model-based control, parameter uncertainties, stability analysis.**

I. Introduction

THE LYAPUNOV approach is the most common approach to investigate the stability of fuzzy control systems based on the Takagi–Sugeno–Kang (TSK) fuzzy plant models [1], [2]. In [3], the fuzzy control system, which is a weighted sum of some linear systems, is guaranteed to be stable if a common solution to some Lyapunov equations exists. Less conservative stability conditions [4]–[8] were derived under the requirement that the rules of the fuzzy controller share the same premises as those of the fuzzy plant model. This requirement implies that the nonlinear system must be known. To deal with parameter uncertainties, a fuzzy plant model subject to parameter uncertainties can be used. Stability conditions for this class of plant models were derived [9], [10].

Switching fuzzy systems and hierarchical fuzzy systems (HFSs) had been investigated. Switching controllers [11]–[13] designed based on fuzzy plant models were also reported. In [12], [13], a switching fuzzy plant model was employed to describe the dynamic behaviors of a nonlinear plant. A switching fuzzy controller was then designed based on this switching model. Owing to the switching elements, an undesirable chattering effect occurs in the control signal. Although the chattering effect can be alleviated by adding the switching function with a saturation function, a steady-state error will be introduced in the system states. In all the above mentioned fuzzy systems, the number of the rules will increase exponentially with the number of input variables. To alleviate this problem, an HFS [14] was proposed. The number of rules of the HFS is a linear function of the input variables. However, when we consider the same degree of model accuracy, we cannot conclude that the number of rules of the HFS must be smaller than that of the TSK fuzzy plant model. Furthermore, there is not a systematic way to represent a nonlinear system as an HFS, and the form of the HFS is not favorable for doing stability analysis. Consequently, the hierarchical fuzzy approach is usually used to construct the controller only [15], [16].

In some published work, fuzzy logic has been combined with traditional sliding-mode controllers to merge their advantages together. In [17], a fuzzy sliding-mode controller used the sliding-surface function as the single input of the fuzzy system, and the number of fuzzy rules can be greatly reduced. In [18], [19], a fuzzy system was employed to estimate the values of the gains of the sliding-mode controller. Adaptive laws were derived to update the rules of the fuzzy systems. As the switching function of the sliding-mode controller is approximated by a continuous function, the chattering effect can be alleviated. In [20], an adaptive fuzzy controller was proposed to generate the control signals by estimating the values of the unknown parameters of the system. Based on these estimated parameter values, tracking control can be achieved by using the sliding-mode control. However, in these approaches, the way to determine the fuzzy rules is still an open question. Furthermore, the approximation error of the fuzzy systems will introduce steady-state errors to the system states or even cause the system to become unstable. In [21], switching elements were still used in the controller for compensating the approximation error of the fuzzy system.

In this paper, a fuzzy combined TSK model, which is a fuzzy combination of the global and local fuzzy plant models, is proposed to represent a nonlinear system subject to parameter uncertainties. The global fuzzy plant model is valid to model the full operating domain of the nonlinear plant while the local fuzzy plant model is only valid to model a small domain of the nonlinear plant near the origin of the state space. A rule base with two fuzzy rules will be employed to combine the local and global fuzzy plant models to form the fuzzy combined TSK

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model. In [11] and [12], the authors have proposed a global switching state-feedback controller which is able to stabilize the nonlinear plant. However, the switching elements will introduce an undesired chattering effect to the system. In order to alleviate the chattering effect, a local fuzzy state-feedback controller which is able to offer smooth control signals to the nonlinear plant is proposed. However, the local fuzzy controller is only capable of stabilizing the nonlinear system operating in a small domain. Based on the fuzzy combined TSK plant model, a fuzzy combined state-feedback controller that integrates the advantages of a global switching control and a local fuzzy control is proposed in this paper. Stability conditions and switching laws will be derived based on the Lyapunov approach. The derived stability conditions are related to the system parameters of either the local or the global closed-loop systems, but not both in a single expression. Hence, the local fuzzy and global switching state-feedback controllers can be designed separately. By properly designing the membership functions of the fuzzy combined TSK model and the controller, the chattering effect can be removed when the system states are inside the valid operating domain of the local fuzzy plant model. In order to alleviate the chattering effect in the transient period, a saturation function is employed to replace the switching function of the global switching state-feedback controller. Thanks to the favorable property of the fuzzy combined controller, the steady-state error introduced by the saturation function can be eliminated.

This paper is organized as follows. In Section II, the fuzzy combined TSK model and the fuzzy combined state-feedback controller will be presented. In Section III, stability analysis will be carried out to investigate the stability of the closed-loop system formed by the fuzzy combined TSK model and the fuzzy combined state-feedback controller. In Section IV, an application example will be given. A conclusion will be drawn in Section V.

II. Fuzzy Combined TSK Model and Fuzzy Combined State-Feedback Controller

The details of the fuzzy combined TSK model and fuzzy combined state-feedback controller will be presented in the following sections.

A. Fuzzy Plant Model

The global fuzzy plant model is used to describe the dynamic behavior of the nonlinear plant in its full operating domain. On the other hand, the local fuzzy plant model is to describe the dynamic behavior of the nonlinear plant in a small operating domain around the origin of the state space.

1) Global Fuzzy Plant Model: Let p be the number of fuzzy rules describing the nonlinear plant, the ith rule is of the following format:

$$\text{Rule } i: \text{IF } f_1(\mathbf{x}(t)) \text{ is } \mathrm{M}_1^i \text{ AND} \ldots \text{AND } f_\Psi(\mathbf{x}(t)) \text{ is } \mathrm{M}_\Psi^i \\ \text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \tag{1}$$

where M_κ^i is a fuzzy term of rule i corresponding to the function $f_\kappa(\mathbf{x}(t))$ containing the parameter uncertainties of the nonlinear plant, $\kappa = 1, 2, \ldots, \Psi$; $i = 1, 2, \ldots, p$; Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices respectively; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The global fuzzy plant model models the dynamic behavior of the nonlinear plant in its full operating domain, e.g. $x_\gamma(t) \in [x_{\gamma_{\min}} \;\; x_{\gamma_{\max}}]$, $\gamma = 1, 2, \ldots, n$. The inferred system is given by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\right) \tag{2}$$

$$\text{where } \sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1,\; w_i(\mathbf{x}(t)) \in [0 \;\; 1] \text{ for all } i. \tag{3}$$

We have that (4), shown at the bottom of the page, is a nonlinear function of $\mathbf{x}(t)$. $\mu_{\mathrm{M}_\kappa^i}(f_\kappa(\mathbf{x}(t)))$ is the membership function corresponding to M_κ^i. If $f_\kappa(\mathbf{x}(t))$ is related to the parameter uncertainties of the nonlinear plant, the value of $\mu_{\mathrm{M}_\kappa^i}(f_\kappa(\mathbf{x}(t)))$ is unknown.

2) Local Fuzzy Plant Model: The small operating domain of the local fuzzy plant model around the origin is characterized by $x_{\gamma_{\min}} \le \tilde{x}_{\gamma_{\min}} \le x_\gamma(x) \le \tilde{x}_{\gamma_{\max}} \le x_{\gamma_{\max}}$, $\gamma = 1, 2, \ldots, n$. The local fuzzy plant model is only valid inside this operating domain. To differentiate the differences, a "$\sim$" at the top of the variables is used to denote the variables of the local fuzzy plant model. The rules of the local fuzzy plant model has the following form:

$$\text{Rule } i: \text{IF } \tilde{f}_1(\mathbf{x}(t)) \text{ is } \tilde{\mathrm{M}}_1^i \text{ AND} \ldots \text{AND } \tilde{f}_\Psi(\mathbf{x}(t)) \text{ is } \tilde{\mathrm{M}}_\Psi^i \\ \text{THEN } \dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}_i\mathbf{x}(t) + \tilde{\mathbf{B}}_i\mathbf{u}(t). \tag{5}$$

The inferred local system is given by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} \tilde{w}_i(\mathbf{x}(t))\left(\tilde{\mathbf{A}}_i\mathbf{x}(t) + \tilde{\mathbf{B}}_i\mathbf{u}(t)\right) \tag{6}$$

where $\sum_{i=1}^{p} \tilde{w}_i(\mathbf{x}(t)) = 1$, $\tilde{w}_i(\mathbf{x}(t)) \in [0 \;\; 1]$, $i = 1, 2, \ldots, p$ and where we have the equation shown at the bottom of the next page.

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\left(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\right)} \tag{4}$$

3) Fuzzy Combined TSK Model: The proposed fuzzy combined TSK model, which is a fuzzy combination of the global and local fuzzy plant models, has two rules in the following format:

$$\text{Rule 1 : IF } q(\mathbf{x}(t)) \text{ is ZE}$$
$$\text{THEN } \dot{\mathbf{x}}(t) = \sum_{i=1}^{p} \widetilde{w}_i(\mathbf{x}(t)) \left(\widetilde{\mathbf{A}}_i \mathbf{x}(t) + \widetilde{\mathbf{B}}_i \mathbf{u}(t)\right) \quad (7)$$
$$\text{Rule 2 : IF } q(\mathbf{x}(t)) \text{ is NZ}$$
$$\text{THEN } \dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t)) (\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t)) \quad (8)$$

where rule 1 is for the local fuzzy plant model and rule 2 is for the global fuzzy plant model. ZE (zero) and NZ (nonzero) are the fuzzy terms. The function $q(\mathbf{x}(t))$ is defined as

$$q(\mathbf{x}(t)) = \max\left(\frac{|x_1(t)|}{\min(|\widetilde{x}_{1_{\min}}|, |\widetilde{x}_{1_{\max}}|)}, \frac{|x_2(t)|}{\min(|\widetilde{x}_{2_{\min}}|, |\widetilde{x}_{2_{\max}}|)} \cdots \frac{|x_n(t)|}{\min(|\widetilde{x}_{n_{\min}}|, |\widetilde{x}_{n_{\max}}|)}\right) \geq 0 \quad (9)$$

where $\min(\cdot)$ and $\max(\cdot)$ denote the minimum and the maximum values of the input arguments, respectively. $q(\mathbf{x}(t)) \leq 1$ implies that the system is operating inside the small domain characterized by $x_{\gamma_{\min}} \leq \widetilde{x}_{\gamma_{\min}} \leq x_\gamma(x) \leq \widetilde{x}_{\gamma_{\max}} \leq x_{\gamma_{\max}}$ for all γ; otherwise, it is outside the small domain. The inferred plant model is given by

$$\dot{\mathbf{x}}(t) = m_1(\mathbf{x}(t)) \sum_{i=1}^{p} \widetilde{w}_i(\mathbf{x}(t)) \left(\widetilde{\mathbf{A}}_i \mathbf{x}(t) + \widetilde{\mathbf{B}}_i \mathbf{u}(t)\right) + m_2(\mathbf{x}(t)) \sum_{i=1}^{p} w_i(\mathbf{x}(t)) (\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t)) \quad (10)$$

where

$$\sum_{k=1}^{2} m_k(\mathbf{x}(t)) = 1; \quad m_k(\mathbf{x}(t)) \in [0\ \ 1], \quad k = 1, 2. \quad (11)$$

$m_1(\mathbf{x}(t)) = \mu_{\mathrm{ZE}}(q(\mathbf{x}(t)))/(\mu_{\mathrm{ZE}}(q(\mathbf{x}(t))) + \mu_{\mathrm{NZ}}(q(\mathbf{x}(t))))$ and $m_2(\mathbf{x}(t)) = \mu_{\mathrm{NZ}}(q(\mathbf{x}(t)))/(\mu_{\mathrm{ZE}}(q(\mathbf{x}(t))) + \mu_{\mathrm{NZ}}(q(\mathbf{x}(t))))$; $\mu_{\mathrm{ZE}}(q(\mathbf{x}(t)))$ and $\mu_{\mathrm{NZ}}(q(\mathbf{x}(t)))$ are the membership functions corresponding to the fuzzy terms ZE and NZ, respectively. The membership function $\mu_{\mathrm{ZE}}(q(\mathbf{x}(t)))$ corresponding to ZE will be designed to cover the region of $0 \leq q(\mathbf{x}(t)) \leq 1$, while the membership function $\mu_{\mathrm{NZ}}(q(\mathbf{x}(t)))$ of NZ will cover the region of $q(\mathbf{x}(t)) > 0$. It should be noted that (10) is equivalent to (2) for any values of $m_1(\mathbf{x}(t))$ and $m_2(\mathbf{x}(t))$. The fuzzy combined TSK model has the property that for both values of $m_1(\mathbf{x}(t))$ and $m_2(\mathbf{x}(t))$ not equal to zero, the local fuzzy plant model is equivalent to the global fuzzy plant model, i.e.,

$$\sum_{i=1}^{p} \widetilde{w}_i(\mathbf{x}(t)) \left(\widetilde{\mathbf{A}}_i \mathbf{x}(t) + \widetilde{\mathbf{B}}_i \mathbf{u}(t)\right) = \sum_{i=1}^{p} w_i(\mathbf{x}(t)) (\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t)) \quad \text{if both } m_1(\mathbf{x}(t)) \text{ and } m_2(\mathbf{x}(t)) \neq 0. \quad (12)$$

The proof of the property of (12) is given in Appendix A.

B. Fuzzy Controller

1) Local Fuzzy State-Feedback Controller: The jth rule of the local fuzzy state-feedback controller is of the following format:

$$\text{Rule } j : \text{IF } g_1(\mathbf{x}(t)) \text{ is } \mathrm{N}_1^j \text{ AND} \ldots \text{AND } g_\Omega(\mathbf{x}(t)) \text{ is } \mathrm{N}_\Omega^j$$
$$\text{THEN } \mathbf{u}(t) = \widetilde{\mathbf{G}}_j \mathbf{x}(t) \quad (13)$$

where N_β^j is a fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta = 1, 2, \ldots, \Omega$; $j = 1, 2, \ldots, c$; Ω is a positive integer; $\widetilde{\mathbf{G}}_j \in \Re^{m \times n}$ is the feedback gain of rule j to be designed. The inferred output of the fuzzy controller is given by

$$\mathbf{u}(t) = \sum_{j=1}^{c} v_j(\mathbf{x}(t)) \widetilde{\mathbf{G}}_j \mathbf{x}(t) \quad (14)$$

$$\text{where} \quad \sum_{j=1}^{c} v_j(\mathbf{x}(t)) = 1, \quad v_j(\mathbf{x}(t)) \in [0\ \ 1], \quad j = 1, 2, \ldots, c. \quad (15)$$

We have that (16), shown at the bottom of the page, is a nonlinear function of $\mathbf{x}(t)$, and $\mu_{\mathrm{N}_\beta^j}(g_\beta(\mathbf{x}(t)))$ is the membership function corresponding to N_β^j to be designed.

2) Global Switching State-Feedback Controller: The global switching state-feedback controller is defined as follows:

$$\mathbf{u}(t) = \sum_{j=1}^{p} n_j(\mathbf{x}(t)) \mathbf{G}_j \mathbf{x}(t) \quad (17)$$

$$\widetilde{w}_i(\mathbf{x}(t)) = \frac{\mu_{\widetilde{\mathrm{M}}_1^i}\left(\widetilde{f}_1(\mathbf{x}(t))\right) \times \mu_{\widetilde{\mathrm{M}}_2^i}\left(\widetilde{f}_2(\mathbf{x}(t))\right) \times \cdots \times \mu_{\widetilde{\mathrm{M}}_\Psi^i}\left(\widetilde{f}_\Psi(\mathbf{x}(t))\right)}{\sum_{k=1}^{p}\left(\mu_{\widetilde{\mathrm{M}}_1^k}\left(\widetilde{f}_1(\mathbf{x}(t))\right) \times \mu_{\widetilde{\mathrm{M}}_2^k}\left(\widetilde{f}_2(\mathbf{x}(t))\right) \times \cdots \times \mu_{\widetilde{\mathrm{M}}_\Psi^k}\left(\widetilde{f}_\Psi(\mathbf{x}(t))\right)\right)}$$

$$v_j(\mathbf{x}(t)) = \frac{\mu_{\mathrm{N}_1^j}(g_1(\mathbf{x}(t))) \times \mu_{\mathrm{N}_2^j}(g_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{N}_\Omega^j}(g_\Omega(\mathbf{x}(t)))}{\sum_{k=1}^{c}\left(\mu_{\mathrm{N}_1^k}(g_1(\mathbf{x}(t))) \times \mu_{\mathrm{N}_2^k}(g_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{N}_\Omega^k}(g_\Omega(\mathbf{x}(t)))\right)} \quad (16)$$

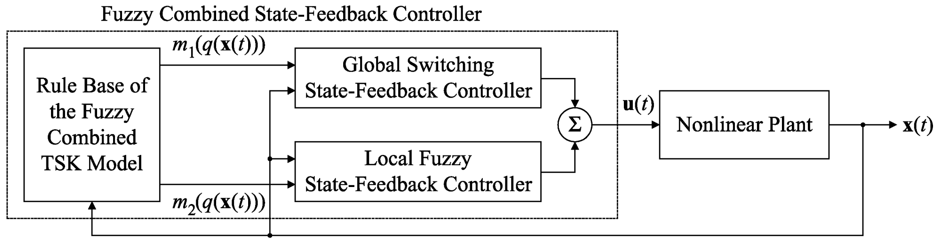


Fig. 1. Block diagram of the closed-loop system.

where $\mathbf{G}_j \in \Re^{m\times n}$, $j = 1, 2, \ldots, p$ are the feedback gains; $n_j(\mathbf{x}(t))$ takes the value of $-(K/\alpha_{\min})$ or $K/\alpha_{\min}$ according to a switching scheme to be derived later, and $K \geq 1$.

3) Fuzzy Combined State-Feedback Controller: The proposed fuzzy combined state-feedback controller as shown in Fig. 1 has two rules in the following format:

$$\text{Rule 1 : IF } q(\mathbf{x}(t)) \text{ is ZE}$$

$$\text{THEN } \mathbf{u}(t) = \sum_{j=1}^{c} v_j(\mathbf{x}(t))\widetilde{\mathbf{G}}_j\mathbf{x}(t) \tag{18}$$

$$\text{Rule 2 : IF } q(\mathbf{x}(t)) \text{ is NZ}$$

$$\text{THEN } \mathbf{u}(t) = \sum_{j=1}^{p} n_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \tag{19}$$

where rule 1 is for the local fuzzy state-feedback controller and rule 2 is for the global switching state-feedback controller. The inferred output of the fuzzy combined state-feedback controller is given by

$$\mathbf{u}(t) = m_1(\mathbf{x}(t))\sum_{j=1}^{c} v_j(\mathbf{x}(t))\widetilde{\mathbf{G}}_j\mathbf{x}(t) + m_2(\mathbf{x}(t))\sum_{j=1}^{p} n_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t). \tag{20}$$

The contribution of each controller to the nonlinear plant is determined by the values of $m_1(\mathbf{x}(t))$ and $m_2(\mathbf{x}(t))$. The local fuzzy state-feedback controller will become dominant over the global switching state-feedback controller when the system state is inside the operating domain near the origin (i.e. $m_1(\mathbf{x}(t)) \gg m_2(\mathbf{x}(t))$). Inside this domain, the chattering effect introduced by the global switching state-feedback controller will decrease and then vanish when $m_2(\mathbf{x}(t)) = 0$.

III. Stability Analysis

In this section, the system stability of the closed-loop system will be investigated. In the following, $w_i(\mathbf{x}(t))$, $\widetilde{w}_i(\mathbf{x}(t))$, $v_j(\mathbf{x}(t))$, $n_j(\mathbf{x}(t))$, and $m_k(\mathbf{x}(t))$ are written as w_i, $\widetilde{w}_i$, v_j, n_j and m_k, respectively, for simplicity. From (10) and (20), and with the property of $\sum_{i=1}^{p} w_i = \sum_{i=1}^{p}\widetilde{w}_i = \sum_{j=1}^{c} v_j = \sum_{k=1}^{2} m_k = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{c}\sum_{l=1}^{2} w_i\widetilde{w}_j v_k m_l = 1$, the closed-loop system can be written as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= m_1\sum_{i=1}^{p}\widetilde{w}_i\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i\left(m_1\sum_{j=1}^{c} v_j\widetilde{\mathbf{G}}_j\mathbf{x}(t) + m_2\sum_{j=1}^{p} n_j\mathbf{G}_j\mathbf{x}(t)\right)\right)\\
&\quad + m_2\sum_{i=1}^{p} w_i\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\left(m_1\sum_{j=1}^{c} v_j\widetilde{\mathbf{G}}_j\mathbf{x}(t) + m_2\sum_{j=1}^{p} n_j\mathbf{G}_j\mathbf{x}(t)\right)\right)\\
&= m_1m_1\sum_{i=1}^{p}\widetilde{w}_i\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i\sum_{j=1}^{c} v_j\widetilde{\mathbf{G}}_j\mathbf{x}(t)\right)\\
&\quad + m_1m_2\sum_{i=1}^{p}\widetilde{w}_i\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i\sum_{j=1}^{p} n_j\mathbf{G}_j\mathbf{x}(t)\right)\\
&\quad + m_2m_1\sum_{i=1}^{p} w_i\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\sum_{j=1}^{c} v_j\widetilde{\mathbf{G}}_j\mathbf{x}(t)\right)\\
&\quad + m_2m_2\sum_{i=1}^{p} w_i\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\sum_{j=1}^{p} n_j\mathbf{G}_j\mathbf{x}(t)\right).\end{aligned} \tag{21}$$

It should be noted that the second and the third terms of (21) vanish for either m_1 or $m_2 = 0$. For both m_1 and $m_2 \neq 0$ and from the property of (12), we have

$$\begin{aligned}m_1m_2\sum_{i=1}^{p}\widetilde{w}_i\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i\sum_{j=1}^{p} n_j\mathbf{G}_j\mathbf{x}(t)\right)\\
= m_1m_2\sum_{i=1}^{p} w_i\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\sum_{j=1}^{p} n_j\mathbf{G}_j\mathbf{x}(t)\right)\end{aligned} \tag{22}$$

$$\begin{aligned}m_2m_1\sum_{i=1}^{p} w_i\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\sum_{j=1}^{c} v_j\widetilde{\mathbf{G}}_j\mathbf{x}(t)\right)\\
= m_2m_1\sum_{i=1}^{p}\widetilde{w}_i\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i\sum_{j=1}^{c} v_j\widetilde{\mathbf{G}}_j\mathbf{x}(t)\right).\end{aligned} \tag{23}$$

From (21) to (23)

$$\begin{aligned}\dot{\mathbf{x}}(t) &= m_1m_1\sum_{i=1}^{p}\widetilde{w}_i\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i\sum_{j=1}^{c} v_j\widetilde{\mathbf{G}}_j\mathbf{x}(t)\right)\\
&\quad + m_1m_2\sum_{i=1}^{p} w_i\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\sum_{j=1}^{p} n_j\mathbf{G}_j\mathbf{x}(t)\right)\\
&\quad + m_2m_1\sum_{i=1}^{p}\widetilde{w}_i\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i\sum_{j=1}^{c} v_j\widetilde{\mathbf{G}}_j\mathbf{x}(t)\right)\end{aligned}$$

$$+ m_2 m_2 \sum_{i=1}^{p} w_i \left(\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \sum_{j=1}^{p} n_j \mathbf{G}_j \mathbf{x}(t) \right)$$
$$= m_1 \sum_{i=1}^{p} \widetilde{w}_i \left(\widetilde{\mathbf{A}}_i \mathbf{x}(t) + \widetilde{\mathbf{B}}_i \sum_{j=1}^{c} v_j \widetilde{\mathbf{G}}_j \mathbf{x}(t) \right) + m_2 \sum_{i=1}^{p} w_i \left(\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \sum_{j=1}^{p} n_j \mathbf{G}_j \mathbf{x}(t) \right). \quad (24)$$

It is assumed that

$$\mathbf{B}(\mathbf{x}(t)) = \sum_{i=1}^{p} w_i(\mathbf{x}(t)) \mathbf{B}_i = \alpha(\mathbf{x}(t)) \mathbf{B}_m \quad (25)$$

where $\mathbf{B}_m \in \Re^{n \times m}$ is a constant matrix. In this paper, $\alpha(\mathbf{x}(t))$ is a bounded, unknown (because $w_i(\mathbf{x}(t))$ is unknown), nonzero, and single-signed scalar but with a known form. As $\alpha(\mathbf{x}(t))$ is in terms of the system states and parameters, the bounds and sign of $\alpha(\mathbf{x}(t))$, i.e., $|\alpha(\mathbf{x}(t))| = [\alpha_{\min} \;\; \alpha_{\max}]$ can be estimated. Based on the form and parameter information of $|\alpha(\mathbf{x}(t))|$, its minimum and the maximum values can be found analytically. However, when $\alpha(\mathbf{x}(t))$ is a complicated function, some optimization methods such as genetic algorithm [9] can be employed to help find the boundary values of $|\alpha(\mathbf{x}(t))|$. It should be noted that because $\alpha(\mathbf{x}(t)) \neq 0$ is required, $\mathbf{B}(\mathbf{x}(t)) \neq \mathbf{0}$ is assumed. From (24) and (25) and writing $\alpha(\mathbf{x}(t))$ as α, we have

$$\dot{\mathbf{x}}(t) = m_1 \sum_{i=1}^{p} \sum_{j=1}^{c} \widetilde{w}_i v_j \left(\widetilde{\mathbf{A}}_i + \widetilde{\mathbf{B}}_i \widetilde{\mathbf{G}}_j \right) \mathbf{x}(t) + m_2 \left(\sum_{i=1}^{p} w_i (\mathbf{A}_i + \mathbf{B}_m \mathbf{G}_i) \mathbf{x}(t) + \sum_{j=1}^{p} (\alpha n_j - w_j) \mathbf{B}_m \mathbf{G}_j \mathbf{x}(t) \right)$$
$$= m_1 \sum_{i=1}^{p} \sum_{j=1}^{c} \widetilde{w}_i v_j \widetilde{\mathbf{H}}_{ij} \mathbf{x}(t) + m_2 \times \left(\sum_{i=1}^{p} w_i \mathbf{H}_i \mathbf{x}(t) + \sum_{j=1}^{p} (\alpha n_j - w_j) \mathbf{B}_m \mathbf{G}_j \mathbf{x}(t) \right) \quad (26)$$

where

$$\widetilde{\mathbf{H}}_{ij} = \widetilde{\mathbf{A}}_i + \widetilde{\mathbf{B}}_i \widetilde{\mathbf{G}}_j, \quad i = 1, 2, \ldots, p; j = 1, 2, \ldots, c \quad (27)$$
$$\mathbf{H}_i = \mathbf{A}_i + \mathbf{B}_m \mathbf{G}_i, \quad i = 1, 2, \ldots, p. \quad (28)$$

To investigate the stability of (26), the following Lyapunov function is considered:

$$V = \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \quad (29)$$

where $\mathbf{P} \in \Re^{n \times n}$ is a symmetric positive definite matrix. From (29)

$$\dot{V} = \dot{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \dot{\mathbf{x}}(t). \quad (30)$$

From (26) and (30)

$$\dot{V} = \left(m_1 \sum_{i=1}^{p} \sum_{j=1}^{c} \widetilde{w}_i v_j \widetilde{\mathbf{H}}_{ij} \mathbf{x}(t) + m_2 \left(\sum_{i=1}^{p} w_i \mathbf{H}_i \mathbf{x}(t) + \sum_{j=1}^{p} (\alpha n_j - w_j) \mathbf{B}_m \mathbf{G}_j \mathbf{x}(t) \right) \right)^{\mathrm{T}} \times \mathbf{P} \mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \times \left(m_1 \sum_{i=1}^{p} \sum_{j=1}^{c} \widetilde{w}_i v_j \widetilde{\mathbf{H}}_{ij} \mathbf{x}(t) + m_2 \left(\sum_{i=1}^{p} w_i \mathbf{H}_i \mathbf{x}(t) + \sum_{j=1}^{p} (\alpha n_j - w_j) \mathbf{B}_m \mathbf{G}_j \mathbf{x}(t) \right) \right)$$
$$= m_1 \sum_{i=1}^{p} \sum_{j=1}^{c} \widetilde{w}_i v_j \mathbf{x}(t)^{\mathrm{T}} \left(\widetilde{\mathbf{H}}_{ij}^{\mathrm{T}} \mathbf{P} + \mathbf{P} \widetilde{\mathbf{H}}_{ij} \right) \mathbf{x}(t) + m_2 \sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}} \left(\mathbf{H}_i^{\mathrm{T}} \mathbf{P} + \mathbf{P} \mathbf{H}_i \right) \mathbf{x}(t) + 2 m_2 \sum_{j=1}^{p} (\alpha n_j - w_j) \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_m \mathbf{G}_j \mathbf{x}(t). \quad (31)$$

Let

$$\widetilde{\mathbf{Q}}_{ij} = -\left(\widetilde{\mathbf{H}}_{ij}^{\mathrm{T}} \mathbf{P} + \mathbf{P} \widetilde{\mathbf{H}}_{ij} \right) > 0, \quad i = 1, 2, \ldots, p; \; j = 1, 2, \ldots, c \quad (32)$$
$$\mathbf{Q}_i = -\left(\mathbf{H}_i^{\mathrm{T}} \mathbf{P} + \mathbf{P} \mathbf{H}_i \right) > 0, \quad i = 1, 2, \ldots, p \quad (33)$$

and the switching law be

$$n_j = -\frac{K \operatorname{sgn}(\alpha) \operatorname{sgn}\left(\mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_m \mathbf{G}_j \mathbf{x}(t) \right)}{\alpha_{\min}}, \quad j = 1, 2, \ldots, p \quad (34)$$

where $K \geq 1$. From (31) to (34)

$$\dot{V} \leq -m_1 \sum_{i=1}^{p} \sum_{j=1}^{c} \widetilde{w}_i v_j \mathbf{x}(t)^{\mathrm{T}} \widetilde{\mathbf{Q}}_{ij} \mathbf{x}(t) - m_2 \sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}} \mathbf{Q}_i \mathbf{x}(t) + 2 m_2 \sum_{j=1}^{p} \left(-\frac{K|\alpha| \left| \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_m \mathbf{G}_j \mathbf{x}(t) \right|}{\alpha_{\min}} + w_j \left| \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_m \mathbf{G}_j \mathbf{x}(t) \right| \right)$$
$$= -m_1 \sum_{i=1}^{p} \sum_{j=1}^{c} \widetilde{w}_i v_j \mathbf{x}(t)^{\mathrm{T}} \widetilde{\mathbf{Q}}_{ij} \mathbf{x}(t) - m_2 \sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}} \mathbf{Q}_i \mathbf{x}(t) + 2 m_2 \sum_{j=1}^{p} \left(-\frac{K|\alpha|}{\alpha_{\min}} + w_j \right) \times \left| \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_m \mathbf{G}_j \mathbf{x}(t) \right|. \quad (35)$$

As $K \geq 1$ and $(|\alpha|/\alpha_{\min}) \geq 1$, therefore $-(K|\alpha|/\alpha_{\min}) \leq -1$. Because $w_j \in [0 \;\; 1]$, $-(K|\alpha|/\alpha_{\min}) + w_j \leq 0$. Based on (32), (33) and the fact that $-(K|\alpha|/\alpha_{\min}) + w_j \leq 0$, $\dot{V} \leq 0$. Equality holds when $\mathbf{x}(t) = \mathbf{0}$. It can be concluded that the closed-loop system is asymptotically stable, i.e., $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. In order to alleviate the chattering effect, the switching function $\mathrm{sgn}(\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{B}_m\mathbf{G}_j\mathbf{x}(t))$ in (34) can be replaced by a saturation function. Equation (34) then becomes

$$n_j = -\frac{K\mathrm{sgn}(\alpha)\mathrm{sat}\left(\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{B}_m\mathbf{G}_j\mathbf{x}(t)\right)}{\alpha_{\min}}, \quad j = 1, 2, \ldots, p \tag{36}$$

where

$$sat(z) = \begin{cases} 1, & \text{for } \frac{z}{T} \geq 1 \\ -1, & \text{for } \frac{z}{T} \leq -1 \\ \frac{z}{T}, & \text{otherwise} \end{cases} \tag{37}$$

where T is a nonzero positive scalar to be designed. As α is a single-signed scalar, $\mathrm{sgn}(\alpha)$ is not a switching signal. The saturation function may introduce steady-state error to the system states. The magnitude of the steady-state-error is affected by the value of T. During the design, the value of T should be designed such that the system states can be driven into the valid modeling domain of the local fuzzy plant model. Once the system states are inside the valid modeling domain of the local fuzzy plant model, the fuzzy state-feedback controller will gradually replace the global switching state-feedback controller. As a result, the chattering effect and the steady-state error will be eliminated eventually when the fuzzy state-feedback controller completely dominates the control process. The analysis results are summarized into the following Lemma.

Lemma 1: The closed-loop system of (21) formed by the fuzzy combined TSK model of (10) and the fuzzy combined state-feedback controller of (20) is guaranteed to be asymptotically stable if the following conditions are satisfied.

i) $\widetilde{\mathbf{Q}}_{ij} = -(\widetilde{\mathbf{H}}_{ij}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\widetilde{\mathbf{H}}_{ij}) > 0, i = 1, 2, \ldots, p; j = 1, 2, \ldots, c.$

ii) $\mathbf{Q}_i = -(\mathbf{H}_i^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_i) > 0, i = 1, 2, \ldots, p.$

iii) $n_j = -(K\mathrm{sgn}(\alpha)\mathrm{sgn}(\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{B}_m\mathbf{G}_j\mathbf{x}(t))/\alpha_{\min})$, $j = 1, 2, \ldots, p$ where $K \geq 1$ and $|\alpha(\mathbf{x}(t))| = [\alpha_{\min} \;\; \alpha_{\max}]$. To alleviate the chattering effect in the transient period, the switching law with a saturation function shown as follows can be employed:

$$n_j = -\frac{K\mathrm{sgn}(\alpha)\mathrm{sat}\left(\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{B}_m\mathbf{G}_j\mathbf{x}(t)\right)}{\alpha_{\min}}, \quad j = 1, 2, \ldots, p.$$

iv) For the fuzzy combined TSK model and the fuzzy combined state-feedback controller, the membership function corresponding to ZE is designed to cover the region of $0 \leq q(\mathbf{x}(t)) \leq 1$, while that of NZ will cover the region that $q(\mathbf{x}(t)) > 0$, with $q(\mathbf{x}(t))$ as defined in (9).

The design procedure of the fuzzy combined state-feedback controller can be summarized into the following steps.

Step I) Obtain the local and global fuzzy plant models of the nonlinear plant by means of the method shown in [10] or other fuzzy modeling methods [1], [2].

Step II) Determine the membership functions and $q(\mathbf{x}(t))$ of the fuzzy combined TSK model of (10) according to the condition iv) of Lemma 1.

Step III) Determine the number of rules, membership functions and feedback gains of the local fuzzy state-feedback controller of (14).

Step IV) Determine the feedback gains of the global switching state-feedback controller of (17). Design n_j as given in condition iii) of Lemma 1 and determine the value of K.

Step V) Combine the local fuzzy and global switching state-feedback controllers using the membership function of the fuzzy combined TSK model determined in Step II to form the fuzzy combined state-feedback controller.

Step VI) Find the solution $\mathbf{P}$ to the conditions i) and ii) of Lemma 1 to guarantee the stability of the closed-loop system. The solution can be solved using some linear matrix inequality (LMI) software tool.

Step VII) This step applies to the fuzzy combined state-feedback controller using saturation function. Determine the value of T such that the weighted global switching state-feedback controller using the saturation function, i.e. $\mathbf{u}(t) = m_2(\mathbf{x}(t))\sum_{j=1}^{p} n_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t)$, is able to drive the system states inside the valid modeling domain of the local fuzzy plant model. Apply this value of T to the fuzzy combined state-feedback controller with the saturation function.

IV. Application Example

An application example on stabilizing a cart-pole typed inverted pendulum [11] subject to parameter uncertainties is given in this section.

Step I) The dynamics of the inverted pendulum is given by

$$\ddot{\theta}(t) = \frac{g\sin(\theta(t)) - \frac{aml\dot{\theta}(t)^2\sin(2\theta(t))}{2} - a\cos(\theta(t))\,u(t)}{\frac{4l}{3} - aml\cos^2(\theta(t))} \tag{38}$$

where $\theta(t)$ is the angular displacement of the pendulum, $g = 9.8\ \mathrm{m/s^2}$ is the acceleration due to gravity, $m \in [m_{\min} \;\; m_{\max}] = [2 \;\; 5]\mathrm{kg}$ is the mass of the pendulum, $M \in [M_{\min} \;\; M_{\max}] = [8 \;\; 10]\mathrm{kg}$ is the mass of the cart, $a = 1/(m+M)$, $2l = 1$ m is the length of the pendulum, and $u(t)$ is the force applied to the cart. Based on the methodology shown in [10], the global and local fuzzy plant models of the nonlinear plant of (38) can be obtained.

To obtain the global fuzzy plant model, the nonlinear plant of (38) can be represented by the following fuzzy rules:

$$\begin{aligned} &\text{Rule } i: \text{IF } f_1(\mathbf{x}(t)) \text{ is } \mathrm{M}_1^i \text{ AND } f_2(\mathbf{x}(t)) \text{ is } \mathrm{M}_2^i \\ &\quad \text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t) \quad \text{for } i = 1, 2, 3, 4. \end{aligned} \tag{39}$$

The system dynamical behavior is described by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \sum_{i=1}^{4} w_i(\mathbf{x}(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)) \\ &= \sum_{i=1}^{4} w_i(\mathbf{x}(t))(\mathbf{A}_i\mathbf{x}(t) + \alpha(\mathbf{x}(t))\mathbf{B}_m u(t)) \end{aligned} \tag{40}$$

where $\mathbf{x}(t) = [x_1(t) \;\; x_2(t)]^{\mathrm{T}} = [\theta(t) \;\; \dot{\theta}(t)]^{\mathrm{T}}$, $\theta(t) \in [-(22\pi/45) \;\; 22\pi/45]$ and $\dot{\theta}(t) \in [-5 \;\; 5]$; $f_1(\mathbf{x}(t)) = ((g - amlx_2(t)^2 \cos(x_1(t)))/(4l/3 - aml\cos^2(x_1(t))))(\sin(x_1(t))/x_1(t))$ and $f_2(\mathbf{x}(t)) = \alpha(\mathbf{x}(t)) = -(a\cos(x_1(t))/(4l/3 - aml\cos^2(x_1(t))))$; $\mathbf{A}_1 = \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ f_{1_{\min}} & 0 \end{bmatrix}$ and $\mathbf{A}_3 = \mathbf{A}_4 = \begin{bmatrix} 0 & 1 \\ f_{1_{\max}} & 0 \end{bmatrix}$; $\mathbf{B}_1 = \mathbf{B}_3 = \begin{bmatrix} 0 \\ f_{2_{\min}} \end{bmatrix}$, $\mathbf{B}_2 = \mathbf{B}_4 = \begin{bmatrix} 0 \\ f_{2_{\max}} \end{bmatrix}$ and $\mathbf{B}_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $f_{1_{\min}} = 9.4047$ and $f_{1_{\max}} = 20.6595$, $f_{2_{\min}} = -0.1765$ and $f_{2_{\max}} = -0.0034$; $w_i(\mathbf{x}(t)) = \mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t)))/\sum_{l=1}^{4}(\mu_{\mathrm{M}_1^l}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^l}(f_2(\mathbf{x}(t))))$; $\mu_{\mathrm{M}_1^\beta}(f_1(\mathbf{x}(t))) = (-f_1(\mathbf{x}(t)) + f_{1_{\max}})/(f_{1_{\max}} - f_{1_{\min}})$ for $\beta = 1, 2$; $\mu_{\mathrm{M}_1^\delta}(f_1(\mathbf{x}(t))) = 1 - \mu_{\mathrm{M}_1^1}(f_1(\mathbf{x}(t)))$ for $\delta = 3, 4$; $\mu_{\mathrm{M}_2^\kappa}(f_2(\mathbf{x}(t))) = (-f_2(\mathbf{x}(t)) + f_{2_{\max}})/(f_{2_{\max}} - f_{2_{\min}})$ for $\kappa = 1, 3$ and $\mu_{\mathrm{M}_2^\phi}(f_2(\mathbf{x}(t))) = 1 - \mu_{\mathrm{M}_2^1}(f_2(\mathbf{x}(t)))$ for $\phi = 2, 4$.

To obtain the local fuzzy plant model, consider the operating domain that $x_1(t) \in [\tilde{x}_{1_{\min}} \;\; \tilde{x}_{1_{\max}}] = [-0.5 \;\; 0.5]$ and $x_2(t) \in [\tilde{x}_{2_{\min}} \;\; \tilde{x}_{2_{\max}}] = [-2.5 \;\; 2.5]$. The local system dynamical behavior can be described by a local fuzzy plant model with the following four rules:

$$\begin{aligned} &\text{Rule } i: \text{IF } \tilde{f}_1(\mathbf{x}(t)) \text{ is } \widetilde{\mathrm{M}}_1^i \text{ AND } \tilde{f}_2(\mathbf{x}(t)) \text{ is } \widetilde{\mathrm{M}}_2^i \\ &\quad \text{THEN } \dot{\mathbf{x}}(t) = \widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i u(t) \quad \text{for } i = 1,2,3,4 \end{aligned} \tag{41}$$

so that the system dynamical behavior is described by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{4} \tilde{w}_i(\mathbf{x}(t))\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i u(t)\right) \tag{42}$$

where $\tilde{f}_1(\mathbf{x}(t)) = f_1(\mathbf{x}(t))$ and $\tilde{f}_2(\mathbf{x}(t)) = f_2(\mathbf{x}(t))$; $\widetilde{\mathbf{A}}_1 = \widetilde{\mathbf{A}}_2 = \begin{bmatrix} 0 & 1 \\ \tilde{f}_{1_{\min}} & 0 \end{bmatrix}$ and $\widetilde{\mathbf{A}}_3 = \widetilde{\mathbf{A}}_4 = \begin{bmatrix} 0 & 1 \\ \tilde{f}_{1_{\max}} & 0 \end{bmatrix}$; $\widetilde{\mathbf{B}}_1 = \widetilde{\mathbf{B}}_3 = \begin{bmatrix} 0 \\ \tilde{f}_{2_{\min}} \end{bmatrix}$ and $\widetilde{\mathbf{B}}_2 = \widetilde{\mathbf{B}}_4 = \begin{bmatrix} 0 \\ \tilde{f}_{2_{\max}} \end{bmatrix}$; $\tilde{f}_{1_{\min}} = 14.8691$ and $\tilde{f}_{1_{\max}} = 20.6595$, $\tilde{f}_{2_{\min}} = -0.1765$ and $\tilde{f}_{2_{\max}} = -0.1086$; $\tilde{w}_i(\mathbf{x}(t)) = (\mu_{\widetilde{\mathrm{M}}_1^i}(\tilde{f}_1(\mathbf{x}(t))) \times \mu_{\widetilde{\mathrm{M}}_2^i}(\tilde{f}_2(\mathbf{x}(t))))/\sum_{l=1}^{4}(\mu_{\widetilde{\mathrm{M}}_1^l}(\tilde{f}_1(\mathbf{x}(t))) \times \mu_{\widetilde{\mathrm{M}}_2^l}(\tilde{f}_2(\mathbf{x}(t))))$; $\mu_{\widetilde{\mathrm{M}}_1^\beta}(\tilde{f}_1(\mathbf{x}(t))) = (-\tilde{f}_1(\mathbf{x}(t)) + \tilde{f}_{1_{\max}})/(\tilde{f}_{1_{\max}} - \tilde{f}_{1_{\min}})$ for $\beta = 1, 2$; $\mu_{\widetilde{\mathrm{M}}_1^\delta}(\tilde{f}_1(\mathbf{x}(t))) = 1 - \mu_{\widetilde{\mathrm{M}}_1^1}(\tilde{f}_1(\mathbf{x}(t)))$ for $\delta = 3, 4$; $\mu_{\widetilde{\mathrm{M}}_2^\kappa}(\tilde{f}_2(\mathbf{x}(t))) = (-\tilde{f}_2(\mathbf{x}(t)) + \tilde{f}_{2_{\max}})/(\tilde{f}_{2_{\max}} - \tilde{f}_{2_{\min}})$ for $\kappa = 1, 3$ and $\mu_{\widetilde{\mathrm{M}}_2^\phi}(\tilde{f}_2(\mathbf{x}(t))) = 1 - \mu_{\widetilde{\mathrm{M}}_2^1}(\tilde{f}_2(\mathbf{x}(t)))$ for $\phi = 2, 4$.

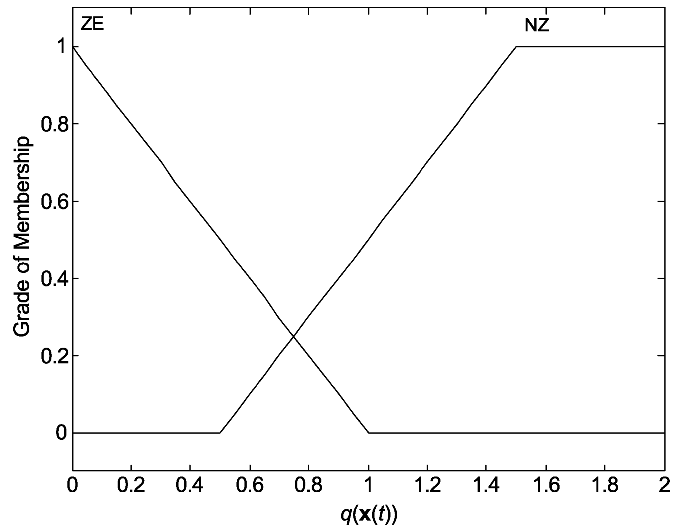


Fig. 2. Membership functions of ZE and NZ.

Step II) The fuzzy combined TSK model having the following two rules are employed to model the nonlinear plant of (38):

$$\begin{aligned} &\text{Rule 1: IF } q(\mathbf{x}(t)) \text{ is ZE} \\ &\quad \text{THEN } \dot{\mathbf{x}}(t) = \sum_{i=1}^{4} \tilde{w}_i(\mathbf{x}(t))\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i u(t)\right) \end{aligned} \tag{43}$$

$$\begin{aligned} &\text{Rule 2: IF } q(\mathbf{x}(t)) \text{ is NZ} \\ &\quad \text{THEN } \dot{\mathbf{x}}(t) = \sum_{i=1}^{4} w_i(\mathbf{x}(t))\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)\right). \end{aligned} \tag{44}$$

The function $q(\mathbf{x}(t))$ is defined as

$$\begin{aligned} q(\mathbf{x}(t)) &= \max\left(\frac{|x_1(t)|}{\min(|\tilde{x}_{1_{\min}}|, |\tilde{x}_{1_{\max}}|)}, \frac{|x_2(t)|}{\min(|\tilde{x}_{2_{\min}}|, |\tilde{x}_{2_{\max}}|)}\right) \\ &= \max\left(\frac{|x_1(t)|}{0.5}, \frac{|x_2(t)|}{2.5}\right) \geq 0. \end{aligned} \tag{45}$$

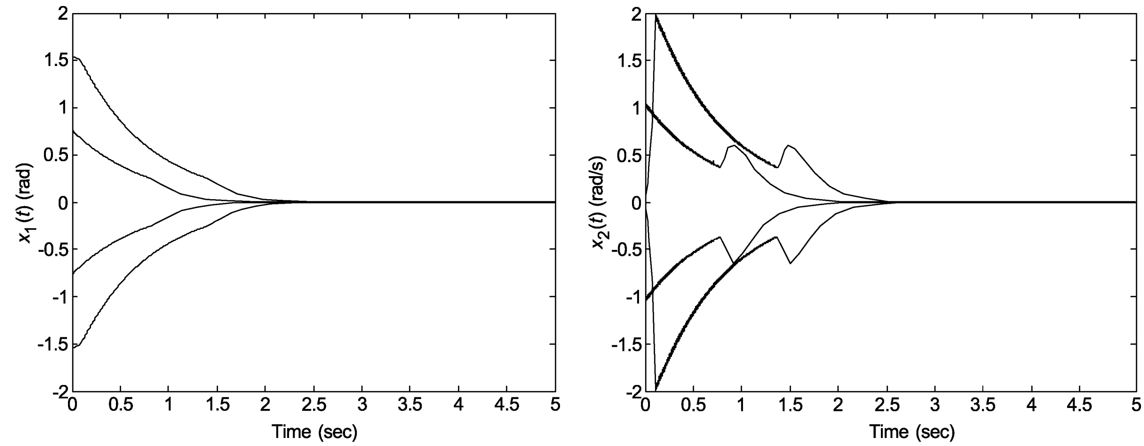
The inferred system is given by

$$\begin{aligned} \dot{\mathbf{x}}(t) = {} & m_1(\mathbf{x}(t)) \sum_{i=1}^{4} \tilde{w}_i(\mathbf{x}(t))\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i u(t)\right) \\ & + m_2(\mathbf{x}(t)) \sum_{i=1}^{4} w_i(\mathbf{x}(t))\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)\right) \end{aligned} \tag{46}$$

The membership functions corresponding to ZE and NZ are designed as shown in Fig. 2.

Step III) A four-rule local fuzzy state-feedback controller is designed based on the local fuzzy plant model. The rule is of the following format:

$$\begin{aligned} &\text{Rule } j: \text{IF } x_1(t) \text{ is } \mathrm{N}_1^j \text{ AND } x_2(t) \text{ is } \mathrm{N}_2^j \\ &\quad \text{THEN } u(t) = \widetilde{\mathbf{G}}_j\mathbf{x}(t), \quad j = 1,2,3,4 \end{aligned} \tag{47}$$



(a) (b)

Fig. 3. System responses with the fuzzy combined state-feedback controller using the switching function for $m = 2$ kg and $M = 8$ kg. (a) $x_1(t)$. (b) $x_2(t)$.

The inferred output of the fuzzy controller is given by

$$u(t) = \sum_{j=1}^{4} v_j\left(\mathbf{x}(t)\right)\widetilde{\mathbf{G}}_j\mathbf{x}(t) \tag{48}$$

where $v_j(\mathbf{x}(t)) = (\mu_{\mathrm{N}_1^j}(x_1(t)) \times \mu_{\mathrm{N}_2^j}(x_2(t)))/\sum_{l=1}^{4}(\mu_{\mathrm{N}_1^l}(x_1(t)) \times \mu_{\mathrm{N}_2^l}(x_2(t)))$; $\mu_{\mathrm{N}_1^\beta}(x_1(t) = (-|x_1(t)| + \widetilde{x}_{1_{\max}})/(\widetilde{x}_{1_{\max}} - \widetilde{x}_{1_{\min}})$ for $\beta = 1,\ 2$; $\mu_{\mathrm{N}_1^\delta}(x_1(t) = 1 - \mu_{\mathrm{N}_1^1}(x_1(t))$ for $\delta = 3,\ 4$; $\mu_{\mathrm{N}_2^\kappa}(x_2(t) = (-|x_2(t)| + \widetilde{x}_{2_{\max}})/(\widetilde{x}_{2_{\max}} - \widetilde{x}_{2_{\min}})$ for $\kappa = 1,\ 3$ and $\mu_{\mathrm{N}_2^\phi}(x_2(t) = 1 - \mu_{\mathrm{N}_2^1}(x_2(t))$ for $\phi = 2,\ 4$; The feedback gains are chosen as $\widetilde{\mathbf{G}}_1 = [265.5473\ \ 67.9887]$, $\widetilde{\mathbf{G}}_2 = [431.5755\ \ 110.4972]$, $\widetilde{\mathbf{G}}_3 = [298.3541\ \ 67.9887]$ and $\widetilde{\mathbf{G}}_4 = [484.8941\ \ 110.4972]$ such that the eigenvalues of $\widetilde{\mathbf{H}}_{ii}$, $i = 1$, 2, 3 and 4 are equal to -4 and -8.

Step IV) The following global switching state-feedback controller is designed based on the global fuzzy plant model

$$u(t) = \sum_{j=1}^{4} n_j\left(\mathbf{x}(t)\right)\mathbf{G}_j\mathbf{x}(t). \tag{49}$$

The switching law is defined as

$$n_j\left(\mathbf{x}(t)\right) = -\frac{K\mathrm{sgn}(\alpha)\mathrm{sgn}\left(\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{B}_m\mathbf{G}_j\mathbf{x}(t)\right)}{\alpha_{\min}}, \quad j = 1, 2, 3 \text{ and } 4 \tag{50}$$

where $K = 1.5$, $\alpha_{\min} = |f_{2_{\max}}| = 0.0034$ which is obtained by genetic algorithm [9]; The feedback gains are chosen as $\mathbf{G}_1 = \mathbf{G}_2 = [-18.8691\ \ -4.0000]$ and $\mathbf{G}_3 = \mathbf{G}_4 = [-24.6595\ \ -4.0000]$ such that the eigenvalues of $\mathbf{H}_i$, $i = 1$, 2, 3, and 4 are equal to -2 and -2.

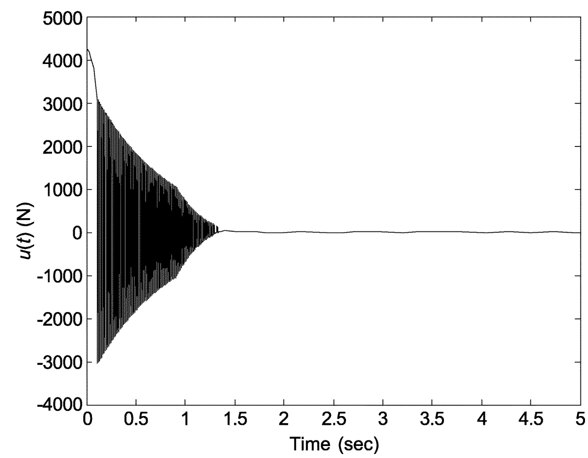


Fig. 4. Control signal of the fuzzy combined state-feedback controller using the switching function for $m = 2$ kg and $M = 8$ kg.

Step V) The fuzzy combined state-feedback controller with the following two rules is designed based on the fuzzy combined TSK model:

Rule 1 : IF $q\left(\mathbf{x}(t)\right)$ is ZE

$$\text{THEN } u(t) = \sum_{j=1}^{4} v_j\left(\mathbf{x}(t)\right)\widetilde{\mathbf{G}}_j\mathbf{x}(t) \tag{51}$$

Rule 2 : IF $q\left(\mathbf{x}(t)\right)$ is NZ

$$\text{THEN } u(t) = \sum_{j=1}^{4} n_j\left(\mathbf{x}(t)\right)\mathbf{G}_j\mathbf{x}(t). \tag{52}$$

The inferred output of the fuzzy combined state-feedback controller is defined as

$$u(t) = m_1\left(\mathbf{x}(t)\right)\sum_{j=1}^{4} v_j\left(\mathbf{x}(t)\right)\widetilde{\mathbf{G}}_j\mathbf{x}(t) + m_2\left(\mathbf{x}(t)\right)\sum_{j=1}^{4} n_j\left(\mathbf{x}(t)\right)\mathbf{G}_j\mathbf{x}(t). \tag{53}$$

Step VI) The LMI toolbox of the Matlab software can be employed to find the solution $\mathbf{P}$ to the stability conditions i) and ii) of Lemma 1. It is found that

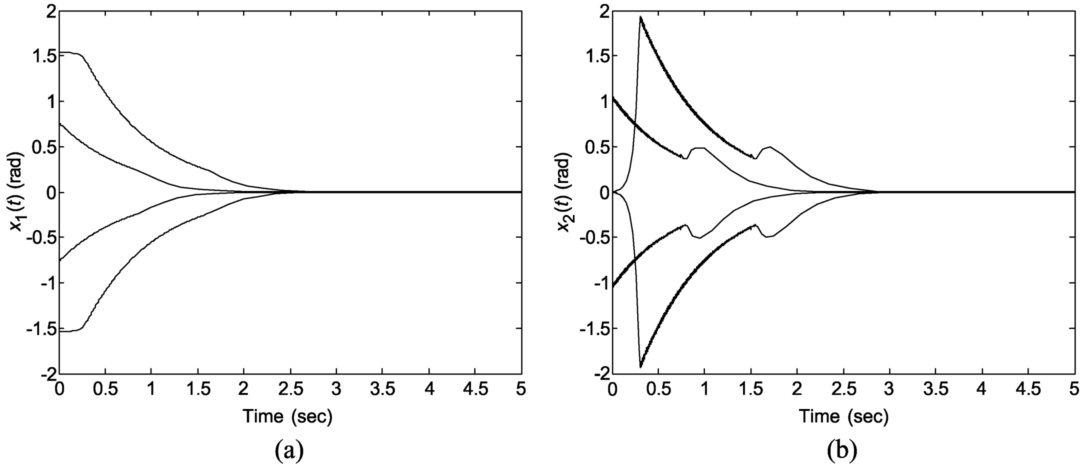


Fig. 5. System responses with the fuzzy combined state-feedback controller using the switching function for $m = 5$ kg and $M = 10$ kg. (a) $x_1(t)$. (b) $x_2(t)$.

when $\mathbf{P} = \begin{bmatrix} 5.1158 & 0.4197 \\ 0.4197 & 0.3082 \end{bmatrix}$, the conditions i) and ii) of Lemma 1 are satisfied. Hence, the stability of the closed-loop system is guaranteed.

Step VII) The fuzzy combined state-feedback controller of (53) is employed to control the uncertain nonlinear system of (38). Fig. 3 shows the system responses with the fuzzy combined state-feedback controller using the switching function of (50) under the initial state conditions of $\mathbf{x}(0) = [22\pi/45 \quad 0]^{\mathrm{T}}$, $\mathbf{x}(0) = [11\pi/45 \ \ 0]^{\mathrm{T}}$, $\mathbf{x}(0) = [-(11\pi/45) \ \ 0]^{\mathrm{T}}$ and $\mathbf{x}(0) = [-(22\pi/45) \ \ 0]^{\mathrm{T}}$ for $m = 2$ kg and $M = 8$ kg. Fig. 4 shows the corresponding control signal for $\mathbf{x}(0) = [22\pi/45 \ \ 0]^{\mathrm{T}}$. To test the robustness property of the fuzzy combined state-feedback controller, we set $m = 5$ kg and $M = 10$ kg. Fig. 5 shows the system responses with the fuzzy combined state-feedback controller using the switching function under different initial state conditions for $m = 5$ kg and $M = 10$ kg. Fig. 6 shows the corresponding control signal for $\mathbf{x}(0) = [22\pi/45 \ \ 0]^{\mathrm{T}}$. It can be seen from these figures that the nonlinear system subject to parameter uncertainties can be stabilized successfully. The chattering effect vanishes when the system states are near the origin. To alleviate the chattering effect in the transient state, a saturation function is employed to replace the switching function in the switching law of (50). As a result, the switching law using the saturation function with $T = 1$ is defined as

$$n_j(\mathbf{x}(t)) = -\frac{K\mathrm{sgn}(\alpha)\mathrm{sat}\left(\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{B}_m\mathbf{G}_j\mathbf{x}(t)\right)}{\alpha_{\min}}, \quad j = 1, 2, 3, 4 \quad (54)$$

Figs. 7 and 8 show the system responses and control signals with the fuzzy combined state-feedback controller using the saturation function for $m = 2$ kg and $M = 8$ kg, and $m = 5$ kg and $M = 10$ kg, respectively, under different initial state conditions. It can be seen from these figures that the chattering effect is significantly reduced and no steady-state error is found.

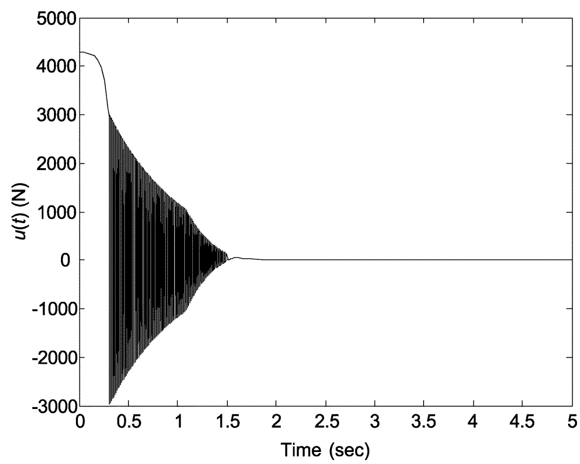


Fig. 6. Control signal of the fuzzy combined state-feedback controller using the switching function for $m = 5$ kg and $M = 10$ kg.

For comparison, a published fuzzy controller [3], [4] having the following rules is employed to handle the nonlinear plant based on the global fuzzy plant model:

$$\text{Rule } j : \text{IF } x_1(t) \text{ is } \mathrm{V}_1^j \text{ AND } x_2(t) \text{ is } \mathrm{V}_2^j \text{ THEN } u(t) = \overline{\mathbf{G}}_j\mathbf{x}(t), \quad j = 1, 2, 3, 4 \quad (55)$$

where $\overline{\mathbf{G}}_j \in \Re^{1\times 2}$ is the feedback gain. The fuzzy controller is defined as

$$u(t) = \sum_{j=1}^{4} v_j\overline{\mathbf{G}}_j\mathbf{x}(t) \quad (56)$$

where $v_j = (\mu_{\mathrm{V}_1^j}(x_1(t)) \times \mu_{\mathrm{V}_2^j}(x_2(t)))/\sum_{l=1}^{4}(\mu_{\mathrm{V}_1^l}(x_1(t)) \times \mu_{\mathrm{V}_2^l}x_2(t))$. The membership functions are defined as $\mu_{\mathrm{V}_1^\beta}(x_1(t)) = (-|x_1(t)| + x_{1_{\max}})/(x_{1_{\max}} - x_{1_{\min}})$ for $\beta = 1, 2$; $\mu_{\mathrm{V}_1^\delta}(x_1(t)) = 1 - \mu_{\mathrm{V}_1^1}(x_1(t))$ for $\delta = 3, 4$; $\mu_{\mathrm{V}_2^\kappa}(x_3(t)) = (-|x_2(t)| + x_{2_{\max}})/(x_{2_{\max}} - x_{2_{\min}})$ for

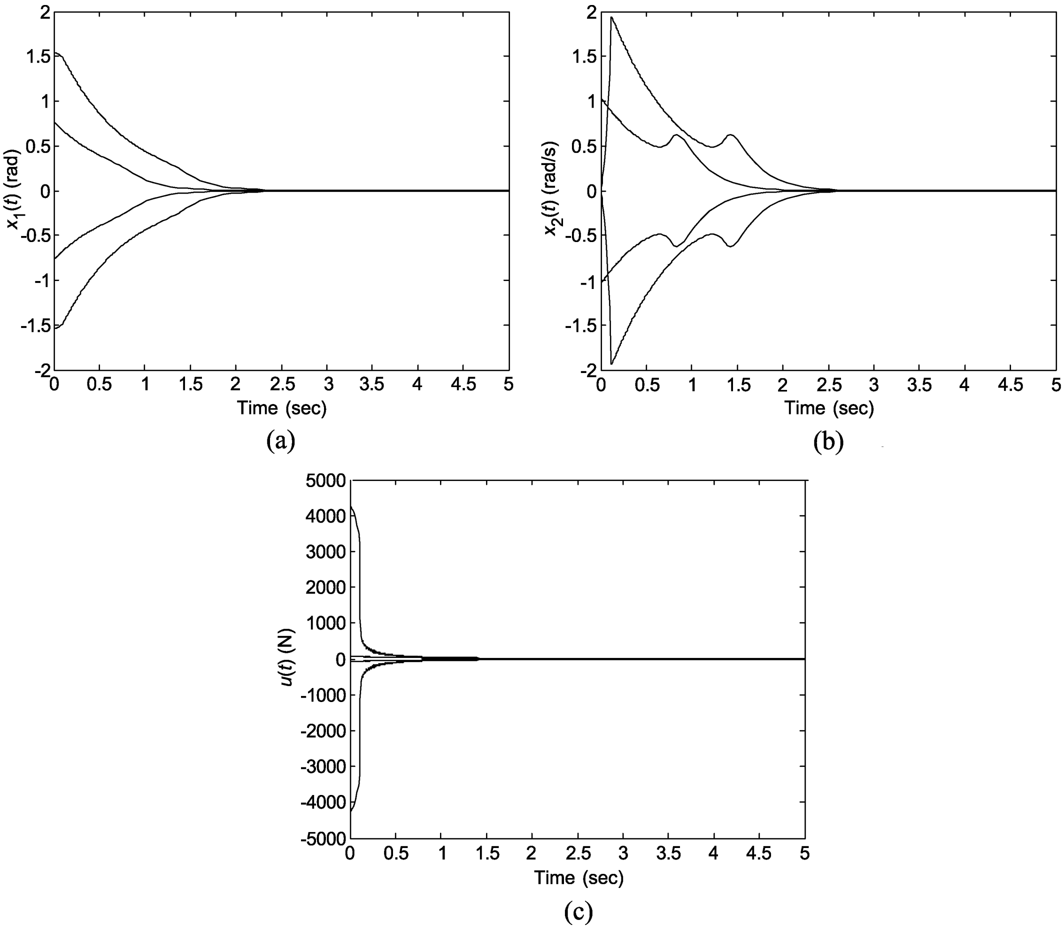


Fig. 7. System responses and control signals with the fuzzy combined state-feedback controller using the saturation function for $m = 2$ kg and $M = 8$ kg. (a) $x_1(t)$. (b) $x_2(t)$. (c) $u(t)$.

$\kappa = 1, 3$ and $\mu_{\mathrm{N}_2^{\phi}}(x_2(t)) = 1 - \mu_{\mathrm{V}_2^1}(x_2(t))$ for $\phi = 2, 4$. By applying the parallel-distribution compensation (PDC) design approach proposed in [4], and designing the feedback gains $\overline{\mathbf{G}}_j$ under the same design criterion as that of the global switching state-feedback controller, i.e., the eigenvalues of $\overline{\mathbf{H}}_{ii} = \mathbf{A}_i + \mathbf{B}_i\overline{\mathbf{G}}_i$ are both assigned to be -2 for all i, we have $\overline{\mathbf{G}}_1 = [75.9474 \;\; 22.6629]$, $\overline{\mathbf{G}}_2 = [3942.5647 \;\; 1176.4735]$, $\overline{\mathbf{G}}_3 = [139.7140 \;\; 22.6629]$ and $\overline{\mathbf{G}}_4 = [7252.8000 \;\; 1176.4735]$. From [3], [4], the system is guaranteed to be stable if there exists a common solution $\overline{\mathbf{P}} > 0$ to the following LMIs:

$$\overline{\mathbf{H}}_{ij}^{\mathrm{T}}\overline{\mathbf{P}} + \overline{\mathbf{P}}\overline{\mathbf{H}}_{ij} < 0, \quad i = 1, 2, 3, 4; \quad j = 1, 2, 3, 4 \tag{57}$$

where $\overline{\mathbf{H}}_{ij} = \mathbf{A}_i + \mathbf{B}_i\overline{\mathbf{G}}_j$. It can be shown that $\overline{\mathbf{P}}$ cannot be found by using of the MATLAB LMI toolbox. Fig. 9 shows the system responses with the published fuzzy controller for $m = 2$ kg and $M = 8$ kg under different initial state conditions. Referring to this figure, it can be seen that the system cannot be stabilized by the fuzzy controller of (56). Furthermore, it can be shown that some relaxed stability conditions [4], [5], [8] are not applicable in this example because of the constraint that the fuzzy controller shares the same premises of the rules of the fuzzy plant model is not satisfied. In this example, the membership functions of the global fuzzy plant model are subject to parameter uncertainties. Hence, the membership functions of the fuzzy plant model cannot be used to design the fuzzy controller [4], [5], [8], which requires the grades of membership to be known. Referring to [11], it has been shown that the linear state-feedback controller designed based on the linearized model of the nonlinear plant is not able to stabilize the inverted pendulum. The simulation results of the closed-loop system with only the global switching state-feedback controller can also be found in [11] and [12]. By comparing the results in [11] and [12], the effectiveness of the proposed fuzzy combined controller can be seen. In [11] and [12], the chattering effect in the transient period is very significant, which still appears near the origin. The chattering effect of the proposed approach during the transient period is now significantly reduced, and is eliminated when the system states are near the origin.

V. Conclusion

A fuzzy combined TSK model has been proposed to model the nonlinear plants subject to parameter uncertainties. Based on this fuzzy combined TSK model, a fuzzy combined state-feedback controller has been designed. The stability conditions and the switching law have been derived. Thanks to the properties of the proposed fuzzy combined TSK model and fuzzy combined state-feedback controller, the stability conditions are in terms of the closed-loop system matrices of the local and global closed-loop systems only. As the nonlinearity of the local fuzzy plant

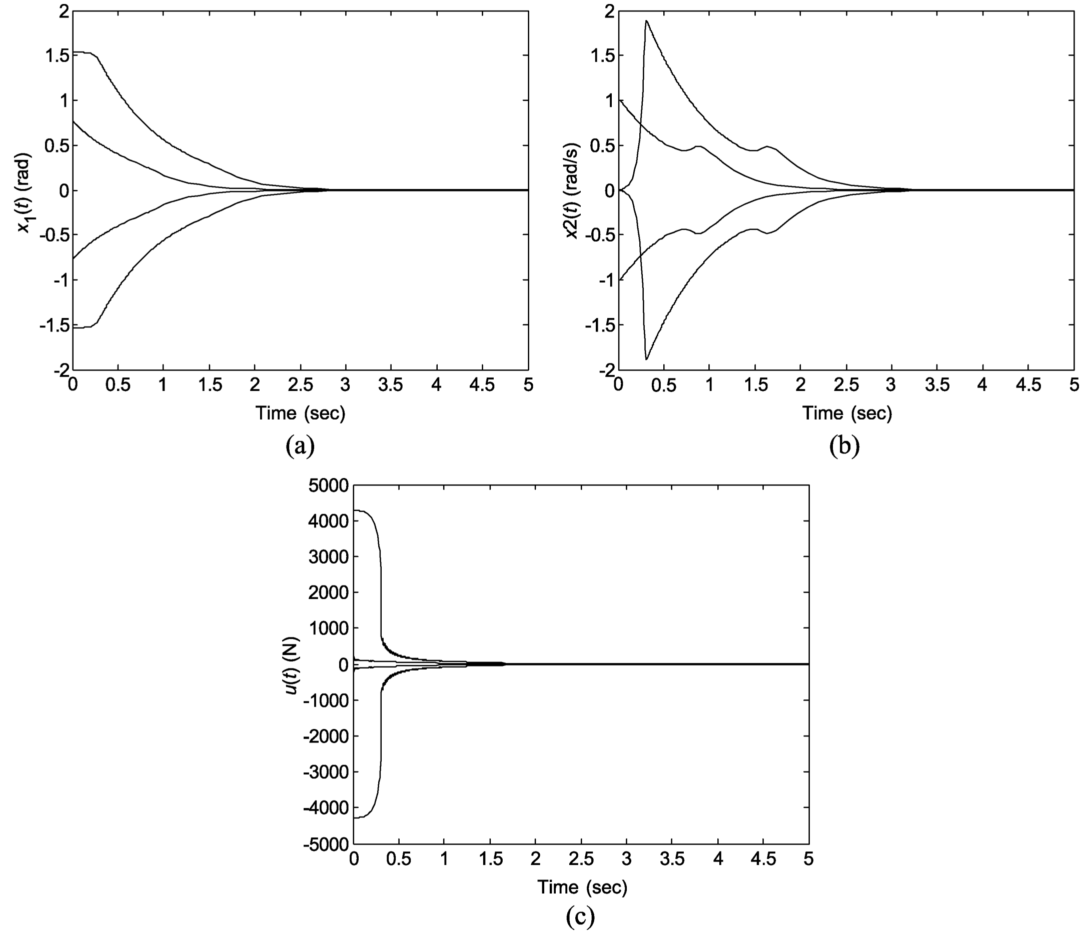


Fig. 8. System responses and control signals with the fuzzy combined state-feedback controller using the saturation function for $m = 5$ kg and $M = 10$ kg. (a) $x_1(t)$. (b) $x_2(t)$. (c) $u(t)$.

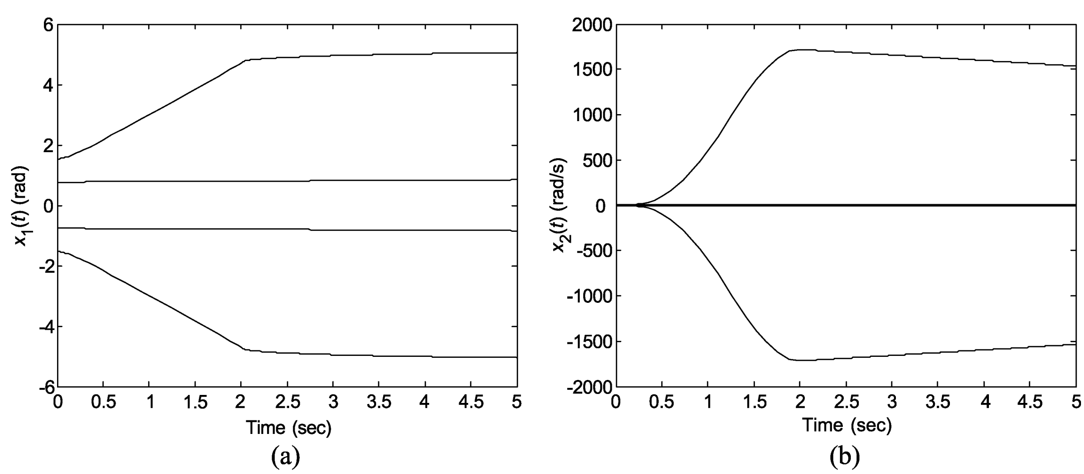


Fig. 9. System responses with the published fuzzy controller for $m = 2$ kg and $M = 8$ kg. (a) $x_1(t)$. (b) $x_2(t)$.

model is less complex than that of the global fuzzy plant model, a stable local fuzzy state-feedback controller is easier to be obtained. Furthermore, the chattering effect and the steady-state error introduced by the global switching state-feedback controller will gradually vanish when the system states are inside the valid modeling domain of the local fuzzy plant model. An application example has been given to illustrate the design procedure and the merits of the proposed approach.

APPENDIX A

From (2), the dynamics of a class of nonlinear systems can be represented by a global fuzzy plant model defined as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i\left(\mathbf{x}(t)\right)\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\right). \quad \text{(A1)}$$

The procedure to obtain the fuzzy plant model is detailed in [10]. In [13], it has been shown that the dynamics of the same class of nonlinear systems can also be represented by a union of Φ local fuzzy plant models (the local fuzzy plant models can be obtained by the procedure in [10]). The operating domain of each local fuzzy plant models can be freely designed such that the union of the operating domains is the same as that of the nonlinear system. The fuzzy plant model in [13] is defined as follows:

$$\dot{\mathbf{x}}(t) = \sum_{j=1}^{\Phi} v_j\left(\mathbf{x}(t)\right) \sum_{i=1}^{p} w_{ji}\left(\mathbf{x}(t)\right)\left(\mathbf{A}_{ji}\mathbf{x}(t) + \mathbf{B}_{ji}\mathbf{u}(t)\right) \quad \text{(A2)}$$

where $\mathbf{A}_{ji}$ and $\mathbf{B}_{ji}$ denote the constant system and input matrices of the i-th rule of the j-th local fuzzy plant model; $v_j(\mathbf{x}(t))$ takes the value of 1 or 0. It takes the value of 1 when the system states are inside the operating domain of the j-th local fuzzy plant model characterized by the function of $v_j(\mathbf{x}(t))$; otherwise, it takes the value of zero. It should be noted that there is only one $v_j(\mathbf{x}(t))$ equal to 1, all others are equal to zero, at any instant. As the dynamics of the same nonlinear system can be represented by (A1) or (A2), (A1) and (A2) are equivalent. Let the local fuzzy plant model of (A2) with $j = 1$ be designed as the local fuzzy plant model of (6) which is restated as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} \widetilde{w}_i\left(\mathbf{x}(t)\right)\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i\mathbf{u}(t)\right). \quad \text{(A3)}$$

From (A2) and (A3), (A2) can be rewritten as

$$\begin{aligned}\dot{\mathbf{x}}(t) = &\sum_{i=1}^{p} \widetilde{w}_i\left(\mathbf{x}(t)\right)\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i\mathbf{u}(t)\right)\\ &+\sum_{j=2}^{\Phi} v_j\left(\mathbf{x}(t)\right) \sum_{i=1}^{p} w_{ji}\left(\mathbf{x}(t)\right)\left(\mathbf{A}_{ji}\mathbf{x}(t) + \mathbf{B}_{ji}\mathbf{u}(t)\right). \quad \text{(A4)}\end{aligned}$$

It should be noted that the operating domain of (A3) covers the region near the origin while those of other local fuzzy plant models of (A4), for $j > 1$, cover the rest of the operating domain of the nonlinear systems. Based on the fact that (A1), (A2), and (A4) are equivalent, we have

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{p} w_i\left(\mathbf{x}(t)\right)\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\right)\\ &= \sum_{i=1}^{p} \widetilde{w}_i\left(\mathbf{x}(t)\right)\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i\mathbf{u}(t)\right)\\ &\quad + \sum_{j=2}^{\Phi} v_j\left(\mathbf{x}(t)\right) \sum_{i=1}^{p} w_{ji}\left(\mathbf{x}(t)\right)\\ &\quad \times\left(\mathbf{A}_{ji}\mathbf{x}(t) + \mathbf{B}_{ji}\mathbf{u}(t)\right). \quad \text{(A5)}\end{aligned}$$

From (A5), and considering that the system is operating inside the operating domain near the origin, i.e. $v_j(\mathbf{x}(t)) = 0$ for $j > 1$, it can be seen that

$$\begin{aligned}&\sum_{i=1}^{p} w_i\left(\mathbf{x}(t)\right)\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\right)\\ &\qquad = \sum_{i=1}^{p} \widetilde{w}_i\left(\mathbf{x}(t)\right)\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i\mathbf{u}(t)\right). \quad \text{(A6)}\end{aligned}$$

Consider (10), which is restated as follows:

$$\begin{aligned}\dot{\mathbf{x}}(t) = &m_1\left(\mathbf{x}(t)\right)\sum_{i=1}^{p} \widetilde{w}_i\left(\mathbf{x}(t)\right)\left(\widetilde{\mathbf{A}}_i\mathbf{x}(t) + \widetilde{\mathbf{B}}_i\mathbf{u}(t)\right)\\ &+m_2\left(\mathbf{x}(t)\right)\sum_{i=1}^{p} w_i\left(\mathbf{x}(t)\right)\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\right) \quad \text{(A7)}\end{aligned}$$

where $m_1(\mathbf{x}(t))$ is a value to indicate if the system states are working inside the operating domain near the origin. $1 \geq m_1(\mathbf{x}(t)) > 0$ indicates that the system states are inside the operating domain of the local fuzzy model of (A3); otherwise, $m_1(\mathbf{x}(t)) = 0$ indicates that the local fuzzy model is invalid. Based on this fact, and from (A6) and (A7), it can be seen that (A6) holds when $1 \geq m_1(\mathbf{x}(t)) > 0$, and $m_2(\mathbf{x}(t)) = 1 - m_1(\mathbf{x}(t)) \geq 0$. (A6) holds as a special case that both $m_1(\mathbf{x}(t))$ and $m_2(\mathbf{x}(t)) \neq 0$. Consequently, the property of (12) is proven. **QED**

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