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Robust Filtering by Fictitious Noises

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Abstract

In this paper, a new approach is presented for robust filtering of a linear discrete-time signal by applying fictitious noise. Modeling errors, in both the numerator and denominator of the transfer functions, are parameterized by using random variables with zero mean and known covariance. The robust performance is obtained by minimizing the mean square estimation error over all of the random parameter and noise. To derive a robust estimator, the uncertainties in the model are incorporated into two mutually uncorrelated fictitious noises with zero means. The covariances of the fictitious noises are computed by using two formulas that are presented in this paper. An illustrative example shows the effectiveness of our approach.

Index Terms -Robust, Filtering, Stochastic, Uncertainty, Innovation, Fictitious Noise.

I. Introduction

Input signal filtering (deconvolution) is concerned with the estimation of a signal through a measured output. This problem has many applications including seismology, speech processing and equalization to numerical differentiation (for example, see [11], [15] and the references therein). Much attention has been paid to this problem that is based on Kalman filtering formulation or polynomial method [4], [9], [10], [14], [15]. In most cases, the parameters of the systems and the statistics of the noises are assumed to be known exactly a priori. However, these details are often not known precisely and the associated values may be time varying due to a perturbation in the transmission medium, linearization and model reduction. Generally, these values may be deterministic in some case and stochastic in other cases [1]. In this way, the performance of the optimal filtering that is associated with some nominal system will degrade.

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The averaged minimum mean square error (MMSE) approach has been used previously in the literature [16], [17]. The estimators are obtained by minimizing the averaged mean square error. The criterion takes into account not only the effect, but also the likehood of different modeling errors. It offers a simpler design for the estimator than any of the minimax schemes. It also avoids the drawbacks of the worst-case designs. This paper is aim to study the robust input signal filtering problem by applying the averaged MMSE approach, where the parametric uncertainties in the numerators and denominators of the transfer function are modeled as white noise [1], [19], [20], [21]. Such models of the uncertainties are encountered in many areas of application such as signal processing [1], [22], control [21], [23], nuclear fission and heat transfer, population models, etc. [20]. By applying the technique of fictitious noise [18], the robust filtering is converted into a standard optimal filtering problem for a system with fictitious noises. Two formulas that are used for computing the covariances of the fictitious noises are presented.

II. Problem Statement

The source signal, $u(t)$, and the noise corrupted measurement, $y(t)$, are described respectively by [1], [9], [4]

$$\left[D(q^{-1}) + D_t(q^{-1})\right] u(t) = \left[C(q^{-1}) + C_t(q^{-1})\right] e(t), \quad (1)$$

$$\begin{aligned}\left[A(q^{-1}) + A_t(q^{-1})\right] y(t) &= \left[B(q^{-1}) + B_t(q^{-1})\right] u(t-k) \\ &\quad + \left[P(q^{-1}) + P_t(q^{-1})\right] v(t).\end{aligned} \quad (2)$$

The polynomials $D(q^{-1})$, $C(q^{-1})$, $A(q^{-1})$, $B(q^{-1})$ and $P(q^{-1})$ are pre-specified, and they have the form

$$X(q^{-1}) = x_0 + x_1 q^{-1} + \ldots + x_{n_x} q^{-n_x}. \quad (3)$$

The polynomials $D_t(q^{-1})$, $C_t(q^{-1})$, $A_t(q^{-1})$, $B_t(q^{-1})$ and $P_t(q^{-1})$ represent the time-varying uncertainties, and they have the form

$$X_t(q^{-1}) = x_0(t) + x_1(t) q^{-1} + \cdots x_{n_x}(t) q^{-n_x}. \quad (4)$$

In practice we may always assume that [1] $d_0 = a_0 = 1$, $d_0(t) = 0$ and $a_0(t) = 0$.

For the sake of convenience in the following discussion, we use the notations:

$$\begin{aligned} \mathcal{D}(t) &\triangleq [d_1(t)\cdots,d_{n_d}(t)]^T, \\ \mathcal{C}(t) &\triangleq [c_0(t)\cdots,c_{n_c}(t)]^T, \\ \mathcal{A}(t) &\triangleq [a_1(t)\cdots,a_{n_a}(t)]^T, \\ \mathcal{B}(t) &\triangleq [b_0(t)\cdots,b_{n_b}(t)]^T, \\ \mathcal{P}(t) &\triangleq [p_0(t)\cdots,p_{n_p}(t)]^T. \end{aligned} \quad (5)$$

In addition, we have

$$\mathcal{D}_c(t) \triangleq [-\mathcal{D}^T(t) \quad \mathcal{C}^T(t)]^T, \quad (6)$$

$$\mathcal{A}_{bp}(t) \triangleq [-\mathcal{A}^T(t) \quad \mathcal{B}^T(t) \quad \mathcal{P}^T(t)]^T, \quad (7)$$

where $\mathcal{A}(t)$, $\mathcal{B}(t)$, $\mathcal{P}(t)$, $\mathcal{D}(t)$ and $\mathcal{C}(t)$ are defined in (5).

The following assumptions are made throughout this paper.

Assumption 2.1 *$e(t)$, $v(t)$, $\mathcal{A}_{bp}(t)$ and $\mathcal{D}_c(t)$ are mutually independent white noises, with zero means and known covariances:*

$$\bar{E}[e^2(t)] = \sigma_e, \quad \bar{E}[v^2(t)] = \sigma_v, \quad (8)$$

$$\tilde{E}\left[\mathcal{A}_{bp}(t)\mathcal{A}_{bp}^T\right] = \mathcal{R}_a, \quad \tilde{E}\left[\mathcal{D}_c(t)\mathcal{D}_c^T\right] = \mathcal{R}_d, \quad (9)$$

where $\bar{E}$ and $\tilde{E}$ are the mathematical expectations of the white noises and the uncertainties in the model, respectively. It should be noted that Assumption 2.1 is a standard assumption (see [1], [19] and the references therein).

Assumption 2.2 The polynomials $A(q^{-1})$ and $D(q^{-1})$ are stable, i.e., all zeros of $A(z^{-1})$, $D(z^{-1})$ are in $|z| < 1$.

The problem to be addressed in this paper is stated as follows:

- **Problem:** From the data $y(t)$, up to time $t-m$, a time-invariant estimator, $\hat{u}(t \mid t-m)$, is sought. This estimator is designed to minimize the following averaged mean square error:

$$\tilde{E}\bar{E}\left[u(t) - \hat{u}(t \mid t-m)\right]^2, \quad (10)$$

where $\bar{E}$ is the mathematical expectation over $e(t)$ and $v(t)$, and $\tilde{E}$ is over $\mathcal{D}_c(t)$ and $\mathcal{A}_{bp}(t)$.

Remark 2.1 Since the white noises represented by $e(t)$ and $v(t)$ are independent of $\mathcal{D}_c(t)$ and $\mathcal{A}_{bp}(t)$, the objective function defined by (10) is obviously equivalent to

$$E\left[u(t) - \hat{u}(t \mid t-m)\right]^2, \quad (11)$$

where E is the mathematical expectation over $e(t)$, $v(t)$, $\mathcal{D}_c(t)$ and $\mathcal{A}_{bp}(t)$.

III. Robust Filtering

In this section, we derive the robust estimator by using a projection formula and an innovation analysis approach. It is shown that the robust estimation involves computing two covariance matrices of fictitious, one spectral factorization and one polynomial equation. This is slightly more complicated than the standard optimal design.

A. Fictitious Noise

Note (4), (1) and (2) are re-written as

$$D(q^{-1})u(t) = C(q^{-1})e(t) + e_0(t), \quad (12)$$

$$A(q^{-1})y(t) = B(q^{-1})u(t-k) + P(q^{-1})v(t) + v_0(t), \quad (13)$$

where

$$e_0(t) = -D_t(q^{-1})u(t) + C_t(q^{-1})e(t), \quad (14)$$

$$v_0(t) = -A_t(q^{-1})y(t) + B_t(q^{-1})u(t-k) + P_t(q^{-1})v(t). \quad (15)$$

$e_0(t)$ and $v_0(t)$ are termed as *fictitious noises*. Substituting (12) into (13) yields

$$ADy(t) = DPv(t) + BCe(t-k) + Be_0(t-k) + Dv_0(t), \quad (16)$$

where the operator q^{-1} has been omitted in the polynomials A, B, P, C and D. Now, we have the following results for the fictitious noises, which will play an important role in the design of the robust deconvolution.

Theorem 3.1

1. $e(t)$, $v(t)$, $e_0(t)$ and $v_0(t)$ are mutually uncorrelated.
2. $e_0(t)$ and $v_0(t)$ are white noises with zero means and certain covariances, which are computed by

$$\sigma_e^0 \triangleq E[e_0^2(t)] = \gamma_1(1-\gamma_0)^{-1}, \quad (17)$$

$$\sigma_v^0 \triangleq E[v_0^2(t)] = (\sigma_e^0\lambda_1 + \lambda_2)(1-\lambda_0)^{-1}, \quad (18)$$

where

$$\gamma_0 = \frac{1}{2\pi i}\oint_{|z|=1}(DD_*)^{-1}\tilde{E}\left[D_tD_{t*}\right]\frac{dz}{z}, \quad (19)$$

$$\gamma_1 = \frac{\sigma_e}{2\pi i}\oint_{|z|=1}(DD_*)^{-1} \tilde{E}\left[(-CD_t + DC_t)(-CD_t + DC_t)_*\right]\frac{dz}{z}, \quad (20)$$

$$\lambda_0 = \frac{1}{2\pi i}\oint_{|z|=1}(AA_*)^{-1}\tilde{E}\left[A_tA_{t*}\right]\frac{dz}{z}, \quad (21)$$

$$\lambda_1 = \frac{1}{2\pi i}\oint_{|z|=1}(AA_*DD_*)^{-1} \tilde{E}\left[(-BA_t + AB_t)(-BA_t + AB_t)_*\right]\frac{dz}{z}, \quad (22)$$

$$\lambda_2 = \frac{1}{2\pi i}\oint_{|z|=1}(AA_*DD_*)^{-1} \times\tilde{E}[(-BA_t + AB_t)(-BA_t + AB_t)_*CC_*\sigma_e + (-PA_t + AP_t)(-PA_t + AP_t)_*DD_*\sigma_v]\frac{dz}{z}, \quad (23)$$

where $\tilde{E}$ is the mathematical expectation for D_t, C_t, A_t, B_t and P_t.

Remark 3.1 Note that $\tilde{E}(D_tD_{t*})$, $\tilde{E}(D_tC_{t*})$, $\tilde{E}(C_tD_{t*})$ and $\tilde{E}(C_tC_{t*})$ in (19)-(20) can be computed by using $\mathcal{R}_d$. $\tilde{E}(A_tA_{t*})$, $\tilde{E}(A_tB_{t*})$, $\tilde{E}(B_tA_{t*})$, $\tilde{E}(B_tB_{t*})$, $\tilde{E}(A_tP_{t*})$, $\tilde{E}(P_tA_{t*})$ and $\tilde{E}(P_tP_{t*})$ in (21)-(23) can be computed by using $\mathcal{R}_a$. $\mathcal{R}_d$ and $\mathcal{R}_a$ are as defined in Assumption 2.1.

B. Spectral Factorization and ARMA Innovation Model

From (16), since $e(t)$, $v(t)$, $e_0(t)$ and $v_0(t)$ are mutually uncorrelated, the spectral density of the output, $y(t)$, is computed as

$$[AA_*DD_*]^{-1}\mathcal{W}(z,z^{-1}), \qquad (24)$$

where

$$\begin{aligned}\mathcal{W}(z,z^{-1}) &= BB_*CC_*\sigma_e + DD_*PP_*\sigma_v \\ &\quad + DD_*\sigma_v^0 + BB_*\sigma_e^0. \end{aligned} \qquad (25)$$

The following assumption in made.

Assumption 3.1 The spectral $\mathcal{W}(z,z^{-1})$ is definite positive on $\mid z \mid = 1$.

Under Assumption **Assumption 3.1**, a unique stable spectral factor β exists, with an order of

$$n_\beta = max\{n_b + n_c, n_d + n_a\}$$

and satisfies the following [13]

$$\sigma_\varepsilon \beta\beta_* = \mathcal{W}. \qquad (26)$$

Or, equivalently, we have

$$\begin{aligned}\beta\varepsilon(t) &= DPv(t) + BCe(t-k) + Be_0(t-k) \\ &\quad + Dv_0(t), \end{aligned} \qquad (27)$$

where $\varepsilon(t)$ is white noise with variance σ_ε. Combining (16) with (27) yields

$$ADy(t) = \beta\varepsilon(t). \qquad (28)$$

It can easily be verified from the above equation that $\varepsilon(t)$ is the innovation of the observation. That is $\varepsilon(t) = y(t) - \hat{y}(t \mid t-1)$, and (28) is referred to as the "ARMA innovation model" [10].

C. Derivation of the Estimator

From the theory associated with projection [10], the robust estimator, $\hat{u}(t \mid t-m)$, that minimize (10) is in fact the projection of $u(t)$ onto the linear space that is generated by the innovation, $\{\varepsilon(t-m), \varepsilon(t-m-1), \cdots\}$. Following this idea, one new derivation method for the estimator is given by applying a simple projection formula. It is different from the previous work in [9], [14].

Lemma 3.1 Consider the models of the signal and measurement described by (1) and (2) and satisfying Assumptions 2.1 and 3.1. The robust deconvolution estimate, $\hat{u}(t \mid t-m)$, that minimizes (10) is given by

$$\hat{u}(t \mid t-m) = \beta^{-1}(q^{-1})L(q^{-1})A(q^{-1})y(t-m), \qquad (29)$$

where the unknown polynomial $L(q^{-1})$ has the form

$$L(q^{-1}) = R(q^{-1}) - S(q^{-1}), \qquad (30)$$

and

$$\begin{aligned}R(q^{-1}) &= r_m + r_{m-1}q^{-1} + \cdots r_{n_c-k}q^{-(n_c-m-k)}, \\ S(q^{-1}) &= s_0 + s_1q^{-1} + \cdots + s_{n_d-1}q^{-(n_d-1)}, \end{aligned} \qquad (31)$$

while the coefficients r_i and s_i are expressed as

$$\begin{aligned} r_i &= E\{[Ce(t) + e_0(t)]\varepsilon(t-i)\}\sigma_\varepsilon^{-1} \\ s_i &= \sum_{j=i}^{n_d} d_j E[u(t-j)\varepsilon(t-m-i)]\sigma_\varepsilon^{-1}. \end{aligned} \qquad (32)$$

The unknown polynomial $L(q^{-1})$ of the estimator in (29) is solved in the following theorem.

Theorem 3.2 Consider the system defined by (1) and (2) and satisfying Assumptions 2.1, 2.2 and 3.1. The robust state estimator is given by (29). Here, the unknown polynomial L with an order of $max\{n_c - m - k, n_d - 1\}$ is, together with M_*, the unique solution to

$$L\beta_*\sigma_\varepsilon + M_*Dz = z^{m+k}\left[CC_*B_*\sigma_e + B_*\sigma_e^0\right], \qquad (33)$$

where M_* is a polynomial with an order of

$$\partial M = max(n_b + n_c + m + k, n_\beta) - 1. \qquad (34)$$

The minimal estimation error is given by

$$\begin{aligned} Ez^2(t)_{min} &= \frac{1}{2\pi i}\oint \frac{1}{DD_*}\left[CC_*\sigma_e + \sigma_e^0\right]\frac{1}{z}dz \\ &- \frac{1}{2\pi i}\oint \frac{1}{\beta\beta_*DD_*\sigma_\varepsilon} \times \\ &[(CC_*B_*\sigma_e + B_*\sigma_e^0)(CC_*B_*\sigma_e + B_*\sigma_e^0)_* \\ &- MM_*DD_*]\frac{1}{z}dz. \end{aligned} \qquad (35)$$

Proof: Omitted.

Remark

(33) is a bilateral polynomial equation matrix. Note that D and β are stable. Thus, D and β_* have no common factors. This implies that the invariant polynomials of D are coprime with all of those of β_* and a solution always exists. Furthermore, it is easy to show that the solution is unique.

IV. Illustrative Example

In this section, we present a numerical example to demonstrate the design of the robust estimator from the previous sections. Then the performance of the robust estimator is examined.

Consider the models of the signal and measurement in (1) and (2), where

$$D(q^{-1}) = 1 - 0.8q^{-1}, \quad D_t(q^{-1}) = d(t)q^{-1},$$

$$C(q^{-1}) = 1, \quad C_t(q^{-1}) = c(t)q^{-1},$$

$$A(q^{-1}) = 1, \quad A_t(q^{-1}) = a(t)q^{-1},$$
$$B(q^{-1}) = 1 - 0.6q^{-1}, \quad B_t(q^{-1}) = b_0(t) + b_1(t)q^{-1},$$
$$P(q^{-1}) = 1, \quad P_t(q^{-1}) = p(t), \quad k = 1,$$

where $d(t) = c(t) = \xi(t)$ and $a(t) = b_0(t) = b_t(t) = p(t) = \zeta(t)$. $\xi(t)$, $\zeta(t)$, the system's noises, $e(t)$, and the observation's noise, $v(t)$, are assumed to be mutually independent white noises with zero means and variances of $\sigma_\xi = 0.16$, $\sigma_\zeta = 0.64$, $\sigma_e = 1$ and $\sigma_v = 0.81$.

Applying Theorem 3.1, we have $\sigma_e^0 = 0.512$ and $\sigma_v^0 = 9.0506$. Then, the spectral factorization is defined as

$$\sigma_\varepsilon(1 + \beta q^{-1})(1 + \beta q) = -8.7956q^{-1} - 8.7956q + 18.2276,$$

which is easily calculated by using

$$\beta(q^{-1}) = 1 - 0.7648q^{-1}, \qquad \sigma_\varepsilon = 11.5009.$$

Having obtained the averaged spectral factorization, the robust estimator for filtering ($m = 0$) is obtained as

$$\hat{u}(t \mid t) = (1 - 0.7648q^{-1})^{-1}L(q^{-1})y(t),$$

where the polynomial $L(q^{-1})$ is the solution to (33), which is given by

$$L(q^{-1}) = 0.1409.$$

For the purpose of comparison, we give the nominal design. In this case, we do not care about the uncertainties $D_t(q^{-1})$, $C_t(q^{-1})$, $A_t(q^{-1})$, $B_t(q^{-1})$ and $P_t(q^{-1})$. So we can assume that

$$\sigma_\xi = \sigma_\zeta = 0.$$

Thus, the spectral density is

$$\mathcal{W}^0(q, q^{-1}) = -1.248q^{-1} - 1.248q + 2.6884,$$

and the spectral factorization is

$$\beta^0(q^{-1}) = 1 - 0.6769q^{-1}, \qquad \sigma_\varepsilon^0 = 1.8436.$$

From (33), $L^0(q^{-1})$ is solved as

$$L^0(q^{-1}) = 0.4922.$$

The non-robust estimator for filtering ($m = 0$) is

$$\hat{u}^0(t \mid t) = (1 - 0.6769q^{-1})^{-1}0.4922y(t).$$

V. Conclusions

In this paper, we have presented a new approach for robust filtering that is based on the stochastic description of the errors in the model, where both of the numerators and denominators of the transfer functions contain unknown stochastic parameters. By applying fictitious noises, the robust estimation has been converted into a standard optimal H_2 estimation problem. The covariances of the fictitious noises, which are the key to a robust estimator, have been computed by using two formulas developed in this paper. The model of the system that is considered is more general than the one in [1], where the model of the signal that is considered is assumed to be pre-specified and the observation contains white noise. The presented results can be extended to multivariable system (MIMO) with a similar discussion as in this paper and [9], which will be investigated separately.

VI. References

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