Fuzzy State Feedback Controller for Nonlinear Systems: Stability Analysis and Design [1]

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***Abstract* - This paper presents the stability analysis of the fuzzy control system consisting of a nonlinear plant and a fuzzy controller and the design of the membership functions of the fuzzy controller. The nonlinear plant is represented by a fuzzy model having *p* rules. A *c*-rule fuzzy state-feedback controller is used to close the feedback loop. A design methodology for the fuzzy controller will be provided. Under this design, the stability condition is reduced to *p* linear matrix inequalities (LMI). An application example on stabilizing a mass-spring-damper system will be given.**

I. Introduction

Fuzzy control has been found capable of tackling ill-defined nonlinear plant [1-2]. However, without carrying out analyses, the design of the fuzzy controller may come with no guarantee of system stability. By using a fuzzy plant model [3, 7], we can express a nonlinear system as a weighted sum of some linear sub-systems. Under this structure, some linear control techniques and stability analysis methods can be applied. Some authors proposed a fuzzy controller to control this class of nonlinear system. The fuzzy controller is a weighted sum of some linear state feedback controllers [4, 6]. A $p \times c$ linear matrix inequality (LMI) problem was derived in [4], where p and c are the numbers of fuzzy rules of the fuzzy plant model and the fuzzy controller respectively. In case, $p = c$, and the premises of the fuzzy plant model and fuzzy controller are the same, $p(p+1)$ LMI conditions were derived in [6]. In this paper, we further reduce the number of LMI condition to p which is independent of the number rules of the fuzzy controller. We also provide a methodology for designing the membership functions of the fuzzy controller. An application example on stabilizing a nonlinear mass-spring-damper system will be given to verify the analysis results of this paper.

II. Fuzzy Plant Model and Fuzzy Controller

We consider an uncertain multivariable nonlinear control system. The plant is represented by a fuzzy plant model. A fuzzy state feedback controller is to be designed to close the feedback loop.

A. Fuzzy Plant Model

Let p be the number of fuzzy rules describing the nonlinear plant [3, 7]. The i-th rule is of the following format,

Rule i : IF $x_1(t)$ is M_1^i and … and $x_n(t)$ is M_n^i

$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(t) + \mathbf{B}_i\mathbf{u}(t) \tag{1}$$

where M_k^i is a fuzzy term of rule i corresponding to the state $x_k(t)$, $k = 1, 2, \ldots, n$, $i = 1, 2, \ldots, p$; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are the system matrix and input matrix respectively; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The system dynamics is described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\right) \tag{2}$$

where,

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1,\ w_i(\mathbf{x}(t)) \in \begin{bmatrix}0 & 1\end{bmatrix} \text{ for all } i \tag{3}$$

is a known nonlinear function of $\mathbf{x}(t)$ and

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(x_1(t)) \times \mu_{\mathrm{M}_2^i}(x_2) \times \cdots \times \mu_{\mathrm{M}_n^i}(x_n)}{\sum_{j=1}^{p}\left(\mu_{\mathrm{M}_1^j}(x_1(t)) \times \mu_{\mathrm{M}_2^j}(x_2(t)) \times \cdots \times \mu_{\mathrm{M}_n^j}(x_n(t))\right)} \tag{4}$$

$\mu_{\mathrm{M}_k^i}(x_k(t))$ is the grade of membership of the fuzzy term M_k^i.

B. Fuzzy State Feedback Controller

A fuzzy state feedback controller having c fuzzy rules as is to be designed for the plant. The j-th rule of the fuzzy controller is of the following format:

$$\text{Rule } j\text{: IF } \mathbf{x}(t) \text{ is } \mathrm{N}^j \text{ THEN } \mathbf{u}(t) = \mathbf{G}_j\mathbf{x}(t) \tag{5}$$

where N^j is a fuzzy term of rule j corresponding to the state vecotr $\mathbf{x}(t)$, $j = 1, 2, \ldots, c$; $\mathbf{G}_j \in \Re^{m\times n}$ is the feedback gain of rule j, to be designed. Then, the inferred output of the fuzzy state feedback controller is given by,

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \tag{6}$$

where,

$$\sum_{j=1}^{c} m_j(\mathbf{x}(t)) = 1 \tag{7}$$

$m_j(\mathbf{x}(t))$ is a nonlinear function of $\mathbf{x}(t)$ defined as,

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$$m_j(\mathbf{x}(t)) = \frac{\mu_{N^j}(\mathbf{x}(t))}{\sum_{k=1}^{c}\left(\mu_{N^k}(\mathbf{x}(t))\right)} \tag{8}$$

$\mu_{N^j}(\mathbf{x}(t))$, which is to be designed, is the grade of membership of the fuzzy term $\mathbf{N}^j$.

III. Stability Analysis and Design

In this section, stability of the fuzzy control system formed by the fuzzy plant model and fuzzy state feedback controller connected in closed-loop will be investigated. A design methodology of $m_j(\mathbf{x}(t))$, $j = 1, 2, \ldots, c$, will be provided under the consideration of the closed-loop system stability. For simplicity, we write $w_i(\mathbf{x}(t))$ as w_i, $m_j(\mathbf{x}(t))$ as m_j. From (2), (6) and the property of $\sum_{i=1}^{p} w_i = \sum_{j=1}^{c} m_j = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j = 1$, the fuzzy control system becomes,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{p} w_i\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\sum_{j=1}^{c} m_j\mathbf{G}_j\mathbf{x}(t)\right)\\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\mathbf{x}(t)\\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{H}_{ij}\mathbf{x}(t)\end{aligned} \tag{9}$$

where,

$$\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j \tag{10}$$

To investigate the stability of (9), we employ the following Lyapunov function in quadratic form,

$$V = \frac{1}{2}\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) \tag{11}$$

where $(\cdot)^{\mathrm{T}}$ denotes the transpose of a vector or matrix, $\mathbf{P} \in \Re^{n\times n}$ is a symmetric positive definite matrix. Differentiating (11), we have,

$$\dot{V} = \frac{1}{2}\left(\dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}}(t)\right) \tag{12}$$

From (9) and (12), we have,

$$\begin{aligned}\dot{V} &= \frac{1}{2}\left[\left(\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\mathbf{H}_{ij}\mathbf{x}(t)\right)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\mathbf{H}_{ij}\mathbf{x}(t)\right]\\ &= \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\mathbf{x}(t)^{\mathrm{T}}\left(\mathbf{H}_{ij}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ij}\right)\mathbf{x}(t)\end{aligned} \tag{13}$$

Let,

$$\mathbf{Q}_{ij} = -\left(\mathbf{H}_{ij}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ij}\right) \tag{14}$$

From (13),

$$\begin{aligned}\dot{V} &= -\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_{ij}\mathbf{x}(t)\\ &= -\frac{1}{2}\sum_{j=1}^{c} m_j\left(\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)\right)\end{aligned} \tag{15}$$

We design the membership functions $\mu_{N^j}(\mathbf{x}(t))$ in (8) as follows,

$$\mu_{N^1}(\mathbf{x}(t)) = \begin{cases}\left(1 - \dfrac{\sum_{j=2}^{c}\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)}{\sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)\right|}\right) & \text{if } \sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)\right| \neq 0\\ \dfrac{1}{c} & \text{otherwise}\end{cases} \tag{16}$$

$$\mu_{N^j}(\mathbf{x}(t)) = \begin{cases}\dfrac{\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)}{\sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)\right|} & \text{if } \sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)\right| \neq 0\\ \dfrac{1}{c} & \text{otherwise}\end{cases}$$

$$\text{for } j = 2, 3, \ldots, c \tag{17}$$

From (8), (16) and (17), it can be seen that $\sum_{j=1}^{c} m_j = \sum_{j=1}^{c}\mu_{N^j}(\mathbf{x}(t)) = 1$. From (15) to (17) and considering the case that $\sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)\right| \neq 0$, we have,

$$\begin{aligned}\dot{V} &\leq -\frac{1}{2}\left(1 - \frac{\sum_{j=2}^{c}\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{i1}\mathbf{x}(t)}{\sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)\right|}\right)\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{i1}\mathbf{x}(t)\\ &\quad - \frac{1}{2}\sum_{j=2}^{c}\frac{\left(\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)\right)^2}{\sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)\right|}\\ &\leq -\frac{1}{2}\left(1 - \frac{\sum_{j=2}^{c}\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{i1}\mathbf{x}(t)}{\sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{ij}\mathbf{x}(t)\right|}\right)\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i\mathbf{Q}_{i1}\mathbf{x}(t)\end{aligned} \tag{18}$$

We choose,

$$\mathbf{Q}_{i1} > \mathbf{0} \text{ for } i = 1, 2, \ldots, p \tag{19}$$

Then, $\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{i1}\mathbf{x}(t) > 0$ when $\mathbf{x}(t) \neq \mathbf{0}$ and $\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{i1}\mathbf{x}(t) = 0$ when $\mathbf{x}(t) = \mathbf{0}$. From (18) and (19) and as $1 - \frac{\sum_{j=2}^{c}\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{i1}\mathbf{x}(t)}{\sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t)\right|} > 0$, we can conclude that,

$$\dot{V} \le -\frac{1}{2}\left(1 - \frac{\sum_{j=2}^{c}\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{i1}\mathbf{x}(t)}{\sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t)\right|}\right)\sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}}\mathbf{Q}_{i1}\mathbf{x}(t) \le 0 \quad (20)$$

Equalities holds when $\mathbf{x} = \mathbf{0}$.

Next, we consider the case that $\sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t)\right| = 0$. From (19), as $\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{i1}\mathbf{x}(t) = 0$ only when $\mathbf{x}(t) = \mathbf{0}$, it implies that $\sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t)\right| = 0$ occurs only when $\mathbf{x}(t) = \mathbf{0}$. Hence, we can conclude that the system is asymptotically stable. The analysis can be summarized by the following Lemma.

Lemma 1: *The fuzzy control system of* (9) *is guaranteed to be asymptotically stable if the following p LMI conditions are satisfied,*

$$\mathbf{Q}_{i1} = -\left(\mathbf{H}_{i1}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{i1}\right) < \mathbf{0} \textit{ for all } i = 1, 2, \ldots, p$$

and the membership functions are designed as follows,

$$\mu_{N^1}(\mathbf{x}(t)) = \begin{cases} \left(1 - \frac{\sum_{j=2}^{c}\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t)}{\sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t)\right|}\right) & \textit{if } \sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t)\right| \neq 0 \\ \frac{1}{c} & \textit{otherwise} \end{cases}$$

$$\mu_{N^j}(\mathbf{x}(t)) = \begin{cases} \frac{\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t)}{\sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t)\right|} & \textit{if } \sum_{j=1}^{c}\left|\mathbf{x}(t)^{\mathrm{T}}\sum_{i=1}^{p} w_i \mathbf{Q}_{ij}\mathbf{x}(t)\right| \neq 0 \\ \frac{1}{c} & \textit{otherwise} \end{cases}$$

for $j = 2, 3, \ldots, c$

From Lemma 1, we can see that the number of LMI conditions is p. The procedure for finding the fuzzy controller can be summarized as follows.

- Step I) Obtain the mathematical model of the nonlinear plant to be controlled.
- Step II) Obtain the fuzzy plant model for the system stated in step I) by means of a fuzzy modeling method, for example, as proposed in [3, 7].
- Step III) Choose the gains ($\mathbf{G}_j$) of the fuzzy controller.
- Step IV) Find $\mathbf{P}$ by solving the p linear matrix inequalities of (19). If $\mathbf{P}$ cannot be found, go back to step III) and choose other gains ($\mathbf{G}_j$) of the fuzzy controller.
- Step V) Design the membership functions of the fuzzy controller based on Lemma 1.

IV. Application Example

An application example based on a nonlinear mass-spring-damper system [5] is given in this section to illustrate the design procedures of the fuzzy state feedback controller.

Step I) Fig. 1 shows the diagram of a mass-spring-damper system. Its dynamic equation is given by,

$$M\ddot{x}(t) + g\left(x(t), \dot{x}(t)\right) + f(x(t)) = \phi(\dot{x}(t))u(t) \quad (21)$$

where M is the mass and u is the force, $f(x(t))$ depicts the spring nonlinearity, $g(x(t), \dot{x}(t))$ depicts the damper nonlinearity, and $\phi(\dot{x}(t))$ depicts the input nonlinearity. Let,

$$\begin{aligned} g(x(t), \dot{x}(t)) &= c_1 x(t) + c_2 \dot{x}(t) \\ f(x(t)) &= c_3 x(t) + c_4 x(t)^3 \\ \phi(\dot{x}(t)) &= 1 + c_5 \dot{x}(t)^2 \end{aligned} \quad (22)$$

The operating range of the states is assumed to be within the interval $\left[-1.5 \quad 1.5\right]$ (the fuzzy plant model derived later will not be valid when the system states are outside this interval). The parameters are chosen as follows: $M = 1.0$, $c_1 = 0$, $c_2 = 1$, $c_3 = 0.01$, $c_4 = 0.1$ and $c_5 = 0.13$. The system then becomes,

$$\begin{aligned} \ddot{x}(t) = &-\dot{x}(t) - 0.01x(t) - 0.1x(t)^3 \\ &+ \left(1.4387 - 0.13\dot{x}(t)^2\right)u(t) \end{aligned} \quad (23)$$

Step II) The nonlinear plant can be represented by a fuzzy model. The i-th rules is given by,

Rule i: IF $x(t)$ is M_1^i AND $\dot{x}(t)$ is M_2^i
THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)$, $i = 1, 2, 3, 4$ (24)

where the membership functions of M_k^i, $k = 1, 2$, $i = 1, 2, 3, 4$, are shown in Fig. 2 and 3 respectively, are given by,

$$\mu_{M_1^1}(x(t)) = \mu_{M_1^2}(x(t)) = 1 - \frac{x(t)^2}{2.25}, \mu_{M_1^3}(x(t)) = \mu_{M_1^4}(x(t)) = \frac{x(t)^2}{2.25}$$

$$\mu_{M_2^1}(\dot{x}(t)) = \mu_{M_2^3}(\dot{x}(t)) = 1 - \frac{\dot{x}(t)^2}{6.75}, \mu_{M_2^2}(x(t)) = \mu_{M_2^4}(x(t)) = \frac{\dot{x}(t)^2}{6.75} \quad (25)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \qquad \mathbf{A}_1 = \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ -0.01 & -1 \end{bmatrix},$$

$$\mathbf{A}_3 = \mathbf{A}_4 = \begin{bmatrix} 0 & 1 \\ -0.235 & -1 \end{bmatrix}; \qquad \mathbf{B}_1 = \mathbf{B}_3 = \begin{bmatrix} 0 \\ 1.4387 \end{bmatrix},$$

$\mathbf{B}_2 = \mathbf{B}_4 = \begin{bmatrix} 0 \\ 0.5613 \end{bmatrix}$. (Details about the derivation of the fuzzy plant model for the mass-spring-damper system can be found in [5].)

Step III) A two-rule fuzzy state feedback controller is designed for the plant of (24).

Rule 1: IF $\mathbf{x}(t)$ is N^1 THEN $\mathbf{u}(t) = \mathbf{G}_1\mathbf{x}(t)$

Rule 2: IF $\mathbf{x}(t)$ is N^1 THEN $\mathbf{u}(t) = \mathbf{G}_2\mathbf{x}(t)$ (26)

The feedback gains of the fuzzy controller are arbitrarily chosen as $\mathbf{G}_1 = [-2.7732 \quad -2.0852]$ and $\mathbf{G}_2 = [-6.7076 \quad -5.3447]$ (so that $\mathbf{H}_{11} = \mathbf{H}_{24}$.)

Step IV) We choose $\mathbf{P} = \begin{bmatrix} 1.1486 & 0.1580 \\ 0.1586 & 0.2225 \end{bmatrix}$ such that $\mathbf{Q}_{il} = -(\mathbf{H}_{il}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{il}) < 0$ for i = 1, 2, 3 4.

Step V) The membership functions of the fuzzy controller are chosen according to Lemma 1,

$$\mu_{N^1}(\mathbf{x}(t)) = \begin{cases} \left(1 - \dfrac{\sum_{j=2}^{2} \mathbf{x}(t)^{\mathrm{T}} \sum_{i=1}^{4} w_i \mathbf{Q}_{ij} \mathbf{x}(t)}{\sum_{j=1}^{2} \left| \mathbf{x}(t)^{\mathrm{T}} \sum_{i=1}^{4} w_i \mathbf{Q}_{ij} \mathbf{x}(t) \right|} \right) & \text{if } \sum_{j=1}^{2} \left| \mathbf{x}(t)^{\mathrm{T}} \sum_{i=1}^{4} w_i \mathbf{Q}_{ij} \mathbf{x}(t) \right| \neq 0 \\ \dfrac{1}{2} & \text{otherwise} \end{cases}$$

$$\mu_{N^2}(\mathbf{x}(t)) = \begin{cases} \dfrac{\mathbf{x}(t)^{\mathrm{T}} \sum_{i=1}^{4} w_i \mathbf{Q}_{i2} \mathbf{x}(t)}{\sum_{j=1}^{2} \left| \mathbf{x}(t)^{\mathrm{T}} \sum_{i=1}^{4} w_i \mathbf{Q}_{ij} \mathbf{x}(t) \right|} & \text{if } \sum_{j=1}^{2} \left| \mathbf{x}(t)^{\mathrm{T}} \sum_{i=1}^{p} w_i \mathbf{Q}_{ij} \mathbf{x}(t) \right| \neq 0 \\ \dfrac{1}{2} & \text{otherwise} \end{cases}$$

Fig. 4 and 5 show the system responses (solid lines) of $x_1(t)$ and $x_2(t)$ of the mass-spring-damper system under the control of the two-rule fuzzy controller with the same initial condition. The responses are compared with those of a single rule fuzzy controller (dotted lines), i.e. a state feedback controller ($\mathbf{u}(t) = \mathbf{G}_1\mathbf{x}(t)$), with $\mathbf{x}(0) = [1.5 \quad 0]^{\mathrm{T}}$. We can see that the responses of the two-rule fuzzy controller are better.

V. Conclusion

A c-rule fuzzy state feedback controller has been proposed to close the feedback loop of a nonlinear system represented by a p-rule fuzzy plant model. The system stability of this fuzzy control system has been presented. p LMI conditions, independent of the number of rule of the fuzzy controller, have been derived. A design methodology of the membership function of the fuzzy controller has been given under the consideration of the system stability. An application example has been given to show the merits and the design procedures of the proposed fuzzy state feedback controller.

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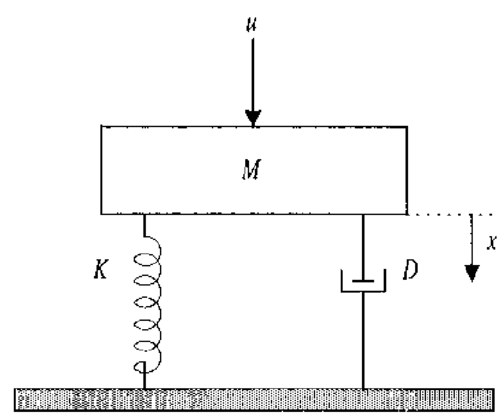

Fig. 1. A mass-spring-damper system.

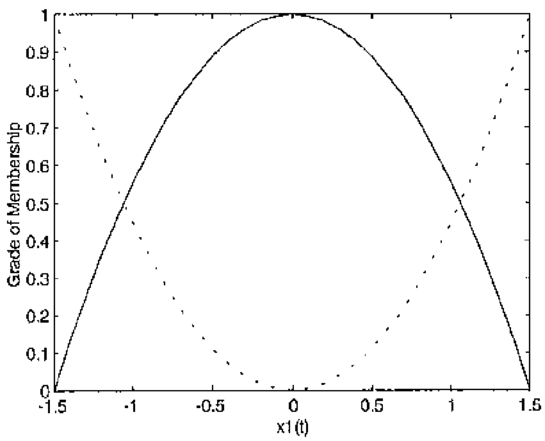

Fig. 2. Membership functions of the fuzzy plant model of the nonlinear mass-spring-damper system:

$$\mu_{M_1^1}(x) = \mu_{M_1^2}(x) = 1 - \frac{x^2}{2.25} \text{ (solid line)},$$

$$\mu_{M_1^3}(x) = \mu_{M_1^4}(x) = \frac{x^2}{2.25} \text{ (dotted line)}.$$

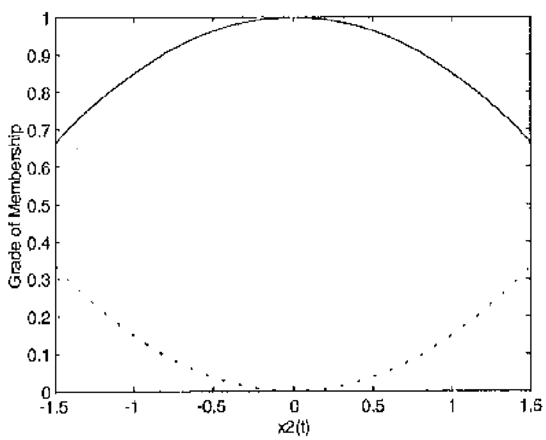

Fig. 3. Membership functions of the fuzzy plant model of the nonlinear mass-spring-damper system:

$$\mu_{M_2^1}(\dot{x}) = \mu_{M_2^3}(\dot{x}) = 1 - \frac{\dot{x}^2}{6.75} \text{ (solid line)},$$

$$\mu_{M_2^2}(\dot{x}) = \mu_{M_2^4}(\dot{x}) = \frac{\dot{x}^2}{6.75} \text{ (dotted line)}.$$

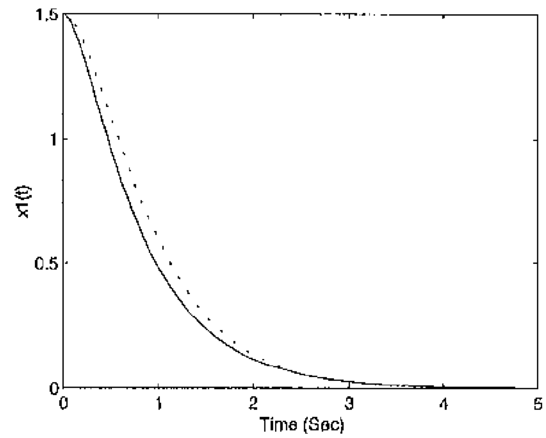

Fig. 4. Responses of $x_1(t)$ of mass-spring-damper system with a single rule fuzzy controller, state feedback controller, (dotted line) and a 2-rule fuzzy controller (solid line).

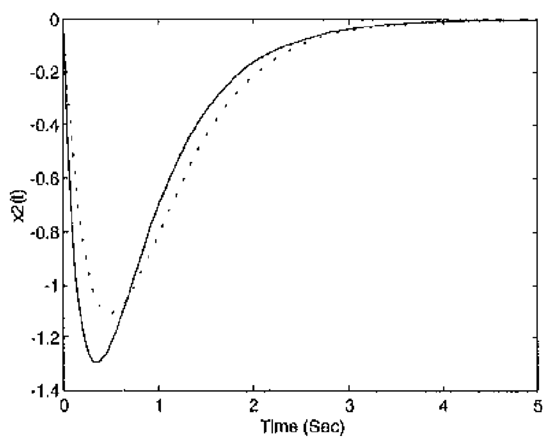

Fig. 4. Responses of $x_2(t)$ of mass-spring-damper system with a single rule fuzzy controller, state feedback controller, (dotted line) and a 2-rule fuzzy controller (solid line).