

Necessary and Sufficient Condition for Finite Horizon H_∞ Estimation of Time Delay Systems

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Abstract—This paper is concerned with the problems of finite horizon H_∞ filtering, prediction and fixed-lag smoothing for linear continuous-time systems with multiple delays. By applying an innovation approach in Krein space, a necessary and sufficient condition for the existence of an H_∞ filter, predictor or smoother is derived. The estimator is given in terms of the solution of a partial differential equation with boundary conditions. The innovation approach in Krein space enables us to convert the very complicated deterministic estimation problem into a stochastic one to which a simple H_2 innovation analysis method can be adapted. The result of this paper demonstrates that the Krein space approach is powerful in solving otherwise very complicated H_∞ problems. Our result is in contrast with many recent sufficient conditions for H_∞ filtering of delay systems.

I. INTRODUCTION

Linear estimation has important applications in many fields of science and engineering and has attracted significant interest in the past 50 years; see, e.g. [6], [11], [16], [10]. It is usually classified based on the performance criterion under which the estimation problem is addressed. H_∞ estimation has gained popularity in the past decades [17], [19] which is to find a linear estimator such that the energy gain between the noise inputs and the estimation error is less than a prescribed level. As compared to the minimum variance estimation, the H_∞ filter does not require the knowledge of the statistics of noise signals. In addition, an H_∞ filter is also less sensitive than the H_2 counterpart to uncertainty in system parameters [19].

The H_∞ filtering problem for continuous-time systems has been addressed in [17] and the solution is given in terms of the existence of a bounded solution to a Riccati differential equation. Recently, the H_∞ prediction and fixed-lag smoothing problems for linear continuous-time systems without delays were investigated in [21]. For linear time-invariant systems with state delays, a sufficient condition has been given in [5] for the H_∞ filtering in the infinite horizon case using a linear matrix inequality (LMI) approach. For

time-varying systems with delayed measurement, the H_∞ filtering has been addressed in [18] which gives a sufficient condition in terms of a Riccati differential equation in the finite horizon case and an algebraic Riccati equation in the infinite horizon case.

In this paper we consider linear time-varying systems with known delays in both the state and output equations. The problem to be investigated is the design of an estimator such that a given H_∞ performance is achieved in the finite horizon case. We first discuss the H_∞ fixed-lag smoothing problem. By converting it into an indefinite quadratic optimization problem, an innovation approach in Krein space [7] is proposed to give a necessary and sufficient condition for the existence of an H_∞ smoother in terms of a bounded solution of a Riccati type of partial differential equation. A smoother is then constructed. The case of filtering is in fact a special case of the smoothing. It can be seen from the derivation of the result that the Krein space innovation approach is very powerful in dealing with complicated problems like the H_∞ fixed-lag smoothing problem for delay systems. Our result can be considered as the H_∞ counterpart of the H_2 result in [13] even though a different derivation method is applied. We further show that the H_∞ prediction problem can be approached in a similar way and hence a necessary and sufficient condition is obtained. As special cases, solutions to the H_∞ fixed-lag smoothing and prediction for systems without delays are also given.

II. PROBLEM STATEMENTS

We consider the linear time-varying system with multiple time delays described by

$$\dot{x}(t) = \sum_{i=0}^k \Phi_i(t)x(t-h_i) + \Gamma(t)u(t), \quad (1)$$

$$y(t) = \sum_{i=0}^k H_i(t)x(t-h_i) + v(t), \quad (2)$$

$$z(t) = L(t)x(t), \quad (3)$$

where $x(t) \in R^n$, $u(t) \in R^r$, $y(t) \in R^m$, $v(t) \in R^m$ and $z(t) \in R^p$ represent the state, input noise, measurement output, measurement noise and the signal to be estimated, respectively and $\Phi_i(t)$, $\Gamma(t)$, $H_i(t)$ and $L(t)$ are bounded time-varying matrices with appropriate dimension. It is assumed that the input and measurement noises are deterministic and are from $L_2[0, T]$ where $T > 0$ is the time-horizon. The scalar quantities $0 = h_0 < h_1 < \dots < h_k$ are known constant time delays of the system. Denote $h \triangleq \max\{h_0, \dots, h_k\}$. The initial condition $x(s) (-h \leq s \leq 0)$ is unknown.

The general H_∞ estimation problem including filtering, prediction and fixed-lag smoothing is stated as:

• **H_∞ Fixed-lag Smoothing and filtering:** Given the desired noise attenuation level $\gamma > 0$, the smoothing lag $\theta \geq 0$ and the observation $\{y(s), 0 \leq s \leq t\}$, find an estimate $\hat{z}(t, \theta)$ of $z(t - \theta)$, if it exists, such that the following inequality is satisfied:

$$\frac{\sup_{\{x(s) | -h \leq s \leq 0, u, v\} \neq 0} \int_{\theta}^T [\hat{z}(t, \theta) - z(t - \theta)]' [\hat{z}(t, \theta) - z(t - \theta)] dt}{\int_{-h}^0 x'(t) \Pi_t^{-1} x(t) dt + \int_0^T u'(t) u(t) dt + \int_0^T v'(t) v(t) dt} < \gamma^2 \quad (4)$$

where “ $'$ ” stands for the transposition and Π_t , $t \in [-h, 0]$ is a given positive definite matrix function which reflects the relative uncertainty of the initial state $x(t)$, $-h \leq t \leq 0$ about the origin to the input and measurement noises.

We note that when $\theta = 0$, the above defines an H_∞ filtering problem.

• **H_∞ Prediction:** Given the desired noise attenuation level $\gamma > 0$, the prediction lead $\theta > 0$ and the observation $\{y(s), 0 \leq s \leq t - \theta\}$, find an estimate $\hat{z}(t - \theta, -\theta)$ of $z(t)$, if it exists, such that the following inequality is satisfied:

$$\frac{\sup_{\{x(s) | -h \leq s \leq 0, u, v\} \neq 0} \int_0^T [\hat{z}(t - \theta, -\theta) - z(t)]' [\hat{z}(t - \theta, -\theta) - z(t)] dt}{\int_{-h}^0 x'(t) \Pi_t^{-1} x(t) dt + \int_0^T u'(t) u(t) dt + \int_0^{T-\theta} v'(t) v(t) dt} < \gamma^2, \quad (5)$$

where Π_t has the same meaning as in the fixed-lag smoothing.

III. H_∞ SMOOTHING

In view of (4), we define

$$\mathcal{J}_S(T) \triangleq \int_{-h}^0 x'(s) \Pi_s^{-1} x(s) ds + \int_0^T u'(s) u(s) ds + \int_0^T v'(s) v(s) ds - \gamma^{-2} \int_{\theta}^T v_z'(s) v_z(s) ds$$

$$= \int_{-h}^0 x'(s) \Pi_s^{-1} x(s) ds + \int_0^T u'(s) u(s) ds + \int_0^T \begin{bmatrix} v(s) \\ v_z(s) \end{bmatrix}' \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^{-2} I_p \end{bmatrix} \begin{bmatrix} v(s) \\ v_z(s) \end{bmatrix} ds, \quad (6)$$

where

$$\begin{aligned} v_z(s) &= \hat{z}(s, \theta) - z(s - \theta) \\ &= \hat{z}(s, \theta) - L(s - \theta)x(s - \theta), \end{aligned} \quad (7)$$

and $v_z(s) = 0$ for $s < \theta$.

It follows from [7] that an estimate $\hat{z}(t, \theta)$ ($\theta \leq t \leq T$) that achieves (4) exists if and only if: 1) the above quadratic function $\mathcal{J}_S(T)$ has a minimum \mathcal{J}_S^{min} with respect to $u(t)$ ($0 \leq t \leq T$) and $x(t)$ ($-h \leq t \leq 0$); and 2) a $\hat{z}(t, \theta)$ ($\theta \leq t \leq T$) can be chosen such that the minimum is positive for all $y(\cdot)$.

A. Stochastic system in Krein space

In association with system (1)-(3) and the cost function (6), we introduce the stochastic system below.

$$\dot{\mathbf{x}}(t) = \sum_{i=0}^k \Phi_i(t) \mathbf{x}(t - h_i) + \Gamma(t) \mathbf{u}(t), \quad (8)$$

$$\mathbf{y}(t) = \sum_{i=0}^k H_i(t) \mathbf{x}(t - h_i) + \mathbf{v}(t), \quad (9)$$

$$\dot{\mathbf{z}}(t, \theta) = L(t - \theta) \mathbf{x}(t - \theta) + \mathbf{v}_z(t), \quad t \geq \theta \quad (10)$$

where the initial state $\mathbf{x}(\tau)$ ($-h \leq \tau \leq 0$) and $\mathbf{u}(t)$, $\mathbf{v}(t)$ and $\mathbf{v}_z(t)$ ¹ are mutually uncorrelated white noises with zero means and known covariance matrices Π_τ , $Q_u(t)$, $Q_v(t)$ and $Q_{v_z}(t)$, with $Q_u(t) = I_r$, $Q_v(t) = I_m$ for $t \geq 0$ and

$$Q_{v_z}(t) = \begin{cases} 0, & t < \theta \\ -\gamma^2 I_p, & t \geq \theta. \end{cases}$$

Note that $\dot{\mathbf{z}}(t, \theta)$ can be considered as a 'fictitious' measurement at time t for the linear combination $L(t - \theta) \mathbf{x}(t - \theta)$.

Denote

$$\mathbf{y}_z(t) \triangleq \begin{cases} \mathbf{y}(t), & 0 \leq t < \theta \\ \begin{bmatrix} \mathbf{y}(t) \\ \dot{\mathbf{z}}(t, \theta) \end{bmatrix}, & t \geq \theta \end{cases} \quad (11)$$

which is the measurement of the stochastic system (8)-(10) in an indefinite linear space. It follows from (11) that for $0 \leq t < \theta$

$$\mathbf{y}_z(t) = \sum_{i=0}^k H_i(t) \mathbf{x}(t - h_i) + \mathbf{v}(t), \quad (12)$$

¹Whenever the Krein Space elements [7] and the Euclidean space elements satisfy the same set of constraints, we shall denote them by the same letters with the former identified by bold faces and the latter by normal faces.

and for $t \geq \theta$,

$$\mathbf{y}_z(t) = \begin{bmatrix} \sum_{i=0}^k H_i(t)\mathbf{x}(t-h_i) \\ L(t-\theta)\mathbf{x}(t-\theta) \end{bmatrix} + \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{v}_z(t) \end{bmatrix}. \quad (13)$$

The measurements up to time t are collected as

$$\{\mathbf{y}_z(s), 0 \leq s \leq t\}. \quad (14)$$

Similar to the case in Hilbert space, define $\hat{\mathbf{y}}_z(s, 0)$ the projection [7] of $\mathbf{y}_z(s)$ onto $\mathcal{L}\{\mathbf{y}_z(r), r < s\}$. It should be noted that unlike in Hilbert space a projection in Krein space may not exist. If the projection $\hat{\mathbf{y}}_z(s, 0)$ exists, we define the innovation of the observation $\mathbf{y}_z(s)$ as

$$\mathbf{w}_z(s) \triangleq \mathbf{y}_z(s) - \hat{\mathbf{y}}_z(s, 0). \quad (15)$$

Note that $\mathbf{w}_z(s)$ is in fact the prediction error of the observation. Obviously, the linear space $\mathcal{L}\{\mathbf{y}_z(s), s < t\}$ is equivalent to $\mathcal{L}\{\mathbf{w}_z(s), s < t\}$ [11].

Lemma 1: Suppose that there exist the projections $\hat{\mathbf{x}}(t, \theta)$ and $\hat{\mathbf{x}}(t, h_i)$ of $\mathbf{x}(t - \theta)$ and $\mathbf{x}(t - h_i)$, respectively, onto the linear space of $\mathcal{L}\{\mathbf{y}_z(s), s < t\}$. Then, the innovation $\mathbf{w}_z(t)$, defined by (15), can be given as

$$\mathbf{w}_z(t) = \begin{cases} \mathbf{y}(t) - \sum_{i=0}^k H_i(t)\hat{\mathbf{x}}(t, h_i), & 0 \leq t < \theta \\ \begin{bmatrix} \mathbf{y}(t) \\ \hat{\mathbf{z}}(t, \theta) \end{bmatrix} - \begin{bmatrix} \sum_{i=0}^k H_i(t)\hat{\mathbf{x}}(t, h_i) \\ L(t-\theta)\hat{\mathbf{x}}(t, \theta) \end{bmatrix}, & t \geq \theta \end{cases} \quad (16)$$

and the innovation covariance matrix

$$Q_{w_z}(t) \triangleq \langle \mathbf{w}_z(t), \mathbf{w}_z(t) \rangle \quad (17)$$

is given by

$$Q_{w_z}(t) = \begin{cases} I_m, & 0 \leq t < \theta \\ \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_p \end{bmatrix}, & t \geq \theta. \end{cases} \quad (18)$$

B. Sufficient and necessary condition for the existence of an H_∞ smoother

Theorem 1: Consider the system (1)-(3) and the associated performance criterion (4). Then, for a given scalar $\gamma > 0$, a smoother $\hat{\mathbf{z}}(t, \theta)$ ($\theta \leq t \leq T$) that achieves (4) exists if and only if

- $\hat{\mathbf{x}}(t, \theta)$ exists for $0 \leq t \leq T$,
- $\hat{\mathbf{x}}(t, h_i)$ exists for $0 \leq t \leq T$ and $0 \leq i \leq k$,

where $\hat{\mathbf{x}}(t, \theta)$ and $\hat{\mathbf{x}}(t, h_i)$ are respectively given from the projections of $\mathbf{x}(t - \theta)$ and $\mathbf{x}(t - h_i)$ onto $\{\mathbf{y}_z(s), s < t\}$. In this situation, a suitable H_∞ smoother (central estimator) $\hat{\mathbf{z}}(t, \theta)$ is given by

$$\hat{\mathbf{z}}(t, \theta) = L(t - \theta)\hat{\mathbf{x}}(t, \theta). \quad (19)$$

C. Calculation of H_∞ estimator $\hat{\mathbf{z}}(t, \theta)$

Define the cross-covariance matrix of the estimates of $\mathbf{x}(t - \tau_1)$ and $\mathbf{x}(t - \tau_2)$ as

$$\begin{aligned} & P(t, \tau_1, \tau_2) \\ & \triangleq \langle \mathbf{x}(t - \tau_1) - \hat{\mathbf{x}}(t, \tau_1), \mathbf{x}(t - \tau_2) - \hat{\mathbf{x}}(t, \tau_2) \rangle, \end{aligned} \quad (20)$$

where $\tau_1 \geq 0$, $\tau_2 \geq 0$ and $\hat{\mathbf{x}}(t, \tau_i)$ ($i = 1, 2$) is the projection of $\mathbf{x}(t - \tau_i)$ onto $\mathcal{L}\{\mathbf{y}_z(s), s < t\}$, $\mathbf{y}_z(s)$ is as in (11). It is obvious that $P(t, \tau_1, \tau_2)' = P(t, \tau_2, \tau_1)$. We have the following result

Theorem 2: The matrix $P(t, \tau_1, \tau_2)$ ($\tau_1 \geq 0$, $\tau_2 \geq 0$) as defined in (20) is the solution to the following Riccati type of partial differential equation and boundary conditions

$$\begin{aligned} & \frac{\partial P(t, \tau_1, \tau_2)}{\partial t} \\ & + \frac{\partial P(t, \tau_1, \tau_2)}{\partial \tau_1} + \frac{\partial P(t, \tau_1, \tau_2)}{\partial \tau_2} \\ & = - \sum_{i,j=0}^k P(t, \tau_1, h_i) H_i'(t) H_j(t) P(t, h_j, \tau_2) \\ & + \gamma^{-2} P(t, \tau_1, \theta) L'(t - \theta) L(t - \theta) P(t, \theta, \tau_2), \end{aligned} \quad (21)$$

$$\begin{aligned} & \frac{\partial P(t, \tau_1, 0)}{\partial t} + \frac{\partial P(t, \tau_1, 0)}{\partial \tau_1} = \sum_{i=0}^k P(t, \tau_1, h_i) \Phi_i(t)' \\ & - \sum_{i,j=0}^k P(t, \tau_1, h_i) H_i'(t) H_j(t) P(t, h_j, 0) \\ & + \gamma^{-2} P(t, \tau_1, \theta) L'(t - \theta) L(t - \theta) P(t, \theta, 0), \end{aligned} \quad (22)$$

$$\begin{aligned} & \frac{\partial P(t, 0, 0)}{\partial t} = \sum_{i=0}^k \Phi_i(t) P(t, h_i, 0) \\ & + \sum_{i=0}^k P(t, 0, h_i) \Phi_i(t)' + \Gamma(t) \Gamma(t)' \\ & - \sum_{i,j=0}^k P(t, 0, h_i) H_i'(t) H_j(t) P(t, h_j, 0) \\ & + \gamma^{-2} P(t, 0, \theta) L'(t - \theta) L(t - \theta) P(t, \theta, 0), \end{aligned} \quad (23)$$

where $L(t - \theta) = 0$ for $t < \theta$ and

$$P(t, 0, s) = P(t, s, 0). \quad (24)$$

In addition, the initial value $P(0, \tau_1, \tau_2)$, $0 \leq \tau_1, \tau_2 \leq h$ is as

$$P(0, \tau_1, \tau_2) = \langle \mathbf{x}(-\tau_1), \mathbf{x}(-\tau_2) \rangle = \Pi_{-\tau_1} \delta(\tau_1 - \tau_2). \quad (25)$$

Theorem 3: Consider the system (1)-(3) and the associated performance criterion (4). Given the desired noise attenuation $\gamma > 0$ and the smoothing lag $\theta \geq 0$, the H_∞ fixed-lag smoothing problem of (4) is solvable if and only if there exists a bounded matrix solution $P(t, \tau, h_i)$ ($0 \leq i \leq k$) for $0 \leq t \leq T$ and $0 \leq \tau \leq h_{max}$, where $h_{max} = \max\{h, \theta\} = \max\{h_0, \dots, h_k, \theta\}$, to the partial differential equations (21)-(23).

In this case, the central estimator $\hat{z}(t, \theta)$ is given by

$$\hat{z}(t, \theta) = L(t - \theta)\hat{x}(t, \theta), \quad (26)$$

where $\hat{x}(t, \theta)$ is computed as

$$\frac{\partial \hat{x}(t, \tau)}{\partial t} + \frac{\partial \hat{x}(t, \tau)}{\partial \tau} = K(t, \tau, t) \times \left[y(t) - \sum_{i=0}^k H_i(t)\hat{x}(t, h_i) \right], \quad (27)$$

$$\frac{\partial \hat{x}(t, 0)}{\partial t} = \sum_{i=0}^k \Phi_i(t)\hat{x}(t, h_i) + K(t, 0, t) \times \left[y(t) - \sum_{i=0}^k H_i(t)\hat{x}(t, h_i) \right], \quad (28)$$

where

$$K(t, \tau, t) = \sum_{i=0}^k P(t, \tau, h_i)H_i(t)', \quad (29)$$

and initial value $\hat{x}(0, \tau) = 0$ for $\tau \geq 0$.

Remark 1: By setting $\gamma \rightarrow \infty$ in (4), $\hat{z}(t, \theta)$ becomes an H_2 estimator which is the same as in [13] for H_2 optimal filtering in linear systems with time delays with $Q_1(t) = \Gamma(t)\Gamma^T(t)$ and $Q_2(t) = I_m$.

IV. H_∞ PREDICTION

In view of (5), we define

$$\begin{aligned} \mathcal{J}_P(T) &\triangleq \int_{-h}^0 x'(s)\Pi_s^{-1}x(s)ds + \int_0^T u'(s)u(s)ds + \int_0^{T-\theta} v'(s)v(s)ds \\ &\quad - \gamma^{-2} \int_0^T v_z'(s-\theta)v_z(s-\theta)ds \\ &= \int_{-h}^0 x'(s)\Pi_s^{-1}x(s)ds + \int_0^T u'(s)u(s)ds \\ &\quad + \int_0^T \begin{bmatrix} v^0(s) \\ v_z^0(s) \end{bmatrix}' \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^{-2}I_p \end{bmatrix} \begin{bmatrix} v^0(s) \\ v_z^0(s) \end{bmatrix} ds, \end{aligned} \quad (30)$$

where

$$\begin{aligned} v_z(s-\theta) &\triangleq \hat{z}(s-\theta, -\theta) - z(s) \\ &= \hat{z}(s-\theta, -\theta) - L(s)x(s) \end{aligned} \quad (31)$$

and $v(s) = 0$ for $s < 0$, $v^0(s) = v(s-\theta)$ and $v_z^0(s) = v_z(s-\theta)$.

In view of (2) and (31), define the following measurement and fictitious measurement:

$$\begin{aligned} y^0(t) &= \sum_{i=0}^k H_i^0(t)x(t-h_i^0) + v^0(t), \quad t \geq \theta \\ \tilde{z}^0(t, 0) &= L(t)x(t) + v_z^0(t), \end{aligned} \quad (32)$$

where $y^0(t) = y(t-\theta)$, $H_i^0(t) = H_i(t-\theta)$, $\tilde{z}^0(t, 0) = \hat{z}(t-\theta, -\theta)$ and $h_i^0 = h_i + \theta$.

According to the discussion in the last section, $\tilde{z}(t-\theta, -\theta) = \tilde{z}^0(t, 0)$ ($0 \leq t \leq T$) that achieves (5) exists if and only if [7]: 1) $\mathcal{J}_P(T)$ has a minimum \mathcal{J}_P^{min} , with respective $u(t)$ ($0 \leq t \leq T$) and $x(t)$ ($-h \leq t \leq 0$) and 2) $\tilde{z}^0(t, 0)$ can be chosen such that \mathcal{J}_P^{min} is positive for all $y^0(\cdot)$.

Similar to the case as in last section, we introduce the stochastic system below.

$$\dot{x}(t) = \sum_{i=0}^k \Phi_i(t)x(t-h_i) + \Gamma(t)u(t), \quad (33)$$

$$y^0(t) = \sum_{i=0}^k H_i^0(t)x(t-h_i^0) + v^0(t), \quad t \geq \theta, \quad (34)$$

$$\tilde{z}^0(t, 0) = L(t)x(t) + v_z^0(t), \quad (35)$$

where the initial state $x(\tau)$ ($-h \leq \tau \leq 0$) and $u(t)$, $v^0(t)$ ($t \geq \theta$) and $v_z^0(t)$ are mutually uncorrelated white noises with zero means and known covariance matrices Π_τ , $Q_u(t)$, $Q_{v^0}(t)$ and $Q_{v_z^0}(t)$ with $Q_u(t) = I_r$, $Q_{v^0}(t) = -\gamma^2 I_p$ for $t \geq 0$ and

$$Q_{v_z^0}(t) = \begin{cases} 0, & t < \theta \\ I_m, & t \geq \theta. \end{cases}$$

Define the cross-covariance matrix of the estimates of $x(t-\tau_1)$ and $x(t-\tau_2)$ as

$$\begin{aligned} P(t, \tau_1, \tau_2) &\triangleq \langle x(t-\tau_1) - \hat{x}(t, \tau_1), x(t-\tau_2) - \hat{x}(t, \tau_2) \rangle, \end{aligned} \quad (36)$$

where $\tau_1 \geq 0$, $\tau_2 \geq 0$ and $\hat{x}(t, \tau_i)$ ($i = 1, 2$) is the projection of $x(t-\tau_i)$ onto $\mathcal{L}\{y_z^0(s), s < t\}$. The innovation of $y_z^0(s)$, denoted by $w_z^0(t) \triangleq y_z^0(t) - \hat{y}_z^0(t, 0)$, where $\hat{y}_z^0(t, 0)$ is the projection of $y_z^0(t)$ onto $\mathcal{L}\{y_z^0(s), s < t\}$.

Then the innovation covariance matrix of $w_z^0(t)$, denoted by $Q_{w_z^0}(t) \triangleq \langle w_z^0(t), w_z^0(t) \rangle$, is computed by

$$Q_{w_z^0}(t) = \begin{cases} -\gamma^2 I_p, & 0 \leq t < \theta \\ \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_p \end{bmatrix}, & t \geq \theta. \end{cases} \quad (37)$$

Theorem 4: The matrix $P(t, \tau_1, \tau_2)$ ($\tau_1 \geq 0, \tau_2 \geq 0$) of (36) is the solution to the following Riccati type of partial differential equation and boundary conditions

$$\begin{aligned} & \frac{\partial P(t, \tau_1, \tau_2)}{\partial t} + \frac{\partial P(t, \tau_1, \tau_2)}{\partial \tau_1} + \frac{\partial P(t, \tau_1, \tau_2)}{\partial \tau_2} \\ = & - \sum_{i,j=0}^k P(t, \tau_1, h_i + \theta) H'_i(t - \theta) H_j(t - \theta) \\ & \times P(t, h_j + \theta, \tau_2) + \gamma^{-2} P(t, \tau_1, 0) L'(t) L(t) \\ & \times P(t, 0, \tau_2), \end{aligned} \quad (38)$$

$$\begin{aligned} & \frac{\partial P(t, \tau_1, 0)}{\partial t} + \frac{\partial P(t, \tau_1, 0)}{\partial \tau_1} \\ = & \sum_{i=0}^k P(t, \tau_1, h_i) \Phi_i(t)' \\ & - \sum_{i,j=0}^k P(t, \tau_1, h_i + \theta) H'_i(t - \theta) H_j(t - \theta) \\ & \times P(t, h_j + \theta, 0) + \gamma^{-2} P(t, \tau_1, 0) L'(t) L(t) \\ & \times P(t, 0, 0), \end{aligned} \quad (39)$$

$$\begin{aligned} \frac{\partial P(t, 0, 0)}{\partial t} = & \sum_{i=0}^k \Phi_i(t) P(t, h_i, 0) + \\ & \sum_{i=0}^k P(t, 0, h_i) \Phi_i(t)' + \Gamma(t) \Gamma(t)' \\ & - \sum_{i,j=0}^k P(t, 0, h_i + \theta) H'_i(t - \theta) H_j(t - \theta) \\ & \times P(t, h_j + \theta, 0) + \gamma^{-2} P(t, 0, 0) L'(t) \\ & \times L(t) P(t, 0, 0), \end{aligned} \quad (40)$$

where $H_i(t - \theta) = 0$ for $t < \theta$ and

$$P(t, 0, s) = P(t, s, 0)'. \quad (41)$$

In addition, the initial value $P(0, \tau_1, \tau_2)$, $0 \leq \tau_1, \tau_2 \leq h$ is as

$$P(0, \tau_1, \tau_2) = \langle \mathbf{x}(-\tau_1), \mathbf{x}(-\tau_2) \rangle = \Pi_{-\tau_1} \delta(\tau_1 - \tau_2). \quad (42)$$

Now we are in the position to present the main result of this section.

Theorem 5: Consider the system (1)-(3) and the associated performance criterion (5). Given the desired noise attenuation $\gamma > 0$, the prediction time lead $\theta > 0$, the H_∞ prediction problem of (5) is solvable if and only if there exists a bounded matrix solution $P(t, \tau, h_i + \theta)$ ($0 \leq i \leq k$) for $0 \leq t \leq T$ and $0 \leq \tau \leq h_{max}^0$, where $h_{max}^0 = \max\{h_0 + \theta, \dots, h_k + \theta\}$, to the partial differential equations (38)-(40).

In this case, a suitable predictor $\hat{z}(t - \theta, -\theta) = \hat{z}^0(t, 0)$ is given by

$$\hat{z}^0(t, 0) = L(t) \hat{x}(t, 0), \quad (43)$$

where $\hat{x}(t, 0)$ is computed from

$$\begin{aligned} \frac{\partial \hat{x}(t, \tau)}{\partial t} + \frac{\partial \hat{x}(t, \tau)}{\partial \tau} = & K(t, \tau, t) \times \\ & \left[y(t - \theta) - \sum_{i=0}^k H_i(t - \theta) \hat{x}(t, h_i + \theta) \right], \end{aligned} \quad (44)$$

$$\begin{aligned} \frac{\partial \hat{x}(t, 0)}{\partial t} = & \sum_{i=0}^k \Phi_i(t) \hat{x}(t, h_i) + K(t, 0, t) \times \\ & \left[y(t - \theta) - \sum_{i=0}^k H_i(t - \theta) \hat{x}(t, h_i + \theta) \right], \end{aligned} \quad (45)$$

with

$$K(t, \tau, t) = \sum_{i=0}^k P(t, \tau, h_i + \theta) H_i(t - \theta)', \quad (46)$$

and initial value $\hat{x}(0, \tau) = 0$ $\tau \geq 0$.

Remark 2: Theorem 5 presents a solution to the H_∞ prediction for linear time delay systems. In the case when all the delays don't exist, the result gives a solution to the H_∞ prediction problem for linear time-varying systems.

V. CONCLUSIONS

In this paper we have studied the H_∞ estimation problem for linear time-varying systems with multiple delays. A necessary and sufficient condition for the existence of an estimator is obtained in terms of a partial differential equation with boundary conditions. The approach applied in this paper is the innovation analysis in Krein space.

Due to the duality of the control and filtering, we believe that the presented results can be extended to the H_∞ control for time delay systems to give a necessary and sufficient condition.

VI. REFERENCES

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