Lyapunov Function Based Design of Heuristic Fuzzy Logic Controllers

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Abstract

A stability design of fuzzy logic controllers (FLCs) for non-linear systems is proposed in this paper. In heuristic design of FLCs, we often have a lot of rules. Although each rule governing the control of the plant refers to a stable closed-loop sub-system, the overall system stability cannot be guaranteed when all of these rules are put together into a rule base for the FLC. This limitation is tackled in this paper. It is shown that on adding arbitrary rules to the FLC without any restriction on the form of membership functions, the system stability can be ensured if each individual rule applying to the plant results in a stable sub-system in the sense of Lyapunov subject to a common Lyapunov function for all rules. Analytical proof of the result is given and its application on designing a heuristic of FLC is illustrated through an example.

1. Introduction

Though fuzzy logic controllers (FLCs) had been proposed for a long time and were successfully applied in many applications, only a limited work on the proof of stability for the closed-loop control system is found. On designing the FLC, we usually focus on the system responses for some common operating conditions. However, the closed-loop system may go unstable at some unpredictable operation points. Stability proof over the whole operation range is thus necessary for fuzzy logic control systems.

Recently, Tanaka and Sugeno proposed a stability design approach [2] which first model the plant by a Takagi-Sugeno (TS) fuzzy model [1]. The fuzzy model represents the plant as a weighted sum of a set of linear state equations. Then, Lyapunov's direct method can be applied to each fuzzy rule. The stability of the whole system can be ensured if a required positive definite matrix exists. Similar stability design approaches by first modeling the plant as fuzzy system (or neural fuzzy system, which is a fuzzy system in the form of a neural network) can also be found in [3-7]. However, a major drawback of these stability design approaches is the difficulty in finding a common Lyapunov function. When combining the TS fuzzy model of the plant and the FLC, the number of sub-systems generated can be $\frac{k(k+1)}{2}$, where k is the number of rules. It is very difficult to find a common Lyapunov function in general cases to satisfy all these sub-systems. Even worse, the existence of the common Lyapunov function is not guaranteed.

Besides the fuzzy-model-based approaches, some stability analysis approaches which relate the FLC with other controllers like the PID controllers [11, 12] and the variable structure controllers [13, 14] can be found. These approaches involve partitioning the state-space into small parts and analyzing each part for closed-loop stability. However, if the number of rules is large, the number of partitions will become large and the analysis will be very time consuming. Moreover, the reported analysis results in [11-14] are only for second order systems. For higher order systems, the partitioning of state space cannot be represented graphically and the design procedures will be further complicated. In addition, these FLCs usually have some pre-defined structures. For example, the membership functions are of regular triangular or trapezoidal shapes, symmetrical, and equally distributed in the universe of discourse. Consequently, adding a rule of other form will greatly affect the analysis. On the other hand, Wang [8-10] proposed a supervisory control methodology to guarantee the stability of the system. This supervisory control will override the FLC in case the system exits some pre-defined bounds. However, the stability is just ensured through a large control signal only; the stability analysis on applying the FLC is not directly tackled.

The aim of this paper is to develop a simple methodology for designing FLCs with stability consideration. An extra fuzzy model is not required. With reference to a non-linear plant, the stable FLC thus designed is not restricted in terms of regular triangular or trapezoidal shapes, symmetrical input/output membership functions. The system is simply controlled by the FLC

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without any supervisory control. The FLC is designed heuristically which is realized by first gathering expert knowledge that can control the plant well with guaranteed stability. This knowledge is expressed as fuzzy rules of the FLC. The rules are gathered to form a rule base, which becomes the heart of the FLC. The system stability is guaranteed by employing a common Lyapunov function to each rule. Results show that on applying the combined rules in the FLC, a stable closed-loop system can be obtained if each individual rule applied to the plant results in a stable sub-system in the sense of Lyapunov.

In section 2, the fuzzy logic control system used in this paper will be introduced. The proposed stability design method will be discussed in section 3. An example will be shown in section 4 to illustrate the application of the proposed method, and a conclusion will be drawn in section 5.

2. Fuzzy Logic Control System

In this paper, a single input n-th order non-linear system of the following form is considered:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u \qquad (1)$$

where $\mathbf{x} = [x_1, x_2, \ldots\ldots, x_n]^T$ is the state-vector, $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots\ldots, f_n(\mathbf{x})]^T$, $\mathbf{b}(\mathbf{x}) = [b_1(\mathbf{x}), b_2(\mathbf{x}), \ldots\ldots, b_n(\mathbf{x})]^T$ are functions describing the dynamics of the plant, and u is the control input of the plant of which the value is determined by an FLC with inputs depending on $\mathbf{x}$. The i-th **IF-THEN** rule of the fuzzy rule base is of the following form [8]:

Rule i: **IF** x_1 is X_{i1} **AND** x_2 is X_{i2} **AND**...... **AND** x_n is X_{in} **THEN** $u = u_i(\mathbf{x})$ (2)

where $X_{i1}, X_{i2}, \ldots\ldots, X_{in}$ are input fuzzy levels, $u = u_i(\mathbf{x})$ is the output of rule i. $u_i(\mathbf{x})$ can be a single value or a function of $\mathbf{x}$. The shape of the membership functions associated with the input fuzzy levels, the method of fuzzification, and the algorithm of rule inference can be arbitrary because these do not affect the stability design discussed in this paper. A degree of membership $\mu_i \in [0, 1]$ is obtained for each rule i. It is assumed that for any $\mathbf{x}$ in the input universe of discourse $\mathbf{X}$, there exists at least one μ_i among all rules that is not equal to zero. By applying the weighted sum defuzzification method, the output of the FLC is given by:

$$u(\mathbf{x}) = \frac{\sum_{i=1}^{k} \mu_i(\mathbf{x}) u_i(\mathbf{x})}{\sum_{i=1}^{k} \mu_i(\mathbf{x})} \qquad (3)$$

where k is the total number of rules.

Two points should be noted from (3). First, for any input $\mathbf{x}_o \in \mathbf{X}$, if the degree of membership $\mu_i(\mathbf{x}_o)$ corresponding to rule i is zero, this rule i is called an *inactive fuzzy rule* for the input $\mathbf{x}_o$. An inactive fuzzy rule will not affect the controller output $u(\mathbf{x}_o)$. Hence, (3) can be re-written as a combination of all *active fuzzy rules* (where $\mu_i(\mathbf{x}_o) \neq 0$),

$$u(\mathbf{x}_o) = \frac{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\mathbf{x}_o) u_i(\mathbf{x}_o)}{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\mathbf{x}_o)} \qquad (4)$$

Second, since all $u_i(\mathbf{x}_o)$ have definite values, the maximum value $u_{max}(\mathbf{x}_o)$ and the minimum value $u_{min}(\mathbf{x}_o)$ of $u_i(\mathbf{x}_o)$ among all active rules can be found. Consequently,

$$\frac{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\mathbf{x}_o) u_{min}(\mathbf{x}_o)}{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\mathbf{x}_o)} < \frac{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\mathbf{x}_o) u_i(\mathbf{x}_o)}{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\mathbf{x}_o)} < \frac{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\mathbf{x}_o) u_{max}(\mathbf{x}_o)}{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\mathbf{x}_o)}$$

$$\Rightarrow \quad u_{min}(\mathbf{x}_o) \frac{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\mathbf{x}_o)}{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\mathbf{x}_o)} < u(\mathbf{x}_o) < u_{max}(\mathbf{x}_o) \frac{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\mathbf{x}_o)}{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\mathbf{x}_o)}$$

$$\Rightarrow \quad u_{min}(\mathbf{x}_o) < u(\mathbf{x}_o) < u_{max}(\mathbf{x}_o) \qquad (5)$$

As a result, the output of the FLC is bounded by $u_{max}(\mathbf{x}_o)$ and $u_{min}(\mathbf{x}_o)$ if the weighted sum method is employed to derive $u(\mathbf{x}_o)$.

3. Stability Design of FLCs

The premise of the stability criteria in this paper is that on applying each rule to the plant individually, the closed-loop sub-system is stable in the sense of Lyapunov, and each rule shares a common quadratic Lyapunov function $V(\mathbf{x})$ such that

i) $V(\mathbf{x})$ is positive definite and continuously differentiable,
ii) $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x}$ in active region

(6)

An active region is defined as the range of $\mathbf{x}$ such that the rule is an active fuzzy rule. Based on the above premise, and considering the active fuzzy rules for input $\mathbf{x}_o$ with the maximum and the minimum control signal $u_{max}(\mathbf{x}_o)$ and $u_{min}(\mathbf{x}_o)$ respectively, the sub-systems formed by these two rules satisfy the following conditions:

$$\dot{V}(\mathbf{x}_o) \leq 0 \text{ for } u(\mathbf{x}_o) = u_{max}(\mathbf{x}_o) \qquad (7)$$

$$\dot{V}(\mathbf{x}_o) \leq 0 \text{ for } u(\mathbf{x}_o) = u_{min}(\mathbf{x}_o) \qquad (8)$$

These conditions will be applied later. Now a common quadratic Lyapunov function is suggested as follows:

$$V(x) = \frac{1}{2} x^T A x \tag{9}$$

Then $\dot{V}(x) = \dot{x}^T A x + x^T A \dot{x}$ (10)

From (1), $\dot{V}(x) = (f(x) + b(x)u)^T A x + x^T A (f(x) + b(x)u)$
$= F(x) + B(x)u$ (11)

where $F(x) = f(x)^T A\, x + x^T A\, f(x)$
$B(x) = b(x)^T A\, x + x^T A\, b(x)$

In particular, if A is symmetric, $F(x) = 2\, x^T\, A\, f(x)$ and $B(x) = 2\, x^T A\, b(x)$

Here we want to show that $\dot{V}(x_o) \le 0$ for all $u(x_o)$ that lies between $u_{min}(x_o)$ and $u_{max}(x_o)$.

Note that both $F(x)$ and $B(x)$ are scalars. Then, two cases should be considered: $B(x)$ is positive and $B(x)$ is negative for $x = x_o$.

If $B(x_o)$ is positive, by using condition (7),

$$\dot{V}(x_o) = F(x_o) + B(x_o)u_{max}(x_o) \le 0$$
$$\Rightarrow \dot{V}(x_o) = F(x_o) + B(x_o)u(x_o) \le 0 \quad \forall \; u(x_o) < u_{max}(x_o) \tag{12}$$

On the other hand, if $B(x_o)$ is negative, by using condition (8),

$$\dot{V}(x_o) = F(x_o) + B(x_o)u_{min}(x_o) \le 0$$
$$\Rightarrow \dot{V}(x_o) = F(x_o) + B(x_o)u(x_o) \le 0 \quad \forall \; u(x_o) > u_{min}(x_o) \tag{13}$$

From (12) and (13),

$$\dot{V}(x_o) \le 0 \;\; \forall \; B(x_o),\; u(x_o) \text{ such that } u_{min}(x_o) < u(x_o) < u_{max}(x_o) \tag{14}$$

The result in (14) clearly indicates that the closed-loop system is stable in the sense of Lyapunov for any $u(x_o)$ lies between $u_{max}(x_o)$ and $u_{min}(x_o)$ as given in (5), and the closed-loop fuzzy logic control system is stable in the sense of Lyapunov provided that each individual rule applying to the plant leads to a stable sub-system in the sense of Lyapunov under a common Lyapunov function.

In summary, the stability consideration for the design of heuristic fuzzy logic control systems can be stated by the following theorem:

Theorem 3.1: Stability of heuristic fuzzy logic control systems: The stability proof of fuzzy logic control systems can be carried out by first applying Lyapunov direct method to each rule. If every rule individually applying to the plant of (1) results in a stable sub-system in the sense of Lyapunov subject to a common Lyapunov function described by (6), the whole fuzzy logic control system is stable as guaranteed by (5) and (14).

Corollary 3.1: Adding of rules: If there exists an FLC that can guarantee system stability as described in *Theorem 3.1*, an extra fuzzy rule (or expert knowledge) can be added to the FLC with guaranteed system stability provided that on applying this rule to the plant individually, the closed-loop sub-system is stable in the sense of Lyapunov subject to the same Lyapunov function V(x) used in the original FLC.

Proof: From the above statement, we have

$$\dot{V}(x_o) \le 0 \;\text{ for } u(x_o) = u_{new}(x_o) \tag{15}$$

where $u_{new}(x_o)$ is the control output corresponding to the newly added rule. Then, either one of the following three cases should occur:

1. If $u_{new}(x_o)$ lies between $u_{max}(x_o)$ and $u_{min}(x_o)$, by (5) and (14), the new system is stable.
2. If $u_{new}(x_o) > u_{max}(x_o)$, $u_{max}(x_o)$ can be replaced by $u_{new}(x_o)$ and condition (7) is replaced by (15). Following the proposed stability analysis, condition (14) can be obtained by replacing $u_{max}(x_o)$ with $u_{new}(x_o)$.
3. Similarly if $u_{new}(x_o) < u_{min}(x_o)$, condition (8) can be replaced by (15), and (14) can also be obtained by replacing $u_{min}(x_o)$ with $u_{new}(x_o)$.

Hence the new system is stable in the sense of Lyapunov after adding the new rule.

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It can be seen from corollary 3.1 that the new rule added to the FLC is not restricted in terms of shape and symmetry of the input and output membership functions. Whenever a new expert rule is obtained and the condition stated in corollary 3.1 is satisfied, it can be directly added into the FLC rule base without affecting the stability of the existing controller. Thus, a stability design approach for heuristic FLCs has been proposed.

4. Examples

The non-linear system to be controlled by an FLC is given as follows:

$$\ddot{x} + \dot{x}^3 + u = 0 \tag{16}$$

The fuzzy rules are summarized in Table 1. Write (16) in the form of (1), we have

$$\dot{x} = f(x) + b(x)u$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_1 = x$, $x_2 = \dot{x}$;

$$f(x) = \begin{bmatrix} x_2 \\ -x_2^3 \end{bmatrix} \text{ and } b(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

The closed-loop system stability is to be proved by using the proposed approach. In the rule table, the input variables in the rule antecedents are x and $\dot{x}$ and the output variable in the rule consequent is u. P, Z, and N are fuzzy levels representing positive, zero and negative respectively. The membership functions are shown in Fig. 1. The rules in Table 1 are read as, taking rule 5 for example, "Rule 5: IF x is P and $\dot{x}$ is Z, THEN $u = x$". The rules are set heuristically. In rule 1, x is positive but is increasing (since $\dot{x}$ is positive), so a big control action is needed and $u = 2$. Similarly, in rule 2, x is negative and decreasing, so u is set to be -2. In rule 3, x is positive but decreasing, therefore no control is needed, and $u = 0$. The same happens in rule 4. When either one of the states is near the origin in rule 5 to rule 8, a proportional control is applied. When the states are at the origin, as described in rule 9, some damping is needed, and a proportional plus derivative control is desired.

To prove the stability of the system by the proposed approach, the stability of the sub-system on applying each rule based on a common Lyapunov function should be proved first. Select a Lyapunov function as follows:

$$V = \frac{1}{2}(x^2 + \dot{x}^2) \tag{17}$$

which is obviously positive definite and continuously differentiable. Then

$$\begin{aligned} \dot{V} &= x\dot{x} + \dot{x}\ddot{x} \\ &= x\dot{x} + \dot{x}(-\dot{x}^3 - u) \end{aligned} \tag{18}$$

For rule 1, $u = 2$, and from (18),

$$\dot{V} = (x-2)\dot{x} - \dot{x}^4$$

It can be seen from Fig. 1 that x lies between 0 and 1 and $\dot{x}$ is positive, therefore $\dot{V} \le 0$ for rule 1 and the sub-system under this rule is stable. For rule 2, x lies between 0 and -1 and $\dot{x}$ is negative, then $\dot{V} = (x-(-2))\dot{x} - \dot{x}^4 \le 0$, and the sub-system under this rule is stable. For rule 3 and rule 4, $u = 0$,

Rule	Premise		Consequent
	x	$\dot{x}$	u
1	P	P	2
2	N	N	-2
3	P	N	0
4	N	P	0
5	P	Z	x
6	N	Z	x
7	Z	P	x
8	Z	N	x
9	Z	Z	$x+\dot{x}$

Table 1. Fuzzy rule base

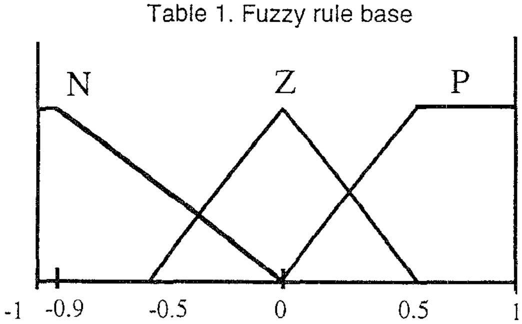

Fig. 1. Membership functions of x and $\dot{x}$

$$\dot{V} = x\dot{x} - \dot{x}^4$$

Since x and $\dot{x}$ are of different sign in rule 3 and rule 4, $\dot{V} \le 0$. Therefore the sub-systems on applying these two rules are stable.

For rule 5 to rule 8, a proportional feedback $u = x$ is used, then

$$\dot{V} = -\dot{x}^4$$

which is always less than or equal to zero. Hence the sub-systems on applying rule 5 to rule 8 are stable. For rule 9, $u = x + \dot{x}$, then

$$\begin{aligned} \dot{V} &= x\dot{x} + \dot{x}(-\dot{x}^3 - x - \dot{x}) \\ &= -\dot{x}^4 - \dot{x}^2 \end{aligned}$$

which is always less than or equal to zero, and the system on applying rule 9 is stable. Hence all the 9 rules in the FLC results in stable sub-systems in the sense of Lyapunov subject to the same Lyapunov function (17). By *Theorem 3.1*, the closed-loop system is stable when all the rules are included into the rule base of the FLC.

The simulation results of the zero-input responses of the states of the closed-loop system with initial values $\mathbf{x}(0) = [1 \ \ -1]^T$ are shown in Fig. 2. It can be seen that the fuzzy logic control system is stable.

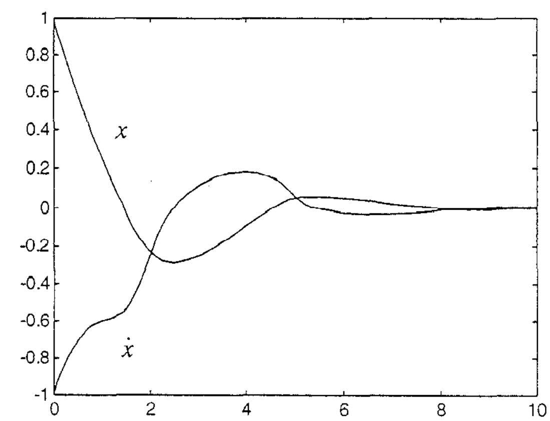

Fig. 2. Responses of x and $\dot{x}$ in the illustrated example

5 Conclusion

An approach for designing heuristic FLC with stability consideration by using Lyapunov direct method is proposed in this paper. The derivation shows that the fuzzy logic control system is stable in the sense of Lyapunov provided that each individual rule applying to the plant results in a stable sub-system in the sense of Lyapunov under a common Lyapunov function. Therefore the stability of the fuzzy logic control system can be guaranteed by examining each individual rule in the FLC. A non-linear system controlled by an FLC is analyzed based on the proposed design approach as an illustrative example. This proposed approach is simpler than other existing approaches like fuzzy-model-based approach, state-space partitioning approach and supervisory control approach, leading to a feasible heuristic design of FLC. Such a design is neither restricted by the shape and symmetry of the input/output membership functions nor the distribution of the fuzzy levels in the universe of discourse.

To probe further, the proposed design approach can be applied to combine controllers. An example on applying this design approach to combine a sliding mode controller (SMC) and a state feedback controller (SFC) is to be reported in [15]. Each rule in this FLC has an SMC or an SFC in the consequent part. The role of the FLC is to schedule different controllers under different antecedents. The stability of the whole system is guaranteed by the proposed design approach. More importantly, the controller thus designed can keep the advantages and remove the disadvantages of the two conventional controllers.

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