ANALYSIS OF BIFURCATION IN SWITCHED DYNAMICAL SYSTEMS WITH PERIODICALLY MOVING BORDERS: APPLICATION TO POWER CONVERTERS

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ABSTRACT

This paper describes a global method for analyzing the bifurcation phenomena in switched dynamical systems whose switching borders are varying periodically with time. The type of systems under study covers most of power electronics circuits. In particular, the complex bifurcation behavior of a voltage feedback buck converter is studied in detail. The analytical method developed in this paper allows bifurcation scenarios to be clearly revealed in any chosen parameter space.

1. INTRODUCTION

In a previous study by Kousaka *et al.* [1], switched dynamical systems with fixed borders are analyzed. Switchings in such systems are determined only by the states of the system and are not affected by external signals. In many engineering applications, however, switchings are determined by the interaction of the states of the system with some external periodic driving signal. Such an operation is found in most of power electronics circuits [2, 3]. In this paper, we extend the previous work to the analysis of switched dynamical systems with their switching borders moving as periodic functions of time. Instead of local theory so far, we consider the switched system from a global viewpoint.

We will first present a system description and a general procedure for analyzing the bifurcation behavior. To illustrate the practicality of the method, we apply the analysis procedure to a popular voltage feedback buck converter [4]. We will systematically describe the bifurcation phenomena in the buck converter, covering the standard period-doubling, tangent bifurcation as well as border collision bifurcation [5, 6]. In particular, the method we develop in this paper permits the types of bifurcations to be clearly and conveniently identified under different choices of parameter values. Hence, the results from the analysis can be used by engineers to develop practically useful design rules for avoiding certain types of bifurcation scenarios.

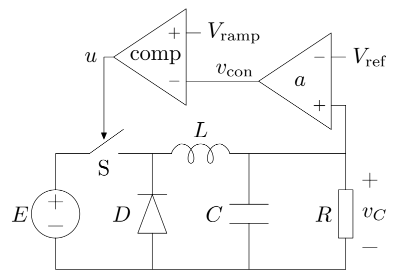


Fig. 1. A voltage feedback buck converter.

2. ANALYTICAL METHOD

2.1. Periodic solution

From an analytical viewpoint, we may look at a switched dynamical system as a set of two or more dynamical systems, each of which defines the system in a finite interval of time. For a voltage buck converter shown in Fig. 1, there are two systems and one border. Switching between two systems is controlled by the switch S, i.e., the output of voltage comparator which compares a control signal $v_{\rm con}$ with a ramp signal $V_{\rm ramp}$. Therefore, a border function can be defined by $\beta(\mathbf{x}, t) = v_{\rm con} - V_{\rm ramp} = 0$. It is a periodic function with period T.

As illustrated in Fig. 2(a), the border divides the state space into two parts. We suppose the solutions in M_1 and M_2 are given by $\varphi(t, \mathbf{x}_0)$ and $\psi(t, \mathbf{x}_0)$, respectively. These two solutions are governed by the state equations of the two respective dynamical systems. Whenever the flow intersects the border B transversally, the system switches. The crossing point can then be regarded as the initial point of the

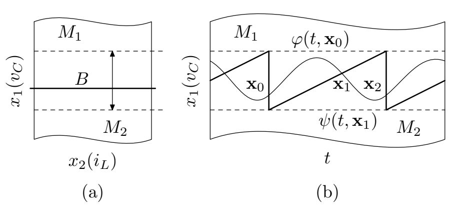


Fig. 2. Periodically moving border and a typical periodic-1 solution of buck converter.

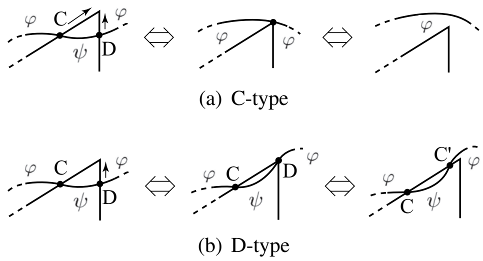


Fig. 3. Classification of border collision in buck converter.

Table 1. System of abbreviations of bifurcation. Each type of bifurcation is abbreviated as $\triangle n \sharp$.

$\triangle$	D	Period-doubling bifurcation
	T	Tangent bifurcation
	Bc	C-type border collision
	Bd	D-type border collision
n	1,2,$\cdots$	Period of solution happening bifurcation
$\sharp$	a,b,$\cdots$	Index

successive flow. Thus, we can describe any trajectory of a switched dynamical system. If we consider a solution shown in Fig. 2(b), we can write the following equations.

$$\mathbf{x}_1 = \varphi(\tau_1, \mathbf{x}_0) \qquad \text{starting at } \mathbf{x}_0\text{, crossing at } \mathbf{x}_1 \quad (1)$$

$$\mathbf{x}_2 = \psi(T - \tau_1, \mathbf{x}_1) \qquad \text{ending at } \mathbf{x}_2 \text{ in one period} \quad (2)$$

$$\beta(\tau_1, \mathbf{x}_1) = 0 \qquad \text{border crossing condition} \quad (3)$$

If $\mathbf{x}_2 = \mathbf{x}_0$, it is obviously a period-1 solution. Then we can solve the above equations to obtain the fixed point using an appropriate numerical method.

2.2. Bifurcation

Apart from standard bifurcations, a special type of bifurcation, known as border collision, is often observed in switched dynamical systems.

Standard bifurcations, such as tangent bifurcation and period-doubling bifurcation, happen if the stability of a fixed point changes. For instance, to determine the stability for the period-1 solution introduced above, the problem is to find $\partial\mathbf{x}_2/\partial\mathbf{x}_0$. From (1) and (3), we can get $\partial\mathbf{x}_1/\partial\mathbf{x}_0$ and $\partial\tau_1/\partial\mathbf{x}_0$ separately. Then, substituting them into the partial derivative of (2) and using appropriate numerical method, we can calculate $\partial\mathbf{x}_2/\partial\mathbf{x}_0$. Thus the standard bifurcation behavior can be analyzed. Note that the above procedure is completely general and system independent.

Unlike standard bifurcations, border collision is a result of *operational change* [2], which is system dependent. For the buck converter under study, at the point where v_{con} "grazes" at the upper or lower tip of the ramp signal V_{ramp}, border collision occurs. According to the actual situation of circuit operation, we classify border collision in this system into "C-type" and "D-type", as depicted in Fig. 3. For any periodic solution running into border collision, except for the earlier set of equations describing the solution, we can write a new equation for the "grazing", which allows us to solve the parameter condition under which a specific border collision occurs.

3. BIFURCATION OF BUCK CONVERTER

In this section, we will investigate the complicated bifurcation behavior exhibited by the buck converter. With the notations in Fig. 1, we fix some of the parameters as follows.

$$L = 20\ \mathrm{mH},\ C = 47\ \mu\mathrm{F},\ a = 8.4,\ V_{\mathrm{ref}} = 11.3\ \mathrm{V},$$
$$V_L = 3.8\ \mathrm{V},\ V_U = 8.2\ \mathrm{V},\ T = 400\ \mu\mathrm{s}$$

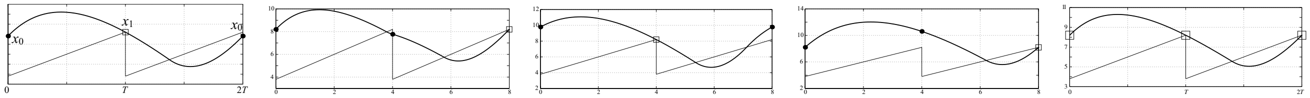
3.1. Bifurcation diagram

Using the analysis methods developed in the foregoing section, together with appropriate numerical calculations, we can obtain a bifurcation diagram in the E–R plane and a blow-up view in Fig. 4. For the sake of clarity and to avoid confusion, we adopt a system for denoting the bifurcation curves, as explained in Table 1. Moveover, we name periodic solutions as P$n(k_1, k_2, \cdots, k_n)$, where n is the period of the solution and $k_1, k_2, \cdots, k_n$ indicates the number of times the solution crosses the border in each period. Then, from these diagrams, in conjunction with Fig. 5, we are able to explain the bifurcation behavior of period-1 and period-2 solutions in the voltage feedback buck converter.

In the dark-grey region (shown as ▭ in Fig. 4(a)), the stable solution is P1(1) for which only period-doubling D1 is observed. Period-2 solution P2(1,1) appears on the right-hand side of D1. For P2(1,1), a period-doubling D2a and border collision are possible.

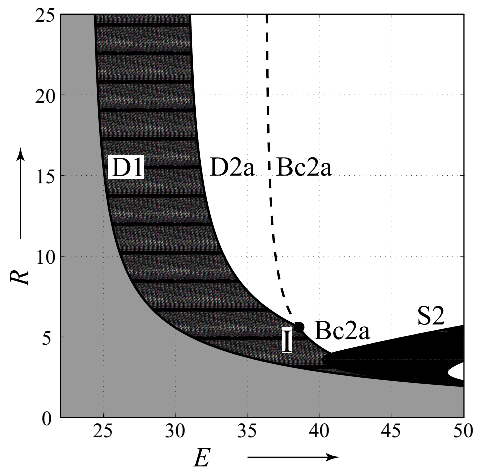
Some interesting bifurcation behavior can be observed around point I on the bifurcation diagram. Crossing the bifurcation curve of Bc2a (see Fig. 5(a)) from left to right, P2(1,1) becomes P2(0,1). However, inspecting the eigenvalues of P2(0,1), we find that P2(0,1) is unstable. Above point I, we see that D2a takes place ahead of Bc2a. For clarity, Bc2a occurring on unstable solution P2(1,1) is shown as a dashed curve in Fig. 4(a). This point will be discussed in the next subsection.

From the blow-up view of Fig. 4(b), we observe that another border collision Bd2a occurs below point J. This bifurcation, corresponding to Fig. 5(b), transmutes P2(1,1) into P2(1,2). Note that P2(1,2) is a stable period-2 solution. Also, P2(1,2) can undergo period-doubling and tangent bifurcation, denoted as D2b and T2 respectively. For ease of reference, the region in which P2(1,1) exists is shown as the

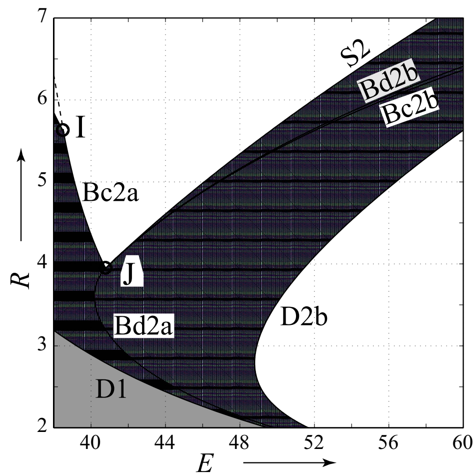


(a) Bc2a (b) Bd2a (c) Bc2b (d) Bd2b (e) Point J

Fig. 5. Conditions of various border collision bifurcations. □ indicates grazing point.



(a)



(b)

Fig. 4. (a) Bifurcation diagram with $C = 47\ \mu$F. (b) An enlarged view. In the figure, ▒, ▒, and ▒ denote the regions where P1(1), P2(1,1) and P2(2,1) exist respectively.

hatched area ▒, and the region in which P2(1,2) exists is shown as the back-hatched area ▒.

Since P2(0,1) is unstable, the Bc2a discussed earlier actually leads to P4(0,1,1,1) and chaos in succession. That is, P2(0,1) is never manifested. This unstable P2(0,1) can undergoe another border collision Bd2b to become P2(0,2), which is stable and only exists in a narrow region between Bd2b and Bc2b. Bc2b, corresponding to Fig. 5(c), transmutes P2(0,2) into P2(1,2). Thus, in the light-gray region in Fig. 4(b)), we actually find a stable period-2 solution coexisting with possible longer periodic solutions or chaos.

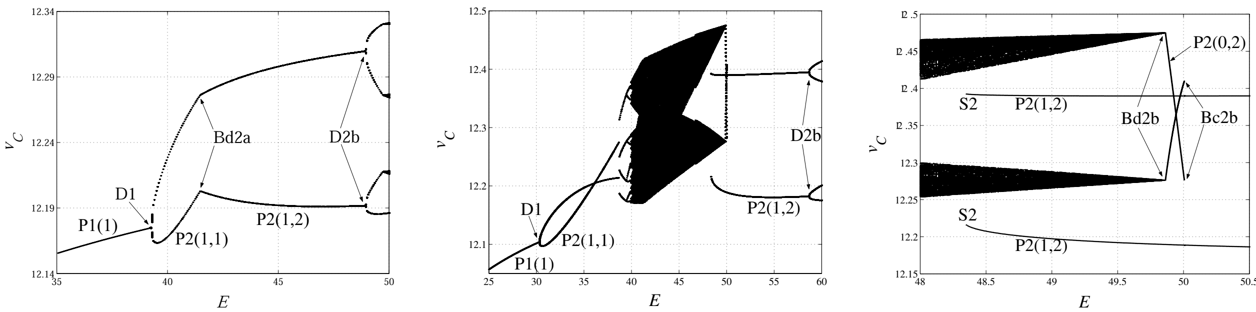
Note that all of four border collision curves meet at the same point J on the bifurcation diagram. The coordinate of J is (40.781 V, 3.946 Ω). At this set of parameters, both C and D types of border collision occur at the same time. This situation is illustrated in Fig. 5(e). For higher periodic solutions, many joint points (like J) of border collision curves can be expected. Finding the position of these points may gives us convenience to determine the complicated higher codimension bifurcation phenomena.

3.2. Discussion

By fixing R at 3 Ω and 5.4 Ω, we obtain the one parameter bifurcation diagrams shown in Fig. 6. These figures are able to reveal further details of the bifurcation behavior.

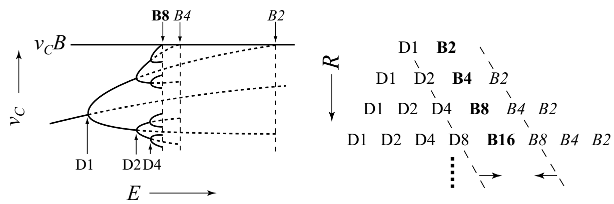
From Fig. 6(c), we can see clearly the coexisting solutions. Furthermore, we observe an important difference between C-type and D-type border collision. The C-type border collision manifests itself as a leap, whereas the D-type manifests as an inflection. This difference can be attributed to the kind of operational change associated with the specific type of border collision. Specifically, in the C-type border collision, the switching sequence is disrupted, giving rise to "skipped" cycles. Moreover, for the D-type, the relative durations of the on and off intervals are disturbed while the same switching sequence is maintained.

Finally, we discuss an interesting interplay between period-doubling bifurcation and border collision. In the normal period-doubling cascade, period-doubling bifurcation continues to generate solutions of doubled periods and to chaos. However, border collision comes into play for the switched dynamical systems. For the buck converter, whenever v_C hits a boundary, border collision occurs, and interrupts the normal period-doubling cascade, as depicted in Fig. 7(a), where dashed curves indicate unstable solutions. We see that the border collision **B8** of stable period-8 solution must happen before the border collision *B4* of the unstable period-4 solution and after the period-doubling D4 of the stable period-4 solution. If R is reduced, the entire period-doubling cascade will move upward. Thus, as R decreases, **B8** and D4 will soon be displaced from the top (disappear) while **B4** will occur for the stable period-4 solution.



(a) $R = 3\,\Omega$ (b) $R = 5.4\,\Omega$ (c) Enlargement of (b)

Fig. 6. Bifurcation diagrams for fixed R, with E serving as the bifurcation parameter.



(a) Schematic bifurcation diagram (b) Typical bifurcation sequence

Fig. 7. Interplay between border collision and period-doubling cascade.

From the above description, we may conceive a general bifurcation pattern, as shown in Fig. 7(b). We now look at the bifurcation sequence with E serving as the parameter and increasing. The first border collision must be located between a period-doubling of a stable solution and a border collision of an unstable solution. Moreover, the first border collisions often represent overtures to prelude the occurrence of chaos. Thus we can intuitively explain (and estimate) the location of the onset of chaos. Referring to the bifurcation diagram of Fig. 4(a) again, we can conclude that chaos occurs between dashed Bc2a and D2a. Here, point I can be interpreted as a critical point where the bifurcation sequence jumps from the second row to the first row in Fig. 7(b). However, we should stress that this simple rule, though helpful in making prediction of the onset of chaos, has assumed the validity of an ideal period-doubling cascade.

4. CONCLUSION

In this paper, we have introduced a method for analyzing the bifurcation behavior of switched dynamical systems with periodically moving borders. By constructing the periodic solutions according to the switching sequences, we can find periodic orbits, evaluate their stability, and study the bifurcation behavior. The method developed in this paper leads to the plotting of detailed bifurcation diagrams on the parameter space that can provide useful practical information for engineers to determine the complex bifurcation behavior of any given switched dynamical system. In particular, we have provided specific bifurcation diagrams for the voltage feedback buck converter and discussed the key features of the bifurcation behavior. In this paper we have shown the rich variety of possible border collision scenarios and their interplay with the main period-doubling cascade. The same method of analysis can be extended to solutions of longer periods, with higher complexity of the numerical solution being the price to pay.

5. REFERENCES

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