Stability and Robustness Analysis and Gain Design for Fuzzy Control Systems Subject to Parameter Uncertainties [1]

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Abstract - **This paper presents the stability and robustness analysis for fuzzy control systems subject to parameter uncertainties, and the design of gains for the fuzzy controller. Based on Lyapunov stability theory, stability and robustness conditions will be derived. The stability conditions involves only $p+1$ linear matrix inequalities, where p is the number of rules of the fuzzy controller. A design methodology for the gains of the fuzzy controller will be given. An application example on stabilizing a nonlinear mass-spring-damper system subject to parameter uncertainties will be given to illustrate the merits of the proposed fuzzy controller.**

I. Introduction

In this paper, we present a stability and robustness analysis of fuzzy model based control systems. Also, we offer a design methodology for the gains of the fuzzy controller. We start with a system that comprises a fuzzy plant model [1] with parameter uncertainties and a fuzzy controller [3] connected in closed-loop. The fuzzy plant model represents a nonlinear system as a weighted sum of a number of sub-systems. Similarly, the fuzzy controller is a weighted sum of a number of sub-controllers. Wang *et. al.* derived a stability condition [3] for this class of nonlinear systems by using Lyapunov stability theory. A sufficient condition for the system stability is obtained by finding a common Lyapunov function for all the fuzzy sub-control systems. For a fuzzy plant model with p rules, a fuzzy controller with p rules is used to close the feedback loop, and $p(p+1)/2$ Lyapunov conditions are required. In this paper, the number of Lyapunov conditions is reduced to $p+1$. Then, the common Lyapunov function is easier to be found. Also, we provide a way of designing the gains of the fuzzy controller. The task of finding the common Lyapunov function can readily be formulated into a linear matrix inequality (LMI) problem [5]. By applying some LMI tools, we can find the design solution easily.

This paper is organized as follows. In section II, the fuzzy plant model and the fuzzy controller will be introduced. In section III, we shall analyze the system stability and robustness of the fuzzy model based control system and provide a way of designing the gains of the fuzzy controller to guarantee system stability. In section IV, an example on stabilizing a mass-spring-damper system will be given to illustrate the merits of the proposed methodology. In section V, a conclusion will be drawn.

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II. Fuzzy Plant Model and Fuzzy Controller

We consider an uncertain multivariable nonlinear control system. The plant is represented by a fuzzy plant model incorporating the information of parameter uncertainties. A fuzzy controller is to be designed to close the feedback loop.

A. Fuzzy Plant Model with Parameter Uncertainties

Let p be the number of fuzzy rules describing the uncertain nonlinear plant. The i-th rule is of the following format,

Rule i : IF $x_1(t)$ is M_1^i and ... and $x_n(t)$ is M_n^i

$$\text{THEN } \dot{\mathbf{x}}(t) = (\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t) + (\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{u}(t) \quad (1)$$

where M_k^i is a fuzzy term of rule i corresponding to the state $x_k(t)$, $k = 1, 2, \ldots, n$, $i = 1, 2, \ldots, p$; $\Delta\mathbf{A}_i \in \Re^{n\times n}$ and $\Delta\mathbf{B}_i \in \Re^{n\times m}$ are the uncertainties of $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ respectively; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The plant dynamics is described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\left[(\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t) + (\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{u}(t)\right] \quad (2)$$

where,

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, \; w_i(\mathbf{x}(t)) \in [0, \; 1] \text{ for all } i \quad (3)$$

is a known nonlinear function of $\mathbf{x}(t)$ and

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(x_1(t)) \times \mu_{\mathrm{M}_2^i}(x_2) \times \cdots \times \mu_{\mathrm{M}_n^i}(x_n)}{\sum_{j=1}^{p}\left(\mu_{\mathrm{M}_1^j}(x_1(t)) \times \mu_{\mathrm{M}_2^j}(x_2(t)) \times \cdots \times \mu_{\mathrm{M}_n^j}(x_n(t))\right)} \quad (4)$$

$\mu_{\mathrm{M}_k^i}(x_k(t))$ is the grade of membership of the fuzzy term M_k^i.

B. Fuzzy Controller

A fuzzy controller having p fuzzy rules is to be designed for the plant. The j-th rule of the fuzzy controller is of the following format:

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Rule j: IF $x_1(t)$ is M_1^j and … and $x_n(t)$ is M_n^j

THEN $\mathbf{u}(t) = m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t)$ (5)

where $\mathbf{G}_j \in \Re^{m \times n}$ is the feedback gain of rule j, $j = 1, 2, \ldots, p$; $m_j(\mathbf{x}(t))$ is a scalar and has the following property,

$$\sum_{j=1}^{p} w_j(\mathbf{x}(t))m_j(\mathbf{x}(t)) = 1 \tag{6}$$

Both $\mathbf{G}_j$ and $m_j(\mathbf{x}(t))$ are to be designed. Then, the inferred output of the fuzzy controller is given by

$$\mathbf{u}(t) = \sum_{j=1}^{p} w_j(\mathbf{x}(t))m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \tag{7}$$

III. Stability and Robustness Analysis and Gain Design

In the following, we are going to analyze the stability and robustness of the fuzzy control system and give the design of $m_j(\mathbf{x}(t))$. From (2) and (7), writing $w_i(\mathbf{x}(t))$ as w_i and $m_j(\mathbf{x}(t))$ as m_j, the closed-loop system (fuzzy control system) is given by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i\left[(\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t) + (\mathbf{B}_i + \Delta\mathbf{B}_i)\sum_{j=1}^{p} w_j m_j \mathbf{G}_j\mathbf{x}(t)\right] \tag{8}$$

By making use of the properties of (3) and (6), $\sum_{i=1}^{p} w_i = \sum_{j=1}^{p} w_j m_j = 1$. Then (8) becomes,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j m_j\left[(\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t) + (\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{G}_j\mathbf{x}(t)\right]$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j m_j\left[\left(\mathbf{H}_{ij} + \Delta\mathbf{H}_{ij}\right)\mathbf{x}(t)\right] \tag{9}$$

where,

$$\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j \tag{10}$$
$$\Delta\mathbf{H}_{ij} = \Delta\mathbf{A}_i + \Delta\mathbf{B}_i\mathbf{G}_j \tag{11}$$

From (9) and making use of the property that $\sum_{i=1}^{p} w_i = 1$, we have,

$$\dot{\mathbf{x}}(t) = \mathbf{H}_m\mathbf{x}(t) + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j m_j\left[\left(\mathbf{H}_{ij} - \mathbf{H}_m + \Delta\mathbf{H}_{ij}\right)\mathbf{x}(t)\right] \tag{12}$$

where $\mathbf{H}_m \in \Re^{n \times n}$ is a stable matrix to be designed. To investigate the stability and robustness of the fuzzy control system of (12), the following Lyapunov function is employed,

$$V(\mathbf{x}(t)) = \frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) \tag{13}$$

where $\mathbf{P} \in \Re^{n \times n}$ is a symmetric positive definite matrix. Then,

$$\dot{V}(\mathbf{x}(t)) = \frac{1}{2}\left(\dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}}(t)\right) \tag{14}$$

From (12) and (14), we have,

$$\dot{V}(\mathbf{x}(t)) =$$
$$\frac{1}{2}\left\{\left(\mathbf{H}_m\mathbf{x}(t) + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j m_j\left[\left(\mathbf{H}_{ij} - \mathbf{H}_m + \Delta\mathbf{H}_{ij}\right)\mathbf{x}(t)\right]\right)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t)\right.$$
$$\left. + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\left(\mathbf{H}_m\mathbf{x}(t) + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j m_j\left[\left(\mathbf{H}_{ij} - \mathbf{H}_m + \Delta\mathbf{H}_{ij}\right)\mathbf{x}(t)\right]\right)\right\}$$
$$= \frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\left(\mathbf{H}_m{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_m\right)\mathbf{x}(t)$$
$$+ \frac{1}{2}\left\{\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j m_j \mathbf{x}(t)^{\mathrm{T}}\left[\left(\mathbf{H}_{ij} - \mathbf{H}_m + \Delta\mathbf{H}_{ij}\right)^{\mathrm{T}}\mathbf{P}\right.\right.$$
$$\left.\left. + \mathbf{P}\left(\mathbf{H}_{ij} - \mathbf{H}_m + \Delta\mathbf{H}_{ij}\right)\right]\mathbf{x}(t)\right\}$$

$$= -\frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_m\mathbf{x}(t) - \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j m_j \mathbf{x}(t)^{\mathrm{T}}\left(\mathbf{Q}_{ij} - \Delta\mathbf{Q}_{ij}\right)\mathbf{x}(t) \tag{15}$$

where,

$$\mathbf{Q}_m = -\mathbf{H}_m{}^{\mathrm{T}}\mathbf{P} - \mathbf{P}\mathbf{H}_m \tag{16}$$
$$\mathbf{Q}_{ij} = -\left(\mathbf{H}_{ij} - \mathbf{H}_m\right)^{\mathrm{T}}\mathbf{P} - \mathbf{P}\left(\mathbf{H}_{ij} - \mathbf{H}_m\right) \tag{17}$$
$$\Delta\mathbf{Q}_{ij} = \Delta\mathbf{H}_{ij}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{H}_{ij} \tag{18}$$

$\mathbf{Q}_m$ and $\mathbf{Q}_{ij}$ are a symmetric positive definite matrix and a symmetric matrix respectively.

From (15), we have,

$$\dot{V}(\mathbf{x}(t)) = -\frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_m\mathbf{x}(t) - \frac{1}{2}\sum_{j=1}^{p} w_j m_j \mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)$$
$$+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j m_j \mathbf{x}(t)^{\mathrm{T}}\Delta\mathbf{Q}_{ij}\mathbf{x}(t)$$
$$\le -\frac{1}{2}\lambda_{\min}\left(\mathbf{Q}_m\right)\|\mathbf{x}(t)\|^2 - \frac{1}{2}\sum_{j=1}^{p} w_j m_j \mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)$$
$$+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j m_j \left\|\Delta\mathbf{Q}_{ij}\right\|\|\mathbf{x}(t)\|^2 \tag{19}$$

where $\lambda_{\min}\left(\mathbf{Q}_m\right)$ denotes the minimum eigenvalue of $\mathbf{Q}_m$, $\|\cdot\|$ denotes the l_2 norm for vectors and l_2 induced norm for matrices. From (19) and the property that $\sum_{i=1}^{p} w_i = \sum_{j=1}^{p} w_j m_j = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j m_j = 1$, we have,

$$\dot{V}(\mathbf{x}(t)) \le -\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j m_j\left(\lambda_{\min}\left(\mathbf{Q}_m\right) - \left\|\Delta\mathbf{Q}_{ij}\right\|\right)\|\mathbf{x}(t)\|^2$$
$$- \frac{1}{2}\sum_{j=1}^{p} w_j m_j \mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)$$

$$\le -\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j m_j \left(\lambda_{\min}(\mathbf{Q}_m) - \max_{i,j}\left\|\Delta\mathbf{Q}_{ij}\right\|_{\max}\right)\left\|\mathbf{x}(t)\right\|^2 - \frac{1}{2}\sum_{j=1}^{p} w_j m_j \mathbf{x}(t)^{\mathrm{T}} \sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t) \tag{20}$$

where,

$$\max_{i,j}\left\|\Delta\mathbf{Q}_{ij}\right\|_{\max} \ge \left\|\Delta\mathbf{Q}_{ij}\right\|_{\max} \ge \left\|\Delta\mathbf{Q}_{ij}\right\| \text{ for all } i \text{ and } j \tag{21}$$

From (20), we have,

$$\dot{V}(\mathbf{x}(t)) \le -\frac{1}{2}\left(\lambda_{\min}(\mathbf{Q}_m) - \max_{i,j}\left\|\Delta\mathbf{Q}_{ij}\right\|_{\max}\right)\left\|\mathbf{x}(t)\right\|^2 - \frac{1}{2}\sum_{j=1}^{p} w_j m_j \mathbf{x}(t)^{\mathrm{T}} \sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t)$$

$$\le -\frac{1}{2}\varepsilon\left\|\mathbf{x}(t)\right\|^2 - \frac{1}{2}\sum_{j=1}^{p} w_j m_j \mathbf{x}(t)^{\mathrm{T}} \sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t) \tag{22}$$

where,

$$\lambda_{\min}(\mathbf{Q}_m) - \max_{i,j}\left\|\Delta\mathbf{Q}_{ij}\right\|_{\max} = \varepsilon \tag{23}$$

Let ε be a nonzero positive scalar and we design m_j as follows,

$$m_j = \begin{cases} 1 & \text{if } w_j = 0 \\ \dfrac{n_j}{w_j \sum_{k=1}^{p} n_k} & \text{otherwise} \end{cases} \tag{24}$$

$$n_j = \begin{cases} \dfrac{1}{\sum_{k=1}^{p} sign(w_k)} & \text{if } \sum_{k=1}^{p} sign(w_k)\left(\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ik}\mathbf{x}(t)\right) = 0 \text{ and } w_j \ne 0 \\ 0 & \text{if } w_j = 0 \\ \mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t) & \text{otherwise} \end{cases} \tag{25}$$

$$sign(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

and

$$\sum_{k=1}^{p} \mathbf{Q}_{ik} > \mathbf{0} \text{ for all } i = 1, 2, \ldots, p \tag{27}$$

It should be noted that the denominator of $\dfrac{1}{\sum_{k=1}^{p} sign(w_k)}$ in (25) will never be equal to zero as at least one of the w_k is greater than zero (this is a property of the fuzzy plant model). By choosing the m_j as (24), we can see that the property of (6) is satisfied.

From (22) to (25) and considering the case that all $w_j \ne 0$ such that $m_j = \dfrac{\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t)}{w_j \sum_{k=1}^{p}\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ik}\mathbf{x}(t)}$, then, we have,

$$\dot{V}(\mathbf{x}(t)) \le -\frac{1}{2}\varepsilon\left\|\mathbf{x}(t)\right\|^2 - \frac{1}{2}\frac{\sum_{j=1}^{p}\left(\sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_{ij}\mathbf{x}(t)\right)^2}{\sum_{k=1}^{p}\sum_{i=1}^{p}\mathbf{x}(t)^{\mathrm{T}} w_i \mathbf{Q}_{ik}\mathbf{x}(t)}$$

$$= -\frac{1}{2}\varepsilon\left\|\mathbf{x}(t)\right\|^2 - \frac{1}{2}\frac{\sum_{j=1}^{p}\left(\sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_{ij}\mathbf{x}(t)\right)^2}{\sum_{i=1}^{p}\mathbf{x}(t)^{\mathrm{T}} w_i \sum_{k=1}^{p}\mathbf{Q}_{ik}\mathbf{x}(t)} \tag{28}$$

From (27), as $\sum_{k=1}^{p}\mathbf{Q}_{ik}$, i = 1, 2, …, p, are positive definite matrices, $\dfrac{\sum_{j=1}^{p}\left(\sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_{ij}\mathbf{x}(t)\right)^2}{\sum_{i=1}^{p}\mathbf{x}(t)^{\mathrm{T}} w_i \sum_{k=1}^{p}\mathbf{Q}_{ik}\mathbf{x}(t)}$ in (28) is semi-positive definite. Thus, from (28), we have,

$$\dot{V}(\mathbf{x}(t)) \le -\frac{1}{2}\varepsilon\left\|\mathbf{x}(t)\right\|^2 \le 0 \tag{29}$$

Hence, we can concluse that the uncertain fuzzy control system of (12) is asymptotically stable, i.e., $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. It can be found that the same result is obtained under different cases of m_j. The analysis result and the design of the m_j are summarized by the following Lemma.

Lemma 1: *An uncertain fuzzy control system as shown in* (12) *is guaranteed to be asyptopically stable if the following conditions are satisfied,*

(i). There exists a solution $\mathbf{P}$ *for the following (p+1) LMIs,*

$$\begin{cases} \mathbf{Q}_m > \mathbf{0} \\ \sum_{k=1}^{p}\mathbf{Q}_{ik} > \mathbf{0} & \text{for all } i \end{cases}$$

where $\mathbf{Q}_m = -\mathbf{H}_m^{\mathrm{T}}\mathbf{P} - \mathbf{P}\mathbf{H}_m$, $\mathbf{H}_m$ *is a stable matrix,* $\mathbf{Q}_{ik} = -(\mathbf{H}_{ik} - \mathbf{H}_m)^{\mathrm{T}}\mathbf{P} - \mathbf{P}(\mathbf{H}_{ik} - \mathbf{H}_m)$, $\Delta\mathbf{Q}_{ij} = \Delta\mathbf{H}_{ij}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{H}_{ij}$, $\mathbf{H}_{ik} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_k$.

(ii). There exists a nonzero positive scalar $\varepsilon = \lambda_{\min}(\mathbf{Q}_m) - \max_{i,j}\left\|\Delta\mathbf{Q}_{ij}\right\|_{\max}$

where $\max_{i,j}\left\|\Delta\mathbf{Q}_{ij}\right\|_{\max} \ge \left\|\Delta\mathbf{Q}_{ij}\right\|_{\max} \ge \left\|\Delta\mathbf{Q}_{ij}\right\|$ *for all i and j,* $\Delta\mathbf{H}_{ij} = \Delta\mathbf{A}_i + \Delta\mathbf{B}_i\mathbf{G}_j$, $\lambda_{\min}(\mathbf{Q}_m)$ *denotes the minimum eigenvalue of* $\mathbf{Q}_m$. $\|\cdot\|$ *denotes the* l_2 *norm for vectors and* l_2 *induced norm for matrices.*

(iii). The gains of the fuzzy controller sub-controllers, m_j, j = *1, 2, …, p, are designed as follows,*

$$m_j = \begin{cases} 1 & \text{if } w_j = 0 \\ \dfrac{n_j}{w_j \sum_{k=1}^{p} n_k} & \text{otherwise} \end{cases}$$

$$n_j = \begin{cases} \dfrac{1}{\sum_{k=1}^{p} sign(w_k)} & \text{if } \sum_{k=1}^{p} sign(w_k)\left(\mathbf{x}(t)^T \sum_{i=1}^{p} w_i \mathbf{Q}_{ik}\mathbf{x}(t)\right) = 0 \text{ and } w_j \neq 0 \\ 0 & \text{if } w_j = 0 \\ \mathbf{x}(t)^T \sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t) & \text{otherwise} \end{cases}$$

$$sign(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{otherwise} \end{cases}$$

From Lemma 1, we can see that the number of linear matrix inequalities is $p+1$ instead of $p(p+1)/2$ as stated in [3]. The procedure for finding the fuzzy controller can be summarized as follows.

Step I) Obtain the mathematical model of the nonlinear plant to be controlled.

Step II) Obtain the fuzzy plant model for the system stated in step I) by means of a fuzzy modeling method, for example, as that proposed in [1, 4].

Step III) Choose the gains ($\mathbf{G}_j$) of the fuzzy controller so that $\mathbf{H}_{ij}$ are stable.

Step IV) Choose a stable matrix $\mathbf{H}_m$ and, find $\mathbf{P}$ by solving the $p+1$ linear matrix inequalities and $\lambda_{\min}(\mathbf{Q}_m) - \max_{i,j}\|\Delta\mathbf{Q}_{ij}\|_{\max} > 0$ as stated in Lemma 1. If $\mathbf{P}$ cannot be found, go back to step III) and choose other gains ($\mathbf{G}_j$) of the fuzzy controller.

Step V) Design the membership functions of the fuzzy controller based on Lemma 1.

IV. Application Example

An application example on stabilizing a mass-spring-damper system [2] is given in this section. A fuzzy controller will be used to control the plant, and simulation results will be given.

Step I) Fig. 1 shows the diagram of a mass-spring-damper system. Its dynamic equation is given by,

$$M\ddot{x}(t) + g(x(t), \dot{x}(t)) + f(x(t)) = \phi(\dot{x}(t))u(t) \tag{30}$$

where M is the mass and u is the force, $f(x(t))$ depicts the spring nonlinearity, $g(x(t), \dot{x}(t))$ depicts the damper nonlinearity, and $\phi(\dot{x}(t))$ depicts the input nonlinearity. Let,

$$\begin{aligned} g(x(t), \dot{x}(t)) &= c_1 x(t) + c_2 \dot{x}(t) + d_1(t)x(t) + d_2(t)\dot{x}(t) \\ f(x(t)) &= c_3 x(t) + c_4 x(t)^3 \\ \phi(\dot{x}(t)) &= 1 + c_5 \dot{x}(t)^2 + d_3\cos(\dot{x}(t)) \end{aligned} \tag{31}$$

The operating range of the states is assumed to be within the interval $[-1.5 \quad 1.5]$ (the fuzzy plant model derived later will not be valid when the system states are outside this interval). The parameters are chosen as follows: $M = 1.0$, $c_1 = 0$, $c_2 = 1$, $c_3 = 0.01$, $c_4 = 0.1$ and $c_5 = 0.13$. The system then becomes,

$$\begin{aligned} \ddot{x}(t) = &-\dot{x}(t) - 0.01x(t) - 0.1x(t)^3 - d_1(t)x(t) - d_2(t)\dot{x}(t) \\ &+ \left(1.4387 - 0.13\dot{x}(t)^2 + d_3(t)\cos(\dot{x}(t))\right)u(t) \end{aligned} \tag{32}$$

where $d_1(t)$, $d_2(t)$ and $d_3(t)$ are the parameter uncertainties. Practically, they should be unknown within given bounds. In this example, we assumed that they are defined as follows,

$d_1(t) = \dfrac{d_1^U + d_1^L}{2} + (c_1^L - \dfrac{d_1^U + d_1^L}{2})\cos(t)$ so that

$$d_1(t) \in [d_1^L, d_1^U] \tag{33}$$

$d_2(t) = \dfrac{d_2^U + d_2^L}{2} + (c_2^L - \dfrac{d_2^U + d_2^L}{2})\cos(t)$ so that

$$d_2(t) \in [d_2^L, d_2^U] \tag{34}$$

$d_3(t) = \dfrac{d_3^U + d_3^L}{2} + (c_3^L - \dfrac{d_3^U + d_3^L}{2})\sin(t)$ so that

$$d_3(t) \in [d_3^L, d_3^U] \tag{35}$$

$d_1^L = -0.4$, $d_1^U = 0.4$, $d_2^L = -0.03$, $d_2^U = 0.03$, $d_3^L = -0.01$ and $d_3^U = 0.01$.

Step II) The nonlinear plant can be represented by a fuzzy model with the following fuzzy rules,

Rule i: IF $x(t)$ is M_1^i AND $\dot{x}(t)$ is M_2^i

THEN $\dot{\mathbf{x}}(t) = (\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t) + (\mathbf{B}_i + \Delta\mathbf{B}_i)u(t)$, i = 1, 2, 3, 4 (36)

where the membership functions of M_j^i, i = 1, 2, 3, 4, j = 1, 2, are shown in Fig. 2 and 3 respectively;

$$\begin{aligned} &\mu_{\mathrm{M}_1^1}(x(t)) = \mu_{\mathrm{M}_1^2}(x(t)) = 1 - \frac{x(t)^2}{2.25}, \mu_{\mathrm{M}_1^3}(x(t)) = \mu_{\mathrm{M}_1^4}(x(t)) = \frac{x(t)^2}{2.25} \\ &\mu_{\mathrm{M}_2^1}(\dot{x}(t)) = \mu_{\mathrm{M}_2^3}(\dot{x}(t)) = 1 - \frac{\dot{x}(t)^2}{6.75}, \mu_{\mathrm{M}_2^2}(x(t)) = \mu_{\mathrm{M}_2^4}(x(t)) = \frac{\dot{x}(t)^2}{6.75} \end{aligned} \tag{37}$$

$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$, $\mathbf{A}_1 = \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ -0.01 & -1 \end{bmatrix}$,

$\mathbf{A}_3 = \mathbf{A}_4 = \begin{bmatrix} 0 & 1 \\ -0.235 & -1 \end{bmatrix}$;

$\Delta\mathbf{A}_1 = \Delta\mathbf{A}_2 = \Delta\mathbf{A}_3 = \Delta\mathbf{A}_4 = \begin{bmatrix} 0 & 0 \\ -d_1(t) & -d_2(t) \end{bmatrix}$;

$\mathbf{B}_1 = \mathbf{B}_3 = \begin{bmatrix} 0 \\ 1.4387 \end{bmatrix}$, $\mathbf{B}_2 = \mathbf{B}_4 = \begin{bmatrix} 0 \\ 0.5613 \end{bmatrix}$,

$\Delta\mathbf{B}_1 = \Delta\mathbf{B}_2 = \Delta\mathbf{B}_3 = \Delta\mathbf{B}_4 = \begin{bmatrix} 0 \\ d_3(t)\cos(\dot{x}(t)) \end{bmatrix}$. (Details about the derivation of the fuzzy plant model for the mass-spring-damper system can be found in [2].)

Step III) When a four-rule fuzzy controller is designed for the plant of (32), we have,

Rule j: IF $x(t)$ is M_1^j AND $\dot{x}(t)$ is M_2^j

$$\text{THEN } \mathbf{u}(t) = m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t),\ j = 1, 2, 3, 4 \tag{38}$$

The feedback gains of the fuzzy controller are arbitrarily chosen as $\mathbf{G}_1 = [-2.7732 \quad -2.0852]$, $\mathbf{G}_2 = [-7.1085 \quad -5.3447]$, $\mathbf{G}_3 = [-2.6169 \quad -2.0852]$ and $\mathbf{G}_4 = [-6.7076 \quad -5.3447]$ (so that $\mathbf{H}_{11} = \mathbf{H}_{22} = \mathbf{H}_{33} = \mathbf{H}_{44} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$.)

Step IV) To check the stability, we choose $\mathbf{H}_m = \begin{bmatrix} 0 & 0.25 \\ -1.3 & -1.3 \end{bmatrix}$ which is a stable matrix. We have 5 linear matrix inequalities. One common solution of them is $\mathbf{P} = \begin{bmatrix} 0.2267 & 0.0197 \\ 0.0197 & 0.0259 \end{bmatrix}$. Under this $\mathbf{P}$, we can find that $\lambda_{\min}(\mathbf{Q}_m) = 0.0310$. The values of $\|\Delta\mathbf{Q}_{ij}\|_{\max}$, $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$, are tabulated in Table I. In this table, we can see that $\max_{i,j}\|\Delta\mathbf{Q}_{ij}\|_{\max} = 0.0305$. By Lemma 1, we can conclude that the closed-loop system is stable.

Step V) According to Lemma 1, the membership functions are designed as,

$$m_j = \begin{cases} 1 & \text{if } w_j = 0 \\ \dfrac{n_j}{w_j \sum_{k=1}^{4} n_k} & \text{otherwise} \end{cases} \tag{39}$$

$$n_j = \begin{cases} \dfrac{1}{\sum_{k=1}^{4} sign(w_k)} & \text{if } \sum_{k=1}^{4} sign(w_k)\left(\mathbf{x}(t)^T \sum_{i=1}^{4} w_i \mathbf{Q}_{ik}\mathbf{x}(t)\right) = 0 \text{ and } w_j \neq 0 \\ 0 & \text{if } w_j = 0 \\ \mathbf{x}(t)^T \sum_{i=1}^{4} w_i \mathbf{Q}_{ij}\mathbf{x}(t) & \text{otherwise} \end{cases} \tag{40}$$

for $j = 1, 2, 3, 4$

Fig. 4 to 7 show the responses of the system states with (dotted lines) and without (solid lines) parameter uncertainties under the initial conditions of $\mathbf{x}(0) = [1.5 \quad 0]^T$ and $\mathbf{x}(0) = [-1.5 \quad 0]^T$ respectively. We can see that the mass-spring-damper system with and without parameter uncertainties can be stabilized by the designed fuzzy controller.

V. Conclusion

The stability and robustness of fuzzy control systems subject to parameter uncertainties have been analyzed. Stability and robustness conditions have been derived. The stability condition involves p+1 LMIs, instead of $p(p+1)/2$ in [2]. A design of the gains of fuzzy sub-controllers has been presented. An application example has been given to show the stabilizability, the robustness and the design procedure of the fuzzy controller.

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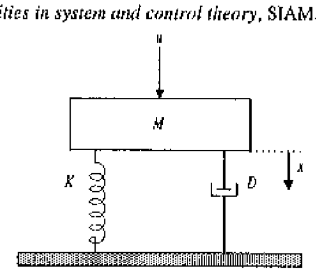

Fig. 1. A mass-spring-damper system.

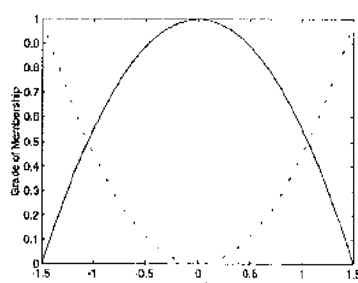

Fig. 2. Membership functions of the fuzzy plant model of the nonlinear mass-spring-damper system: $\mu_{\mathrm{M}_1^1}(x) = \mu_{\mathrm{M}_1^2}(x) = 1 - \dfrac{x^2}{2.25}$ (solid line), $\mu_{\mathrm{M}_1^3}(x) = \mu_{\mathrm{M}_1^4}(x) = \dfrac{x^2}{2.25}$ (dotted line).

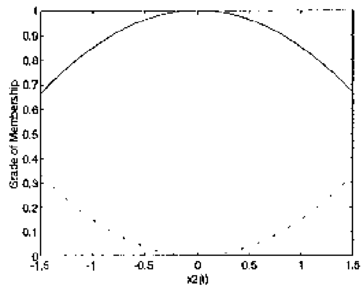

Fig. 3. Membership functions of the fuzzy plant model of the nonlinear mass-spring-damper system: $\mu_{M_1^1}(\dot{x}) = \mu_{M_1^2}(\dot{x}) = 1-\frac{\dot{x}^2}{6.75}$ (solid line), $\mu_{M_1^3}(\dot{x}) = \mu_{M_1^4}(\dot{x}) = \frac{\dot{x}^2}{6.75}$ (dotted line).

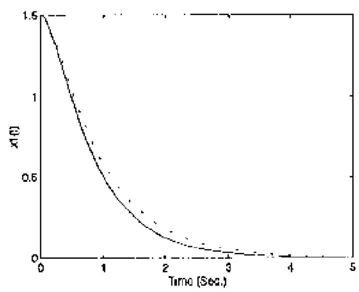

Fig. 4. Responses of $x_1(t)$ of the nonlinear mass-spring-damper system with (dotted line) and without (solid line) parameter uncertainties under the initial condition $\mathbf{x}(0) = [1.5 \quad 0]^T$.

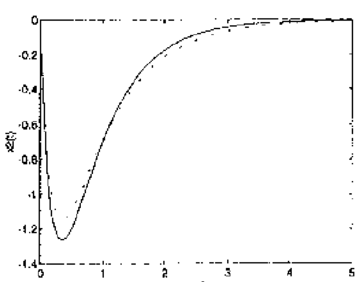

Fig. 5. Responses of $x_2(t)$ of the nonlinear mass-spring-damper system with (dotted line) and without (solid line) parameter uncertainties under the initial condition $\mathbf{x}(0) = [-1.5 \quad 0]^T$.

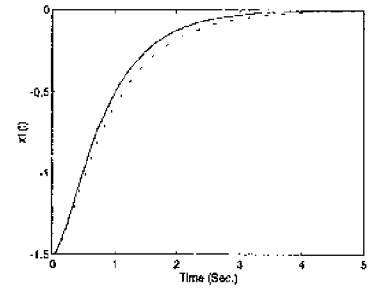

Fig. 6. Responses of $x_1(t)$ of the nonlinear mass-spring-damper system with (dotted line) and without (solid line) parameter uncertainties under the initial condition $\mathbf{x}(0) = [-1.5 \quad 0]^T$.

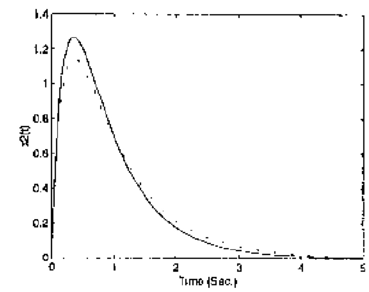

Fig. 7. Responses of $x_2(t)$ of the nonlinear mass-spring-damper system with (dotted line) and without (solid line) parameter uncertainties under the initial condition $\mathbf{x}(0) = [1.5 \quad 0]^T$.

i, j	$\|\Delta \mathbf{Q}_{ij}\|_{max}$
1, 1	0.0268
1, 2	0.0305
1, 3	0.0267
1, 4	0.0302
2, 1	0.0268
2, 2	0.0305
2, 3	0.0267
2, 4	0.0302
3, 1	0.0268
3, 2	0.0305
3, 3	0.0267
3, 4	0.0302
4, 1	0.0268
4, 2	0.0305
4, 3	0.0267
4, 4	0.0302

Table 1. Values of $\|\Delta \mathbf{Q}_{ij}\|_{max}$.