Stability Analysis and Performance Deign for Fuzzy-Model-Based Control System under Imperfect Premise Matching

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Abstract—**This paper presents the stability analysis and performance design for nonlinear systems. The T-S fuzzy model is employed to represent the nonlinear plant to facilitate the stability analysis. A fuzzy controller, under imperfect premise matching such that the T-S fuzzy model and the fuzzy controller do not share the same membership functions, is proposed to perform the control task. Consequently, the design flexibility can be enhanced and simple membership functions can be employed to lower the structural complexity of the fuzzy controller. However, the favourable characteristic given by perfect premise matching will vanish, which leads to conservative stability conditions. In this paper, under imperfect premise matching, the information of membership functions of the fuzzy model and controller is considered during the stability analysis. LMI-based stability conditions are derived to guarantee the system stability using the Lyapunov-based approach. Free matrices are introduced to alleviate the conservativeness of the stability conditions. LMI-based performance conditions are also derived to guarantee the system performance. Simulation examples are given to illustrate the effectiveness of the proposed approach.**

I. Introduction

Fuzzy control technique is good for handling mathematically ill-defined nonlinear systems. To investigate the system stability, the T-S fuzzy model [1]-[2] was proposed to provide a general framework to represent the nonlinear plant as a weighted sum of some linear sub-systems. Each linear sub-system effectively models the dynamics of the nonlinear plant in a local operating domain. As the linear and nonlinear parts of the nonlinear plant are extracted, the T-S fuzzy model exhibits a semi-linear characteristic that facilitates the system analysis and controller design.

Based on the T-S fuzzy model, a fuzzy controller [3]-[4] was proposed to control the nonlinear plant. The fuzzy controller is a weighted sum of some linear sub-controllers. A fuzzy-model-based control system is formed by connecting the fuzzy plant model and fuzzy controller in a feedback loop. By employing the Lyapunov stability theory, the system is guaranteed to be asymptotically stable if there exists a solution to a number of inequalities [3]-[4]. If the membership functions of the fuzzy controller in the premise of the fuzzy rules are allowed to be designed arbitrarily, greater design flexibility can be achieved. Furthermore, the fuzzy-model-based control system will remains stable if the nonlinear plant has parameter uncertainties that are only reflected in the membership functions of the T-S fuzzy plant model. However, the stability conditions obtained are relatively conservative. When the premise rules of both the T-S fuzzy model and the fuzzy controller are different, it is referred to as imperfect premise-matching in this paper.

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A parallel compensation distribution (PDC) design technique was proposed in [5] to facilitate the stability analysis. Under the PDC design technique, the fuzzy controller shares the same premise rules as those of the T-S fuzzy plant model, resulting in the same grades of membership. Under this case, the stability conditions can be relaxed by grouping inequalities sharing the same membership grades. However, parameter uncertainties are not allowed in the membership functions of the fuzzy plant model. Moreover, when the premise membership functions of the T-S fuzzy model are complicated, the structural complexity of the fuzzy controller will be increased. Further relaxed stability conditions were obtained in [6]-[11]. The stability conditions are cast as some linear matrix inequalities (LMIs) [12] of which the solution can be solved numerically using some convex programming techniques. In [5]-[11], as the premise rules of both T-S fuzzy model and fuzzy controller are the same, it is referred to as perfect premise matching in this paper.

To retain the design flexibility and robust property of the fuzzy-model-based control systems, a fuzzy controller which does not share the same premise rules as those of the T-S fuzzy plant model is proposed to control nonlinear plants. Under imperfect premise matching, the advantages of perfect premise matching shown in [5]-[11] vanish, which leads to conservative stability analysis result. To alleviate the conservativeness problem, the approach proposed by the same authors in [13] is employed and further enhanced in this paper. The membership functions of the fuzzy controller are designed to satisfy a design condition that facilitates the stability analysis. Under such a design condition, arbitrary free matrices can be introduced to compensate the unstable elements. LMI-based stability conditions are derived using the Lyapunov stability theory. To ensure the system performance, a scalar cost function is employed to measure quantitatively how good the system performs. LMI-based performance conditions are derived to guarantee the scalar cost function to be minimized to a prescribed level. With the LMI-based stability and performance conditions, a stable and well-performed fuzzy-model-based control system can be designed.

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II. Fuzzy Model and Fuzzy Controller

A fuzzy-model-based control system comprising a nonlinear plant represented by a fuzzy model and a fuzzy controller connected in closed-loop is considered.

A. Fuzzy Plant Model

Let p be the number of fuzzy rules describing the behaviour of the nonlinear plant. The i-th rule is of the following format:

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

THEN $\dot{\mathbf{x}}(t)=\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t)$, $i = 1, 2, ..., p$ (1)

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, ..., \Psi$; $i = 1, 2, ..., p$; Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices respectively; $\mathbf{x}(t)\in\Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t)\in\Re^{m\times 1}$ is the input vector. The system dynamics are described by,

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t)\big) \quad (2)$$

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t))=1,\ w_i(\mathbf{x}(t))\in[0\quad 1],\ i = 1, 2, ..., p \quad (3)$$

$$w_i(\mathbf{x}(t))=\frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\Big(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\Big)} \quad (4)$$

is the normalized grade of membership which is a nonlinear function of $\mathbf{x}(t)$. $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, $\alpha = 1, 2, …, \Psi$, is the grade of membership corresponding to the fuzzy term M_α^i.

B. Fuzzy Controller

A fuzzy controller with p rules is considered. The j-th rule of the fuzzy controller is defined as follows.

Rule j: IF $g_1(\mathbf{x}(t))$ is N_1^j AND … AND $g_\Omega(\mathbf{x}(t))$ is N_Ω^j

THEN $\mathbf{u}(t)=\mathbf{G}_j\mathbf{x}(t)$, $j = 1, 2, ..., p$ (5)

where N_β^j is a fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta = 1, 2, ..., \Omega$; $j = 1, 2, ..., p$; Ω is a positive integer; $\mathbf{G}_j\in\Re^{m\times n}$ is the feedback gain of rule j. The inferred output of the fuzzy controller is given by,

$$\mathbf{u}(t)=\sum_{j=1}^{p} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \quad (6)$$

$$\sum_{j=1}^{p} m_j(\mathbf{x}(t))=1,\ m_j(\mathbf{x}(t))\in[0\quad 1],\ j = 1, 2, ..., p \quad (7)$$

$$m_j(\mathbf{x}(t))=\frac{\mu_{\mathrm{N}_1^j}(g_1(\mathbf{x}(t)))\times\mu_{\mathrm{N}_2^j}(g_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{N}_\Omega^j}(g_\Omega(\mathbf{x}(t)))}{\sum_{k=1}^{p}\Big(\mu_{\mathrm{N}_1^k}(g_1(\mathbf{x}(t)))\times\mu_{\mathrm{N}_2^k}(g_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{N}_\Omega^k}(g_\Omega(\mathbf{x}(t)))\Big)} \quad (8)$$

is the normalized grade of membership which is a nonlinear function of $\mathbf{x}(t)$. $\mu_{\mathrm{N}_\beta^j}(g_\beta(\mathbf{x}(t)))$, $j = 1, 2, ..., p$, is the grade of membership corresponding to the fuzzy term N_β^j.

III. Stability Analysis and Performance Design

From (2) and (6), the fuzzy-model-based control system is defined as follows.

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))\big(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\big)\mathbf{x}(t) \quad (9)$$

It can be seen from (9) that the fuzzy model and fuzzy controller do not share the same membership functions, which lead to imperfect premise matching. Under such a condition, various simple membership functions can be employed to enhance the design flexibility and lower the structural complexity of the fuzzy controller. As a result, we are able to widen the applicability of fuzzy control approach in practice. However, it was shown in [3]-[4] that conservative stability conditions were achieved under imperfect premise matching. Furthermore, analysis techniques [5]-[11] under perfect premise matching cannot be employed to facilitate the stability analysis. In this paper, membership functions of both the fuzzy plant model and the fuzzy controller are employed to alleviate the conservativeness under imperfect premise matching. In the following, for simplicity, $w_i(\mathbf{x}(t))$ and $m_j(\mathbf{x}(t))$ are denoted by w_i and m_j. The property of $\sum_{i=1}^{p} w_i$ = $\sum_{j=1}^{p} m_j$ = $\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j$ = 1 is employed during the analysis.

A. Stability Analysis

The following Lyapunov function candidate is considered:

$$V(t)=\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) \quad (10)$$

where $\mathbf{P}=\mathbf{P}^{\mathrm{T}}\in\Re^{n\times n}>0$. From (9), we have,

$$\dot{V}(t)=\dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}\mathbf{x}(t)+\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}}(t)$$
$$=\sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j \mathbf{x}(t)^{\mathrm{T}}\Big(\big(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\big)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\big(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\big)\Big)\mathbf{x}(t) \quad (11)$$

Remark 1: It was reported in [3]-[4] that the fuzzy-model-based control system of (9) is asymptotically stable if there exists a symmetric positive definite matrix $\mathbf{P}$ such that $\big(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\big)^{\mathrm{T}}\mathbf{P}+\mathbf{P}\big(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\big)<0$ for all i and j.

From (10), letting $\mathbf{X}=\mathbf{P}^{-1}$ and $\mathbf{z}(t)=\mathbf{X}^{-1}\mathbf{x}(t)$, we have,

$$\dot{V}(t)=\sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j \mathbf{z}(t)^{\mathrm{T}}\Big(\mathbf{X}\big(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\big)^{\mathrm{T}}+\big(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\big)\mathbf{X}\Big)\mathbf{z}(t) \quad (12)$$

To alleviate the conservativeness of the stability conditions, some free matrices will be introduced to compensate the unstable elements. Considering $\sum_{i=1}^{p}\sum_{j=1}^{p} w_i(w_j-m_j)\mathbf{\Lambda}_i$ = $\sum_{i=1}^{p} w_i\sum_{j=1}^{p}(w_j-m_j)\mathbf{\Lambda}_i$ = $\sum_{i=1}^{p} w_i(1-1)\mathbf{\Lambda}_i$ = $\mathbf{0}$, where $\mathbf{\Lambda}_i=\mathbf{\Lambda}_i^{\mathrm{T}}\in\Re^{n\times n}>0$, $i = 1, 2, ..., p$, are arbitrary matrices; these terms are introduced to (12) as follows.

$$\dot{V}(t)=\sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j \mathbf{z}(t)^{\mathrm{T}}\Big(\mathbf{X}\big(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\big)^{\mathrm{T}}+\big(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j\big)\mathbf{X}\Big)\mathbf{z}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p} w_i(w_j-m_j+\rho_j w_j-\rho_j w_j)\mathbf{z}(t)^{\mathrm{T}}\mathbf{\Lambda}_i\mathbf{z}(t)$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j \mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{X}(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)^{\mathrm{T}}+(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)\mathbf{X}\right)\mathbf{z}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p} w_i(w_j-\rho_j w_j)\mathbf{z}(t)^{\mathrm{T}}\mathbf{\Lambda}_i\mathbf{z}(t)-\sum_{i=1}^{p}\sum_{j=1}^{p} w_i(m_j-\rho_j w_j)\mathbf{z}(t)^{\mathrm{T}}\mathbf{\Lambda}_i\mathbf{z}(t) \quad (13)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p} w_i\rho_j w_j\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{V}_{ij}-\mathbf{V}_{ij}\right)\mathbf{z}(t)$$

where $\mathbf{V}_{ij}=\mathbf{V}_{ij}^{\mathrm{T}}\in\Re^{n\times n}$; $i, j = 1, 2, ..., p$, are arbitrary matrices and $0 < \rho_j < 1$, $j = 1, 2, ..., p$, are designed such that $m_j-\rho_j w_j\geq 0$ for all j and $\mathbf{x}(t)$ (w_j and m_j are function of $\mathbf{x}(t)$). These are additional matrices and conditions introduced to alleviate the conservativeness under imperfect premise matching. From (13), we have,

$$\dot{V}(t)=\sum_{i=1}^{p}\sum_{j=1}^{p} w_i\left(m_j+\rho_j w_j-\rho_j w_j\right)\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{X}(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)^{\mathrm{T}}+(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)\mathbf{X}\right)\mathbf{z}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p} w_i(w_j-\rho_j w_j)\mathbf{z}(t)^{\mathrm{T}}\mathbf{\Lambda}_i\mathbf{z}(t)-\sum_{i=1}^{p}\sum_{j=1}^{p} w_i(m_j-\rho_j w_j)\mathbf{z}(t)^{\mathrm{T}}\mathbf{\Lambda}_i\mathbf{z}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p} w_i\rho_j w_j\mathbf{z}(t)^{\mathrm{T}}\mathbf{V}_{ij}\mathbf{z}(t)-\sum_{i=1}^{p}\sum_{j=1}^{p} w_i\rho_j w_j\mathbf{z}(t)^{\mathrm{T}}\mathbf{V}_{ij}\mathbf{z}(t)$$
$$=\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\mathbf{z}(t)^{\mathrm{T}}\rho_j\left(\mathbf{X}(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)^{\mathrm{T}}+(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)\mathbf{X}-\mathbf{\Lambda}_i-\mathbf{V}_{ij}\right)\mathbf{z}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p} w_i\left(m_j-\rho_j w_j\right)\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{X}(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)^{\mathrm{T}}+(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)\mathbf{X}-\mathbf{\Lambda}_i\right)\mathbf{z}(t) \quad (14)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{\Lambda}_i+\rho_j\mathbf{V}_{ij}\right)\mathbf{z}(t)$$

With the membership function design condition that $m_j-\rho_j w_j\geq 0$ for all j and $\mathbf{x}(t)$, and letting

$$\mathbf{X}(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)^{\mathrm{T}}+(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)\mathbf{X}-\mathbf{\Lambda}_i<0,\ i, j = 1, 2, ..., p, \quad (15)$$

the time derivative of $V(t)$ in (14) can be reduced to:

$$\dot{V}(t)\leq\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\mathbf{z}(t)^{\mathrm{T}}\rho_j\left(\mathbf{X}(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)^{\mathrm{T}}+(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)\mathbf{X}-\mathbf{\Lambda}_i-\mathbf{V}_{ij}\right)\mathbf{z}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{\Lambda}_i+\rho_j\mathbf{V}_{ij}\right)\mathbf{z}(t)$$
$$=\sum_{i=1}^{p} w_i^2\mathbf{z}(t)^{\mathrm{T}}\rho_i\left(\mathbf{X}(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_i)^{\mathrm{T}}+(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_i)\mathbf{X}-\mathbf{\Lambda}_i-\mathbf{V}_{ii}\right)\mathbf{z}(t)$$
$$+\sum_{j=1}^{p}\sum_{i<j} w_i w_j\mathbf{z}(t)^{\mathrm{T}}\begin{pmatrix}\rho_j\left(\mathbf{X}(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)^{\mathrm{T}}+(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)\mathbf{X}-\mathbf{\Lambda}_i-\mathbf{V}_{ij}\right)\\+\rho_i\left(\mathbf{X}(\mathbf{A}_j+\mathbf{B}_j\mathbf{G}_i)^{\mathrm{T}}+(\mathbf{A}_j+\mathbf{B}_j\mathbf{G}_i)\mathbf{X}-\mathbf{\Lambda}_j-\mathbf{V}_{ji}\right)\end{pmatrix}\mathbf{z}(t) \quad (16)$$
$$+\sum_{i=1}^{p} w_i^2\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{\Lambda}_i+\rho_i\mathbf{V}_{ii}\right)\mathbf{z}(t)+\sum_{j=1}^{p}\sum_{i<j} w_i w_j\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{\Lambda}_i+\rho_j\mathbf{V}_{ij}+\mathbf{\Lambda}_j+\rho_i\mathbf{V}_{ji}\right)\mathbf{z}(t)$$

Let

$$\mathbf{R}_{ii}>\rho_i\left(\mathbf{X}(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_i)^{\mathrm{T}}+(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_i)\mathbf{X}-\mathbf{\Lambda}_i-\mathbf{V}_{ii}\right),\ i = 1, 2, ..., p \quad (17)$$

$$\mathbf{R}_{ij}+\mathbf{R}_{ij}^{\mathrm{T}}\geq\rho_j\left(\mathbf{X}(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)^{\mathrm{T}}+(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)\mathbf{X}-\mathbf{\Lambda}_i-\mathbf{V}_{ij}\right)+\rho_i\left(\mathbf{X}(\mathbf{A}_j+\mathbf{B}_j\mathbf{G}_i)^{\mathrm{T}}+(\mathbf{A}_j+\mathbf{B}_j\mathbf{G}_i)\mathbf{X}-\mathbf{\Lambda}_j-\mathbf{V}_{ji}\right),\ j = 1, 2, ..., p;\ i<j \quad (18)$$

$$\mathbf{S}_{ii}>\mathbf{\Lambda}_i+\rho_i\mathbf{V}_{ii},\ i = 1, 2, ..., p \quad (19)$$

$$\mathbf{S}_{ij}+\mathbf{S}_{ji}\geq\mathbf{\Lambda}_i+\rho_j\mathbf{V}_{ij}+\mathbf{\Lambda}_j+\rho_i\mathbf{V}_{ji},\ j = 1, 2, ..., p;\ i<j \quad (20)$$

where $\mathbf{R}_{ij}=\mathbf{R}_{ji}^{\mathrm{T}}\in\Re^{n\times n}$ and $\mathbf{S}_{ij}=\mathbf{S}_{ji}^{\mathrm{T}}\in\Re^{n\times n}$, $i, j = 1, 2, ..., p$. From (16) to (20), we have,

$$\dot{V}(t)\leq\begin{bmatrix}w_1\mathbf{z}(t)\\w_2\mathbf{z}(t)\\\vdots\\w_n\mathbf{z}(t)\end{bmatrix}^{\mathrm{T}}\mathbf{R}\begin{bmatrix}w_1\mathbf{z}(t)\\w_2\mathbf{z}(t)\\\vdots\\w_n\mathbf{z}(t)\end{bmatrix}+\begin{bmatrix}w_1\mathbf{z}(t)\\w_2\mathbf{z}(t)\\\vdots\\w_n\mathbf{z}(t)\end{bmatrix}^{\mathrm{T}}\mathbf{S}\begin{bmatrix}w_1\mathbf{z}(t)\\w_2\mathbf{z}(t)\\\vdots\\w_n\mathbf{z}(t)\end{bmatrix} \quad (21)$$

where $\mathbf{R}=\begin{bmatrix}\mathbf{R}_{11}&\mathbf{R}_{12}&\cdots&\mathbf{R}_{1p}\\\mathbf{R}_{21}&\mathbf{R}_{22}&\cdots&\mathbf{R}_{2p}\\\vdots&\vdots&\vdots&\vdots\\\mathbf{R}_{p1}&\mathbf{R}_{p2}&\vdots&\mathbf{R}_{pp}\end{bmatrix}$ and $\mathbf{S}=\begin{bmatrix}\mathbf{S}_{11}&\mathbf{S}_{12}&\cdots&\mathbf{S}_{1p}\\\mathbf{S}_{21}&\mathbf{S}_{22}&\cdots&\mathbf{S}_{2p}\\\vdots&\vdots&\vdots&\vdots\\\mathbf{S}_{p1}&\mathbf{S}_{p2}&\vdots&\mathbf{S}_{pp}\end{bmatrix}$. It can be seen from (21) that if the conditions of $\mathbf{X} > 0$, (15), (17) to (20), $\mathbf{R} < 0$ and $\mathbf{S} < 0$ are satisfied, $\dot{V}(t)\leq 0$ (equality holds when $\mathbf{x}(t) = \mathbf{z}(t) = \mathbf{0}$), which implies the asymptotic stability of the fuzzy-model-based control system of (9). The stability analysis result is summarized in the following theorem:

Theorem 1: *The fuzzy-model-based control system of* (9) *is asymptotically stable if the membership functions of the fuzzy model and fuzzy controller are designed such that* $m_j(\mathbf{x}(t))-\rho_j w_j(\mathbf{x}(t))\geq 0$ *for all* j *and* $\mathbf{x}(t)$ *, where* $0<\rho_j<1$ *, and there exist matrices* $\mathbf{X}=\mathbf{X}^{\mathrm{T}}\in\Re^{n\times n}$ *,* $\mathbf{R}_{ij}=\mathbf{R}_{ji}^{\mathrm{T}}\in\Re^{n\times n}$ *,* $\mathbf{S}_{ij}=\mathbf{S}_{ji}^{\mathrm{T}}\in\Re^{n\times n}$ *,* $\mathbf{V}_{ij}=\mathbf{V}_{ij}^{\mathrm{T}}\in\Re^{n\times n}$ *and* $\mathbf{\Lambda}_j=\mathbf{\Lambda}_j^{\mathrm{T}}\in\Re^{n\times n}$*, and pre-defined feedback gains* $\mathbf{G}_j\in\Re^{n\times n}$ *such that the following LMIs are satisfied: (i)* $\mathbf{X}>0$*, (ii) conditions* (15), (17), (18), (19) *and* (20) *hold, and (iii)* $\mathbf{R} < 0$ *and* $\mathbf{S} < 0$.

B. Performance Design

System performance is an important issue to be considered. A scalar performance index [14] is employed to measure quantitatively the system performance and is defined as follows.

$$J=\int_0^{\infty}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{u}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{J}_1&\mathbf{0}\\\mathbf{0}&\mathbf{J}_2\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{u}(t)\end{bmatrix}dt \quad (22)$$

where $\mathbf{J}_1=\mathbf{J}_1^{\mathrm{T}}\in\Re^{n\times n}>0$ and $\mathbf{J}_2=\mathbf{J}_2^{\mathrm{T}}\in\Re^{m\times m}>0$ are predefined weighting matrices. It can be seen that J is regarded as the integral of the energy of the system states and control signals. The contribution of each term is governed by the corresponding weighting matrix of $\mathbf{J}_1$ or $\mathbf{J}_2$. From (6) and (22), we have,

$$J=\int_0^{\infty}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\sum_{j=1}^{p}m_j\mathbf{G}_j\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{J}_1&\mathbf{0}\\\mathbf{0}&\mathbf{J}_2\end{bmatrix}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\sum_{k=1}^{p}m_k\mathbf{G}_k\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t)\end{bmatrix}dt \quad (23)$$

It is assumed that

$$J<\eta\int_0^{\infty}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{X}^{-1}&\mathbf{0}\\\mathbf{0}&\mathbf{X}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t)\end{bmatrix}dt \quad (24)$$

where η is a non-zero positive scalar. The objective of (24) is to attenuate the scalar performance index J to a prescribed level governed by the value of η. From (23) and (24), letting the feedback gains of the fuzzy controller be designed as $\mathbf{G}_j=\mathbf{N}_j\mathbf{X}^{-1}$ where $\mathbf{N}_j\in\Re^{m\times n}$, j =1, 2, ..., p, we have,

$$\int_0^{\infty}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{X}^{-1}&\mathbf{0}\\\mathbf{0}&\mathbf{X}^{-1}\end{bmatrix}\left(\begin{bmatrix}\mathbf{X}&\mathbf{0}\\\mathbf{0}&\sum_{j=1}^{p}m_j\mathbf{N}_j^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{J}_1&\mathbf{0}\\\mathbf{0}&\mathbf{J}_2\end{bmatrix}\begin{bmatrix}\mathbf{X}&\mathbf{0}\\\mathbf{0}&\sum_{k=1}^{p}m_k\mathbf{N}_k\end{bmatrix}-\eta\begin{bmatrix}\mathbf{X}&\mathbf{0}\\\mathbf{0}&\mathbf{X}\end{bmatrix}\right)\begin{bmatrix}\mathbf{X}^{-1}&\mathbf{0}\\\mathbf{0}&\mathbf{X}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t)\end{bmatrix}dt<0 \quad (25)$$

The inequality of (25) holds when the following inequality holds.

$$\mathbf{W}=\begin{bmatrix}\mathbf{X} & \mathbf{0}\\ \mathbf{0} & \sum_{j=1}^{p}m_j\mathbf{N}_j^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{J}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_2\end{bmatrix}\begin{bmatrix}\mathbf{X} & \mathbf{0}\\ \mathbf{0} & \sum_{k=1}^{p}m_k\mathbf{N}_k\end{bmatrix}-\eta\begin{bmatrix}\mathbf{X} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}\end{bmatrix}<0 \quad (26)$$

By Schur complement, and using the property of $\sum_{j=1}^{p}m_j=1$, $\mathbf{W}<0$ is equivalent to the follow inequality.

$$\overline{\mathbf{W}}=\sum_{j=1}^{p}m_j\overline{\mathbf{W}}_j<0 \quad (27)$$

where $\overline{\mathbf{W}}_j=\begin{bmatrix}-\eta\mathbf{X} & \mathbf{0} & \mathbf{X} & \mathbf{0}\\ \mathbf{0} & -\eta\mathbf{X} & \mathbf{0} & \mathbf{N}_j^{\mathrm{T}}\\ \mathbf{X} & \mathbf{0} & -\mathbf{J}_1^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{N}_j & \mathbf{0} & -\mathbf{J}_2^{-1}\end{bmatrix}$, $j = 1, 2, ..., p$. The inequality of (27) holds when $\overline{\mathbf{W}}_j<0$ for all j, which are regarded as the LMI-based performance conditions. The analysis result is summarized in the following theorem:

Theorem 2: *The fuzzy-model-based control system of* (9) *is asymptotically stable and its system performance is guaranteed by the performance requirement of* (24) *if the membership functions of the fuzzy plant model and controller are designed such that* $m_j(\mathbf{x}(t))-\rho_j w_j(\mathbf{x}(t))\geq 0$ *for all j and* $\mathbf{x}(t)$, *where* $0<\rho_j<1$, *and there exist pre-defined non-zero positive scalar* η, *matrices* $\mathbf{X}=\mathbf{X}^{\mathrm{T}}\in\Re^{n\times n}$, $\mathbf{N}_j\in\Re^{m\times n}$, $\mathbf{R}_{ij}=\mathbf{R}_{ji}^{\mathrm{T}}\in\Re^{n\times n}$, $\mathbf{S}_{ij}=\mathbf{S}_{ji}^{\mathrm{T}}\in\Re^{n\times n}$, $\mathbf{V}_{ij}=\mathbf{V}_{ij}^{\mathrm{T}}\in\Re^{n\times n}$ *and* $\mathbf{\Lambda}_j=\mathbf{\Lambda}_j^{\mathrm{T}}\in\Re^{n\times n}$, *and pre-defined weighting matrices* $\mathbf{J}_1=\mathbf{J}_1^{\mathrm{T}}\in\Re^{n\times n}>0$ *and* $\mathbf{J}_2=\mathbf{J}_2^{\mathrm{T}}\in\Re^{m\times m}>0$ *such that the following LMI-based stability and performance conditions are satisfied:*

Stability Conditions:

$\mathbf{X}>0$; $\mathbf{X}\mathbf{A}_i^{\mathrm{T}}+\mathbf{N}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{X}+\mathbf{B}_i\mathbf{N}_j-\mathbf{\Lambda}_i<0$, $i, j = 1, 2, ..., p$;

$\mathbf{R}_{ii}>\rho_i(\mathbf{X}\mathbf{A}_i^{\mathrm{T}}+\mathbf{N}_i^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{X}+\mathbf{B}_i\mathbf{N}_i)$, $i = 1, 2, ..., p$;

$$\mathbf{R}_{ij}+\mathbf{R}_{ij}^{\mathrm{T}}\geq\rho_j\left(\mathbf{X}\mathbf{A}_i^{\mathrm{T}}+\mathbf{N}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{X}+\mathbf{B}_i\mathbf{N}_j-\mathbf{\Lambda}_i-\mathbf{V}_{ij}\right)+\rho_i\left(\mathbf{X}\mathbf{A}_j^{\mathrm{T}}+\mathbf{N}_i^{\mathrm{T}}\mathbf{B}_j^{\mathrm{T}}+\mathbf{A}_j\mathbf{X}+\mathbf{B}_j\mathbf{N}_i-\mathbf{\Lambda}_j-\mathbf{V}_{ji}\right), j = 1, 2, ..., p;$$

$i<j$; $\mathbf{S}_{ii}>\mathbf{\Lambda}_i+\rho_i\mathbf{V}_{ii}$, $i = 1, 2, ..., p$;

$\mathbf{S}_{ij}+\mathbf{S}_{ji}\geq\mathbf{\Lambda}_i+\rho_j\mathbf{V}_{ij}+\mathbf{\Lambda}_j+\rho_i\mathbf{V}_{ji}$, $j = 1, 2, ..., p$; $i<j$;

$$\mathbf{R}=\begin{bmatrix}\mathbf{R}_{11} & \mathbf{R}_{12} & \cdots & \mathbf{R}_{1p}\\ \mathbf{R}_{21} & \mathbf{R}_{22} & \cdots & \mathbf{R}_{2p}\\ \vdots & \vdots & \vdots & \vdots\\ \mathbf{R}_{p1} & \mathbf{R}_{p2} & \vdots & \mathbf{R}_{pp}\end{bmatrix}<0 ; \quad \mathbf{S}=\begin{bmatrix}\mathbf{S}_{11} & \mathbf{S}_{12} & \cdots & \mathbf{S}_{1p}\\ \mathbf{S}_{21} & \mathbf{S}_{22} & \cdots & \mathbf{S}_{2p}\\ \vdots & \vdots & \vdots & \vdots\\ \mathbf{S}_{p1} & \mathbf{S}_{p2} & \vdots & \mathbf{S}_{pp}\end{bmatrix}<0$$

and the feedback gains are designed as $\mathbf{G}_j=\mathbf{N}_j\mathbf{X}^{-1}$, j =1, 2, ..., p.

Performance Conditions:

$$\overline{\mathbf{W}}_j=\begin{bmatrix}-\eta\mathbf{X} & \mathbf{0} & \mathbf{X} & \mathbf{0}\\ \mathbf{0} & -\eta\mathbf{X} & \mathbf{0} & \mathbf{N}_j^{\mathrm{T}}\\ \mathbf{X} & \mathbf{0} & -\mathbf{J}_1^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{N}_j & \mathbf{0} & -\mathbf{J}_2^{-1}\end{bmatrix}>0, j = 1, 2, ..., p.$$

It should be noted that the values of η and the weighting matrices of $\mathbf{J}_1$ and $\mathbf{J}_2$ have to be determined prior to applying Theorem 2.

IV. Simulation Examples

A. Simulation Example 1

In this example, the stabilization ability of the fuzzy controllers with different values of ρ_j is tested using the LMI-based stability in Theorem 1. Consider the following fuzzy plant model [10],

Rule i: IF $x_1(t)$ is M_1^i

$$\text{THEN } \dot{\mathbf{x}}(t)=\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_iu(t), i = 1, 2 \quad (28)$$

where $\mathbf{A}_1=\begin{bmatrix}2 & -10\\ 1 & 0\end{bmatrix}$, $\mathbf{A}_2=\begin{bmatrix}a & -10\\ 1 & 3\end{bmatrix}$, $\mathbf{B}_1=\begin{bmatrix}1\\ 0\end{bmatrix}$ and $\mathbf{B}_2=\begin{bmatrix}b\\ 0\end{bmatrix}$; $1\leq a\leq 3$ and $1\leq b\leq 3$. It is assumed that the membership functions of the fuzzy plant model and fuzzy controller are different. Considering ρ_1 = 0.75 and ρ_2 = 0.85, and employing the design criterion in [10], the feedback gains of the fuzzy controller are designed such that the eigenvalues of $\mathbf{A}_1+\mathbf{B}_1\mathbf{G}_1$ and $\mathbf{A}_2+\mathbf{B}_2\mathbf{G}_2$ are all located at –2. Fig. 1(a) shows the stability region for Theorem 1. For comparison purpose, we choose ρ_1 = 0.85 and ρ_2 = 0.9 with other parameters unchanged. Fig. 1(b) shows the stability region under such case. It can been seen that the stability region with ρ_1 = 0.85 and ρ_2 = 0.9 is larger. The larger values of ρ_j indicate that the membership functions of the fuzzy plant model and fuzzy controller are closer to each other. Under the imperfect premise matching, the published stability conditions in [3]-[4] cannot provide feasible solutions. Furthermore, the stability conditions in [5]-[11] for perfect premise matching cannot be applied.

B. Simulation Example 2

An inverted pendulum on a cart [15] is employed to illustrate the design flexibility of the proposed fuzzy controller. Simple membership functions, which lead to low structural complexity, are employed for the proposed fuzzy controller. Under such a design, the published stability conditions [5]-[11] for perfect premise matching cannot be applied to guarantee the system stability. The proposed LMI-based stability conditions in Theorem 2 show an effective approach to help design a stable fuzzy controller under imperfect premise matching. Furthermore, LMI-performance conditions are employed to guarantee the system performance of the fuzzy-model-based control system.

Step I) The dynamic equations of the inverted pendulum on a cart [15] is given by,

$$\dot{x}_1(t)=x_2(t) \quad (29)$$

$$\dot{x}_2(t)=\frac{\begin{pmatrix}-F_1(M+m)x_2(t)-m^2l^2x_2(t)^2\sin x_1(t)\cos x_1(t)+F_0mlx_4(t)\cos x_1(t)\\ +(M+m)mgl\sin x_1(t)-ml\cos x_1(t)u(t)\end{pmatrix}}{(M+m)(J_o+ml^2)-m^2l^2(\cos x_1(t))^2} \quad (30)$$

$$\dot{x}_3(t)=x_4(t) \quad (31)$$

$$\dot{x}_4(t)=\frac{\begin{pmatrix}F_1mlx_2(t)\cos x_1(t)+(J+ml^2)mlx_2(t)^2\sin x_1(t)-F_0(J_o+ml^2)x_4(t)\\ -m^2gl^2\sin x_1(t)\cos x_1(t)+(J_o+ml^2)u(t)\end{pmatrix}}{(M+m)(J_o+ml^2)-m^2l^2(\cos x_1(t))^2} \quad (32)$$

where $x_1(t)$ and $x_2(t)$ denote the angular displacement (rad) and the angular velocity (rad/s) of the pendulum from vertical respectively, $x_3(t)$ and $x_4(t)$ denote the displacement (m) and the velocity (m/s) of the cart respectively, g = 9.8 m/s^2 is the acceleration due to gravity, $m = 0.22$ kg is the mass of the pendulum, $M = 1.3282$ kg is the mass of the cart, $l = 0.304$ m is the length from the center of mass of the pendulum to the shaft axis, $J_o = ml^2/3$ kgm^2 is the moment of inertia of the pendulum around the center of mass, $F_0 = 22.915$ N/m/s and $F_1 = 0.007056$ N/rad/s are the friction factors of the cart and the pendulum respectively, and u(t) is the force (N) applied to the cart. The nonlinear plant can be represented by a fuzzy model with two fuzzy rules [15]. The i-th rule is given by,

Rule i: IF $x_1(t)$ is M_1^i

THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)$ for $i = 1, 2$ (33)

The system dynamics are described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} w_i\big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)\big) \tag{34}$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^T$;

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (M+m)mgl/a_1 & -F_1(M+m)/a_1 & 0 & F_0 ml/a_1 \\ 0 & 0 & 1 & 0 \\ -m^2gl^2/a_1 & F_1 ml/a_1 & 0 & -F_0(J_o+ml^2)/a_1 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{3\sqrt{3}}{2\pi}(M+m)mgl/a_2 & -F_1(M+m)/a_2 & 0 & F_0 ml\cos(\pi/3)/a_2 \\ 0 & 0 & 1 & 0 \\ -\frac{3\sqrt{3}}{2\pi}m^2gl^2\cos(\pi/3)/a_2 & F_1 ml\cos(\pi/3)/a_2 & 0 & -F_0(J_o+ml^2)/a_1 \end{bmatrix};$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ -ml/a_1 \\ 0 \\ (J_o+ml^2)/a_1 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ -ml\cos(\pi/3)/a_2 \\ 0 \\ (J_o+ml^2)/a_2 \end{bmatrix};$$

$a_1 = (M+m)(J+ml^2) - m^2l^2$,

$a_2 = (M+m)(J_o+ml^2) - m^2l^2\cos(\pi/3)^2$. The membership functions are defined as $w_1(x_1(t)) = \mu_{M_1^1}(x_1(t)) = \left(1 - \frac{1}{1+e^{-7(x_1(t)-\pi/6)}}\right)\frac{1}{1+e^{-7(x_1(t)+\pi/6)}}$ and $w_2(x_1(t)) = \mu_{M_1^2}(x_1(t)) = 1 - \mu_{M_1^1}(x_1(t))$ which are shown in Fig. 2.

Step II)A two-rule fuzzy controller is proposed to control the nonlinear plant to achieve $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. The j-th rule is given by,

Rule j: IF $x_1(t)$ is N_1^j THEN $u(t) = \mathbf{G}_j\mathbf{x}(t)$, $j = 1, 2$ (35)

From (6), the fuzzy controller is defined as,

$$u(t) = \sum_{j=1}^{2} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \tag{36}$$

The membership functions are designed as $m_1(x_1(t)) = \mu_{N_1^1}(x_1(t)) = 0.99e^{-\frac{x_1(t)}{2\times 1.5^2}}$ and $m_2(x_1(t)) = \mu_{N_1^2}(x_1(t)) = 1 - \mu_{N_1^1}(x_1(t))$. The membership functions of the fuzzy controller are shown in Fig. 2. It can be seen that simpler membership functions compared with those of the fuzzy plant model are employed to implement the fuzzy controller, which can lower the structural complexity.

Step III) Based on the membership functions of the fuzzy plant model and the fuzzy controller, if $\rho_1 = \rho_2 = 0.82$, we have $m_j(x_1(t)) - \rho_j w_j(x_1(t)) > 0$ for all $x_1(t)$. The stability conditions in Theorem 2 are employed to help design a stable fuzzy controller for the inverted pendulum. By solving the conditions with the help of MATLAB, we obtain the feedback gains $\mathbf{G}_1 = [1834.1728 \;\; 111.5387 \;\; 29.4702 \;\; 84.0377]$ and $\mathbf{G}_2 = [1854.1876 \;\; 112.6633 \;\; 29.7438 \;\; 84.6040]$. The fuzzy controller with these feedback gains are referred to as fuzzy controller 1. To illustrate the effectiveness of the LMI-based performance conditions, they are employed to realize the system performance design. We choose $\eta = 0.01$ and the weighting matrices $\mathbf{J}_1 = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}$ and $\mathbf{J}_2 = 1$. By solving the LMI-based stability and performance conditions in Theorem 2, we have $\mathbf{G}_1 = [529.0839 \;\; 36.8266 \;\; 0.0513 \;\; 23.6628]$ and $\mathbf{G}_2 = [602.6933 \;\; 41.9423 \;\; 0.2107 \;\; 26.9586]$. The fuzzy controller with these feedback gains are referred to as fuzzy controller 2. To illustrate the effect of the weighting matrices, we choose $\mathbf{J}_1 = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&5&0\\0&0&0&1 \end{bmatrix}$ with other parameter values being unchanged. We obtain the feedback gains as $\mathbf{G}_1 = [594.2144 \;\; 41.3972 \;\; 0.6793 \;\; 26.5554]$ and $\mathbf{G}_2 = [853.6374 \;\; 59.4738 \;\; 2.5276 \;\; 38.2191]$. The fuzzy controller with these feedback gains are referred to as fuzzy controller 3.

The fuzzy controllers 1 to 3 are employed to stabilize the inverted pendulum described in (29) to (32). Fig. 3 shows the system state responses and the control signals under the initial system state condition of $\mathbf{x}(0) = \begin{bmatrix} \frac{15\pi}{36} & 0 & 0 & 0 \end{bmatrix}^T$. Referring to this figure, it can be seen that the inverted pendulum can be stabilized by all fuzzy controllers. The fuzzy controller 1 offers a very fast transient response at the cost of large control signals. With the LMI-based performance conditions, $\mathbf{J}_2$ is employed to constrain the control signal. It can be seen from the Fig. 3 that the magnitudes of the control signals offered by fuzzy controllers 2 and 3 are lower. Comparing with fuzzy controller 2, the fuzzy controller 3 (with $\mathbf{J}_1 = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&5&0\\0&0&0&1 \end{bmatrix}$) places a heavier weight on $x_3(t)$ in the performance index to suppress its magnitude. Consequently, the system state response of $x_3(t)$ of the inverted pendulum with fuzzy controller 3 offers better performance than that with fuzzy controller 2 in terms of transient response and settling time.

Furthermore, in this example, it can be seen that simple membership functions are employed for the fuzzy controller instead of employing the complicated membership functions of the fuzzy plant model under perfect premise matching. As a result, the stability conditions in [5]-[11] need not be applied to aid the design of the fuzzy controller. Under the imperfect premise matching, the proposed LMI-based stability and performance conditions offer a systematic way to realize a stable and well-performed fuzzy-model-based control system.

V. CONCLUSION

The system stability of fuzzy-model-based control systems under imperfect premise matching has been investigated. Under the imperfect premise matching, the advantages of design flexibility and robustness property of the fuzzy-model-based control system can be retained. In order to facilitate the stability analysis approach, membership functions of the fuzzy plant model and fuzzy controller are designed subject to a given condition. This membership function design condition allows the introduction of some arbitrarily free design matrices which effectively compensate the unstable components of the nonlinear system so as to reduce the conservativeness of the stability conditions. LMI-based stability conditions have been derived to guarantee the system stability. To ensure the system performance, LMI-based performance conditions have been derived to guarantee a pre-defined scalar cost function to be attenuated to a prescribed level. Simulation examples have been given to illustrate the merits of the proposed approach.

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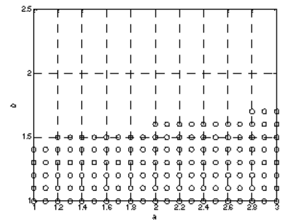

Fig. 1(a).

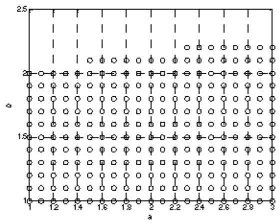

Fig. 1(b).

Fig. 1. Stability regions under Theorem 1. (a) $\rho_1 = 0.75$ and $\rho_2 = 0.85$. (b) $\rho_1 = 0.85$ and $\rho_2 = 0.9$.

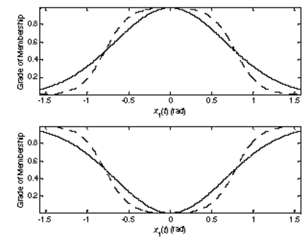

Fig. 2. Membership functions of the fuzzy plant model and fuzzy controller. Upper: $w_1(x_1(t)) = \mu_{M_1^1}(x_1(t))$ (dotted line) and $m_1(x_1(t)) = \mu_{N_1^1}(x_1(t))$ (solid line). Lower: $w_2(x_1(t)) = \mu_{M_1^2}(x_1(t)) = 1-\mu_{M_1^1}(x_1(t))$ (Dotted line) and $m_2(x_1(t)) = \mu_{N_1^2}(x_1(t)) = 1-\mu_{N_1^1}(x_1(t))$ (Solid line).

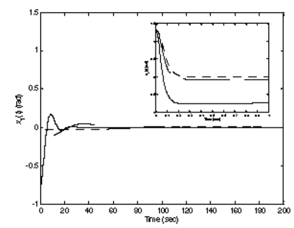

Fig. 3(a). $x_1(t)$.

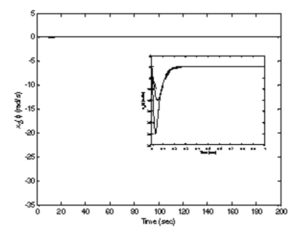

Fig. 3(b). $x_2(t)$.

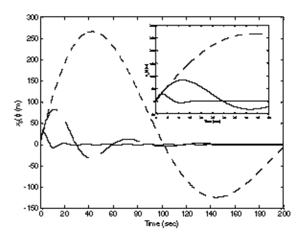

Fig. 3(c). $x_3(t)$.

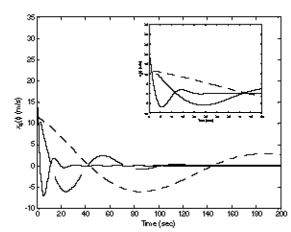

Fig. 3(d). $x_4(t)$.

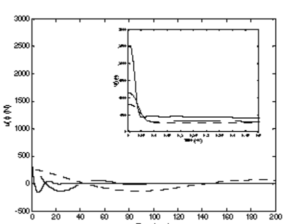

Fig. 3(e). $u(t)$.

Fig. 3. System state responses and control signals of the inverted pendulum with fuzzy controllers 1 (solid lines), 2 (dotted lines) and 3 (dash lines).