Fast Analytical Approach to Finding Steady-State Waveforms for Power Electronics Circuits Using Orthogonal Polynomial Basis Functions

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Abstract— **This paper describes a general analytical procedure for steady-state waveform analysis of power electronics circuits using orthogonal polynomials. The proposed method exploits the time-domain piecewise property of power electronics circuits as well as the orthogonality of certain polynomials in order to maximize the accuracy and computational efficiency. This new analytical approach can provide a fast means to obtain improved and more robust solutions of steady-state waveforms of power electronics circuits.**

I. Introduction

Steady-state waveform analysis of switching converters provides important design information, such as device stress, parasitic ringing frequency, etc., for engineers to ensure reliable circuit operation [1]–[5]. For many nonlinear circuits, finding the steady-state solution is a time consuming task which may also suffer from numerical instabilities, especially for power electronics circuits consisting of slow and fast variations in different parts of the same waveform.

It has been shown previously that wavelet approximation is well suited for steady-state waveform analysis of power electronics circuits because of the time-domain piecewise property of such circuits [7], [8]. Furthermore, instead of applying one wavelet approximation to the whole repetition period, several wavelet approximations can be applied in a piecewise manner to improve the accuracy of approximation and the computational efficiency [9]. The piecewise approach eliminates the difficulty of solving discontinuous differential equations and allows more accurate analytical approach to be taken to the continuous differential equations within a switch state in a piecewise manner. The use of wavelets, however, is a particular realization of the idea of piecewise approximation using wavelets as the basis functions. In this paper we generalize the solution approach and seek appropriate basis functions such that the required steady-state waveforms can be found analytically and speedily. Polynomial functions having certain properties can be used as the basis functions, for which analytical solutions can be derived. The Chebyshev and Legendre polynomials will be used to illustrate the method.

The paper is organized as follows. Section II describes the requirements of polynomials for use as basis functions in the approximation of solutions of differential equations. Section III describes the analytical procedure for finding steady-state waveforms of piecewise switched systems. In Section IV, an application example is presented to evaluate and compare the effectiveness of the proposed analytical piecewise solution with that of the wavelet-based piecewise approximation. A brief description for handling feedback control circuits is discussed in Section V. Finally, we give the conclusion in Section VI.

II. Choice of Polynomials as Basis Functions

In order to permit analytical solutions to be derived for differential equations in terms of polynomial series, the following conditions must be satisfied:

- The derivatives of the polynomial bases can be expressed in terms of the polynomial bases themselves;
- The polynomial bases are orthogonal.

The Chebyshev and Legendre polynomials [6] satisfy these conditions and hence are good candidates for the construction of analytical solutions for differential equations. The Chebyshev polynomials of the first kind $T_n(x)$ and the Legendre polynomials $P_n(x)$ are defined on the close interval $[-1, +1]$ as follows:

$$T_n(x) = \cos n(\arccos x) \quad (1)$$

$$P_n(x) = \frac{d^n}{dx^n}\frac{(x^2-1)^n}{2^n n!} \quad (2)$$

where n is a non-negative integer representing the order of the polynomial. These polynomial bases are orthogonal in the sense that their inner products are given by

$$\langle T_m, T_n\rangle \stackrel{\text{def}}{=} \int_{-1}^{+1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n = 0 \\ \frac{\pi}{2} & \text{if } m = n \neq 0. \end{cases} \quad (3)$$

$$\langle P_m, P_n\rangle \stackrel{\text{def}}{=} \int_{-1}^{+1} P_m(x)P_n(x)dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases} \quad (4)$$

In matrix form, the inner products $\boldsymbol{Q}_T^{(n)}$ and $\boldsymbol{Q}_P^{(n)}$ are each an $(n+1)$ square matrix given by

$$\boldsymbol{Q}_T^{(n)} = \begin{bmatrix} \pi & 0 & \cdots & 0 \\ 0 & \frac{\pi}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\pi}{2} \end{bmatrix}, \boldsymbol{Q}_P^{(n)} = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & \frac{2}{3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2}{2n+1} \end{bmatrix}. \quad (5)$$

The first derivative of the Chebyshev polynomials can be expressed as

$$\frac{dT_n}{dx} = \begin{cases} 0 & \text{for } n = 0 \\ T_0 & \text{for } n = 1 \\ 2n \sum_{m=0}^{n/2-1} T_{2m+1} & \text{for } n \geq 2 \text{ and } n \text{ even} \\ nT_0 + 2n \sum_{m=0}^{(n-1)/2} T_{2m} & \text{for } n \geq 3 \text{ and } n \text{ odd} \end{cases} \tag{6}$$

which can be written in matrix form as

$$\dot{\boldsymbol{T}}_n = \boldsymbol{D}_T^{(n)} \boldsymbol{T}^{(n)} \tag{7}$$

where $\boldsymbol{D}_T^{(n)}$ is an $(n+1)$ square matrix and $\boldsymbol{T}^{(n)}$ is the vector of Chebyshev polynomials.

As for Legendre polynomials, the first derivatives are

$$\frac{dP_n}{dx} = \begin{cases} 0 & \text{for } n = 0 \\ P_0 & \text{for } n = 1 \\ \sum_{m=0}^{(n/2-1)} (4m+3) P_{2m+1} & \text{for } n \geq 2 \text{ and } n \text{ even} \\ \sum_{m=0}^{(n-1)/2} (4m+1) P_{2m} & \text{for } n \geq 3 \text{ and } n \text{ odd} \end{cases} \tag{8}$$

or

$$\dot{\boldsymbol{P}}_n = \boldsymbol{D}_P^{(n)} \boldsymbol{P}^{(n)}. \tag{9}$$

III. Analytical Piecewise Solutions

In this section we present an analytical approach of the piecewise approximation of steady-state waveforms of switched circuits using Chebyshev or Legendre polynomials as basis functions. Our method is an enhancement of the algorithm proposed earlier by Tam *et al.* [9], in which it has been suggested that the solution in each interval (switch state) should be represented by a different set of wavelets and the complete solution waveform is formed by "gluing" together the solutions of individual intervals. By doing so, the piecewise method yields an accurate steady-state waveform description with much less number of wavelet terms. Here, instead of using wavelets as basis functions as proposed in Tam *et al.* [9], we propose the use of polynomial basis functions in order to allow analytical solutions to be derived.

Any power electronics circuit can be described by a number of state equations, each describing the circuit for each switch state. We can therefore represent a s-switch-state converter, for switch state j, as

$$\dot{\boldsymbol{x}}_j = \boldsymbol{A}_j \boldsymbol{x}_j + \boldsymbol{U}_j \quad \text{for } j = 1, 2, \cdots, s \tag{10}$$

where $\boldsymbol{x}_j$ is an m-dim state vector, $\boldsymbol{A}_j$ is a time-invariant matrix, and $\boldsymbol{U}_j$ is the input function.

Thus, when Chebyshev polynomials are used to represent the solution of the system in switch state j, we have

$$\boldsymbol{K}_j \boldsymbol{D}_T \boldsymbol{T} = \boldsymbol{A}_j \boldsymbol{K}_j \boldsymbol{T} + \boldsymbol{U}_j \tag{11}$$

where

$$\boldsymbol{K}_j = \begin{bmatrix} k_{j,1,0} & k_{j,1,1} & \cdots & k_{j,1,n} \\ k_{j,2,0} & k_{j,2,1} & \cdots & k_{j,2,n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{j,m,0} & k_{j,m,1} & \cdots & k_{j,m,n} \end{bmatrix}. \tag{12}$$

Hence, we have

$$\boldsymbol{F}\boldsymbol{K} = -\boldsymbol{U}. \tag{13}$$

Here, $\boldsymbol{F}$ is a $sm(n+1)$ square matrix and $\boldsymbol{K} = (k_{j,i,h})$, where $j = 1, \ldots, s$, $i = 1, \ldots, m$ and $h = 0, \ldots, n$, is a $sm(n+1)$-dim vector, i.e.,

$$\boldsymbol{F} = \begin{bmatrix} \langle \boldsymbol{F}_1, \boldsymbol{T}_n \rangle + \boldsymbol{F}_a & \cdots & \boldsymbol{0} & \boldsymbol{F}_b \\ \boldsymbol{F}_b & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \langle \boldsymbol{F}_{s-1}, \boldsymbol{T}_n \rangle + \boldsymbol{F}_a & \boldsymbol{0} \\ \underbrace{\boldsymbol{0}}_{(n+1)m \text{ columns}} & \cdots & \boldsymbol{F}_b & \langle \boldsymbol{F}_s, \boldsymbol{T}_n \rangle + \boldsymbol{F}_a \end{bmatrix} \tag{14}$$

$$\boldsymbol{U} = \left[\begin{bmatrix} \langle \boldsymbol{U}_1, \boldsymbol{T}_n \rangle \\ \langle \boldsymbol{U}_2, \boldsymbol{T}_n \rangle \\ \vdots \\ \underbrace{\langle \boldsymbol{U}_s, \boldsymbol{T}_n \rangle}_{1 \text{ column}} \end{bmatrix} \right\} (n+1)ms \text{ rows} \tag{15}$$

where

$$\langle \boldsymbol{F}_j, \boldsymbol{T}_n \rangle = \begin{bmatrix} a_{j,11}\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{D}^T & \cdots & a_{j,1i}\boldsymbol{Q} & \cdots & a_{j,1m}\boldsymbol{Q} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j,i1}\boldsymbol{Q} & \cdots & a_{j,ii}\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{D}^T & \cdots & a_{j,im}\boldsymbol{Q} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j,m1}\boldsymbol{Q} & \cdots & a_{j,mi}\boldsymbol{Q} & \cdots & a_{j,mm}\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{D}^T \end{bmatrix} \tag{16}$$

$$\langle \boldsymbol{U}_j, T_n \rangle = \begin{bmatrix} u_{j,1}\boldsymbol{Q}(:,1) \\ \vdots \\ u_{j,i}\boldsymbol{Q}(:,1) \\ \vdots \\ u_{j,m}\boldsymbol{Q}(:,1) \end{bmatrix} \tag{17}$$

$$\boldsymbol{F}_a = \begin{bmatrix} \underbrace{\begin{matrix} \boldsymbol{M}(-1) & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{M}(-1) \end{matrix}}_{(n+1)m \text{ square matrix}} \end{bmatrix} \tag{18}$$

$$\boldsymbol{F}_b = \begin{bmatrix} \underbrace{\begin{matrix} -\boldsymbol{M}(+1) & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & -\boldsymbol{M}(+1) \end{matrix}}_{(n+1)m \text{ square matrix}} \end{bmatrix} \tag{19}$$

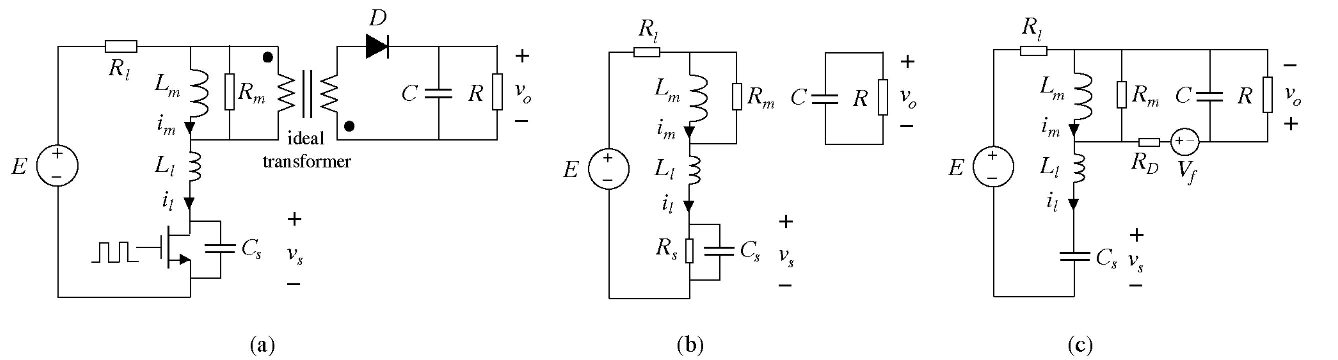

Fig. 1. (a) Flyback converter model with transformer leakage inductance and device parasitic capacitance; (b) on-time model; (c) off-time model.

$$M(y) = \begin{bmatrix} [T(y)]^T \\ \vdots \\ \underbrace{[T(y)]^T}_{(n+1)\text{ square matrix}} \end{bmatrix} \quad (20)$$

where $Q(:, j)$ represents the j-column of the matrix Q.

Finally, all the coefficients necessary for generating the solution for each switch state can be obtained by solving (13), and hence the complete solution can be found. In a likewise manner, we may derive the solution based on Legendre polynomials.

IV. Simulations and Evaluations

As an example, we consider the flyback converter circuit shown in Fig. 1 (a). This is a realistic model including the parasitic capacitance across the switch and leakage inductance of the transformer. We can identify two switch states in the circuit operation as follows. When the switch is turned on, current flows through the magnetizing inductance L_m and the leakage inductance L_l, with the transformer secondary opened and the diode not conducting. When the switch is turned off, the transformer secondary conducts through the diode, clamping the primary voltage (i.e., voltage across L_m) to the output network (assuming a 1:1 turns ratio). Thus, L_m discharges through the transformer primary, while the leakage L_l and the parasitic capacitance C_s form a damped resonant loop around the input voltage source. Figs. 1 (b) and (c) show the detailed models for the on-time and off-time durations, respectively. The state equation of this converter is given by

$$\dot{x} = A(t)x + U(t) \quad (21)$$

where $x = [i_m \; i_l \; v_s \; v_o]^T$, and $A(t)$ and $U(t)$ are given by

$$A(t) = A_1(1 - s(t)) + A_2 s(t) \quad (22)$$

$$U(t) = U_1(1 - s(t)) + U_2 s(t) \quad (23)$$

with $s(t)$ defined as

$$s(t) = \begin{cases} 0 & \text{for } 0 \le t \le T_D \\ 1 & \text{for } T_D \le t \le T \\ s(t-T) & \text{for all } t > T. \end{cases} \quad (24)$$

and the U's and A's being derivable from the circuit topologies.

Simulation results are evaluated using the *mean absolute error* (MAE), which are defined as

$$\text{MAE} = \frac{1}{N}\sum_{i=1}^{N} |\hat{x}_i - x_i| \quad (25)$$

where N is the total number of points sampled along the interval $[-1, 1]$ for error calculation. In the following, we use uniform sampling (i.e., equal spacing) with $N = 1001$, including boundary points.

The parameters for simulation are listed in Table I. We will compare the waveforms of the switch voltage obtained from the numerical wavelet-based and analytical polynomial-based piecewise approaches.

Figs. 2 (a) and (b) show the solution waveforms using the wavelet-based piecewise and analytical polynomial-based piecewise methods, each with 65 terms. Both methods give a very good waveform approximation, but from Fig. 3 (a), we clearly see that the proposed analytical approach provides a superior accuracy at the beginning of switch waveform as the artifactual high frequency components are removed.

From the MAEs and simulation times shown in Table II, the advantage of using the proposed analytical piecewise method is clearly demonstrated.

We also show in Fig. 3 (b) the MAEs using Chebyshev and Legendre polynomials up to different orders. Although Chebyshev and Legendre polynomials perform with different accuracy in terms of MAEs for different maximum polynomial orders, they converge to a result when polynomials of order up to 74 are used.

V. Solutions for Closed-Loop Systems

To find the solution of closed-loop systems where the duty cycle D has to be found by iteration, we first approximate the output voltage waveform with an initial value of D of 0.5 using polynomials of lower order (e.g., $n = 10$). The averaged output voltage is then compared with the desired voltage reference. Then, we adjust the D using an iterative algorithm such as the bi-section method until the approximated output voltage is within the designed tolerance level. Finally, the waveforms are approximated with the final value of D using polynomials up to an appropriate order.

TABLE I

COMPONENT AND PARAMETER VALUES FOR SIMULATION.

Component/Parameter	Value
Magnetizing (storage) inductance, L_m	0.4 mH
Leakage inductance, L_l	1 μH
Equivalent transformer primary resistance, R_m	1 MΩ
Output capacitance, C	0.1 mF
Load resistance, R	12.5 Ω
Input voltage, E	16 V
Diode forward drop, V_f	0.8 V
Switching period, T	100 μs
On-time, T_D	45 μs
Equivalent loop resistance, R_l	0.4 Ω
Switch on-resistance, R_S	0.001 Ω
Switch capacitance, C_S	200 nF
Diode on-resistance, R_D	0.001 Ω

TABLE II

COMPARISON OF MAES AND SIMULATION TIMES OF THE SWITCH VOLTAGE WAVEFORM FROM WAVELET-BASED PIECEWISE METHOD AND POLYNOMIAL-BASED ANALYTICAL PIECEWISE METHOD

Orders (up to)	MAE for v_s (wavelet)	Time (s) (wavelet)	MAE for v_s (analytical)	Time (s) (analytical)
16	1.255486	0.156	1.016074	0.031
32	0.844064	0.609	1.078683	0.235
64	0.386687	3.782	0.288449	2.047
128	0.340033	22.579	0.322014	16.594

The duty cycle D in most feedback circuits can be calculated using an integrator in the feedback loop. The duty cycle D can therefore be approximated using polynomials of low order. Since the analytical method is able to extract the desired low-frequency details without knowledge of the high-frequency components, the approximation can be done much faster for each iteration. The final complete waveform can thus be calculated using the final converged value of D using polynomials of higher order. With this hybrid approach, we can then solve the steady-state waveforms of closed-loop power electronics circuits easily and speedily.

VI. CONCLUSION

An analytical approach for finding steady-state waveforms for power electronics circuits based on piecewise approximation is presented. The proposed approach exploits the time-domain piecewise property of power electronics circuits and finds the solution in each switch state separately using a set of polynomials which allow the complete solution to be found analytically. An example has demonstrated the effectiveness and computational efficiency of using Chebyshev or Legendre polynomials for finding steady-state waveforms of power electronics circuits.

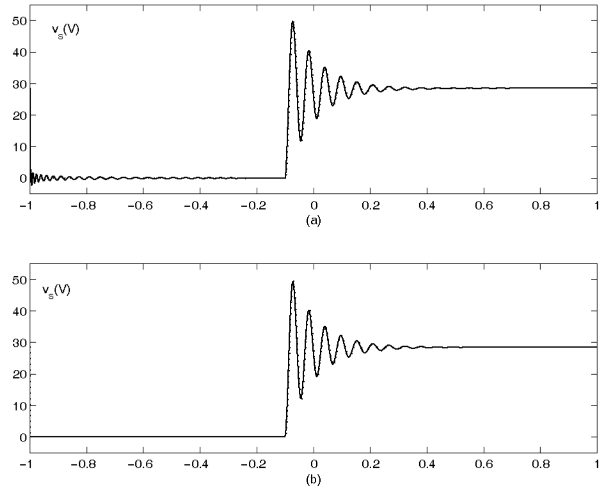

Fig. 2. Switch voltage waveforms of flyback converter. Dashed line is waveform from SPICE simulation, and solid line is approximated waveform (a) using wavelet-based piecewise method with 65 terms; (b) using Chebyshev polynomial analytical piecewise method with 65 terms.

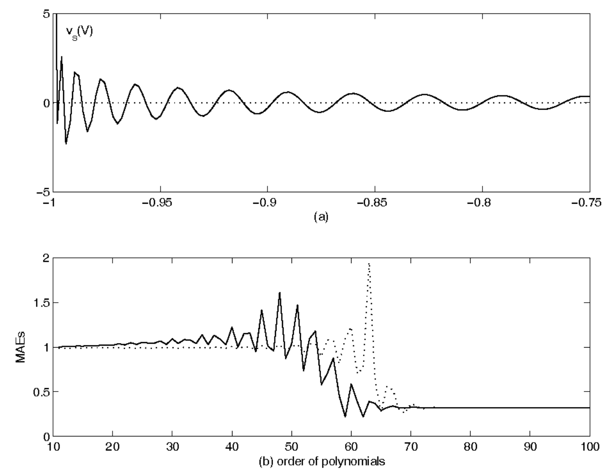

Fig. 3. (a) Closed-up view of switch voltage waveforms of flyback converter using wavelet-based piecewise (solid line) and analytical Chebyshev polynomial (dotted line); (b) comparison of MAEs with polynomials up to the order used. Solid and dotted lines are the MAEs from analytical Chebyshev and Legendre polynomial solutions, respectively.

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