Lyapunov Function Based Design of Robust Fuzzy Controllers for Uncertain Nonlinear Systems: Distinct Lyapunov Functions[1]

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Abstract

This paper presents the stability and robustness analyses of an uncertain fuzzy control system which is formed by an uncertain fuzzy plant model and a fuzzy controller. The fuzzy plant model with parameter uncertainties describes exactly the behavior of an uncertain nonlinear plant. Three design approaches are introduced to close the feedback loop. Based on the Lyapunov's stability theory, new stability criteria and robust areas are to be derived without resorting to a common Lyapunov function. An application example on stabilizing an uncertain nonlinear mass-spring-damper system will be given to illustrate the merit.

1. Introduction

As fuzzy control was found capable of tackling uncertain nonlinear systems, it has become a hot topic of research. The design is usually by heuristic methods. Although they are simple and easy to understand, the stability and performance are not guaranteed. To prove the stability, researchers had derived different conditions based on sliding mode control technique [3] and adaptive technique [4]. One significant work [1] proposed the use of a fuzzy model. If this fuzzy model can describe exactly the system dynamics, a stability condition can be derived by finding a common Lyapunov's function [2, 5]. However, the condition is valid to systems without parameter uncertainties. In this paper, we analyze the stability and robustness of an uncertain fuzzy control system. The fuzzy model is modified to one with parameter uncertainties such that it can exactly describe the behavior of an uncertain nonlinear system [6]. Based on this modified fuzzy model, stability conditions and robust area are derived by applying Lyapunov's stability theory. Unlike [1], the Lyapunov's functions used no longer need to be common.

2. Fuzzy plant model and fuzzy controller

An uncertain multivariable fuzzy control system can be regarded as consisting of a fuzzy plant model and a fuzzy controller closing the feedback loop.

2.1. Fuzzy plant model with uncertainties

Let p be the number of fuzzy rules describing the uncertain nonlinear plant. The i-th rule is of the following format,

Rule i: IF x_1 is M_1^i and … and x_n is M_n^i

$$\text{THEN } \dot{\mathbf{x}} = (\mathbf{A}^i + \Delta\mathbf{A}^i)\mathbf{x} + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{u} \tag{1}$$

where M_k^i is a fuzzy term of rule i corresponding to the state x_k, $k = 1, \ldots, n$, $i = 1, \ldots, p$; $\Delta\mathbf{A}^i \in \Re^{n\times n}$ and $\Delta\mathbf{B}^i \in \Re^{n\times m}$ are the uncertainties of $\mathbf{A}^i \in \Re^{n\times n}$ and $\mathbf{B}^i \in \Re^{n\times m}$ respectively; $\mathbf{x} \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u} \in \Re^{m\times 1}$ is the input vector. The inferred system states are given by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w^i(\mathbf{x})\big((\mathbf{A}^i + \Delta\mathbf{A}^i)\mathbf{x}(t) + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{u}(t)\big) \tag{2}$$

$$\sum_{i=1}^{p} w^i(\mathbf{x}) = 1,\ w^i(\mathbf{x}) \in [0,\ 1] \text{ for all } i \tag{3}$$

$$w^i(\mathbf{x}) = \frac{\mu_{M_1^i}(x_1) \circ \mu_{M_2^i}(x_2) \circ \cdots \circ \mu_{M_n^i}(x_n)}{\sum_{i=1}^{p}\big(\mu_{M_1^i}(x_1) \circ \mu_{M_2^i}(x_2) \circ \cdots \circ \mu_{M_n^i}(x_n)\big)} \tag{4}$$

$\mu_{M_k^i}(x_k)$ is the grade of membership and '$\circ$' denotes the t-norm operator.

2.2. Fuzzy controller

A fuzzy controller with c fuzzy rules is to be designed for the plant. The j-th rule of the fuzzy controller is of the following format:

Rule j: IF x_1 is N_1^j and … and x_n is N_n^j

$$\text{THEN } \mathbf{u} = \mathbf{G}^j\mathbf{x} + \mathbf{r} \tag{5}$$

where N_β^j is a fuzzy term of rule j corresponding to the state x_β, $\beta = 1, \ldots, n$, $j = 1, \ldots, c$; $\mathbf{G}^j \in \Re^{m\times n}$ is the feedback gain of rule j, $\mathbf{r} \in \Re^{m\times 1}$ is the input vector. The inferred output of the fuzzy controller is given by

$$\mathbf{u} = \sum_{j=1}^{c} m^j(\mathbf{G}^j\mathbf{x} + \mathbf{r}) \tag{6}$$

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$$\sum_{j=1}^{c} m^j(\mathbf{x}) = 1,\ m^j(\mathbf{x}) \in [0,\ \ 1] \quad \text{for all } j \tag{7}$$

$$m^j(\mathbf{x}) = \frac{\mu_{N_1^j}(x_1) \circ \mu_{N_2^j}(x_2) \circ \cdots \circ \mu_{N_n^j}(x_n)}{\sum_{j=1}^{c} (\mu_{N_1^j}(x_1) \circ \mu_{N_2^j}(x_2) \circ \cdots \circ \mu_{N_n^j}(x_n))} \tag{8}$$

$\mu_{N_l^j}(x_l)$ is the grade of membership.

3. Stability and robustness analysis

The stability and robustness of the uncertain fuzzy control system are to be analyzed in this section. Three cases of controller design approaches will be investigated.

3.1. General Design Approach (GDA)

General design approach allows differences in the rule antecedents between the fuzzy plant model and the fuzzy controller. This approach gives designers the largest freedom on controller design. From (1) to (8), the closed-loop fuzzy system is given by,

$$\dot{\mathbf{x}} = \sum_{i=1}^{p} \sum_{j=1}^{c} w^i m^j ((\mathbf{H}^{ij} + \Delta\mathbf{H}^{ij})\mathbf{x} + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}) \tag{9}$$

$$\mathbf{H}^{ij} = \mathbf{A}^i + \mathbf{B}^i\mathbf{G}^j,\ \Delta\mathbf{H}^{ij} = \Delta\mathbf{A}^i + \Delta\mathbf{B}^i\mathbf{G}^j \tag{10}$$

3.2. Parallel Design Approach (PDA)

Parallel design approach uses the same rule antecedents of the plant model in the fuzzy controller. Hence, some of the terms in (9) can be grouped together. This makes the stability criterion to be satisfied more easily. The closed-loop fuzzy system is given by,

$$\dot{\mathbf{x}} = \sum_{i=1}^{p} w^i w^i (\mathbf{H}^{ii} + \Delta\mathbf{H}^{ii})\mathbf{x} + 2\sum_{i<j}^{p} w^i w^j (\mathbf{J}^{ij} + \Delta\mathbf{J}^{ij})\mathbf{x} + \sum_{i=1}^{p} w^i (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r} \tag{11}$$

$$\mathbf{J}^{ij} = \frac{\mathbf{H}^{ij} + \mathbf{H}^{ji}}{2},\ \Delta\mathbf{J}^{ij} = \frac{\Delta\mathbf{H}^{ij} + \Delta\mathbf{H}^{ji}}{2} \tag{12}$$

$$\mathbf{H}^{ij} = \mathbf{A}^i + \mathbf{B}^i\mathbf{G}^j,\ \Delta\mathbf{H}^{ij} = \Delta\mathbf{A}^i + \Delta\mathbf{B}^i\mathbf{G}^j \tag{13}$$

3.3. Simplified Design Approach (SDA)

Simplified design approach that requires the sub-system in each rule of the fuzzy plant model has a common input matrix **B**, and the fuzzy controller has the same number of rules with the same antecedents as the fuzzy plant model. The closed-loop fuzzy system is given by,

$$\dot{\mathbf{x}} = \sum_{j=1}^{p} w^j ((\mathbf{H}^j + \Delta\mathbf{H}^j)\mathbf{x} + (\mathbf{B}^j + \Delta\mathbf{B}^j)\mathbf{r}) \tag{14}$$

$$\mathbf{H}^j = \mathbf{A}^j + \mathbf{B}\mathbf{G}^j,\ \Delta\mathbf{H}^j = \Delta\mathbf{A}^j + \Delta\mathbf{B}\mathbf{G}^j \tag{15}$$

Because the input matrices are common, we have,

$$\mathbf{B} = \mathbf{B}^j,\ \Delta\mathbf{B} = \Delta\mathbf{B}^j \text{ or,} \tag{16}$$

$$\sum_{j=1}^{p} w^j \mathbf{B}^j = \mathbf{B},\ \sum_{j=1}^{p} w^j \Delta\mathbf{B}^j = \Delta\mathbf{B} \tag{17}$$

3.4. Stability and Robustness Analysis

In this section, the system stability and the robustness are analyzed. Theorem 1 to 4 summarize the analysis results for the three design approaches respectively. Theorem 1 and 2 are directly extended from [1], whereas Theorem 3 and 4 describe the less conservative conditions.

Theorem 1. *Under GDA, the fuzzy control system as given by* (9) *without uncertainty, i.e.* $\Delta\mathbf{H}^{ij} = 0$, *is stable if the following inequality holds:*

$\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}^{ij} < -\varepsilon\mathbf{I}$ *for all i and j*

where $\mathbf{P} \in \Re^{n \times n}$ *is a symmetric positive definite matrix,* ε *is a positive value and* $\mathbf{I} \in \Re^{n \times n}$ *is an identity matrix,* $\|\cdot\|$ *denotes the* l_2 *vector norm or induced matrix norm.*

Under PDA, the fuzzy control system as given by (12) *without uncertainty, i.e.* $\Delta\mathbf{H}^{ii} = 0$ *and* $\Delta\mathbf{J}^{ij} = 0$, *is stable if the following inequalities hold:*

$$\begin{cases} \mathbf{H}^{ii\,\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}^{ii} < -\varepsilon\mathbf{I} \text{ for all } i \\ \mathbf{J}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{J}^{ij} < -\varepsilon\mathbf{I} \text{ for all } i<j \end{cases}$$

Under SDA, the fuzzy control system of (15) *without uncertainty, i.e.* $\Delta\mathbf{H}^j = 0$, *is stable if the following inequality holds:*

$\mathbf{H}^{j\,\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}^j < -\varepsilon\mathbf{I}$ *for all j*

Definition 1. *The robust area of a fuzzy control system is defined as the area in the parameter space inside which uncertainties are allowed to exist without affecting the system stability.*

Theorem 2. *Under GDA, with the uncertain fuzzy control system given by* (9), *the robust area is governed by,*

$\left\|\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}^{ij}\right\|_{\text{Robust area}} \mathbf{I} < -(\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}^{ij}) - \varepsilon\mathbf{I}$ *for all i and j.*

The uncertain fuzzy control system is stable if the uncertainty $\left\|\Delta\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{H}^{ij}\right\|$, *with* $\left\|\Delta\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{H}^{ij}\right\|_{\max}$ *as its maximum value, satisfies the following condition:*

$\left\|\Delta\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{H}^{ij}\right\| < \left\|\Delta\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{H}^{ij}\right\|_{\max} < \left\|\Delta\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{H}^{ij}\right\|_{\text{Robust area}}$ for all i and j

Under PDA, with the uncertain fuzzy control system given by (12), *the robust area is governed by,*

$$\begin{cases} \left\|\mathbf{H}^{ii\,\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}^{ii}\right\|_{\text{Robust area}} \mathbf{I} < -(\mathbf{H}^{ii\,\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}^{ii}) - \varepsilon\mathbf{I} \text{ for all } i \\ \left\|\mathbf{J}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{J}^{ij}\right\|_{\text{Robust area}} \mathbf{I} < -(\mathbf{J}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{J}^{ij}) - \varepsilon\mathbf{I} \text{ for all } i<j \end{cases}$$

The uncertain fuzzy control system is stable if the uncertainty $\left\|\Delta\mathbf{H}^{ii\,\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{H}^{ii}\right\|$ and $\left\|\Delta\mathbf{J}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{J}^{ij}\right\|$, *with* $\left\|\Delta\mathbf{H}^{ii\,\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{H}^{ii}\right\|_{\max}$ and $\left\|\Delta\mathbf{J}^{ij\,\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{J}^{ij}\right\|_{\max}$ *as* *their maximum values respectively, satisfy the following conditions:*

$$\begin{cases} \left\|\Delta\mathbf{H}^{ii\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\mathbf{H}^{ii}\right\| < \left\|\Delta\mathbf{H}^{ii\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\mathbf{H}^{ii}\right\|_{\max} < \left\|\Delta\mathbf{H}^{ii\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\mathbf{H}^{ii}\right\|_{\text{Robust area}} & \text{for all } i \\ \left\|\Delta\mathbf{J}^{ij\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\mathbf{J}^{ij}\right\| < \left\|\Delta\mathbf{J}^{ij\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\mathbf{J}^{ij}\right\|_{\max} < \left\|\Delta\mathbf{J}^{ij\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\mathbf{J}^{ij}\right\|_{\text{Robust area}} & \text{for all } i<j \end{cases}$$

Under SDA, with the uncertain fuzzy control system given by (15), *the robust area is governed by*

$$\left\|\Delta\mathbf{H}^{j\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\mathbf{H}^{j}\right\|_{\text{Robust area}}\mathbf{I} < -(\mathbf{H}^{j\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{H}^{j})-\varepsilon\mathbf{I} \text{ for all } j$$

The uncertain fuzzy control system is stable if the uncertainty $\left\|\Delta\mathbf{H}^{j\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\mathbf{H}^{j}\right\|$, *with* $\left\|\Delta\mathbf{H}^{j\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\mathbf{H}^{j}\right\|_{\max}$ *as its maximum value, satisfies the following condition:*

$$\left\|\Delta\mathbf{H}^{j\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\mathbf{H}^{j}\right\| < \left\|\Delta\mathbf{H}^{j\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\mathbf{H}^{j}\right\|_{\max} < \left\|\Delta\mathbf{H}^{j\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\mathbf{H}^{j}\right\|_{\text{Robust area}} \quad \text{for all } j$$

Theorem 3. *Under GDA, the fuzzy control system as given by* (9) *without uncertainty, i.e.* $\Delta\mathbf{H}^{ij}=\mathbf{0}$, *is stable if the following inequalities hold:*

$$\begin{cases} \mathbf{H}^{kl\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\mathbf{H}^{kl} < -\varepsilon\mathbf{I} \\ \dfrac{\mathbf{H}^{kl\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\mathbf{H}^{kl}}{p\times c}+\left(1-\dfrac{1}{p\times c}\right)\max\limits_{i\neq k \text{ and } j\neq l}[\lambda_{\max}(\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\mathbf{H}^{ij}),\ 0] < -\varepsilon\mathbf{I} \end{cases}$$

for all i, k = 1, ..., p; j, l = 1, ..., c.

Under PDA, the fuzzy control system as given by (12) *without uncertainty, i.e.* $\Delta\mathbf{H}^{ii}=\mathbf{0}$ *and* $\Delta\mathbf{J}^{ij}=\mathbf{0}$, *is stable if the following inequalities hold:*

$$\begin{cases} \mathbf{H}^{kk\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{H}^{kk} < -\varepsilon\mathbf{I} \\ \dfrac{\mathbf{H}^{kk\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{H}^{kk}}{p^2}+\left(1-\dfrac{1}{p^2}\right)\max[\max\limits_{k\neq i}\lambda_{\max}(\mathbf{H}^{ii\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{H}^{ii}), \\ \max\limits_{i<j}\lambda_{\max}(\mathbf{J}^{ij\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{J}^{ij}),\ 0]\mathbf{I} < -\varepsilon\mathbf{I} \end{cases}$$

for i, k = 1, ..., p; j = 1, ..., c.

Under SDA, the fuzzy control system of (15) *without uncertainty, i.e.* $\Delta\mathbf{H}^{j}=\mathbf{0}$, *is stable if the following inequalities hold:*

$$\begin{cases} \mathbf{H}^{k\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\mathbf{H}^{k} < -\varepsilon\mathbf{I} \\ \dfrac{\mathbf{H}^{k\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\mathbf{H}^{k}}{p}+\left(1-\dfrac{1}{p}\right)\max\limits_{k\neq i}[\lambda_{\max}(\mathbf{H}^{j\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\mathbf{H}^{j}),\ 0] < -\varepsilon\mathbf{I} \end{cases}$$

for all j, k = 1, ..., p.

Theorem 4. *Under GDA, with the uncertain fuzzy control system given by* (9), *the robust area is governed by,*

$$\begin{cases} \dfrac{\left\|\mathbf{H}^{kl\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\mathbf{H}^{kl}\right\|_{\text{Robust area}}\mathbf{I}}{p\times c}+\left(1-\dfrac{1}{p\times c}\right)\left\|\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\mathbf{H}^{ij}\right\|_{\text{Robust area}}\mathbf{I} \\ < -\dfrac{\mathbf{H}^{kl\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\mathbf{H}^{kl}}{p\times c}-\left(1-\dfrac{1}{p\times c}\right)\lambda_{\max}(\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\mathbf{H}^{ij})-\varepsilon\mathbf{I} \\ \left\|\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\mathbf{H}^{ij}\right\|_{\text{Robust area}}+\lambda_{\max}(\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\mathbf{H}^{ij}) > 0 \end{cases}$$

for all i, k = 1, ..., p; j, l = 1, ..., c, and $ij \neq kl$.

The uncertain fuzzy control system is stable if the uncertainty $\left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|$, *with* $\left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max}$ *as its maximum value, satisfies the following condition:*

$$\left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\| < \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max} < \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\text{Robust area}}$$

for all *i* and *j*.

Under PDA, with the uncertain fuzzy control system given by (12), *the robust area is governed by,*

$$\begin{cases} \dfrac{\left\|\mathbf{H}^{kk\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{H}^{kk}\right\|_{\text{Robustarea}}\mathbf{I}}{p^2}+\left(1-\dfrac{1}{p^2}\right)\left\|\mathbf{H}^{kk\mathrm{T}}\mathbf{P}^{ii}+\mathbf{P}^{kk}\mathbf{H}^{ii}\right\|_{\text{Robustarea}}\mathbf{I} \\ < -\dfrac{\mathbf{H}^{kk\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{H}^{kk}}{p^2}-\left(1-\dfrac{1}{p^2}\right)\lambda_{\max}(\mathbf{H}^{ii\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{H}^{ii})\mathbf{I}-\varepsilon\mathbf{I} \text{ for all } i\neq k \\ \left\|\mathbf{H}^{kk\mathrm{T}}\mathbf{P}^{ii}+\mathbf{P}^{ii}\mathbf{H}^{kk}\right\|_{\text{Robustarea}}+\lambda_{\max}(\mathbf{H}^{ii\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{H}^{ii}) \geq 0 \text{ for all } i\neq k \\ \dfrac{\left\|\mathbf{H}^{kk\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{H}^{kk}\right\|_{\text{Robustarea}}\mathbf{I}}{p^2}+\left(1-\dfrac{1}{p^2}\right)\left\|\mathbf{J}^{ij\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{J}^{ij}\right\|_{\text{Robustarea}}\mathbf{I} \\ < -\dfrac{\mathbf{H}^{kk\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{H}^{kk}}{p^2}-\left(1-\dfrac{1}{p^2}\right)\lambda_{\max}(\mathbf{J}^{ij\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{J}^{ij})\mathbf{I}-\varepsilon\mathbf{I} \text{ for all } k;\ i<j \\ \left\|\mathbf{J}^{ij\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\mathbf{J}^{ij}\right\|_{\text{Robustarea}}+\lambda_{\max}(\mathbf{J}^{ij\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{ij}\mathbf{J}^{kk}) \geq 0 \text{ for all } i<j;\ k \end{cases}$$

The uncertain fuzzy control system is stable if the uncertainty $\left\|\Delta\mathbf{H}^{ii\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\Delta\mathbf{H}^{ii}\right\|$ and $\left\|\Delta\mathbf{J}^{ij\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\Delta\mathbf{J}^{ij}\right\|$, *with* $\left\|\Delta\mathbf{H}^{ii\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\Delta\mathbf{H}^{ii}\right\|_{\max}$ and $\left\|\Delta\mathbf{J}^{ij\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\Delta\mathbf{J}^{ij}\right\|_{\max}$ *as its maximum value respectively, satisfy the following conditions:*

$$\begin{cases} \left\|\Delta\mathbf{H}^{ii\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\Delta\mathbf{H}^{ii}\right\| < \left\|\Delta\mathbf{H}^{ii\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\Delta\mathbf{H}^{ii}\right\|_{\max} < \left\|\Delta\mathbf{H}^{ii\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\Delta\mathbf{H}^{ii}\right\|_{\text{Robust area}} & \text{for all } i,\ k \\ \left\|\Delta\mathbf{J}^{ij\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\Delta\mathbf{J}^{ij}\right\| < \left\|\Delta\mathbf{J}^{ij\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\Delta\mathbf{J}^{ij}\right\|_{\max} < \left\|\Delta\mathbf{J}^{ij\mathrm{T}}\mathbf{P}^{kk}+\mathbf{P}^{kk}\Delta\mathbf{J}^{ij}\right\|_{\text{Robust area}} & \text{for all } i<j;\ k \end{cases}$$

Under SDA, with the uncertain fuzzy control system given by (15), *the robust area is governed by*

$$\begin{cases} \dfrac{\left\|\mathbf{H}^{k\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\mathbf{H}^{k}\right\|_{\text{Robust area}}\mathbf{I}}{p}+\left(1-\dfrac{1}{p}\right)\left\|\mathbf{H}^{i\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\mathbf{H}^{i}\right\|_{\text{Robust area}}\mathbf{I} \\ < -\dfrac{\mathbf{H}^{k\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\mathbf{H}^{k}}{p}-\left(1-\dfrac{1}{p}\right)\lambda_{\max}(\mathbf{H}^{i\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\mathbf{H}^{i})-\varepsilon\mathbf{I} \\ \left\|\mathbf{H}^{i\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\mathbf{H}^{i}\right\|_{\text{Robust area}}+\lambda_{\max}(\mathbf{H}^{i\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\mathbf{H}^{i}) > 0 \end{cases}$$

for all $k \neq i$.

The uncertain fuzzy control system is stable if the uncertainty $\left\|\Delta\mathbf{H}^{i\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\Delta\mathbf{H}^{i}\right\|$, *with* $\left\|\Delta\mathbf{H}^{i\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\Delta\mathbf{H}^{i}\right\|_{\max}$ *as its maximum value, satisfies the following condition:*

$$\left\|\Delta\mathbf{H}^{i\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\Delta\mathbf{H}^{i}\right\| < \left\|\Delta\mathbf{H}^{i\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\Delta\mathbf{H}^{i}\right\|_{\max} < \left\|\Delta\mathbf{H}^{i\mathrm{T}}\mathbf{P}^{k}+\mathbf{P}^{k}\Delta\mathbf{H}^{i}\right\|_{\text{Robust area}}$$

for all $k \neq i$.

Proof: In the following, we will prove the theorems above under GDA only. The proofs for PDA and SDA are similar to those of GDA and are omitted. Consider the Lyapunov's functions,

$$V^{kl} = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{P}^{kl}\mathbf{x} \text{ for } k, = 1, \ldots, p \text{ and } l = 1, \ldots, c,\ k \neq i \text{ and } l \neq j. \tag{18}$$

Differentiate (18), we obtain,

$$\dot{V}^{kl} = \frac{1}{2}(\dot{\mathbf{x}}^{\mathrm{T}}\mathbf{P}^{kl}\mathbf{x} + \mathbf{x}^{\mathrm{T}}\mathbf{P}^{kl}\dot{\mathbf{x}}) \tag{19}$$

From (9) and (19),

$$\dot{V}^{kl} = \frac{1}{2}((\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j (\mathbf{H}^{ij} + \Delta\mathbf{H}^{ij})\mathbf{x} + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r})^{\mathrm{T}}\mathbf{P}^{kl}\mathbf{x} + \mathbf{x}^{\mathrm{T}}\mathbf{P}^{kl}(\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j (\mathbf{H}^{ij} + \Delta\mathbf{H}^{ij})\mathbf{x} + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}))$$

$$\leq \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j \mathbf{x}^{\mathrm{T}}(\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\mathbf{H}^{ij} + \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max})\mathbf{x} + \sum_{i=1}^{p} w^i \left\|\mathbf{P}^{kl}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|\|\mathbf{x}\| \tag{20}$$

where $\left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\| \leq \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max}$.

Let $V^{kl} = V$, $\mathbf{P}^{kl} = \mathbf{P}$ and

$$\mathbf{H}^{ij\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}^{ij} + \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{H}^{ij}\right\|_{\max}\mathbf{I} \leq -\varepsilon\mathbf{I} \text{ for all } i \text{ and } j. \tag{21}$$

From (20) and (21),

$$\dot{V} \leq \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j \mathbf{x}^{\mathrm{T}}(-\varepsilon\mathbf{I})\mathbf{x} + \sum_{i=1}^{p} w^i \left\|\mathbf{P}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|\|\mathbf{x}\|$$

$$\leq -\frac{\varepsilon}{2}\mathbf{x}^{\mathrm{T}}\mathbf{x} + \sum_{i=1}^{p} w^i \left\|\mathbf{P}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|\|\mathbf{x}\| \tag{22}$$

From (22), there are two cases to be investigated: $\mathbf{r} = \mathbf{0}$ and $\mathbf{r} \neq \mathbf{0}$. For the former case, (22) becomes,

$$\dot{V} \leq -\frac{\varepsilon}{2}\mathbf{x}^{\mathrm{T}}\mathbf{x} = -\frac{\varepsilon}{2}\|\mathbf{x}\|^2 \leq 0 \tag{23}$$

From (19), we have

$$V = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x} \geq \frac{1}{2}\lambda_{\min}(\mathbf{P})\|\mathbf{x}\|^2 \Rightarrow \dot{V} \geq \lambda_{\min}(\mathbf{P})\|\mathbf{x}\|\frac{\mathrm{d}\|\mathbf{x}\|}{\mathrm{d}t} \tag{24}$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix. From (23) and (24),

$$\frac{\mathrm{d}\|\mathbf{x}\|}{\mathrm{d}t} \leq -\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}\|\mathbf{x}\| \Rightarrow \|\mathbf{x}(t)\| \leq \|\mathbf{x}(t_o)\| e^{-\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}(t-t_o)} \tag{25}$$

where t_o is an arbitrary initial time. (25) implies $\|\mathbf{x}(t)\| \to 0$ as $t \to \infty$. For $\mathbf{r} \neq \mathbf{0}$, from (22) and (24),

$$\frac{\mathrm{d}\|\mathbf{x}\|}{\mathrm{d}t} \leq -\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}\|\mathbf{x}\| + \frac{\sum_{i=1}^{p} w^i \left\|\mathbf{P}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|}{\lambda_{\min}(\mathbf{P})}$$

$$\leq -\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}\|\mathbf{x}\| + \frac{\sum_{i=1}^{p} w^i \left\|\mathbf{P}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|_{\max}}{\lambda_{\min}(\mathbf{P})}$$

$$\leq -\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}\|\mathbf{x}\| + \frac{\max_i \left\|\mathbf{P}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|_{\max}}{\lambda_{\min}(\mathbf{P})} \tag{26}$$

where,

$$\left\|\mathbf{P}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\| \leq \left\|\mathbf{P}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|_{\max} \leq \max_i \left\|\mathbf{P}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|_{\max}$$

for all k and l. From (26),

$$\left(\frac{\mathrm{d}\|\mathbf{x}(t)\|}{\mathrm{d}t} + \frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}\|\mathbf{x}(t)\|\right) e^{\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}(t-t_o)} \leq \frac{\max_i \left\|\mathbf{P}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|_{\max}}{\lambda_{\min}(\mathbf{P})} e^{\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}(t-t_o)}$$

$$\Rightarrow \frac{\mathrm{d}\|\mathbf{x}(t)\| e^{\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}(t-t_o)}}{\mathrm{d}t} \leq \frac{\max_i \left\|\mathbf{P}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|_{\max}}{\lambda_{\min}(\mathbf{P})} e^{\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}(t-t_o)}$$

$$\|\mathbf{x}(t)\| e^{\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}(t-t_o)} \leq \|\mathbf{x}(t_o)\| + \frac{\max_i \left\|\mathbf{P}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|_{\max}}{\lambda_{\min}(\mathbf{P})} \int_{t_o}^{t} e^{\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}(\tau-t_o)} d\tau$$

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(t_o)\| e^{-\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}(t-t_o)} + \frac{\max_i \left\|\mathbf{P}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\right\|_{\max}}{\varepsilon}\left(1 - e^{-\frac{\varepsilon}{2\lambda_{\min}(\mathbf{P})}(\tau-t_o)}\right) \tag{27}$$

From (27), $\|\mathbf{x}(t)\|$ is bounded if $\mathbf{r}$ is bounded. Hence, we can prove the condition of (21) is a condition for system stability. By assuming that the system has no parameter uncertainty, the stability condition for GDA in Theorem 1 is proved. The robust area under GDA in Theorem 2 can be proved by replacing $\left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{H}^{ij}\right\|_{\max}$ with $\left\|\Delta\mathbf{H}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{H}^{ij}\right\|_{\text{Robust area}}$. This ends the proofs of the Theorem 1 and Theorem 2.

Next, we prove Theorem 3 and Theorem 4. From (20) and assuming that $w^k m^l \geq \frac{1}{p \times c} \geq w^i m^j$ for $i, k, = 1, \ldots, p;\ j, l = 1, \ldots, c$, and $kl \neq ij$, then,

$$\dot{V}^{kl} \leq \frac{1}{2}\mathbf{x}^{\mathrm{T}}(w^k m^l(\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\mathbf{H}^{ij} + \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max}\mathbf{I}) + \sum_{\substack{i=1\\i\neq k}}^{p}\sum_{\substack{j=1\\j\neq l}}^{c} w^i m^j(\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\mathbf{H}^{ij} + \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max}\mathbf{I}))\mathbf{x} + \sum_{i=1}^{p} w^i \mathbf{x}^{\mathrm{T}}\mathbf{P}^{kl}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}$$

$$\leq \frac{1}{2}\mathbf{x}^{\mathrm{T}}(w^k m^l(\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\mathbf{H}^{ij} + \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max}\mathbf{I}) + \sum_{\substack{i=1\\ij\neq kl}}^{p}\sum_{j=1}^{c} w^i m^j((\lambda_{\max}(\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\mathbf{H}^{ij}) + \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max})\mathbf{I}))\mathbf{x} + \sum_{i=1}^{p} w^i \mathbf{x}^{\mathrm{T}}\mathbf{P}^{kl}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}$$

Let

$$\begin{cases} \mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\mathbf{H}^{ij} + \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max}\mathbf{I} \leq -\varepsilon\mathbf{I} \\ \max_{ij\neq kl}(\lambda_{\max}(\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\mathbf{H}^{ij}) + \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max}) \geq 0 \end{cases}$$

for all i, j, k, l and $ij \neq kl$.

$$\Rightarrow \dot{V}^{kl} \leq \frac{1}{2}\mathbf{x}^{\mathrm{T}}\left(\frac{\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\mathbf{H}^{ij} + \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max}\mathbf{I}}{p \times c} + \left(1 - \frac{1}{p \times c}\right)\max_{ij\neq kl}((\lambda_{\max}(\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\mathbf{H}^{ij}) + \left\|\Delta\mathbf{H}^{ij\mathrm{T}}\mathbf{P}^{kl} + \mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max})\mathbf{I})\right)\mathbf{x} + \sum_{i=1}^{p} w^i \mathbf{x}^{\mathrm{T}}\mathbf{P}^{kl}(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}$$

It should be noted that if $\max_{ij\neq kl}(\lambda_{\max}(\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\mathbf{H}^{ij})+\left\|\Delta\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max}\leq 0$, the above condition is reduced to (21).

Let

$$\frac{\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\mathbf{H}^{ij}+\left\|\Delta\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max}}{p\times c} + (1-\frac{1}{p\times c})\max_{ij\neq kl}((\lambda_{\max}(\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\mathbf{H}^{ij}) + \left\|\Delta\mathbf{H}^{ij\,\mathrm{T}}\mathbf{P}^{kl}+\mathbf{P}^{kl}\Delta\mathbf{H}^{ij}\right\|_{\max})\mathbf{I})\leq -\varepsilon\mathbf{I} \quad (28)$$

for all i, j, k, l, $ij\neq kl$.

Condition of (28) will lead to the conditions for GDA in Theorem 3 and Theorem 4. Under (28),

$$\dot{V}^{kl}\leq -\frac{\varepsilon}{2}\mathbf{x}^{\mathrm{T}}\mathbf{x}+\sum_{i=1}^{p}w^{i}\mathbf{x}^{\mathrm{T}}\mathbf{P}^{kl}(\mathbf{B}^{i}+\Delta\mathbf{B}^{i})\mathbf{r} \quad (29)$$

Compare (21) and (28), we find that the condition of (28) is less conservative than (21). (29) is similar to (22). Hence, the stability of the uncertain system can be proved for the two cases of $\mathbf{r}$ mentioned early. Recalling that the inequality (29) holds under the assumption that the product of the grades of membership $w^{k}m^{l}$ is the largest, the system is stable only in a certain range of $\mathbf{x}$ that satisfies the assumption. This range of $\mathbf{x}$ will change as the largest product value of the grades of membership changes during operation. Hence, the stability is guaranteed for a local system corresponding to a particular range of $\mathbf{x}$. To prove the overall system stability, (25) and (27), we can see that the norm $\|\mathbf{x}(t)\|$ is always exponentially decaying. The switching from one local system to another only results in a change of time constant only. Therefore, the system stability is guaranteed globally. This ends the proofs of Theorem 3 and 4. For the Theorem 3 and 4, the total number of Lyapunov's functions involved is $p\times c$.

4. Application Example

An application example on stabilizing an uncertain nonlinear mass-spring-damper system is given [5]. The behavior of this system can be described by

$$M\ddot{x}+g(x,\dot{x})+f(x)=\phi(\dot{x})u \quad (30)$$

where M is the mass and u is the force, $f(x)$ describes the spring nonlinearity and uncertainty, $g(x,\dot{x})$ describes the damper nonlinearity and uncertainty, and $\phi(\dot{x})$ describes the input nonlinearity and uncertainty. Let

$$\begin{aligned} g(x,\dot{x}) &= D(c_1x+c_2\dot{x}+c_3(t)\dot{x}) \\ f(x) &= K(c_4x+c_5x^3+c_6(t)x) \\ \phi(\dot{x}) &= 1.4387+c_7\dot{x}^2+c_8(\dot{x}) \end{aligned} \quad (31)$$

The operating range of the states is assumed to be within −1.5 and 1.5. The parameters are chosen as follows: $M=1.0$, $D=K=1.0$, $c_1=0$, $c_2=1$, $c_3(t)=\frac{c_3^U+c_3^L}{2}+(c_3^L-\frac{c_3^U+c_3^L}{2})\sin(10t)$ so that $c_3(t)\in[c_3^L,c_3^U]$, $c_4=0.01$, $c_5=0.1$, $c_6(t)=\frac{c_6^U+c_6^L}{2}+(c_6^L-\frac{c_6^U+c_6^L}{2})\cos(5t)$ so that $c_6(t)\in[c_6^L,c_6^U]$, $c_7=-0.13$, $c_8(t)=(\frac{c_8^U+c_8^L}{2}+(c_8^L-\frac{c_8^U+c_8^L}{2})\cos(5t))\cos(5\dot{x})$ so that $c_8(\dot{x})\in[c_8^L,c_8^U]$, $c_3^L=-0.1$, $c_6^L=-1.26$, $c_8^L=-0.1$, $c_3^U=0.1$, $c_6^U=1.26$, $c_8^U=0.1$. It can be seen that the parameter uncertainties c_3, c_6 and c_8 are modeled as functions of time in order to show the robustness of the designed controller. The nonlinear system then becomes

$$\ddot{x}=-\dot{x}-0.01x-0.1x^3-c_3(t)\dot{x}-c_6(t)x+(1.4387-0.13\dot{x}^2+c_8(\dot{x}))u \quad (32)$$

which can exactly be represented by the following rules.

Rule i: IF x is M_1^i and $\dot{x}$ is M_2^i

THEN $\dot{\mathbf{x}}=(\mathbf{A}^i+\Delta\mathbf{A}^i)\mathbf{x}+(\mathbf{B}^i+\Delta\mathbf{B}^i)u$, $i=1, 2, 3, 4$ (33)

with the t-norm operation being chosen as the multiplication. The fuzzy rules of the fuzzy controller designed by GDA are defined as follows,

Rule 1: IF x is M_1^j and $\dot{x}$ is M_2^j

THEN $u=\mathbf{G}^j\mathbf{x}$ $j=1, 2, 3, 4$ (34)

where the membership functions of M_α^i, $i=1, 2, 3, 4$, $\alpha=1, 2$, are $\mu_{\mathrm{M}_1^1}(x)=\mu_{\mathrm{M}_1^2}(x)=1-\frac{x^2}{2.25}$, $\mu_{\mathrm{M}_1^3}(x)=\mu_{\mathrm{M}_1^4}(x)=\frac{x^2}{2.25}$, $\mu_{\mathrm{M}_2^1}(\dot{x})=\mu_{\mathrm{M}_2^3}(\dot{x})=1-\frac{\dot{x}^2}{6.75}$, $\mu_{\mathrm{M}_2^2}(\dot{x})=\mu_{\mathrm{M}_2^4}(\dot{x})=\frac{\dot{x}^2}{6.75}$; $\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$, $x_1=x$, $x_2=\dot{x}+2x_1$;

$\mathbf{A}^1=\begin{bmatrix}-2 & 0\\-2.01 & 1\end{bmatrix}$, $\mathbf{A}^2=\begin{bmatrix}-2 & 0\\-2.01 & 1\end{bmatrix}$, $\mathbf{A}^3=\begin{bmatrix}-2 & 0\\-2.235 & 1\end{bmatrix}$, $\mathbf{A}^4=\begin{bmatrix}-2 & 0\\-2.235 & 1\end{bmatrix}$; $\mathbf{B}^1=\begin{bmatrix}0\\1.4387\end{bmatrix}$, $\mathbf{B}^2=\begin{bmatrix}0\\0.5613\end{bmatrix}$, $\mathbf{B}^3=\begin{bmatrix}0\\1.4387\end{bmatrix}$, $\mathbf{B}^4=\begin{bmatrix}0\\0.5613\end{bmatrix}$,

$\Delta\mathbf{A}^1=\Delta\mathbf{A}^2=\Delta\mathbf{A}^3=\Delta\mathbf{A}^4=\begin{bmatrix}0 & 1\\2c_3(t)-c_4(t) & -c_3(t)\end{bmatrix}$,

$\Delta\mathbf{B}^1=\Delta\mathbf{B}^2=\Delta\mathbf{B}^3=\Delta\mathbf{B}^4=\begin{bmatrix}0\\c_8(\dot{x})\end{bmatrix}$. The feedback gains are designed as $\mathbf{G}^{11}$=[1.3971 −2.0852], $\mathbf{G}^{12}$=[3.5810 −5.3447], $\mathbf{G}^{13}$=[1.5535 −2.0852] and $\mathbf{G}^{14}$=[3.9818 −5.3447] so that $\mathbf{H}^{11}=\mathbf{H}^{22}=\mathbf{H}^{33}=\mathbf{H}^{44}=\begin{bmatrix}-2 & 0\\0 & -2\end{bmatrix}$;

$\mathbf{P}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Figure. 1 and 2 show the responses of $x(t)$ and $\dot{x}(t)$ with (solid lines) and without (dotted lines) parameter uncertainties with initial states of $\mathbf{x}=[-1,\ -1]$. The analysis results are tabulated in Table I which shows that the system is stable according to Theorem 2. Moreover, from (25), the system performance can be predicted to lie inside the range:

$$\|\mathbf{x}(t)\| \le \sqrt{2}e^{-\frac{\varepsilon}{2}t} \tag{35}$$

where ε is chosen as 0.01.

5. Conclusions

The stability and robustness of uncertain fuzzy control systems have been analyzed. Three design approaches of fuzzy controller have been introduced and investigated. Stability conditions and robust areas for each design approaches have been derived based on Lyapunov's stability theorem. The Lyapunov's functions for different subsystems are allowed to be distinct, and as a result, less conservative stability conditions have been obtained. An application example on stabilizing an uncertain nonlinear mass-spring-damper system has been given.

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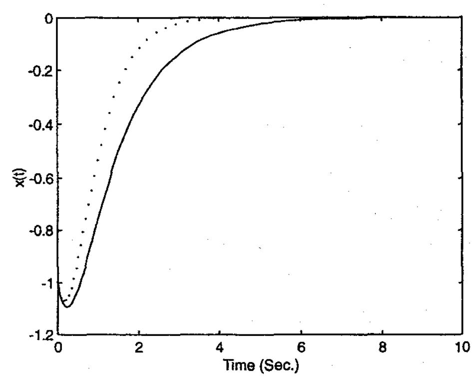

Figure 1. Response of $x(t)$ of system with (solid line) and without (dotted line) parameter uncertainties.

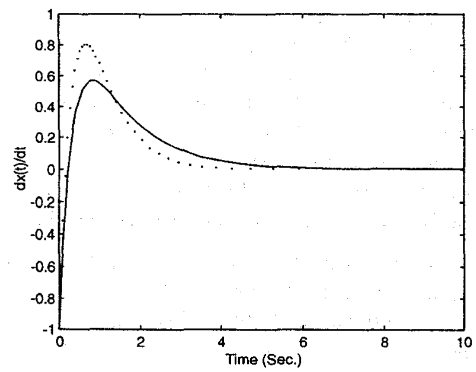

Figure 2. Response of $\dot{x}(t)$ of system with (solid line) and without (dotted line) parameter uncertainties.

i,j	$\lambda_{\max}(\mathbf{H}^{ii\,T}\mathbf{P}+\mathbf{P}\mathbf{H}^{ii})$ or $\lambda_{\max}(\mathbf{J}^{ij\,T}\mathbf{P}+\mathbf{P}\mathbf{J}^{ij})$	$\left\|\Delta\mathbf{H}^{ii\,T}\mathbf{P}+\mathbf{P}\Delta\mathbf{H}^{ii}\right\|_{\text{Robust area}}$ or $\left\|\Delta\mathbf{J}^{ij\,T}\mathbf{P}+\mathbf{P}\Delta\mathbf{J}^{ij}\right\|_{\text{Robust area}}$	$\left\|\Delta\mathbf{H}^{ii\,T}\mathbf{P}+\mathbf{P}\Delta\mathbf{H}^{ii}\right\|_{\max}$ or $\left\|\Delta\mathbf{J}^{ij\,T}\mathbf{P}+\mathbf{P}\Delta\mathbf{J}^{ij}\right\|_{\max}$	Column 2 minus Column 3
1, 1	–4.0000	2.8644	2.7280	–0.1364
1, 2	–3.7087	3.4948	3.3284	–0.1664
1, 3	–4.0000	2.8807	2.7435	–0.1372
1, 4	–3.6050	3.5357	3.3673	–0.1684
2, 2	–4.0000	3.1755	3.0243	–0.1512
2, 3	–3.7459	3.1836	3.0320	–0.1516
2, 4	–4.0000	3.2043	3.0517	–0.1526
3, 3	–4.0000	2.8807	2.7435	–0.1372
3, 4	–3.6468	3.2043	3.0517	–0.1526
4, 4	–4.0000	3.5357	3.3673	–0.1684

Table I. The stability and robustness analysis result.