A Combined Wavelet-Element Free Galerkin Method for Numerical Calculations of Electromagnetic Fields

Shiyou Yang, Guangzheng Ni, J. R. Cardoso, S. L. Ho, and J. M. Machado

***Abstract*—A combined wavelet-element free Galerkin (EFG) method is proposed for solving electromagnetic (EM) field problems. The bridging scales are used to preserve the consistency and linear independence properties of the entire bases. A detailed description of the development of the discrete model and its numerical implementations is given to facilitate the reader to understand the proposed algorithm. A numerical example to validate the proposed method is also reported.**

***Index Terms*—Bridge scales, meshless method, moving least squares approximation, wavelet method.**

I. Introduction

RECENT advances in meshless methods have provided a new alternative for solving boundary value problems in engineering. Since the approximations arising from these methods are based entirely on a set of nodes and a connectivity of the elements is not required, the meshless methods can be very promising in studies involving variable geometries or in problems calling for hierarchical solution procedures. Indeed, different meshless methods such as the wavelet-based method [1], the diffuse element method [2], the element free Galerkin method [3], the reproducing kernel method [4], and the meshless local Petrov–Galerkin approach [5] have all been proposed and used successfully to solve some typical engineering field problems. Among the aforementioned procedures, the wavelet method is truly meshless since this method does not require any mesh or integration "cell" in numerical implementations [1]. However, the inherent inefficiency of the wavelet methods in imposing boundary and interface conditions often outweighs their advantages. Hence, wavelet methods are not used widely in engineering studies. On the other hand, the finite-element method (FEM) is well developed and employed in engineering. FEM is also very efficient in enforcing boundary conditions. To take full advantages of both the wavelet and the FEM, a combined wavelet-element free Galerkin (wavelet-EFG) method using FEM to enforce essential boundary conditions is proposed. To retain the required mathematical properties of the shape functions such as consistency and linear independence of the proposed method, the bridge scales are generalized and introduced. Extensive numerical simulations have also been carried out to validate and demonstrate the advantages and shortcomings of the proposed method.

Manuscript received June 18, 2002. This work was supported by the National Science Foundation of China (NSFC) under Grant 50277036.

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Digital Object Identifier 10.1109/TMAG.2003.810370

II. A Combined Wavelet-EFG Method

A. Wavelet Approximations

For any function $u(x,y) \in \Omega$, its approximation using wavelets can be given as

$$w(x,y) = \sum_{i,j} c_{i,j}\phi_{i,j}^J(x,y) = \sum_{i,j} c_{i,j}\phi_i^J(x)\phi_j^J(y) \quad (1)$$

where J is the resolution or scale parameter and $\phi_i^J(z) = 2^{J/2}\phi(2^J Z - i)$ is the one-dimensional (1-D) scale function of the wavelets, and it can be determined from the following two scale relations:

$$\phi(x) = \sum_{k=0}^{L-1} p_k\phi(2x-k). \quad (2)$$

In the proposed algorithm, the Daubechies' scale function is used, and L is an even integer.

B. A Combined Wavelet-EFG Method

As similar to other meshless methods, the wavelet algorithm suffers from the inefficiency in enforcing boundary and interface conditions when they are used to solve boundary value problems. On the other hand, FEM implements boundary and interface conditions readily. To make full use of the FEM and meshless method, a combined wavelet-EFG method is proposed. In the combined algorithm, the FEM is used to impose essential boundary conditions, and the Galerkin approach is employed to derive the discrete model.

For the proposed algorithm to work, the entire domain of the problem is divided into three subregions as schematically demonstrated in Fig. 1, i.e., Ω_f contains only finite elements (FEs), Ω_w contains only wavelets contributing to the approximation of the solution variable, and Ω_f^w contains both FEs and wavelets having mutual influences. In region Ω_f or Ω_w, the interpolation of the solution variable is the same as the standard form of the FE or the wavelet method as formulated in (1). To develop a general interpolation formula in region Ω_f^w for the solution variable $u(x,y)$ using both FE shape functions and wavelets, one begins with

$$u(x,y) = \sum_i u_i N_i^{\mathrm{fem}}(x,y) + \sum_{i,j} c_{i,j}\phi_i^J(x)\phi_j^J(y) \quad (3)$$

where $N_i^{\mathrm{fem}}(x,y)$ is the standard FE shape function.

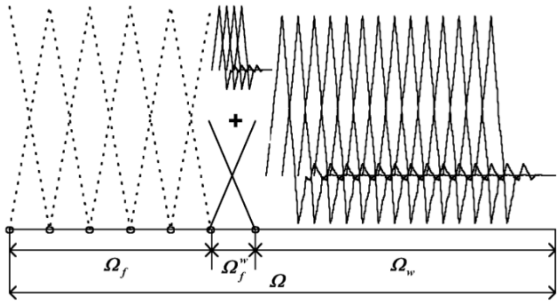

Fig. 1. Division of the solution domain and the shape functions.

To retain the required mathematical properties of the entire bases in terms of the consistency and the linear independence, the bridging scale concept as proposed in [6] is used to modify the wavelets. The basic concept of the bridging scales is based on a hierarchical decomposition of a function u which is dependent on some projection operator P to represent the projection of u onto the span of some set of basis functions. To decompose the solution variable into two different parts, i.e., the one that is approximated by wavelets and the one that is represented by FE shape functions, one employs the property of a projection operator such that multiple projections of the function will leave the function unchanged [6], i.e., $PPu = Pu$. By using this concept, the total function u of (3) can now be reformulated as

$$u = Pu + w - Pw \tag{4}$$

where Pw is the bridging scale term.

The objective to include the bridging scale term Pw is to make the wavelet decomposition part $w - Pw$ to contain only the parts of u which are not included in Pu, thereby ensuring a hierarchical decomposition of u. In other words, by taking projection P to both sides of (4), one obtains

$$Pu = PPu + Pw - PPw = Pu. \tag{5}$$

For the proposed combined wavelet-EFG method, in region Ω_f^w where both FE and wavelets have mutual influences, Pu, Pw, and w become, respectively

$$Pu(x,y) = \sum_i u_i N_i^{\text{fem}}(x,y) \tag{6}$$

$$w(x,y) = \sum_{i,j} c_{i,j}\phi_i^J(x)\phi_j^J(y) \tag{7}$$

$$\begin{aligned} Pw(x,y) &= \sum_i N_i^{\text{fem}}(x,y) w(x_i,y_i) \\ &= \sum_i N_i^{\text{fem}}(x,y)\left\{\sum_{j,k} c_{j,k}\phi_j^J(x_i)\phi_k^J(y_i)\right\}. \end{aligned} \tag{8}$$

Substituting (6)–(8) into (4), in region Ω_f^w, the approximation of $u(x,y)$ becomes

$$u(x,y) = \sum_i u_i N_i^{\text{fem}}(x,y) + \sum_{i,j} c_{i,j}\tilde{\phi}_{i,j}^J(x,y). \tag{9}$$

Accordingly, the modified wavelets based on the introduction of bridging scales are defined as

$$\tilde{\phi}_{j,k}^J(x,y) = \phi_j^J(x)\phi_k^J(y) - \sum_i N_i^{\text{fem}}(x,y)\phi_j^J(x_i)\phi_k^J(y_i). \tag{10}$$

It can be seen from (10) that, by introducing the bridge scales, the modified wavelets $\tilde{\phi}_{j,k}^J(x,y)$ are decoupled from the nodes of the FE meshes, thereby allowing the proposed algorithm to deal with the essential boundary conditions in a more simple way comparable to that in standard FEM algorithms.

C. Mathematical Properties of the Entire Bases

The mathematical properties of the total bases, including the FE shape functions and the modified wavelet bases which ensure the solution is convergent, such as consistency and linear independence, can be proved by using methods that are similar to that described in [6]. For example, suppose the wavelet bases $\phi_i^J(x)$ (for 1-D case) are reproducing polynomials up to the order n, then

$$\sum_i \phi_i^J(x)x_i^m = x^m \ (0 \le m \le n). \tag{11}$$

By choosing u_i to x_i^m and c_j to x_j^m in (9), one obtains

$$\begin{aligned} u(x) &= \sum_i N_i^{\text{FEM}}(x)x_i^m + x^m \\ &\quad - \sum_i N_i^{\text{FEM}}(x)\sum_j \phi_j^J(x_i)\}x_j^m \\ &= \sum_i N_i^{\text{FEM}}(x)x_i^m + x^m - \sum_i N_i^{\text{FEM}}(x)x_i^m = x^m. \end{aligned} \tag{12}$$

Thus, the consistency of the total bases up to order n is proved.

D. Discretization for a Two-Dimensional Static Problems

One considers the following two-dimensional (2-D) static field problems on the domain Ω and bounded by the boundary $\Gamma = \Gamma_D \bigcup \Gamma_N$ as

$$\Omega : \beta\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^2 u}{\partial y^2} = -f \tag{13}$$

$$\Gamma_D : u = u_0 \tag{14}$$

$$\Gamma_N : \beta\frac{\partial u}{\partial n} = q. \tag{15}$$

Based on the weak form of the partial differential (13) and the boundary conditions (14) and (15) as well as using a Galerkin approach, one can derive the discrete equations as given in

$$\begin{bmatrix} K & N \\ N^T & M \end{bmatrix}\begin{Bmatrix} U \\ C \end{Bmatrix} = \begin{Bmatrix} F \\ G \end{Bmatrix} \tag{16}$$

and

$$\begin{aligned} k_{ij} = \int\int_\Omega \{\beta[(N_j^{FEM})_x(N_i^{FEM})_x \\ + (N_j^{FEM})_y(N_i^{FEM})_y\} dxdy \end{aligned} \tag{17}$$

$$f_i = \int\int_\Omega f N_i^{FEM} dxdy + \int_{\Gamma_N} q N_i^{FEM} ds \tag{18}$$

$$\begin{aligned} n_{p,q;i} = \int\int_{\Omega_{enr}} \beta[(\tilde{\phi}_{p,q}^J)_x(N_i^{FEM})_x \\ + (\tilde{\phi}_{p,q}^J)_y(N_i^{FEM})_y] dxdy \end{aligned} \tag{19}$$

$$\begin{aligned} m_{p,q;i.j} = \int\int_\Omega \beta[(\tilde{\phi}_{p,q}^J(x,y))_x(\tilde{\phi}_{i,j}^J(x,y))_x \\ + (\tilde{\phi}_{p,q}^J(x,y))_y(\tilde{\phi}_{i,j}^J(x,y))_y] dxdy \end{aligned} \tag{20}$$

$$g_{p,q} = \int_{\Gamma_N} \tilde{\phi}_{p,q}^J(x,y) q ds + \int\int_\Omega f\tilde{\phi}_{p,q}^J(x,y) dxdy \tag{21}$$

where $(\phi)_z$ denotes the partial differential of ϕ with respect to variable z.

The computation of the integrals (the connection coefficients in terminology of wavelet methods) in (19)–(21) can be determined by using the iterative algorithms as reported in (1) and (7).

III. Numerical Implementation

A. Partition and Meshing of the Solution Domain

In the proposed combined wavelet-EFG method, the incorporation of FEM is to enforce the essential boundary conditions in a simple way. Consequently, meshes are required in a very thin layer in every essential boundary for the FEM to work. The wavelet bases in these layers have to be modified as formulated in (10). However, the wavelet bases in the residual regions remain unchanged. Thus, throughout most of the solution domain, the proposed method is still classified as a meshless wavelet one, and this kind of discretization as a whole retains the merits of the meshless wavelet methods. To deal with the interface conditions of different materials, the jump function approach is used in the proposed algorithm [1].

B. Decoupling and Solving of Finite-Element and Meshless Wavelet Systems

Since the FE shape functions are effective only in very thin layers in the essential boundaries, and throughout most of the solution domain, the approximation of the solution variable is obtained using purely meshless wavelets, the degree of freedom (DoF) of the FE system generally is very small when compared to that of the meshless wavelet system. Moreover, the ratio between the values of the quantities in the submatrix K and those in the submatrix C of the stiffness matrix of (16) may be too large or too small. Such features are, however, a good recipe for poor matrix conditioning. Hence, if the discrete linear equation set of (16) is to be solved as a whole, some numerical techniques must be designed to guarantee good performances in the numerical computation. To avoid this poor matrix conditioning problem, bearing in mind that the DoFs of FEM are far smaller in general than those of the wavelet bases, the two matrix systems, i.e., the FE and the meshless wavelet matrix systems, are decoupled and solved separately and iteratively in the proposed method. After this numerical treatment, the well-developed numerical techniques of the wavelet algorithm as well as those for FEM such as fast wavelet transform algorithm and the compact storage techniques that can be used for the manipulation of the stiffness matrix of the FE formulation by virtue of the stiffness matrix (16) are still applicable. More specifically

$$KU = F - NC \tag{22}$$

$$MC = G - N^T U. \tag{23}$$

Thus, the iterative solution procedure for the two matrix systems can be described as follows.

Step 1) Equation set (22) is firstly solved by setting a zero initial condition of wavelet coefficients.

Step 2) Equation (23) is then solved with the values of the newly solved U as known variables.

Step 3) Equation (22) is solved again using C of Step 2).

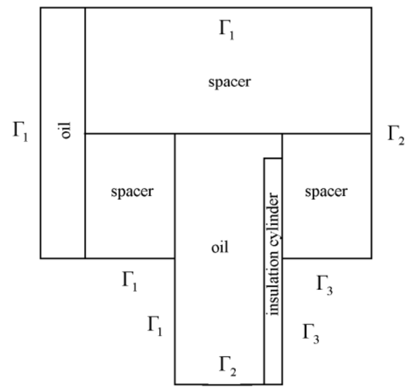

Fig. 2. Schematic diagram of the end region of a power transformer being studied.

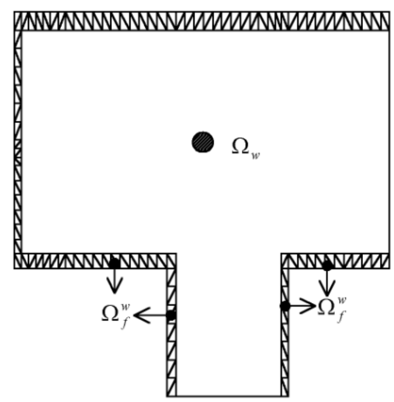

Fig. 3. Partition of the solution domain and the corresponding mesh used by the proposed combined wavelet-EFG method.

Step 4) The solutions of U between the two previous successive iterations are compared. If the error is within a threshold value, the iterative process is stopped; otherwise, go to Step 2) to begin the next iteration cycle.

IV. Numerical Example

To validate the proposed method, it is used to compute the end fields of a power transformer of Fig. 2. The corresponding boundary value problem is formulated as

$$\begin{aligned} \varepsilon\frac{\partial^2\varphi}{\partial x^2} + \varepsilon\frac{\partial^2\varphi}{\partial y^2} &= 0 \\ \varphi|_{\Gamma_1} &= 0, \quad \varphi|_{\Gamma_3} = 1, \quad \frac{\partial\varphi}{\partial n}|_{\Gamma_2} = 0. \end{aligned} \tag{24}$$

In the numerical implementation, the solution domain is divided into two different subregions (Fig. 3), i.e., Ω_f^w: very thin layers near the essential boundaries where both FE and wavelets have mutual influences, Ω_w: most of the solution domain in which the interpolation of the solution variable is obtained using the standard form of the wavelet method. The wavelets used in the proposed method are Daubechies scaling function with $L = 6$, and the first-order triangular elements are used in the FEM. The meshes used by the proposed and standard FEM are plotted, respectively, in Figs. 3 and 4. Fig. 5 gives the accuracy comparison of the computed results of the proposed algorithm and the standard FEM under the meshes as shown, respectively,

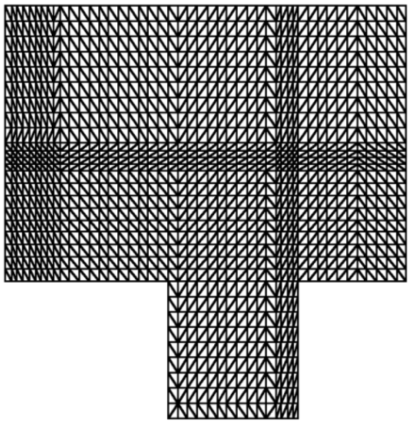

Fig. 4. Mesh used by the standard FEM for comparison purposes.

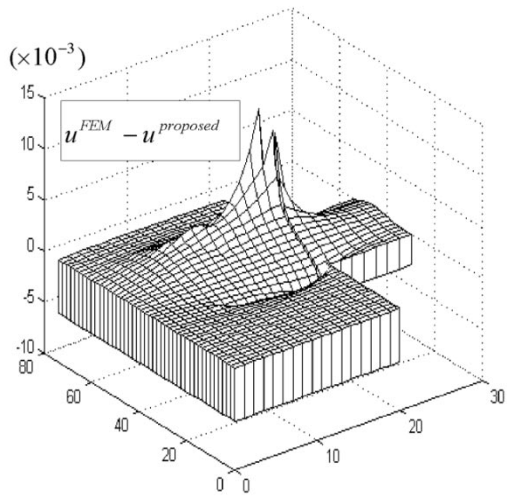

Fig. 5. Comparison of computed results between the proposed algorithm and standard FEM for the power transformer prototype problem.

in Figs. 3 and 4. The corresponding performance comparison results are given in Table I. From these numerical results, one can see the following.

1) Compared with the standard FEM, the proposed algorithm inherits most of the advantages of the wavelet methods of being virtually "meshless" even in the sense of integration cells in the numerical implementations. The proposed algorithm can also produce higher order smooth approximation of the solution variable.
2) Compared with available meshless methods, the most salient and significant feature of the proposed method is that no special technique is needed to impose the essential boundary conditions, making it ideal to take care of the boundary conditions simply and efficiently.

TABLE I
PERFORMANCE COMPARISON OF THE PROPOSED AND STANDARD FEMs

	No FE nodes	No wavelet bases	Total DOFs	CPU (s)
Proposed	210	1044	1254	3.5
FEM	1225	/	1225	3

3) Compared with the standard FEM, the proposed algorithm still requires special techniques to deal with interface conditions of different materials.
4) Compared with the standard FEM, the proposed method is less efficient in the utilization of the CPU time.
5) Since the proposed method can produce higher order smooth approximation of the solution variable, and the FE one used in this paper can only produce first-order smooth approximation of the solution variable, the differences of the numerical solutions between the two methods are very significant in regions where there are some sharp variations of the gradient in the solution variable.

V. CONCLUSION

A combined wavelet-EFG method is introduced in this paper. Extensive numerical results positively confirm the feasibility of using the proposed numerical method in finding the solution of practical field problems. Moreover, its advantages and limitation as compared with the traditional FEMs and the conventional meshless methods are addressed fully. To make the meshless algorithms a feasible alternative to FEM, further development in the corresponding numerical techniques such as those for enforcing interface conditions of different materials are necessary. An efficient solver for the discrete equation set is also essential.

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