Lyapunov-Function-Based Design of Fuzzy Logic Controllers and Its Application on Combining Controllers

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***Abstract*—This paper presents the design of fuzzy logic controllers (FLC's) for nonlinear systems with guaranteed closed-loop stability and its application on combining controllers. The design is based on heuristic fuzzy rules. Although each rule in the FLC refers to a stable closed-loop subsystem, the overall system stability cannot be guaranteed when all these rules are applied together. In this paper, it is proved that if each subsystem is stable in the sense of Lyapunov (ISL) under a common Lyapunov function, the overall system is also stable ISL. Since no fuzzy plant model is involved, the number of subsystems generated is relatively small, and the common Lyapunov function can be found more easily. To probe further, an application of this design approach to an inverted pendulum system that combines a sliding-mode controller (SMC) and a state feedback controller (SFC) is to be reported. Each rule in this FLC has an SMC or an SFC in the consequent part. The role of the FLC is to schedule the final control under different antecedents. The stability of the whole system is guaranteed by the proposed design approach. More importantly, the controller thus designed can keep the advantages and remove the disadvantages of the two conventional controllers.**

***Index Terms*—Combining controllers, fuzzy logic control, Lyapunov, stability.**

I. Introduction

ALTHOUGH fuzzy logic controllers (FLC's) had been proposed for a long time and were successfully applied in many applications [14], [16], a comprehensive work on the proof of stability for the closed-loop control system began only recently. On designing FLC's, we usually focus on the system responses for some common operating conditions [14], [16]. However, these methods cannot guarantee the closed-loop system stability. A proof of stability over the whole operation range is necessary before the fuzzy logic control system is put into real practice.

Recently, Tanaka and Sugeno proposed a stability design approach [2] which first modeled the plant by a Takagi–Sugeno (TS) fuzzy model [1]. This fuzzy model represents the plant as a weighted sum of a set of linear state equations. An FLC is designed based on this fuzzy plant model. Then, Lyapunov's direct method can be applied to each fuzzy subsystem [2], [18], [19] that is formed by each rule of the fuzzy plant model and the FLC. The stability of the whole system can be ensured if a required positive-definite matrix exists. Similar stability design approaches related to this fuzzy-model-based approach can also be found in [3]–[7], [18], and [19]. However, a major drawback of these stability design approaches is the difficulty in finding a common Lyapunov function. When combining the TS fuzzy plant model and the FLC, the number of subsystems generated can be $k(k+1)/2$, where k is the number of rules [2], [19]. It is very difficult to find a common Lyapunov function in general cases to satisfy all these subsystems.

Besides the fuzzy-model-based approaches, some other approaches [11]–[13], [17] involve partitioning of the state space into small parts, and each part is analyzed for closed-loop stability. However, if the number of rules is large, the number of partitions will become large and the analysis will be very time consuming. Moreover, the reported results are mostly for second-order systems. For higher order systems, the partitioning of state space cannot be viewed graphically, and the design procedures will be further complicated. In addition, these FLC's usually have some predefined structures. For example, the input/output membership functions are of regular triangular or trapezoidal shapes, symmetrical, and equally distributed in the universe of discourse. Consequently, adding a rule of another form will greatly affect the analysis.

To guarantee the bounded-input–bounded-output stability of the closed-loop system, Wang [8], [9] proposed a supervisory control methodology. This supervisory control will override the FLC in case the system exits some predefined bounds. If the width between the bounds is reduced to zero, the supervisory control is identical to a sliding-mode control. In all these cases, the stability is maintained by a large control signal only; the stability analysis of the fuzzy logic control system is not directly tackled.

In view of these weaknesses, a simple but more general methodology for designing FLC's that guarantees system stability is proposed in this paper. This design methodology employs a heuristic design concept for FLC's. Expert knowledge that can control the plant well and with guaranteed stability is first gathered. This knowledge is expressed as fuzzy

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rules of the FLC. The rule base formed then becomes the heart of the FLC. When the weighted sum defuzzification method [2]–[4], [8], [9], [16] is used, it will be shown in this paper that the control output of the FLC is bounded. Under this bounded control signal, if each individual subsystem corresponding to each rule of the FLC is stable in the sense of Lyapunov (ISL) subject to a common Lyapunov function, the overall closed-loop system (applying the FLC) can be shown to be also stable ISL [10]. Thus, the system stability is guaranteed by finding a common Lyapunov function for all rules. This common Lyapunov function is easier to be found than that in [2] because the number of fuzzy subsystems involved is only the number of rules k instead of $k(k+1)/2$. Moreover, by using this design approach, the adding of a new fuzzy rule becomes very easy. Unlike the state-space partitioning approaches [11]–[13], [17], no restriction on the shape and distribution of input membership functions is needed. An example on designing a heuristic FLC will be given to illustrate the proposed stability design approach.

Next, we shall illustrate that one useful application of the proposed design approach is to combine controllers into a single FLC. The role of this FLC is effectively to schedule a suitable control with respect to the operating conditions. A lot of conventional control algorithms had been developed for various kinds of systems [15], [20], [21]. However, each control algorithm has its own advantages and limitations. The aim of combining controllers and realizing it as an FLC is to determine the most suitable control from the embedded controller for a given operating condition, so as to ensure the best performance. With proper design, the FLC can retain the advantages and remove the disadvantages of the embedded conventional controllers. More importantly, thanks to the proposed stability design approach, the stability of the closed-loop system is guaranteed if each fuzzy rule of the FLC leads to a stable subsystem ISL subject to a common Lyapunov function in its active region. The term "active region" will be defined in Section II. In this paper, we will combine a sliding-mode controller (SMC) and a state feedback controller (SFC) into a single FLC. The resulting closed-loop system has fast response, due to the SMC. Still, when the states are near the sliding plane, the FLC will gradually be dominated by the SFC to avoid chattering. As a result, the advantages of the two conventional controllers can be kept by this combined controller.

In Section II, the fuzzy logic control system concerned in this paper will be introduced. The proposed stability design approach will be presented in Section III. Section IV will give a simple example demonstrating the design of a heuristic FLC by applying the proposed approach. In Section V, we discuss the designs of an SMC and an SFC for a nonlinear system and the way of combining them into a single FLC with stability consideration. This FLC will be applied to control a car-pole inverted pendulum. A conclusion will be drawn in Section VI.

II. Fuzzy Logic Control System

Consider a single-input nth-order nonlinear system of the following form:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x})u \tag{1}$$

where $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]^T$ is the state vector, $\boldsymbol{f}(\boldsymbol{x}) = [f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \cdots, f_n(\boldsymbol{x})]^T, \boldsymbol{b}(\boldsymbol{x}) = [b_1(\boldsymbol{x}), b_2(\boldsymbol{x}), \cdots, b_n(\boldsymbol{x})]^T$ are functions describing the dynamics of the plant, and u is the control input of the plant, the value of which is determined by an FLC with inputs depending on $\boldsymbol{x}$. The ith **IF-THEN** rule of the fuzzy rule base of the FLC is of the following form:

$$\textit{Rule } i\text{:}\quad \textbf{IF } x_1 \text{ is } X_{i1} \textbf{ AND } x_2 \text{ is } X_{i2} \textbf{ AND } \cdots \textbf{ AND } x_n \text{ is } X_{in} \textbf{ THEN } u = u_i(\boldsymbol{x}) \tag{2}$$

where $X_{i1}, X_{i2}, \cdots, X_{in}$ are input fuzzy levels, and $u = u_i(\boldsymbol{x})$ is the control output of rule i. $u_i(\boldsymbol{x})$ can be a single value or a function of $\boldsymbol{x}$. The shape of the membership functions associated with the input fuzzy levels, the method of fuzzification, and the algorithm of rule inference can be arbitrary, because these do not affect the stability design discussed in this paper. A degree of membership $\mu_i \in [0,1]$ is obtained for each rule i. It is assumed that for any $\boldsymbol{x}$ in the input universe of discourse $\boldsymbol{X}$, there exists at least one μ_i among all rules that is not equal to zero. By applying the weighted sum defuzzification method, the overall output of the FLC is given by

$$u(\boldsymbol{x}) = \frac{\sum_{i=1}^{k} \mu_i(\boldsymbol{x}) \mu_i(\boldsymbol{x})}{\sum_{i=1}^{k} \mu_i(\boldsymbol{x})} \tag{3}$$

where k is the total number of rules. Here we need to define the following terms: 1) active/inactive fuzzy rules and 2) active region of a fuzzy rule.

Definition 2.1: For any input $\boldsymbol{x_O} \in \boldsymbol{X}$, if the degree of membership $\mu_i(\boldsymbol{x_O})$ corresponding to fuzzy rule i is zero, this fuzzy rule i is called an *inactive fuzzy rule* for the input $\boldsymbol{x_O}$; otherwise, it is called an *active fuzzy rule*. An *active region* of a fuzzy rule is defined as a region $\boldsymbol{X_r} \subset \boldsymbol{X}$ such that its membership function $\mu_i(\boldsymbol{x_O})$ is nonzero for all $\boldsymbol{x_O} \in \boldsymbol{X_r}$.

It should be noted that with $\boldsymbol{x} = \boldsymbol{x_O}$, an inactive fuzzy rule will not affect the controller output $u(\boldsymbol{x_O})$. Hence, (3) can be rewritten so as to consider all active fuzzy rules (where $\mu_i(\boldsymbol{x_O}) \neq 0$) only,

$$u(\boldsymbol{x_O}) = \frac{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\boldsymbol{x_O}) u_i(\boldsymbol{x_O})}{\sum_{i=1, \mu_i \neq 0}^{k} \mu_i(\boldsymbol{x_O})}. \tag{4}$$

Now, among all the $u_i(\boldsymbol{x_O})$ of the subsystems corresponding to the active fuzzy rules, the maximum value $u_{\max}(\boldsymbol{x_O})$ and the minimum value $u_{\min}(\boldsymbol{x_O})$ of $u_i(\boldsymbol{x_O})$ can be found. Then,

$$\frac{\sum_{i=1,\mu_i\neq 0}^{k} \mu_i(\boldsymbol{x_O})u_{\min}(\boldsymbol{x_O})}{\sum_{i=1,\mu_i\neq 0}^{k} \mu_i(\boldsymbol{x_O})} \le \frac{\sum_{i=1,\mu_i\neq 0}^{k} \mu_i(\boldsymbol{x_O})u_i(\boldsymbol{x_O})}{\sum_{i=1,\mu_i\neq 0}^{k} \mu_i(\boldsymbol{x_O})} \le \frac{\sum_{i=1,\mu_i\neq 0}^{k} \mu_i(\boldsymbol{x_O})u_{\max}(\boldsymbol{x_O})}{\sum_{i=1,\mu_i\neq 0}^{k} \mu_i(\boldsymbol{x_O})}$$

$$\Rightarrow \quad u_{\min}(\boldsymbol{x_O})\frac{\sum_{i=1,\mu_i\neq 0}^{k} \mu_i(\boldsymbol{x_O})}{\sum_{i=1,\mu_i\neq 0}^{k} \mu_i(\boldsymbol{x_O})} \le u(\boldsymbol{x_O}) \le u_{\max}(\boldsymbol{x_O})\frac{\sum_{i=1,\mu_i\neq 0}^{k} \mu_i(\boldsymbol{x_O})}{\sum_{i=1,\mu_i\neq 0}^{k} \mu_i(\boldsymbol{x_O})}$$

$$\Rightarrow \quad u_{\min}(\boldsymbol{x_O}) \le u(\boldsymbol{x_O}) \le u_{\max}(\boldsymbol{x_O}), \text{ equality holds when } u_i(\boldsymbol{x_O}) = u_{\min}(\boldsymbol{x_O}) = u_{\max}(\boldsymbol{x_O}). \tag{5}$$

In conclusion, the overall FLC output is bounded by $u_{\min}(\boldsymbol{x_O})$ and $u_{\max}(\boldsymbol{x_O})$ among the rules if the weighted sum defuzzification method is employed to derive $u(\boldsymbol{x_O})$.

III. Design of Stable FLCS

The premise of the stability criterion in this paper is that, on applying each rule to the plant individually, the closed-loop subsystem formed is stable ISL in its active region, and each rule shares a common quadratic Lyapunov function $V(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ such that:

1) $V(\boldsymbol{x})$ is positive definite and continuously differentiable;
2) $\dot{V}(\boldsymbol{x}) \le 0$ for all $\boldsymbol{x}$ in its active region. (6)

For an input $\boldsymbol{x_O} \in \boldsymbol{X}$, let the maximum and the minimum control signal among all active fuzzy rules be $u_{\max}(\boldsymbol{x_O})$ and $u_{\min}(\boldsymbol{x_O})$, respectively. From (6), we have the subsystems formed by these two rules satisfying the following conditions:

$$\dot{V}(\boldsymbol{x_O}) \le 0, \qquad \text{for } u(\boldsymbol{x_O}) = u_{\max}(\boldsymbol{x_O}) \tag{7}$$
$$\dot{V}(\boldsymbol{x_O}) \le 0, \qquad \text{for } u(\boldsymbol{x_O}) = u_{\min}(\boldsymbol{x_O}). \tag{8}$$

Lemma 3.1: If a system in the form of (1) satisfies the premise of the stability criterion of (6), we have $\dot{V}(\boldsymbol{x_O})$ for all $\boldsymbol{x_O} \in \boldsymbol{X}$ and FLC output $u(\boldsymbol{x_O}) \in [u_{\min}(\boldsymbol{x_O}), u_{\max}(\boldsymbol{x_O})]$.

Proof:

$$V(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \tag{9}$$
$$\Rightarrow \quad \dot{V}(\boldsymbol{x}) = \dot{\boldsymbol{x}}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{A}\dot{\boldsymbol{x}}. \tag{10}$$

From (1),

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= (\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x})u)^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{A}(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x})u)\\ &= F(\boldsymbol{x}) + B(\boldsymbol{x})u\end{aligned} \tag{11}$$

where $F(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{f}(\boldsymbol{x})$ and $B(\boldsymbol{x}) = \boldsymbol{b}(\boldsymbol{x})^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{b}(\boldsymbol{x})$.

Note that both $F(\boldsymbol{x})$ and $B(\boldsymbol{x})$ are scalars. Then, two cases should be considered: $B(\boldsymbol{x})$ is positive and $B(\boldsymbol{x})$ is negative for $\boldsymbol{x} = \boldsymbol{x_O}$.

Case 1: $B(\boldsymbol{x_O})$ is positive.

By using condition (7),

$$\begin{aligned}\dot{V}(\boldsymbol{x_O})|_{u(\boldsymbol{x_O})=u_{\max}(\boldsymbol{x_O})} &= F(\boldsymbol{x_O}) + B(\boldsymbol{x_O})u_{\max}(\boldsymbol{x_O}) \le 0\\ \Rightarrow \quad \dot{V}(\boldsymbol{x_O}) &= F(\boldsymbol{x_O}) + B(\boldsymbol{x_O})u(\boldsymbol{x_O}) \le 0\\ &\qquad \forall u(\boldsymbol{x_O}) \le u_{\max}(\boldsymbol{x_O})\\ \Rightarrow \quad \dot{V}(\boldsymbol{x_O}) &\le 0 \quad \text{for}\\ &\qquad u(\boldsymbol{x_O}) \in [u_{\min}(\boldsymbol{x_O}), u_{\max}(\boldsymbol{x_O})].\end{aligned} \tag{12}$$

Case 2: $B(\boldsymbol{x_O})$ is negative.

By using condition (8),

$$\begin{aligned}\dot{V}(\boldsymbol{x_O})|_{u(\boldsymbol{x_O})=u_{\min}(\boldsymbol{x_O})} &= F(\boldsymbol{x_O}) + B(\boldsymbol{x_O})u_{\min}(\boldsymbol{x_O}) \le 0\\ \Rightarrow \quad \dot{V}(\boldsymbol{x_O}) &= F(\boldsymbol{x_O}) + B(\boldsymbol{x_O})u(\boldsymbol{x_O}) \le 0\\ &\qquad \forall u(\boldsymbol{x_O}) \le u_{\min}(\boldsymbol{x_O})\end{aligned} \tag{13}$$

$$\begin{aligned}\Rightarrow \quad \dot{V}(\boldsymbol{x_O}) &\le 0 \quad \text{for}\\ &\qquad u(\boldsymbol{x_O}) \in [u_{\min}(\boldsymbol{x_O}), u_{\max}(\boldsymbol{x_O})].\end{aligned} \tag{14}$$

From (12) and (13), the lemma is proved.

Q.E.D.

It has been shown in (5) that, for an arbitrary input state $\boldsymbol{x_O}$, the control output of an FLC is bounded by $u_{\min}(\boldsymbol{x_O})$ and $u_{\max}(\boldsymbol{x_O})$ if the weighted sum defuzzification method is employed. Hence, if all subsystems formed by applying individual rules to the plant satisfy $\dot{V}(\boldsymbol{x}) \le 0$ in the active regions of the rules, by *Lemma 3.1*, $\dot{V}(\boldsymbol{x}) \le 0$ for all $\boldsymbol{x} \in \boldsymbol{X}$, and the closed-loop system is stable ISL under the control of the FLC. In summary, the stability condition for a fuzzy logic control system can be stated as follows:

Summary 3.1: Consider an FLC as described in Section II. If every rule of the FLC applying to the plant of (1) individually gives a stable subsystem ISL in the active region of the fuzzy rule subject to a common Lyapunov function, and the defuzzification method is realized as given by (3), the whole fuzzy logic control system is stable ISL.

A. Adding of Rules

By using the proposed design approach, adding of new fuzzy rules becomes very easy. According to *Summary 3.1*, if we want to add a new rule to the rule base of the FLC, we just need to ensure that, on applying the new fuzzy rule to the plant, a stable subsystem ISL is obtained in the active region of the new fuzzy rule subject to the same Lyapunov function. Should this happen, the whole fuzzy logic control system is also stable ISL when this new rule is added. This can be explained as follows.

Let the output of this new rule be $u(\boldsymbol{x}) = u_{\text{new}}(\boldsymbol{x})$. Since the subsystem corresponding to this rule is stable ISL, we have

$$\dot{V}(\boldsymbol{x_O}) \le 0 \quad \text{for } u(\boldsymbol{x_O}) = u_{\text{new}}(\boldsymbol{x_O}), \boldsymbol{x_O} \in \boldsymbol{X_{r,\text{new}}} \tag{15}$$

where $\boldsymbol{X_{r,\text{new}}}$ is the active region of the new rule. Then three cases come out.

1) If $u_{\text{new}}(\boldsymbol{x_O})$ lies between $u_{\max}(\boldsymbol{x_O})$ and $u_{\min}(\boldsymbol{x_O})$, by (5) and (14), the new system is stable ISL.
2) If $u_{\text{new}}(\boldsymbol{x_O}) > u_{\max}(\boldsymbol{x_O}), u_{\max}(\boldsymbol{x_O})$ is replaced by $u_{\text{new}}(\boldsymbol{x_O})$ and condition (7) is replaced by (15). Following the same analysis procedures, condition (14) can be obtained by replacing $u_{\max}(\boldsymbol{x_O})$ with $u_{\text{new}}(\boldsymbol{x_O})$.
3) If $u_{\text{new}}(\boldsymbol{x_O}) < u_{\min}(\boldsymbol{x_O})$, similar to the second case, condition (8) is replaced by (15), and condition (14) can also be obtained by replacing $u_{\min}(\boldsymbol{x_O})$ with $u_{\text{new}}(\boldsymbol{x_O})$.

Consequently, the new system is stable ISL after adding the new rule. Moreover, unlike that in [11]–[13], the new rule is not restricted in terms of the shape, symmetry, and distribution of the input membership functions.

B. Robustness of the FLC

The robustness of an FLC can be defined as the ability to maintain the system stability under the presence of parameter uncertainties and/or unknown disturbances. Now, let (1) be rewritten as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + b(\boldsymbol{x})u + \boldsymbol{d}.$$

Assuming $B(\boldsymbol{x_O})$ to be positive, when the rule corresponding to $u_i(\boldsymbol{x_O}) = u_{\max}(\boldsymbol{x_O})$ is applied, from (12),

$$\begin{aligned} \dot{V}(\boldsymbol{x_O})|_{u(\boldsymbol{x_O})=u_{\max}(\boldsymbol{x_O})} &= F(\boldsymbol{x_O}) + \boldsymbol{x_O}^T\boldsymbol{Ad} + \boldsymbol{d}^T\boldsymbol{Ax_O} \\ &\quad + B(\boldsymbol{x_O})u_{\max}(\boldsymbol{x_O}) \\ \Rightarrow \quad \dot{V}(\boldsymbol{x_O})|_{u(\boldsymbol{x_O})=u_{\max}(\boldsymbol{x_O})} &\le F(\boldsymbol{x_O}) + |\boldsymbol{x_O}^T\boldsymbol{Ad}| + |\boldsymbol{d}^T\boldsymbol{Ax_O}| \\ &\quad + B(\boldsymbol{x_O})u_{\max}(\boldsymbol{x_O}). \end{aligned}$$

With the worst case consideration, let $|\boldsymbol{x_O}^T\boldsymbol{Ad}|, |\boldsymbol{d}^T\boldsymbol{Ax_O}|$ take their maximum values with respect to $\boldsymbol{x_O}$ and $\boldsymbol{d}$. Since the subsystem is stable ISL on applying this rule, $\dot{V}(\boldsymbol{x_O})|_{u(\boldsymbol{x_O})=u_{\max}(\boldsymbol{x_O})} \le 0$. Then, for a smaller value of $u(\boldsymbol{x_O}), \dot{V}(\boldsymbol{x_O})|_{u(\boldsymbol{x_O})<u_{\max}(\boldsymbol{x_O})}$ will become more negative, resulting in a more robust closed-loop system. We can conclude that $u_{\max}(\boldsymbol{x_O})$ is the least robust rule for positive $B(\boldsymbol{x_O})$. Similarly, if $B(\boldsymbol{x_O})$ is negative, we can also use condition (13) to conclude that $u_{\min}(\boldsymbol{x_O})$ is the least robust rule. In conclusion, either the rule corresponding to $u_{\max}(\boldsymbol{x_O})$ or $u_{\min}(\boldsymbol{x_O})$ is the least robust rule. The robustness of the whole FLC, of which the output lies between $u_{\max}(\boldsymbol{x_O})$ and $u_{\min}(\boldsymbol{x_O})$, will be better than or the same as that offered by the least robust rule.

TABLE I
FUZZY RULE BASE

Rule	Antecedent x	Antecedent $\dot{x}$	Consequent u
1	P	P	–2
2	N	N	2
3	P	N	0
4	N	P	0
5	P	Z	$-x$
6	N	Z	$-x$
7	Z	P	$-x$
8	Z	N	$-x$
9	Z	Z	$-x-\dot{x}$

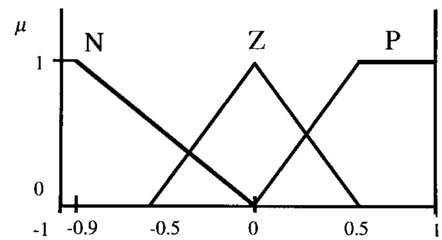

Fig. 1. Membership functions of x and $\dot{x}$.

IV. DESIGN EXAMPLE

In this section, a mass–spring–damper system [18] is to be controlled by a heuristic FLC. The fuzzy rules of the FLC are summarized in Table I. The system to be controlled is given by

$$M\ddot{x} + g(x,\dot{x}) + hx = u \tag{16}$$

where $M = 2.0$ kg is the mass and u is the input force, $h = 0.5$ N·m^{-1} is the spring constant, and $g(x,\dot{x}) = 2\dot{x}^3 + 0.5x$ is a nonlinear function describing the damper. Expressing (16) in the form of (1), we have

$$\dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}) + \boldsymbol{b}(\boldsymbol{z})u$$

where

$$\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad z_1 = x, \quad z_2 = \dot{x}$$

$$\boldsymbol{f}(\boldsymbol{z}) = \begin{bmatrix} z_2 \\ -0.5z_1 - z_2^3 \end{bmatrix}$$

$$\boldsymbol{b}(\boldsymbol{z}) = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}.$$

In Table I, the input variables in the antecedent part of the rules are x and $\dot{x}$, and the output variable in the consequent part is u. P, Z, and N are fuzzy levels representing "positive," "zero," and "negative," respectively. The membership functions are shown in Fig. 1. The rules in Table I are read as, taking rule 5 for example, "Rule 5: IF x is P and $\dot{x}$ is Z, THEN $u = -x$." The rules are set heuristically. In rule 1, x is positive and is increasing (since $\dot{x}$ is positive), so a big control action is needed, such that $u = -2$. Similarly, in rule 2, x is negative and is decreasing, so u is set to be 2. In rule 3, x is positive but decreasing, therefore, no control is needed, and $u = 0$. The similar situation happens in rule 4. When one of the states is near the origin, as in rules 5–8, a proportional control is applied. When the states are at the

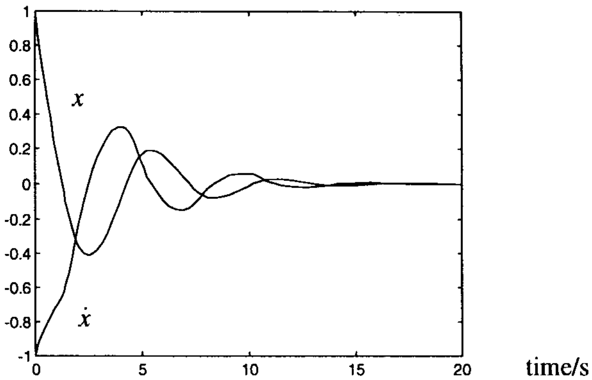

Fig. 2. Responses of x and $\dot{x}$ in the illustrated example.

origin, as described in rule 9, some damping is needed, and a proportional plus derivative control is employed.

To prove the system stability by the proposed design approach, the stability of the subsystem on applying each rule subject to a common Lyapunov function has to be proved.

Proof: Select a Lyapunov function as follows:

$$V = \tfrac{1}{2}(x^2 + \dot{x}^2) \tag{17}$$

which is obviously positive definite and continuously differentiable. Then,

$$\begin{aligned}\dot{V} &= x\dot{x} + \dot{x}\ddot{x} \\ &= x\dot{x} + \dot{x}(-0.5x - \dot{x}^3 + 0.5u)\end{aligned} \tag{18}$$

For rule 1, $u = -2$ and $x \in [0, 1], \dot{x} > 0$

$$\Rightarrow \quad \dot{V} = 0.5(x-2)\dot{x} - \dot{x}^4 \le 0.$$

For rule 2, $u = 2$ and $x \in [-1, 0], \dot{x} < 0$

$$\Rightarrow \quad \dot{V} = 0.5(x+2)\dot{x} - \dot{x}^4 \le 0.$$

For rules 3 and 4, $u = 0$ and x and $\dot{x}$ are of opposite sign

$$\Rightarrow \quad \dot{V} = 0.5x\dot{x} - \dot{x}^4 \le 0.$$

For rules 5–8, $u = -x$

$$\Rightarrow \quad \dot{V} = -\dot{x}^4 \le 0.$$

For rule 9, $u = -x - \dot{x}$

$$\begin{aligned}\Rightarrow \quad \dot{V} &= x\dot{x} + \dot{x}(-0.5x - \dot{x}^3 - 0.5x - 0.5\dot{x}) \\ &= -\dot{x}^4 - 0.5\dot{x}^2 \le 0.\end{aligned}$$

Hence, all of the nine rules in the FLC can lead to stable subsystems ISL subject to the same Lyapunov function (17). From *Summary 3.1,* the closed-loop system is stable ISL when all the rules are included into the rule base of the FLC.

Q.E.D.

Simulation results of the zero-input responses of the closed-loop system with initial values $\boldsymbol{x}(0) = [1\ -1]^T$ are shown in Fig. 2. The stability of the fuzzy logic control system is verified.

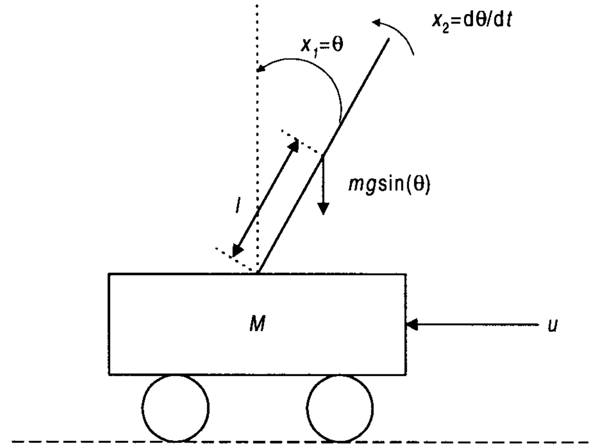

Fig. 3. A car-pole inverted pendulum.

V. Application: Combining Conventional Controllers

In this section, a sliding-mode controller (SMC) and a state feedback controller (SFC) will be combined into a single FLC by applying the proposed designed approach. This FLC will be used to balance a car-pole inverted pendulum system [15], as shown in Fig. 3. When the states are far from the origin of the state plane, the SMC will take a major part of control to give a fast transient response. However, when the states are approaching the equilibrium values (the origin of the state plane), the SMC will gradually be replaced by the SFC, in order to avoid chattering. The equations of motion of the car-pole inverted pendulum are as follows:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g\sin(x_1) - amlx_2^2\sin(2x_1)/2 - a\cos(x_1)u}{4l/3 - aml\cos^2(x_1)}\end{aligned} \tag{19}$$

where x_1 denotes the angular displacement (in rad) of the pendulum from the vertical axis, and x_2 is the angular velocity (in rad s^{-1}), $g = 9.8$ m s^{-2} is the acceleration due to gravity, $m = 2.0$ kg is the mass of the pendulum, $M = 8.0$ kg is the mass of the cart, $2l = 1.0$ m is the length of the pendulum, and u is the force applied to the cart (in N); $a = (m+M)^{-1}$. Reshuffling the terms, (19) can be written in the following form:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x})u$$

where

$$\begin{aligned}\boldsymbol{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_{11}(\boldsymbol{x}) & f_{12}(\boldsymbol{x}) \\ f_{21}(\boldsymbol{x}) & f_{22}(\boldsymbol{x}) \end{bmatrix}\boldsymbol{x} \\ &= \begin{bmatrix} 0 & 1 \\ \dfrac{g\sin(x_1)/x_1 - amlx_2^2\sin(2x_1)/2x_1}{4l/3 - aml\cos^2(x_1)} & 0 \end{bmatrix}\boldsymbol{x} \\ \boldsymbol{b}(\boldsymbol{x}) &= \begin{bmatrix} b_1(\boldsymbol{x}) \\ b_2(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{-a\cos(x_1)}{4l/3 - aml\cos^2(x)1)} \end{bmatrix}.\end{aligned}$$

A. SMC

An SMC can be designed for this car-pole inverted pendulum. Define a sliding plane

$$\sigma = \boldsymbol{s}\boldsymbol{x} = 0 \tag{20}$$

where $s = [1\ 0.1]$. It is obvious that the sliding plane is stable. An equivalent control u_{eq} can be obtained by considering

$$\dot{\sigma} = \boldsymbol{s}\dot{\boldsymbol{x}} = 0 \tag{21}$$
$$\Rightarrow \quad \boldsymbol{s}\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{s}\boldsymbol{b}(\boldsymbol{x})u = 0$$
$$\Rightarrow \quad u = u_{\mathrm{eq}} = -(\boldsymbol{s}\boldsymbol{b}(\boldsymbol{x}))^{-1}\boldsymbol{s}\boldsymbol{f}(\boldsymbol{x}). \tag{22}$$

The final control is realized as $u = u_{\mathrm{eq}} + u_d$, where $u_d = -(\boldsymbol{s}\boldsymbol{b}(\boldsymbol{x}))^{-1}\ k_d \ \mathrm{sgn}(\sigma), k_d$ is a positive constant, and

$$\mathrm{sgn}(\sigma) = \begin{cases} 1, & \text{if } \sigma > 0 \\ 0, & \text{if } \sigma = 0 \\ -1, & \text{if } \sigma < 0. \end{cases}$$

Defining a Lyapunov function

$$V = \tfrac{1}{2}\sigma^2 \tag{23}$$

we have

$$\begin{aligned}\dot{V} &= \sigma\dot{\sigma} \\ &= \sigma[\boldsymbol{s}\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{s}\boldsymbol{b}(\boldsymbol{x})(-(\boldsymbol{s}\boldsymbol{b}(\boldsymbol{x}))^{-1}(\boldsymbol{s}\boldsymbol{f}(\boldsymbol{x}) + k_d \ \mathrm{sgn}(\sigma)))] \\ &= -\sigma k_d \ \mathrm{sgn}(\sigma) \\ &= -k_d|\sigma| \\ &\leq 0. \end{aligned} \tag{24}$$

B. SFC

An SFC can also be employed for the system of (19). Design a state feedback gain

$$\boldsymbol{K_f}(\boldsymbol{x}) = [k_{f1}(\boldsymbol{x})\ k_{f2}(\boldsymbol{x})] = \left[\frac{-10 - f_{21}(\boldsymbol{x})}{b_2(\boldsymbol{x})}\ \frac{-11}{b_2(\boldsymbol{x})}\right]$$

such that the control signal is given by

$$u = \boldsymbol{K_f}(\boldsymbol{x})\boldsymbol{x}. \tag{25}$$

By using the same Lyapunov function V of (23), if the control law of (25) is employed, we have

$$\begin{aligned}\dot{V} &= \sigma\dot{\sigma} \\ &= \boldsymbol{x}^T\boldsymbol{s}^T\boldsymbol{s}\dot{\boldsymbol{x}} \\ &= \boldsymbol{x}^T\boldsymbol{s}^T\boldsymbol{s}(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x})\boldsymbol{K_f}(\boldsymbol{x})\boldsymbol{x}) \\ &= \boldsymbol{x}^T\begin{bmatrix}1\\0.1\end{bmatrix}[1 \quad 0.1] \\ &\quad \left(\begin{bmatrix}0 & 1\\ f_{21}(\boldsymbol{x}) & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ b_2(\boldsymbol{x})k_{f1}(\boldsymbol{x}) & b_2(\boldsymbol{x})k_{f2}(\boldsymbol{x})\end{bmatrix}\right)\boldsymbol{x} \\ &= -\boldsymbol{x}^T\begin{bmatrix}1 & 0.1\\ 0.1 & 0.01\end{bmatrix}\boldsymbol{x} \\ &\leq 0. \end{aligned} \tag{26}$$

C. FLC

Define the rules of an FLC as follows:

Rule 1: IF σ is PE THEN $u = -(\boldsymbol{s}\boldsymbol{b}(\boldsymbol{x}))^{-1}(\boldsymbol{s}\boldsymbol{f}(\boldsymbol{x}) + k_d)$

Rule 2: IF σ is ZE THEN $u = \boldsymbol{K_f}(\boldsymbol{x})\boldsymbol{x}$

Rule 3: IF σ is NE THEN $u = -(\boldsymbol{s}\boldsymbol{b}(\boldsymbol{x}))^{-1}(\boldsymbol{s}\boldsymbol{f}(\boldsymbol{x}) - k_d)$

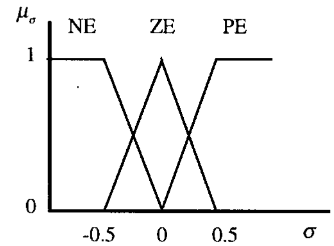

Fig. 4. Fuzzy levels of σ and their membership functions.

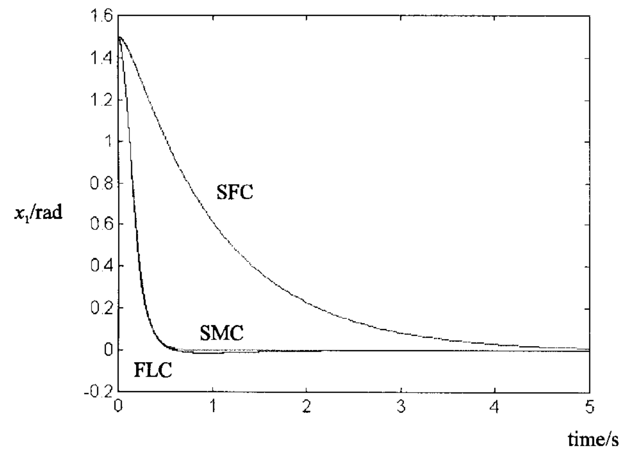

Fig. 5. System response of x_1 for an initial $x = [1.5\ 0]^T$.

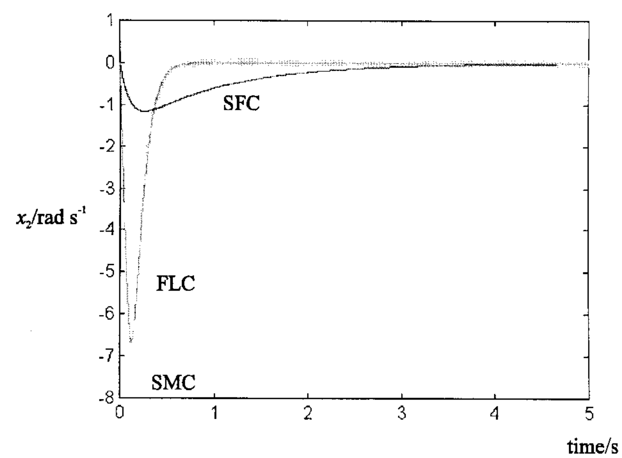

Fig. 6. System response of x_2 for an initial $x = [1.5\ 0]^T$.

where PE, ZE and NE are fuzzy levels of σ of which the membership functions are shown in Fig. 4.

To prove the stability of the fuzzy-controlled inverted pendulum system, we need to ensure $\boldsymbol{x}$ tends to $\mathbf{0}$ for all operation conditions. Although the Lyapunov function of (23) is a function of σ only, but not $\boldsymbol{x}$, thanks to the stability of the sliding plane, $\boldsymbol{x}$ tends to $\mathbf{0}$ if $\sigma = 0$. Hence, from *Summary 3.1*, the proof of system stability is reduced to proving that σ tends to zero on applying each rule to the inverted pendulum. This cannot be reached immediately, because the conditions V is positive definite and $\dot{V} \leq 0$ may imply that σ tends to a finite constant instead of zero. To verify that σ must finally go to zero, we need to prove that $\dot{V} < 0$ at $V \neq 0$ for all operation

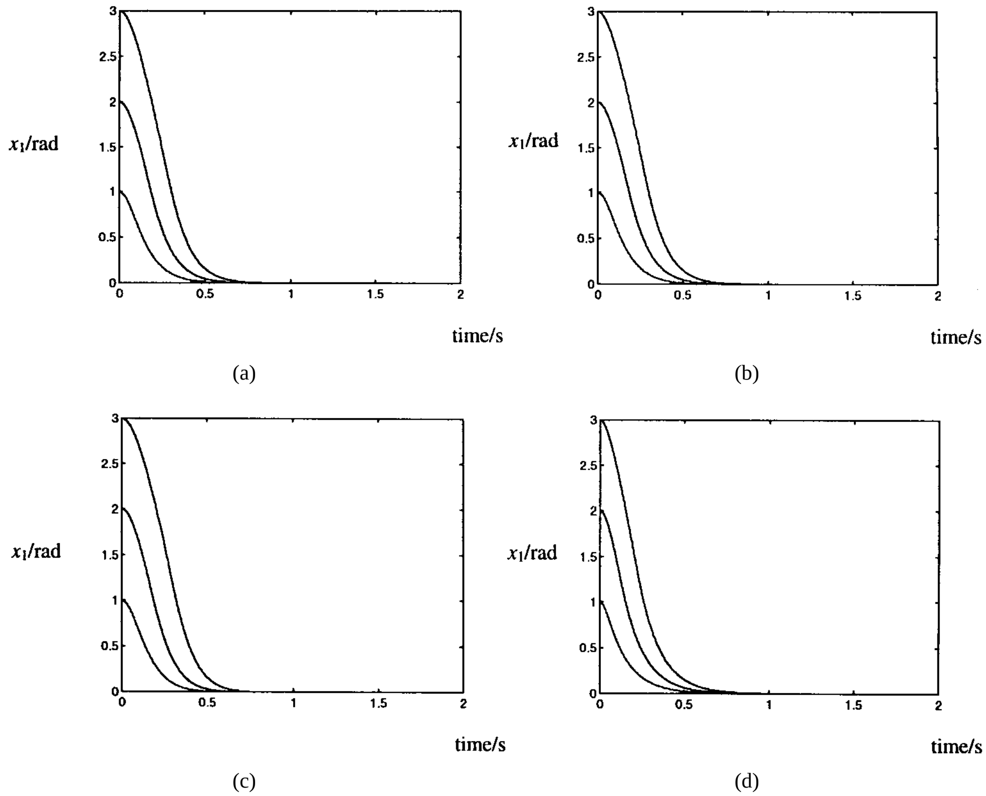

Fig. 7. System response of x_1: (a) for other initial conditions, (b) for m changed to 12 kg, (c) for M changed to 2 kg, and (d) for l changed to 0.25 m.

conditions. This proof is given in the Appendix. Hence, the closed-loop system under the control of the proposed FLC is stable.

D. Simulation Results

The designed FLC is applied to the plant of (19), and simulation results of the closed-loop system responses are obtained. For comparison, the responses on applying the SMC alone and the SFC alone are also taken. The initial state is $\boldsymbol{x}(0) = [1.5\ 0]^T$. The responses of x_1 and x_2 are shown in Figs. 5 and 6, respectively. It can be seen that the settling time of x_1 on applying the FLC is better than that on applying the SFC alone. The responses on using the FLC and the SMC are similar, but no chattering exists in x_2 if the FLC is used.

The transient responses of x_1 on applying the FLC under initial states $\boldsymbol{x}(0) = [1\ 0]^T, [2\ 0]^T$ and $[3\ 0]^T$ are shown in Fig. 7(a). To test the robustness of the FLC, simulations under different plant parameter values are conducted. Fig. 7(b)–(d) shows the same transient responses as those in Fig. 7(a), but with m changed to 12 kg, M changed to 2 kg, and l changed to 0.25 m, respectively. From these responses, we can see that the FLC designed is robust to parameter variations.

VI. Conclusion

An approach for designing stable heuristic FLC's has been proposed in this paper. It has been shown that a fuzzy logic control system is stable ISL, provided that every individual rule applying to the plant gives a stable subsystem ISL in the active region of the fuzzy rule under a common Lyapunov function. Therefore, the stability of the fuzzy logic control systems can be guaranteed by examining each individual rule in the FLC, which is much simpler than the existing approaches. The stability of a nonlinear mass–spring–damper system controlled by an FLC has been analyzed based on the proposed approach as an illustrative example.

A practical application of the proposed design approach is to combine conventional controllers by an FLC and generate the appropriate control according to the operating conditions. In Section V, an FLC combining an SMC and an SFC has been reported. The stability of the FLC is guaranteed from the proposed stability analysis. This FLC is applied to balance a car-pole inverted pendulum system. It has been clearly shown that a good transient response, as well as robustness to parameter variations, can be obtained due to the SMC. However, chattering does not take place, due to the effect of the SFC near the equilibrium point. The combined controller then inherits the merits of the two conventional controllers.

Appendix

This appendix is to prove that $\dot{V} < 0$ at $V \neq 0$ for all operation conditions of the designed FLC. It can be seen from Fig. 4 that, for any value of $\sigma \neq 0$, only one of the following two cases will occur:

1) Either Rule 1 or Rule 3 is active.

When this case happens, $\sigma > 0.5$ or $\sigma < -0.5$, from (24),

$$\begin{aligned}\dot{V} &= -k_d|\sigma| \\ &< 0 \end{aligned} \tag{A1}$$

since $\sigma \neq 0$.

2) Either Rules 1 and 2 or Rules 3 and 2 are active simultaneously.

Consider Rules 1 and 2 are active, i.e., $0 < \sigma \leq 0.5$. Let $u_1(\boldsymbol{x_O})$ and $u_2(\boldsymbol{x_O})$ be the outputs of Rules 1 and 2, respectively. Since the two corresponding subsystems are stable ISL, we have

$$\dot{V}_1(\boldsymbol{x_O}) = F(\boldsymbol{x_O}) + B(\boldsymbol{x_O})u_1(\boldsymbol{x_O}) \leq 0$$
$$\dot{V}_2(\boldsymbol{x_O}) = F(\boldsymbol{x_O}) + B(\boldsymbol{x_O})u_2(\boldsymbol{x_O}) \leq 0.$$

Also, let $\dot{V}(\boldsymbol{x_O}) = F(\boldsymbol{x_O}) + B(\boldsymbol{x_O})u(\boldsymbol{x_O})$, where $u(\boldsymbol{x_O})$ is the output of the FLC; then, one of the following two cases will happen.

Case 1: $u_1(\boldsymbol{x_O}) = u_2(\boldsymbol{x_O})$

In this case, $u(\boldsymbol{x_O}) = u_1(\boldsymbol{x_O}) = u_2(\boldsymbol{x_O})$. Since $u(\boldsymbol{x_O}) = u_1(\boldsymbol{x_O})$, from (A1) ,$\dot{V} < 0$ at $V \neq 0$.

Case 2: $u_1(\boldsymbol{x_O}) \neq u_2(\boldsymbol{x_O})$

Let $u_1(\boldsymbol{x_O}) > u_2(\boldsymbol{x_O})$, from (5)

$$u_2(\boldsymbol{x_O}) < u(\boldsymbol{x_O}) < u_1(\boldsymbol{x_O}).$$

If $B(\boldsymbol{x_O}) > 0, \dot{V}(\boldsymbol{x_O}) < \dot{V}_1(\boldsymbol{x_O}) \leq 0$. Otherwise, if $B(\boldsymbol{x_O}) < 0$, $\dot{V}(\boldsymbol{x_O}) < \dot{V}_2(\boldsymbol{x_O}) \leq 0$. As a result, $\dot{V} < 0$ when $u_1(\boldsymbol{x_O}) > u_2(\boldsymbol{x_O})$. Following the same steps as above, it can be easily proved that $\dot{V} < 0$ when $u_1(\boldsymbol{x_O}) < u_2(\boldsymbol{x_O})$.

Hence, $\dot{V} < 0$ when Rule 1 and Rule 2 are both active.

When Rule 2 and Rule 3 are active, a similar proof can be carried out to show that $\dot{V} < 0$ at $V \neq 0$. Q.E.D.

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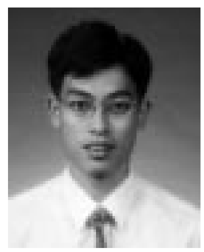

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