Paired-domination in Inflated Graphs

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Abstract

The inflation G_I of a graph G with $n(G)$ vertices and $m(G)$ edges is obtained from G by replacing every vertex of degree d of G by a clique K_d. A set S of vertices in a graph G is a paired dominating set of G if every vertex of G is adjacent to some vertex in S and if the subgraph induced by S contains a perfect matching. The paired domination number $\gamma_p(G)$ is the minimum cardinality of a paired dominating set of G. In this paper, we show that if a graph G has a minimum degree $\delta(G) \geq 2$, then $n(G) \leq \gamma_p(G_I) \leq \frac{4m(G)}{\delta(G)+1}$, and the equality $\gamma_p(G_I) = n(G)$ holds if and only if G has a perfect matching. In addition, we present a linear time algorithm to compute a minimum paired-dominating set for an inflation tree.

Keywords: domination, inflated graphs, perfect matching

1 Introduction

All graphs considered here are finite, undirected, and simple. For standard graph theory terminology not given here we refer to [6]. Let $G = (V, E)$ be a graph with vertex set V of order $n(G)$ and edge set E of size $m(G)$. The degree, neighborhood, close neighborhood of a vertex x of G are respectively denoted by $d_G(x), N_G(x), N_G[x]$ or simply by $d(x), N(x), N[x]$ if there is no ambiguity. For a subset $S \subseteq V$, we define $N[S] = \cup_{x \in S} N[x]$. The subgraph induced by S is denoted by $\langle S \rangle$. The *private neighbor set* of a vertex $v \in S$ with respect to the set S, denoted by $pn[v, S]$, is the set $N[v] - N[S - \{v\}]$. If $pn[v, S] \neq \emptyset$ for some vertex $v \in S$, then every vertex of $pn[v, S]$ is called a *private*

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neighbor of v with respect to S. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$.

A *matching* in a graph G is a set of independent edges in G. The cardinality of a maximum matching in G is called the matching number of G and is denoted by $\beta_1(G)$, or simply by β_1. A *perfect matching* M in G is a matching of G such that every vertex of G is incident to an element of M. A set S of vertices of G is *a dominating set* if every vertex of $V - S$ has at least one neighbor in S. For sets $X, Y \subseteq V$, we say that X dominates Y if every vertex in Y has a neighbor in X, and we write $X \succ Y$. In particular, when S is a dominating set of G, we say that S dominates V and write $S \succ V$. A subset $S \subseteq V$ is a *paired-dominating set* if S is a dominating set of G and the induced subgraph $\langle S \rangle$ has a perfect matching. If $(v_j, v_k) = e_i \in M$, where M is a perfect matching of $\langle S \rangle$, we say that v_j and v_k are paired in S. The *paired-domination number* $\gamma_p(G)$ is defined to be the minimum cardinality of a paired-dominating set S in G. Obviously, every graph without isolated vertices has a paired-dominating set since the end-vertices of any maximal matching form such a set. Paired-domination was introduced by Haynes and Slater [7] with the following application in mind. If we think of each $s \in S \subseteq V$ as the location of a guard capable of protecting each vertex in $N[s]$, then "domination" requires every vertex to be protected. For paired-domination, we require the guards' locations to be selected as adjacent pairs of vertices so that each guard is assigned one other location and they are designated as backup for each other. Paired-domination is also studied in [5, 8, 9]

For the notation for inflated graphs, we follow that of [3]. The *inflation* or *inflated graph* G_I of the graph G without isolated vertices is obtained as follows: each vertex x_i of degree $d(x_i)$ of G is replaced by a clique $X_i \cong K_{d(x_i)}$ and each edge (x_i, x_j) of G is replaced by an edge (u, v) in such a way that $u \in X_i, v \in X_j$, and two different edges of G are replaced by non-adjacent edges of G_I. An obvious consequence of the definition is that $n(G_I) = \sum_{x_i \in V(G)} d_G(x_i) = 2m(G)$, $\Delta(G_I) = \Delta(G)$ and $\delta(G_I) = \delta(G)$. There are two different kinds of edges in G_I. The edges of the clique X_i are colored red and the X_i's are called the *red cliques* (a red clique X_i is reduced to a point if x_i is a pendant vertex of G). The other ones, which correspond to the edges of G, are colored blue and they form a perfect matching of G_I. Every vertex of G_I belongs to exactly one red clique and one blue edge. Two adjacent vertices of G_I are said to *red-adjacent* if they belong to a same red clique, *blue-adjacent* otherwise. In general, we adopt the following notation: if x_i and x_j are two adjacent vertices of G, the end-vertices of the blue edge of G_I replacing the edge (x_i, x_j) of G are called $x_i x_j$ in X_i and $x_j x_i$ in X_j, and this blue edge is $(x_i x_j, x_j x_i)$. Figures 1 and 2 show examples of inflated graphs. Clearly an inflation is claw-free. More precisely, G_I is the line-graph $L(S(G))$ where the subdivision $S(G)$ of G is obtained by replacing each edge of G by a path of length 2. The study of various domination parameters in inflated graphs was originated by Dunbar and Haynes in [2].

Results related to the domination parameters in inflated graphs can be found in [3, 4, 10].

In this paper, we prove that for a graph G with $\delta(G) \geq 2$, $n(G) \leq \gamma_p(G_I) \leq \frac{4m(G)}{\delta(G)+1}$, and $\gamma_p(G) = n(G)$ if and only if G has a perfect matching. In the last section, we give a linear algorithm to compute a paired-dominating set for an inflated tree.

2 Bounds on paired-domination number in inflated graphs

Let G be a graph. For $X,\ Y \subseteq V(G)$, and $X \cap Y = \emptyset$, let $e(X,Y) = |\{(x,y) \in E(G) : x \in X,\ y \in Y\}|$. $G(X,Y)$ denotes the bipartite graph with vertex classes X and Y that contains all edges of G having one end-vertex in X and the other end-vertex in Y. First we recall a result that we will use later.

Lemma 2.1 ([1]) *If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.*

Lemma 2.2 *If G has no isolated vertices, then $\gamma_p(G_I) \leq 2n(G) - 2\beta_1(G)$ and this bound is tight.*

Proof. Let $M = \{(u_1, v_1), (u_2, v_2), \ldots, (u_{\beta_1}, v_{\beta_1})\}$ be the maximum matching of G and Ω be the set of vertices not met by M, where β_1 is the matching number of G. Then Ω is an independent set of vertices of G. For each $x_j \in \Omega$, we choose a vertex $x'_j \in N(x_j)$. Then $S = \{u_i v_i \mid 1 \leq i \leq \beta_1\} \cup \{x_j x'_j \mid x_j \in \Omega\}$ is a paired-dominating set of G_I. So, $\gamma_p(G_I) \leq 2\beta_1(G) + 2(n(G) - 2\beta_1(G)) = 2(n(G) - \beta_1(G))$. The bound can be attained for instance when $G = K_{r,r+1}$ ($r \geq 2$). Figure 1 shows the case for $r = 2$. ■

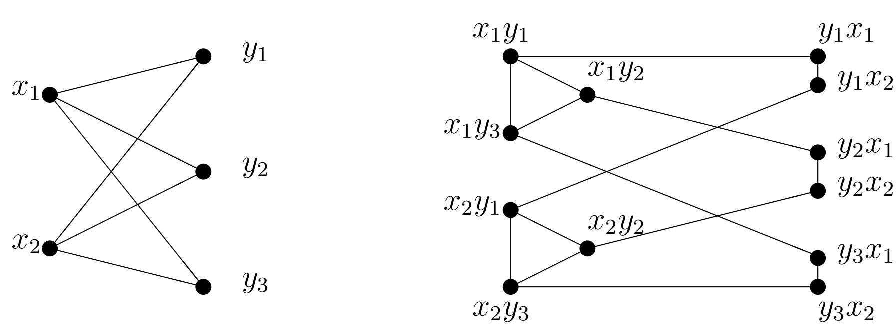

Figure 1: The complete bipartite graph $K_{2,3}$ and its inflation

Theorem 2.3 *If G is a graph with $\delta(G) \geq 2$, then $\gamma_p(G_I) \geq n(G)$ with equality if and only if G has a perfect matching.*

Proof. Let S be a minimum paired-dominating set of G_I. We partition the red cliques of G_I into $\mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2$, where

$$\begin{array}{rcl}\mathcal{U}_0 & = & \{X_i \mid X_i \text{ is a red clique of } G_I \text{ and } |V(X_i) \cap S| = 0\}, \\ \mathcal{U}_1 & = & \{X_i \mid X_i \text{ is a red clique of } G_I \text{ and } |V(X_i) \cap S| = 1\}, \\ \mathcal{U}_2 & = & \{X_i \mid X_i \text{ is a red clique of } G_I \text{ and } |V(X_i) \cap S| \geq 2\}.\end{array}$$

Let $l_0 = |\mathcal{U}_0|, l_1 = |\mathcal{U}_1|$ and $l_2 = |\mathcal{U}_2|$. Then $n(G) = l_0 + l_1 + l_2$. We set

$$\begin{array}{rcl}S_1 & = & \{x_i x_j \in S \mid X_i \in \mathcal{U}_1, \text{ where } x_j \in N_G(x_i)\}, \\ S_2 & = & \{x_i x_j \in S \mid X_i \in \mathcal{U}_2, \text{ where } x_j \in N_G(x_i)\}.\end{array}$$

So

$$|S_1| = |\mathcal{U}_1| = l_1, \; l_2 \leq \lfloor \frac{|S_2|}{2} \rfloor. \tag{1}$$

Let $T = \cup_{X_i \in \mathcal{U}_0} V(X_i)$. We consider the bipartite subgraph $G_I(T, S_2)$ of G_I.

Claim 1. For any $xx' \in T$, $x'x$, the extremity of the blue edge through xx', is in S_2.

Suppose to the contrary that there exists a vertex $uu' \in T$ and $u'u \notin S_2$, that is, $u'u \in S_1$ or $u'u \in V - S_1 \cup S_2$. If $u'u \in V - S_1 \cup S_2$, then uu' cannot be dominated by S, a contradiction. If $u'u \in S_1$, since S is a minimum paired-dominating set of G_I, it follows that $uu' \in S$. But $T \cap S = \emptyset$, again a contradiction. The Claim follows.

Since $\delta(G) \geq 2$, it follows that $|V(X_i)| \geq 2$ in inflation G_I. By Claim 1 and counting the number of edges between S_2 and T, we get

$$2l_0 \leq \sum_{X_i \in \mathcal{U}_0} |V(X_i)| \leq e(S_2, T) \leq |S_2|. \tag{2}$$

So

$$l_0 \leq \lfloor \frac{|S_2|}{2} \rfloor. \tag{3}$$

Using (1) and (3), we have

$$\begin{array}{rcl}\gamma_p(G_I) & = & |S_1| + |S_2| \\ & \geq & |S_1| + \lfloor \frac{|S_2|}{2} \rfloor + \lfloor \frac{|S_2|}{2} \rfloor \\ & \geq & l_1 + l_0 + l_2 \\ & = & n(G).\end{array}$$

Furthermore, if G has a perfect matching, then by Lemma 2.2, we immediately have $\gamma_p(G_I) = n(G)$. Conversely, we will show that if $\gamma_p(G_I) = n(G)$ then G has a perfect matching. Suppose that $\gamma_p(G_I) = n(G)$, then n is even and $|S_2| = |S| - l_1 = n - l_1 = l_0 + l_2$.

We claim that $|S_2|$ is even, thus, l_1 is also even. Otherwise, if $|S_2|$ is odd, then by (1) and (3), we have $|S_2| = l_0 + l_2 \leq \lfloor \frac{|S_2|}{2} \rfloor + \lfloor \frac{|S_2|}{2} \rfloor = |S_2| - 1 < |S_2|$, a contradiction. Let $|S_2| = 2k$. Then $l_0 \leq \lfloor \frac{|S_2|}{2} \rfloor = k$ and $l_2 \leq \lfloor \frac{|S_2|}{2} \rfloor = k$. Combining this with $|S_2| = l_0 + l_2$, it follows that $l_0 = l_2 = k$. This implies that each clique $X_i \in \mathcal{U}_2$ exactly contains two vertices of S.

Claim 2. For any $X_i \in \mathcal{U}_0$ $(i = 1, 2, \ldots, k), |V(X_i)| = 2$ and $|T| = 2k$.

Otherwise, if there exists a $X_{i_0} \in \mathcal{U}_0$ such that $|V(X_{i_0})| \geq 3$, then by (2), we have $2l_0 < e(S_2, T) \leq |S_2| = 2k$, so $l_0 < k$, contradicting the fact that $l_0 = k$. So, $|V(X_i)| = 2$ for any $X_i \in \mathcal{U}_0$ $(i = 1, 2, \ldots, k)$ and $|T| = 2k$.

Claim 3. $N_{G_I}[T] - T = S_2$.

By Claim 1 and Claim 2, every vertex in T is adjacent to a vertex in S_2 and no vertex in S_2 is adjacent more than one vertex in T. So, $|N_{G_I}[T] - T| \geq |T| = 2k = |S_2|$. Since $N_{G_I}[T] - T \subseteq S_2$, it immediately follows that $N_{G_I}[T] - T = S_2$.

By Claim 2 and Claim 3, there is a one-to-one correspondence between the set T and the set S_2 in G_I. Therefore, the vertices of S can be paired as follows: two vertices of S_2 in the same red clique are paired, for $x_ix_j \in S_1$, then $x_jx_i \in S_1$, and x_ix_j and x_jx_i are paired. We set

$$
\begin{aligned}
U_0^* &= \{x_i \in V(G) \mid X_i \in \mathcal{U}_0\} \\
U_2^* &= \{x_j \in V(G) \mid X_j \in \mathcal{U}_2\}.
\end{aligned}
$$

We consider the bipartite subgraph $G(U_0^*, U_2^*)$ of G. Obviously, $G(U_0^*, U_2^*)$ is 2-regular. By Lemma 2.1, $G(U_0^*, U_2^*)$ has a perfect matching M'. Hence, $M = M' \cup \{(x_i, x_j) \mid x_ix_j \in S_1\}$ is a matching of G. Since $|M| = |M'| + \frac{l_1}{2} = k + \frac{l_1}{2} = \frac{n}{2}$, it follows that M is a perfect matching of G. This completes the proof of Theorem 2.3. ■

Theorem 2.4 *If $\delta(G) \geq 2$, then $\gamma_p(G_I) \leq \frac{4m(G)}{\delta(G)+1}$ and the bound is tight.*

Proof. Let $M = \{(u_1, v_1), (u_2, v_2), \ldots, (u_{\beta_1}, v_{\beta_1})\}$ be the maximum matching of G and Ω be the set of vertices not met by M, where β_1 is the matching number of G. Let $U_i(V_i)$ be the red clique of G_I corresponding to $u_i(v_i)$ of G. Let $\Omega = \{x_1, x_2, \cdots, x_s\}$. For each $x_j \in \Omega$, choose two vertices $x_jw_j, x_jw_j' \in X_j$ in G_I. We set

$$
\begin{aligned}
A &= \{u_iv_i \in U_i, v_iu_i \in V_i \mid (u_i, v_i) \in M, 1 \leq i \leq \beta_1\} \\
B &= \{x_jw_j, x_jw_j' \in X_j \mid 1 \leq j \leq s\}.
\end{aligned}
$$

Then $S = A \cup B$ is a paired-dominating set of G_I. Depending on the value $\delta(G)$, we distinguish two cases.

Case 1. $\delta(G) \geq 3$.

For any vertex u_i met by M, if $|V(U_i) \cap N_{G_I}(B)| = |\{u_i x_{i_1}, u_i x_{i_2}, \ldots, u_i x_{i_l}\}| \geq 2$, then $|V(V_i) \cap N_{G_I}(B)| = \emptyset$. Otherwise, if there exists a vertex $v_i x_j \in V(V_i) \cap N_{G_I}(B)$, then $j \neq i_1$ or i_2. Thus, either $x_{i_1} u_i v_i x_j$ or $x_{i_2} u_i v_i x_j$ is a M-augmenting path of G, which contradicts the maximality of M. Hence, $|pn[v_i u_i, S]| \geq \delta(G) - 1$, and thus $|pn[u_i v_i, S]| + |pn[v_i u_i, S]| \geq \delta(G) - 1$. Similarly, if $|V(U_i) \cap N_{G_I}(B)| = 1$, then $|V(V_i) \cap N_{G_I}(B)| \leq 1$. So, $|pn[u_i v_i, S]| + |pn[v_i u_i, S]| \geq 2\delta(G) - 4 \geq \delta(G) - 1$ again. Therefore, for each $1 \leq i \leq \beta_1$, we have $|pn[u_i v_i, S]| + |pn[v_i u_i, S]| \geq \delta(G) - 1$. Note that $|N_{G_I}[B]| \geq (\delta(G) + 1)s$. So, we have

$$(\delta(G) - 1)\beta_1(G) + 2\beta_1(G) + (\delta(G) + 1)s \leq 2m(G).$$

This implies that $\beta_1(G) + s \leq \frac{2m(G)}{\delta(G)+1}$. So, $\gamma_p(G_I) \leq 2(\beta_1(G) + s) \leq \frac{4m(G)}{\delta(G)+1}$.

Case 2. $\delta(G) = 2$.

In G_I, we note that

$$4\beta_1(G) + 2s \leq |N_{G_I}[A]| + |B| \leq 2m(G)$$

and

$$2\beta_1(G) + 4s \leq |A| + |N_{G_I}[B]| \leq 2m(G).$$

It immediately follows that $3\beta_1(G) + 3s \leq 2m(G)$. So, $\gamma_p(G_I) \leq 2(\beta_1(G) + s) \leq \frac{4m(G)}{\delta(G)+1}$. This bound is tight for $G = mK_3$. ■

3 Paired domination of inflated trees

In this section, we turn our attention to trees. For ease of presentation, we consider *rooted trees*. A *rooted tree* T is a directed tree in which there exists a vertex r with the property that there is a directed path in T from r to every other vertex in T. The vertex r is unique with respect to the above-mentioned property and is called the *root* of T. Thus, if T is a rooted tree at r, then all edges of T are directed away from r. For a vertex v of a rooted tree T, the *parent* $p(v)$ of v is the unique vertex such that there is a directed edge from $p(v)$ to v, a *child* of v is a vertex u such that $p(u) = v$, and a *descendant* of v is a vertex u such that there is a directed v-u path in T. We define the notation $D(v) = \{u \in V | \ u \ \text{ is a descendant of } \ v\}$, $D[v] = D(v) \cup \{v\}$. The subtree of T induced by $D[v]$ is denoted by T_v; note that if T is rooted at r, then $T = T_r$. A vertex of T is said to be a *leaf* if it is an endvertex, and a *branch vertex* if it has degree at least 3. A path P in T is said to be a v-L *path* if P joins v to a leaf of T. Denote the length of P by $l(P)$.

Let T_I be the inflated graph of tree T, and we call T_I the *inflated tree*. Let u be a branch vertex in T at the maximum distance from root r, and U is a red clique of T_I corresponding to u of T. We define

$$C^0(U) \;\; = \;\; \{ux \in V(U) | \ x \text{ is a child of } \ u \text{ in } T, \text{ and } T_x \text{ contains a } x\text{-}L \text{ path}$$

$$\begin{aligned} & P \text{ in } T \text{ with } \ l(P) = 0(mod\ 2)\}, \\ C^1(U) \ = \ & \{ux \in V(U) |\ x \ \text{ is a child of } \ u \text{ in } T, \text{ and } T_x \text{ contains a } x\text{-}L \text{ path} \\ & P \text{ in } \ T \text{ with } \ l(P) = 1(mod\ 2)\} \end{aligned}$$

For each uw (w is a child of u in T) in U, we assign a priority to uw, where $uw \in C^0(U)$ has a higher priority than $uw^1 \in C^1(U)$. Let T_{ux} denote the subgraph of T_I that is an isomorphism to the inflated graph of $\langle D[x] \cup \{u\}\rangle$, and let $D[ux] = V(T_{ux})$.

In the following we present a linear time algorithm for finding the minimum paired-domination set in an inflated tree T_I.

Algorithm 1. Minimum paired-domination for inflated trees.

Input: A rooted tree T_r with root r. An inflated graph T_I of the tree T_r.

Step 1. Set $T := T_r, T_I := T_I, S := \emptyset$. We set T_0 to be a dummy empty graph, i.e., a graph with no vertices and no edges.

Step 2. Using the breadth-first method to search all the vertices of T_r and determine the distance $d_{T_I}(x, r)$ for each $x \in V(T_r)$, and simultaneously generate the branch-vertex sequence

$$(u_1, u_2, \ldots, u_s)$$

such that each branch vertex appears exactly once in the sequence, and such that

$$d(u_1, r) \leq d(u_2, r) \leq \ldots \leq d(u_s, r).$$

Set $B(T) := \{u_1, u_2, \ldots, u_s\}, m := s$.

Step 3. If $m = 0$ (i.e., $B(T) = \emptyset$), set $T_0 := T_0 \cup T_I$, go to Step 5; otherwise (i.e., $B(T) \neq \emptyset$), go to Step 4.

Step 4. Set $u := u_m$. For each child x of u in tree T, let x' be the unique vertex in T_x with $d_T(x') = 1$. Set

$$\begin{aligned} C^0(U) &= \{ux \in U |\ d_T(r, x') - d_T(r, x) = 0(mod\ 2)\}, \\ C^1(U) &= \{ux \in U |\ d_T(r, x') - d_T(r, x) = 1(mod\ 2)\}. \end{aligned}$$

Choose a vertex ux (x is a child of u in T) in the red clique U has the lowest priority. If $ux \in C^1(U)$, set $T_I := T_I - D[ux], T := T - D[x], T_0 := T_0 \cup T_{ux}$. If $ux \in C^0(U)$, then there exists another vertex $ux_1 \in C^0(U)$ (x_1 is a child of u in T) in the red clique U. If $d_T(u) = 3$, set $T_I := T_I - D[ux] \cup D[ux_1] \cup \{up(u)\}$, $T := T - T_u$, $T_0 := T_0 \cup (T_{ux} \cup T_{ux_1} \cup \{(ux, ux_1)\})$. If $d_T(u) > 3$, set $T_I := T_I - D[ux] \cup D[ux_1]$, $T := T - D[x] \cup D[x_1], T_0 = T_0 \cup (T_{ux} \cup T_{ux_1} \cup \{(ux, ux_1)\})$, and return to Step 3.

Step 5. If T_0 is a dummy empty graph, then stop; otherwise, go to Step 6.

Step 6. Choose an arbitrary component P of T_0. Suppose that $P = v_{i_1}v_{i_2}\dots v_{i_l}$, $l = 4k + r(0 \le r \le 3)$, where we denote each vertex of P by a simple letter x.

Step 7. If $r = 0$, set $S := S \cup \{v_{i_2}, v_{i_3}, v_{i_6}, v_{i_7}, \dots, v_{i_{l-2}}v_{i_{l-1}}\}$, $T_0 := T_0 - V(P)$, return to Step 6. Otherwise, go to Step 8.

Step 8. Set $S := S \cup \{v_{i_2}, v_{i_3}, v_{i_6}, v_{i_7}, \dots, v_{i_{4k-2}}, v_{i_{4k-1}}, v_{i_{l-1}}, v_{i_l}\}$, $T_0 = T_0 - V(P)$, return to Step 6.

Output: The vertex set S (which is a minimum paired-dominating set of the inflated tree T_I).

The complexity of the above algorithm can be estimated as follows. The time needed to perform Step 2 is clearly $O(|V(T_r)|)$. The time needed to perform Step 4 for a given branch vertex u is $O(|C(u)|)$. Hence, the time taken by the loop from Step 3 to Step 4 is at most $O(|V(T_r)|)$. The loop from Step 7 to Step 8 for determining the minimum paired-dominating set of a path P clearly needs at most $O(|V(P)|)$ time. Thus, the time taken by the loop from Step 6 to Step 8 is at most $O(V(T_r))$. It follows that the total time needed to perform the above algorithm is $O(|V(T_r)|)$. In Figure 2 we show an example of inflated trees and a minimum paired-dominating set (the shaded vertices represent the paired-dominating set) computed using Algorithm 1.

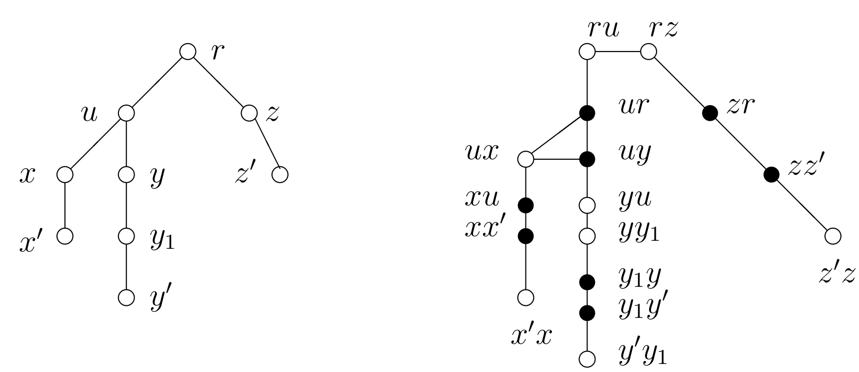

Figure 2: A paired-dominating set computed using Algorithm 1 in the inflation of a tree.

We now verify the validity of Algorithm 1. First, Algorithm 1 leads immediately to the following property.

Property 3.1 *(a) Any branch of the graph T_0 produced by Step 1–Step 4 is a path P. (b) For every component P of T_0, $S \cap P$ is a minimum paired-dominating set of P.*

Lemma 3.2 *Let T_I be an inflated graph of a rooted tree T_r, and u be a branch vertex at the maximum distance from r in T_r, then there exists a minimum paired-dominating set S of T_I containing all the vertices in $C^0(U)$.*

Proof. Let S be a minimum paired-dominating set of T_I. For any vertex $ux \in C^0(U)$, if ux is adjacent to a pendant vertex, then $ux \in S$. So we assume that $D[ux] = \{ux, xu, xx_1, x_1x, x_1x_2 \ldots, x_lx_{l-1}\}$ $(l \geq 2)$ for any $ux \in C^0(U)$. If $ux \notin S$, then $S \cap D[ux] \succ T_{xu}$ and $\gamma_p(T_{xu}) = \gamma_p(T_{xx_1}) + 2$. Let S_1 be a minimum paired-dominating set of T_{xx_1}, then $S' = (S - S \cap D[xu]) \cup (S_1 \cup \{xu, ux\})$ is a minimum paired-dominating set of T_I. The lemma follows. ■

Theorem 3.3 *Given an inflated tree T_I, Algorithm 1 computes in time $O(n)$ a minimum paired-dominating set of T_I.*

Proof. Let T_I be an inflated graph of tree T_r. We proceed by induction on the order of T_I. Let u be a branch vertex at the maximum distance from r in T_r. Let ux (x is a child of u in T) be a vertex in the red clique U of the lowest priority in T_I.

Case 1. $ux \in C^1(U)$.

We consider $T'_I = T_I - D[ux]$. It is easily seen that T'_I is an inflated graph of tree $T' = T - D[x]$. Let S be a paired-dominating set of T_I produced by Algorithm 1, then $S' = S \cap V(T'_I)$ is a paired-dominating set of T'_I produced by Algorithm 1. By the inductive hypothesis, S' is a minimum paired-dominating set of T'_I. Combining with Property 3.1, we have $\gamma_p(T_I) \leq \gamma_p(T'_I) + \gamma_p(T_{ux}) = |S|$. Furthermore, we show that $\gamma_p(T_I) \geq |S|$. Assume $D[ux] = \{ux, xu, xx_1, x_1x, x_1x_2 \ldots, x_lx_{l-1}\}$. Let D be a minimum paired-dominating set of T_I. If $ux \notin D$, then $D \cap D[ux] \succ T_{xu}$ and $D \cap V(T'_I) \succ T'_I$. Combining with $\gamma_p(T_{xu}) = \gamma_p(T_{ux})$, we have $\gamma_p(T_I) = |D| \geq \gamma_p(T'_I) + \gamma_p(T_{ux}) = |S|$. If $ux \in D$, without loss of generality, we assume that ux is paired with a vertex uw in T'_I, then $D \cap D[ux] \succ T_{xx_1}$. But $\gamma_p(T_{xx_1}) = \gamma_p(T_{ux})$. Let D_1 be the minimum paired-dominating set of $T_{x_1x_2}$, $uw' \in N_{T_I}(uw)$ and $uw' \notin D$, then $D' = (D - D \cap D[ux]) \cup (D_1 \cup \{xu, xx_1\}) \cup \{uw'\}$ is a minimum paired-dominating set of T_I. And $D' \cap V(T'_I) \succ T'_I$, $D' \cap D[ux] \succ T_{ux}$. So, $\gamma_p(T_I) = |D'| \geq \gamma_p(T'_I) + \gamma_p(T_{ux}) = |S|$.

Case 2. $ux \in C^0(U)$.

There exists another child x' of u in T such that $ux' \in C^0(U)$. If $d_T(u) > 3$, we consider $T'_I = T_I - D[ux] \cup D[ux']$. Let ux'' (x'' is a child of u) be a vertex in $C^0(U) - \{ux, ux'\}$. It is easily seen that T'_I is an inflated graph of $T' = T - D[x] \cup D[x']$. Let S be a paired-dominating set of T_I produced by Algorithm 1, then $S' = S \cap V(T'_I)$ is a paired-dominating set of T'_I produced by Algorithm 1. By the inductive hypothesis, S' is a minimum paired-dominating set of T'_I. Combining with Property 3.1, we have $\gamma_p(T_I) \leq \gamma_p(T'_I) + \gamma_p(T_{ux} \cup T_{ux'} \cup \{(ux, ux')\}) = |S|$. Furthermore, we show that $\gamma_p(T_I) \geq |S|$. By Lemma 3.2, let D be a minimum paired-dominating set of T_I containing all the vertices in $C^0(U)$, then $ux, ux', ux'' \in D$. Without loss of generality, we assume ux is paired with ux', then $D \cap T'_I \succ T'_I$, and $D \cap (D[ux] \cup D[ux']) \succ T_{ux} \cup T_{ux'} \cup \{(ux, ux')\}$. So, $\gamma_p(T_I) = |D| \geq \gamma_p(T'_I) + \gamma_p(T_{ux} \cup T_{ux'} \cup \{(ux, ux')\}) = |S|$.

If $d(u) = 3$, we consider $T_I' = T_I - (D[ux] \cup D[ux'] \cup \{up(u)\})$. It is easily seen that T_I' is an inflated graph of $T' = T - D[u]$. Let S be a paired-dominating set of T_I produced by Algorithm 1, then $S' = S \cap V(T_I')$ is a paired-dominating set of T_I' produced by Algorithm 1. By the inductive hypothesis, S' is a minimum paired-dominating set of T_I'. So $\gamma_p(T_I) \leq \gamma_p(T_I') + \gamma_p(T_{ux} \cup T_{ux'} \cup \{(ux, ux')\}) = |S|$. Furthermore, we show that $\gamma_p(T_I) \geq |S|$. By Lemma 3.2, let D be a minimum paired-dominating set of T_I containing ux and ux'. Without loss of generality, we may assume $up(u) \notin D$, then $D \cap V(T_I') \succ T_I', D \cap (D[ux] \cup D[ux']) \succ T_{ux} \cup T_{ux'} \cup \{(ux, ux')\}$. So, $\gamma_p(T_I) \geq \gamma_p(T_I') + \gamma_p(T_{ux} \cup T_{ux'} \cup \{(ux, ux')\}) = |S|$. Then, $\gamma_p(T_I) = |S|$. This completes the proof of Theorem 3.3. ■

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