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Dynamics of the spin-2 Bose condensate driven by external magnetic fields

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Dynamic response of the $F=2$ spinor Bose-Einstein condensate (BEC) under the influence of external magnetic fields is studied. A general formula is given for the oscillation period to describe population transfer from the initial polar state to other spin states. We show that when the frequency and the reduced amplitude of the longitudinal magnetic field are related in a specific manner, the population of the initial spin-0 state will be dynamically localized during time evolution. The effects of external noise and nonlinear spin-exchange interaction on the dynamics of the spinor BEC are studied. We show that while the external noise may eventually destroy the Rabi oscillations and dynamic spin localization, these coherent phenomena are robust against the nonlinear atomic interaction.

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I. INTRODUCTION

Recent experiments on ^{23}Na condensates confined in an optical trap [1–3] have stimulated extensive interest in the study of multicomponent spinor Bose-Einstein condensates (BECs). Due to the hyperfine spin of atoms, BECs of alkali-metal atoms have internal degrees of freedom which are frozen in a magnetic trap. Introduction of an optical trap liberates them to allow BEC to be in a superposition of magnetic sublevels. Therefore, the spinor BEC can be represented by a vector order parameter. The $F=1$ spinor BEC was first theoretically studied by Ho [4], Ohmi and Machida [5] by generalizing the Gross-Pitaevskii (GP) equation under the restriction of gauge and spin-rotation symmetry. Within the mean-field theory, they predicted a rich set of new phenomena such as spin textures and topological excitations. Law *et al.* [6] constructed an excellent algebraic representation of the $F=1$ BEC Hamiltonian to study the exact many-body states, and found that spin-exchange interactions cause a set of collective dynamic behavior of BEC. Since the spinor BEC appears feasible by using the $F=2$ multiplet of bosons, such as ^{23}Na, ^{87}Rb, or ^{85}Rb, it is necessary to investigate the ground-state structure and magnetic response of $F=2$ spinor BEC. Recently, Ciobanu *et al.* [7] generalized the approach for the $F=1$ spinor BEC developed by Ho to study the ground-state structure of the $F=2$ spinor BEC. They found that there are three possible phases in zero magnetic field, which are characterized by a pair of parameters describing the ferromagnetic order and the formation of singlet pairs. From current estimates of scattering lengths, they also found that the spinor BEC's of ^{87}Rb and ^{23}Na have a polar ground state, whereas those of ^{85}Rb and ^{83}Rb are cyclic and ferromagnetic, respectively. Koashi and Ueda [8] studied the exact eigenspectra and eigenstates of $F=2$ spinor BEC. They found that, compared to $F=1$ spinor BEC, the $F=2$ spinor BEC exhibits an even richer magnetic response due to quantum correlations among three bosons.

Different from superfluid ^{3}He, a new feature in a spinor BEC is that its response to an external magnetic field is dominated by the electronic rather than the nuclear spin. This opens up possibilities of manipulating the magnetism of superfluid vapors. Observation of spin domains by Stenger *et al.* at MIT [2] offers a remarkable example of such manipulations. Pu *et al.* [9] investigated the effects of external magnetic fields on the dynamics of $F=1$ spinor BEC and found various magnetic-field-induced effects, such as stochastization in population evolution, metastability in spin composition, and dynamic localization in spin space. In this paper, we investigate the time evolution of the $F=2$ spinor BEC in the presence of external magnetic fields with longitudinal and transverse field strengths $B_z(t)$ and B_x, respectively. Here the longitudinal magnetic field lifts the energy degeneracy of the spin states through Zeeman effects, whereas the transverse field appears as a coupling between different spin components. We assume the longitudinal field to be time dependent and that the transverse field is static. Using perturbation theory, a general formula for the oscillation period to describe the passing across the initial spin-polarized state is obtained. Another aspect we are interested in the effects of the external noise and the nonlinear spin-exchange interactions on the quantum coherent behavior of the spinor BEC.

In Sec. II, the Hamiltonian for the spin-2 BEC system is presented. The Rabi oscillation and dynamic spin localization are shown in Sec. III. In Sec. IV, we discuss the effects of external noise on the dynamics of the spinor BEC. A full numerical discussion based on an effective one-dimensional nonlinear Schrödinger equation is given in Sec. V. A summary is given in Sec. VI.

II. THE MODEL: SINGLE-MODE APPROXIMATION

We consider the $F=2$ spinor BEC subject to a spatial-uniform magnetic field $\hat{B}(t)$. The Hamiltonian invariant under spin space rotation and gauge transformation is written in terms of the five-component field operators: $\Psi_{+2},\ldots,\Psi_{-2}$, corresponding to the sublevels m_F

$=+2,\ldots,-2$ of the hyperfine state $F=2$. Namely, it is given [7,8] by $H=H_0+H_B$,

$$H_0=\int d\mathbf{r}\Bigg(\frac{\hbar^2}{2M}\nabla\Psi_\alpha^+\cdot\nabla\Psi_\alpha+U\Psi_\alpha^+\Psi_\alpha+\frac{\overline{c_0}}{2}\Psi_\alpha^+\Psi_\beta^+\Psi_\beta\Psi_\alpha+\frac{\overline{c_1}}{2}\sum_i[\Psi_\alpha^+(F_i)_{\alpha,\beta}\Psi_\beta]^2+\overline{c_2}\Psi_\alpha^+\Psi_{\alpha'}^+\langle 2\alpha;2\alpha'00\rangle\times\langle 00|2\beta;2\beta'\rangle\Psi_\beta\Psi_{\beta'}\Bigg), \tag{1}$$

$$H_B=-\mu_B g_f\int d\mathbf{r}\Psi_\alpha^+[\hat{B}(t)\cdot F]_{\alpha,\beta}\Psi_\beta, \tag{2}$$

where $\overline{c_0}$, $\overline{c_1}$, and $\overline{c_2}$ are related to scattering lengths a_0, a_2, and a_4 of the two colliding bosons, with total angular momenta 0, 2, and 4, by $\overline{c_0}=4\pi\hbar^2(3a_4+4a_2)/7M$, $\overline{c_1}=4\pi\hbar^2(a_4-a_2)/7M$, and $\overline{c_2}=4\pi\hbar^2(3a_4-10a_2+7a_0)/7M$. In addition, we have introduced in Eqs. (1) and (2), 5×5 spin matrices F_i $(i=x,y,z)$,

$$F_x=\begin{pmatrix}0&1&0&0&0\\1&0&\sqrt{3/2}&0&0\\0&\sqrt{3/2}&0&\sqrt{3/2}&0\\0&0&\sqrt{3/2}&0&1\\0&0&0&1&0\end{pmatrix}, \tag{3}$$

$$F_y=-i\begin{pmatrix}0&1&0&0&0\\-1&0&\sqrt{3/2}&0&0\\0&-\sqrt{3/2}&0&\sqrt{3/2}&0\\0&0&-\sqrt{3/2}&0&1\\0&0&0&-1&0\end{pmatrix}, \tag{4}$$

$$F_z=\begin{pmatrix}2&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&-1&0\\0&0&0&0&-2\end{pmatrix}. \tag{5}$$

We assume that the external magnetic field is weak so that the coordinate wave function $\Phi(\mathbf{r})$ is independent of the spin state and solely determined by the first three terms of Eq. (1), namely, $[-\hbar^2\nabla^2/2M+U+\overline{c_0}(N-1)|\Phi|^2]\Phi=\mu\Phi$, with N being the total particle number. Substituting $\Psi_\alpha=\hat{a}_\alpha\Phi$ into Eqs. (1) and (2) and keeping only spin-dependent terms, we obtain

$$H_0'=\frac{c_1}{2}\hat{F}\cdot\hat{F}+\frac{2c_2}{5}\hat{S}_+\hat{S}_- \tag{6}$$

and

$$H_B=-\mu_B g_f\hat{B}(t)\cdot\hat{F}, \tag{7}$$

where $\hat{F}_i=\hat{a}_\alpha^\dagger(F_i)_{\alpha\beta}\hat{a}_\beta$, $\hat{S}_+=\hat{S}_-^\dagger=(\hat{a}_0^\dagger)^2/2-\hat{a}_1^\dagger\hat{a}_{-1}^\dagger+\hat{a}_2^\dagger\hat{a}_2^\dagger$, and $c_i=(\overline{c_i})\int d\mathbf{r}|\Phi|^4$. We further assume that the magnetic fields consist of longitudinal and transverse components. Without loss of generality, the transverse direction of the field is chosen to be along the x axis, i.e., $\hat{B}(t)=B_l(t)\hat{z}+B_x\hat{x}$. In such a case, the second-quantized Hamiltonian of H_B is

$$H_B=-\mu_B g_f B_l(t)(\hat{a}_2^\dagger\hat{a}_2+\hat{a}_1^\dagger\hat{a}_1-\hat{a}_{-1}^\dagger\hat{a}_{-1}-\hat{a}_{-2}^\dagger\hat{a}_{-2})-\mu_B g_f B_x(\hat{a}_2^\dagger\hat{a}_1+\sqrt{3/2}\hat{a}_1^\dagger\hat{a}_0+\sqrt{3/2}\hat{a}_0^\dagger\hat{a}_{-1}+\hat{a}_{-1}^\dagger\hat{a}_{-2}+\text{H.c.}). \tag{8}$$

Similar to Ciobanu *et al.* [7], mean-field approximation is used such that the field operators $\hat{a}_\alpha$ are replaced by c numbers $a_\alpha=\sqrt{P_\alpha}e^{i\theta_\alpha}$, where $P_\alpha=N_\alpha/N$ is the population in spin α, and θ_α the phase of wave function a_α. Furthermore, since this paper deals with the quantum coherent behavior of the system under the influence of the external magnetic fields, we assume that the initial spin state of the BEC is the eigenstate in the absence of external fields, consequently, contribution from Hamiltonian (6) is a constant energy shift, and can be neglected in the dynamics. The semiclassical equations of motion in Heisenberg representation can be derived from the Hamiltonian H_B [we introduce the state vector $a=(a_2,\ldots,a_{-2})^T$]

$$i\dot{a}=H_{eff}(t)a, \tag{9}$$

where

$$H_{eff}(t)=-b_l(t)F_z-b_xF_x. \tag{10}$$

Here we have defined $b_l(t)=\mu_B g_f B_l(t)$ and $b_x=\mu_B g_f B_x$. The time evolution of the system has a rich set of intriguing features with different initial conditions. In this paper, we consider the case that the system begins with unperturbed spin-0 state $a(0)=(0,0,1,0,0)^T$, i.e., the $P0$ polar phase defined by Ciobanu *et al.* [7]. This phase is possible to be realized in ^{87}Rb and ^{23}Na.

III. RABI OSCILLATIONS AND DYNAMIC SPIN LOCALIZATION

In this section, we analyze the time evolution of $F=2$ BEC described by Eq. (9) with an initial polar state $a(0)=(0,0,1,0,0)^T$. Assuming that $a(t)=\exp[i\int_0^t d\tau b_l(\tau)F_z]\varphi(t)$, we obtain

$$i\dot{\varphi}(t)=H_I(t)\varphi(t), \tag{11}$$

where $H_I(t)=-b_x\exp[-i\int_0^t d\tau b_l(\tau)F_z]F_x\exp[i\int_0^t d\tau b_l(\tau)F_z]$. Making use of the identity

$$\exp(i\lambda F_z)F_x\exp(-i\lambda F_z)=F_x\cos\lambda-F_y\sin\lambda, \tag{12}$$

the formal solution of Eq. (11) is obtained as

$$\varphi(t)=\hat{T}\exp\left(i\int_0^t d\tau[X(\tau)F_x+Y(\tau)F_y]\right)\varphi(0), \quad (13)$$

where $X(t)=b_x\cos[\int_0^t d\tau b_l(\tau)]$, $Y(t)=b_x\sin[\int_0^t d\tau b_l(\tau)]$, and $\hat{T}$ denotes time ordering. Considering the situation in which the transverse magnetic field b_x is weak, we may approximate Eq. (13) by the following solution:

$$\varphi(t)=\exp\left(i\int_0^t d\tau[X(\tau)F_x+Y(\tau)F_y]\right)\varphi(0). \quad (14)$$

This solution is valid to the first order in b_x and preserves unitarity. After a straightforward calculation, we obtain the time evolution of the population of spin-α state as

$$P_\alpha(t)=|a_\alpha(t)|^2=\left|\sum_{\beta=-2}^{2} d^2_{\beta\alpha}(\pi/2)d^2_{\beta 0}(\pi/2)\times\exp\left(-i\alpha\int_0^t d\tau b_l(\tau)\right)e^{i\beta\lambda}\right|^2, \quad (15)$$

where $\lambda=\sqrt{[\int_0^t d\tau X(\tau)]^2+[\int_0^t d\tau Y(\tau)]^2}$, and the matrix element is defined as

$$d^2_{\beta\alpha}(\theta)=\langle 2\beta|e^{-i\theta F_y}|2\alpha\rangle \quad (\beta,\alpha=2,\ldots,-2). \quad (16)$$

In particular, the population of the initial spin-0 state at time t is

$$P_0(t)=|a_0(t)|^2=\cos^2\left|\int_0^t d\tau_1 b_x\exp\left(i\int_0^{\tau_1}d\tau_2 b_l(\tau_2)\right)\right|. \quad (17)$$

Thus to the first-order approximation in b_x, we obtain the analytical expression of the evolution probability for the spinor BEC remaining in the initial polar state. Expression (17) is valid for arbitrary time-dependent external magnetic fields. Obviously, $\exp[i\int_0^t d\tau b_l(\tau)]$ can be expanded as a discrete Fourier series, and the time integral is either bounded or increases linearly in time (on top of an oscillatory piece). In the former case, P_0 remains close to unity at all times because of the smallness of b_x. This high population of the initial spin state is just the spin localization. In the latter case, P_0 oscillates between 0 and 1, implying population transfer between P_0 and other P_α ($\alpha=\pm 1, \pm 2$). Hence, we have the spin delocalization, i.e., Rabi oscillation. Thus, we obtain the oscillation period T as

$$T=\frac{\pi}{\lim_{t\to\infty}\left|\frac{1}{t}\int_0^t d\tau_1 b_x\exp\left(i\int_0^{\tau_1}d\tau_2 b_l(\tau_2)\right)\right|}. \quad (18)$$

This result is still valid for any time-dependent external field. We give two examples of special time-dependent external magnetic fields.

(a) Sinusoidal field $b_l(t)=b\cos(\omega t)$. According to Eq. (18), the oscillation period is

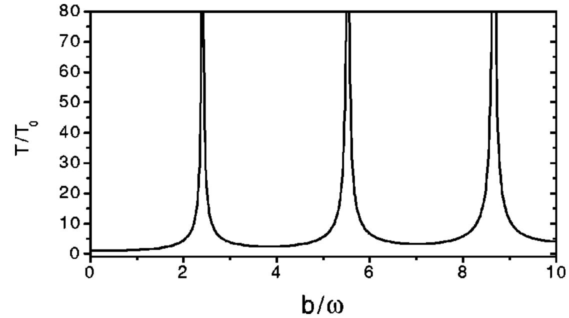

FIG. 1. Scaled oscillation period T/T_0 of the sinusoidal magnetic-field case vs the reduced field strength b/ω.

$$T=\frac{\pi}{b_x J_0(b/\omega)}, \quad (19)$$

where J_0 is the zeroth-order Bessel function of the first kind. From this result, it can be seen that when $J_0(b/\omega)\neq 0$, there exists a finite oscillation period, indicating that the population can transit from an initial polar state to other spin states within the driving process. When $b=0$, the longitudinal field-free oscillation has period $T_0=\pi/b_x$. The fact $|J_0(b/\omega)|\leqslant 1$ implies that $T\geqslant T_0$. This shows that the invasion of the longitudinal magnetic field will suppress the coherent population transfer among the five spin components. The extreme case occurs when the reduced amplitude of the longitudinal magnetic field b/ω is a root of J_0. In this case, we have $T\to\infty$, which implies that the population transfer is totally suppressed. Hence the system will stay in the initial polar state during the whole driving process. This is just the phenomenon of spin localization. The above analysis is illustrated in Fig. 1, where the scaled oscillation period T/T_0 is plotted against b/ω. It can be seen that the oscillation period increases significantly when b/ω approaches the roots of Bessel function J_0.

(b) A combination of static and sinusoidal magnetic fields $b_l(t)=b_0+b\cos(\omega t)$. In this case, the oscillation period is

$$T=\frac{\pi}{\left|b_x\lim_{N\to\infty}\frac{\sin(b_0\pi N/\omega)}{N\sin(b_0\pi/\omega)}\hat{J}_{b_0/\omega}(b/\omega)\right|}, \quad (20)$$

where $\hat{J}$ is the Anger function [10] defined by

$$\hat{J}_a(b)=\frac{1}{\pi}\int_0^\pi dx\cos(ax-b\sin x). \quad (21)$$

There exists an infinite oscillation period when $b_0/\omega\neq k$ (k is an integer), which means that the population time evolution will undergo localization in the initially populated spin-0 state during the whole driving process. When $b_0/\omega=k$, the oscillation period becomes

$$T=\frac{\pi}{b_x J_k(b/\omega)}, \quad (22)$$

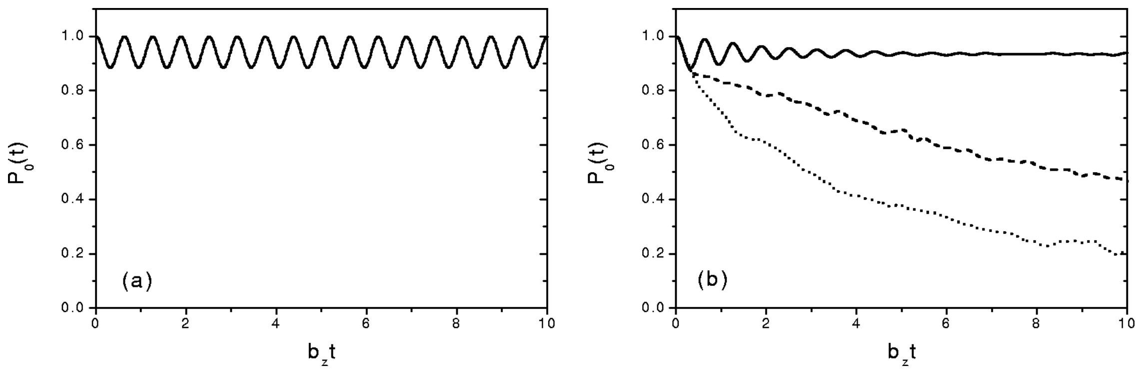

FIG. 2. Time evolution of $P_0(t)$ with and without external noise in the static magnetic field, $b/b_x=10$. (a) The case without the noise, (b) the case with the noise $\Delta/b_x=1.0$ (solid curve), $\Delta/b_x=5.0$ (dashed curve), and $\Delta/b_x=10.0$ (dotted curve). $b_x\tau_c$ is fixed to be 1.0 for the case with the noise.

where J_k are kth order Bessel functions. From this result, we find that we can have a finite oscillation period only when b/ω is not equal to a root of J_k, and in this case the evolution of the initial polar phase is delocalized into other spin states. If b/ω becomes a root of J_k, the oscillation period approaches infinity, and we have the spin localization.

IV. EFFECTS OF EXTERNAL NOISE

In practice, since the external magnetic field may have a fluctuation component, the effects of external noise have to be considered. Another motivation for the introduction of an external noise is to provide the fluctuations required to destroy the coherence of the population transfer process and, in suitable cases, leads to a rate process for the decay of the population. In the following, we study the dynamics of the system when the external magnetic field has a fluctuating component. In the simplest case, the statistical properties of the imposed magnetic field can be prescribed to be independent of the characteristics of the system. As a versatile choice for the noise, an Ornstein-Uhlenbeck (OU) process [11] is used. This permits the investigation of the role of the strength and size of correlation time of the noise on the time evolution of the spinor BEC. Without loss of generality, the external longitudinal magnetic field $b_l(t)$ is assumed to contain two components, a stochastic part $f(t)$ and a systematic part $b_0+b_1\cos(\omega t)$, i.e.,

$$b_l(t)=b_0+b_1\cos(\omega t)+f(t). \tag{23}$$

The noise $f(t)$ is assumed to be characterized by an OU process whereby it has zero average value and correlation function

$$\langle f(t)f(s)\rangle=\Delta^2\exp(-|t-s|/\tau_c). \tag{24}$$

Here the quantities Δ and τ_c are the strength and decay constants of the noise, respectively. When the noise is external to the system, and therefore not necessarily thermal in character, Δ and τ_c can be varied in a controlled manner, and are not restricted by the physical properties of the system. In the absence of an analytic solution, a numerical investigation of population $P_\alpha(t)$ is adapted. The population distribution in the five components of spinor BEC with the external noise are solved numerically by generating trajectories for the different realization of the noise. The procedure [12] to integrate the stochastic equations is as follows. A stochastic term is added at each step with its statistical properties described by an OU process. The OU process is generated by solving a Langevin equation with a δ-correlated noise term. This en-

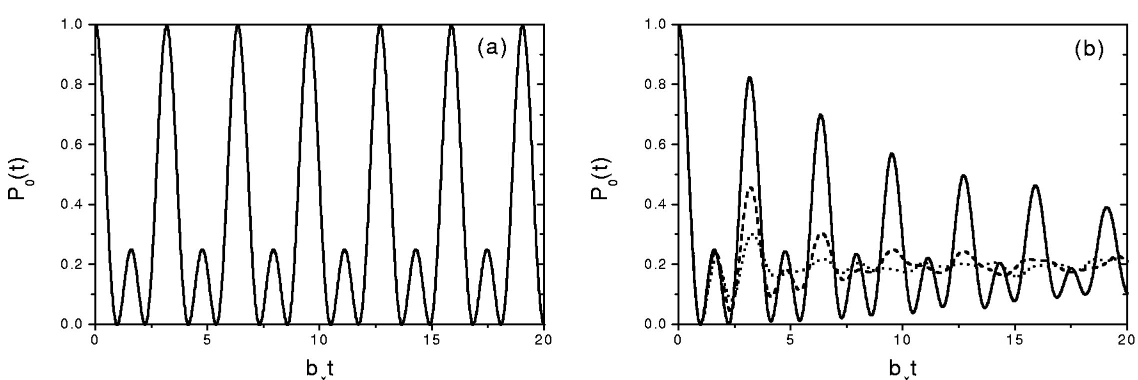

FIG. 3. Time evolution of $P_0(t)$ with and without external noise in sinusoidal magnetic field, $b/b_x=2$, $\omega/b_x=10$. (a) The case without the noise, (b) the case with the noise $\Delta/b_x=0.2$ (solid curve), $\Delta/b_x=0.6$ (dashed curve), and $\Delta/b_x=1.0$ (dotted curve). $b_x\tau_c$ is fixed to be 1.0 for the case with the noise.

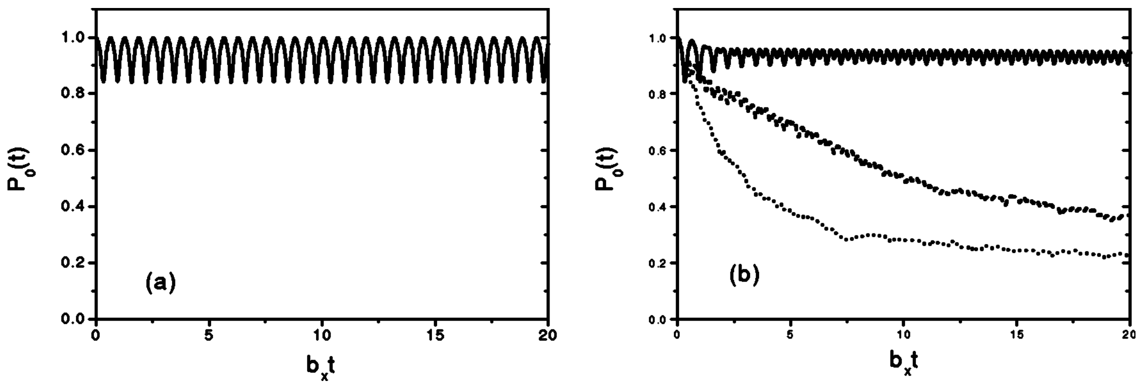

FIG. 4. Time evolution of $P_0(t)$ with and without external noise in sinusoidal magnetic field, $b/b_x = 24.02$, $\omega/b_x = 10$. (a) The case without the noise, (b) the case with the noise $\Delta/b_x = 1.0$ (solid curve), $\Delta/b_x = 5.0$ (dashed curve), and $\Delta/b_x = 10.0$ (dotted curve). $b_x\tau_c$ is fixed to be 1.0 for the case with the noise.

sures that the correlation function of $f(t)$ has the desired statistical property given by Eq. (24).

We present in Fig. 2 the time evolution of the population $P_0(t)$ of spin-0 state in the static magnetic field $[b_1 = 0$ in Eq. (23)] for the value of $b_0 = 10b_x$. In the absence of noise, as shown in Fig. 2(a), the population of spin-0 state is always close to unity during time evolution. This effect is purely the static spin localization. In this case, the longitudinal magnetic field lifts the degeneracy of the five spin components and the consequent energy mismatch hinders the BEC transfer from the initially occupied polar state to its neighbors in the spin space. In the case of the intermediate $(\Delta\tau_c \sim 1)$ noise modulation, as shown in Fig. 2(b) (solid curve), it can be seen that the population of spin-0 state has a damped oscillation with a slow monotonic decay superimposed. The combination of noise and static magnetic field, suppressing population transfer, produces the decay. It also shows in Fig. 2(b) that with increasing the strength of noise, the decay becomes more rapid. Therefore, the decaying time can be controlled by varying the magnetic field or the strength of noise.

Figure 3 shows the time dependency of $P_0(t)$ in the time-dependent longitudinal magnetic field with the systematic part $b_l = b\cos(\omega t)$. The field parameters are chosen as $b = 2b_x$ and $\omega = 10b_x$, corresponding to the Rabi oscillation with scaled period $T = T_0/J_0(0.2)$, as shown in Fig. 3(a). In the case of a weak noise $\Delta/b_x = 0.2$ (dashed curve), the population P_0 is still oscillatory in our scope of time, but its oscillation amplitude decreases with time. When the strength of the noise increases to $\Delta/b_x = 1.0$ (dotted curve), the Rabi oscillation breaks down completely and the system decays fast towards equilibrium. This suggests that the coherent Rabi oscillation for a spinor BEC is sensitive to the external noise and is destroyed even in a weak coupling regime. We also present in Fig. 4 the time evolution of P_0 for the value of $b/\omega = 2.402$, corresponding to the dynamic localization, as shown in Fig. 4(a). In Fig. 4(b), we can see that in the case of intermediate $(\Delta\tau_c \sim 1)$ noise modulation (solid curve), the population P_0 still remains close to unity for an extremely long time. This insensitivity to the presence of weak noise reflects the strength of the systematic field. When the strength of noise increases to a strong coupling regime, a more rapid and less oscillatory decay is observed in the population evolution, as shown in Fig. 4(b) (dotted curve).

Figure 5 shows the time evolution of P_0 in the external longitudinal magnetic fields consisting of static and time-periodic components [Eq. (23)]. The field parameters are chosen as $b_0/b_x = 10$, $\omega/b_x = 10$, and $b_1/b_x = 38.3$, corresponding to the spin dynamic localization, as shown in Fig.

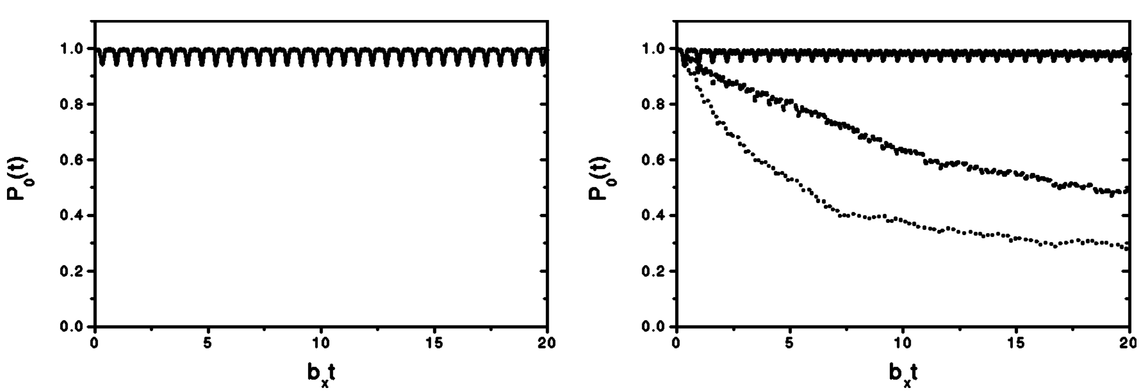

FIG. 5. Time evolution of $P_0(t)$ with and without external noise in a combination of static and sinusoidal magnetic field, $b_0/b_x = 10$, $b_1/b_x = 38.3$, $\omega/b_x = 10$. (a) The case without the noise, (b) the case with the noise $\Delta/b_x = 1.0$ (solid curve), $\Delta/b_x = 5.0$ (dashed curve), and $\Delta/b_x = 10.0$ (dotted curve). $b_x\tau_c$ is fixed to be 1.0 for the case with the noise.

5(a). In the case of intermediate ($\Delta\tau_c \sim 1$) noise modulation, similar to that shown in Fig. 4(b), the population P_0 of spin-0 state remains close to unity in our scope of time. Even in the strong coupling regime (dashed curve), the fast oscillations still exist, suggesting the fundamental interplay between the systematic and noise fields. With further increase in the strength of noise, the system decays rapidly towards equilibrium, and therefore, the spinor BEC initially localized in spin-0 state is delocalized and diffuses among all spin states, as shown in Fig. 5(b) (dotted curve).

V. DYNAMIC SPIN LOCALIZATION BEYOND SINGLE-MODE APPROXIMATION

In the above discussions, the single-mode assumption (SMA) has been used: the wave function for each spin component retains the same spatial profile during the time development. Therefore, the possibility of high-energy mode excitation is neglected and the system dynamics is fully described by the internal population transfer among the spin components. This SMA treatment is valid for a small particle number N. When N is large, the nonlinear spin-mixing processes induce the large energy transfer from the initial ground state to the excited modes, thus destroying the validity of the SMA. However, when the initial state is taken to be the eigenstate of the mean-field GP equation, the SMA is expected to describe well the dynamics in the absence of dissipation and external driving [13]. In this section, we go beyond the SMA by means of a numerical simulation of the time-spatial evolution of the system.

Due to anisotropic nature of the optical trapping potential, the cigar shaped BEC is assumed to be quasi-one-dimensional and hence the wave function associated with the spin-α state may be written as

$$\psi_\alpha(x,y,z,t) = \phi_\perp(x,y)\,\phi_\alpha(z,t)\,e^{-i\omega_\perp t}, \tag{25}$$

where z is the direction of weak confinement, $\phi_\perp(x,y)$ is the ground-state wave function of the two-dimensional harmonic oscillator, and $\omega_\perp$ the tight confinement frequency. Inserting Eq. (25) into Eqs. (1) and (2), we obtain the equations of motion for the longitudinal wave functions $\phi_\alpha(z,t)$ in a dimensionless form

$$i\hbar\frac{\partial\psi_2}{\partial t} = \mathcal{L}\psi_2 + f_-\psi_1 + f_z\psi_2 + \psi_{-2}^* s_- - b_x\psi_1 - 2b_l(t)\,\psi_2, \tag{26a}$$

$$i\hbar\frac{\partial\psi_1}{\partial t} = \mathcal{L}\psi_1 + f_+\psi_2 + \sqrt{\frac{3}{2}}f_-\psi_0 + f_z\psi_1 - \psi_{-1}^* s_- - b_x\left(\psi_1 + \sqrt{\frac{3}{2}}\psi_0\right) - b_l(t)\,\psi_1, \tag{26b}$$

$$i\hbar\frac{\partial\psi_0}{\partial t} = \mathcal{L}\psi_0 + \sqrt{\frac{3}{2}}(f_+\psi_1 + f_-\psi_{-1}) + \psi_0^* s_- - b_x(\psi_1 + \psi_{-1}), \tag{26c}$$

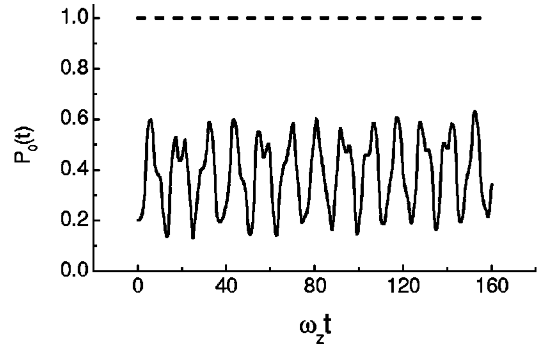

FIG. 6. Time evolution of $P_0(t)$ for two kinds of initial population: $N_i/N = 0.2$, $i = -2, \ldots, 2$ (solid line), $N_0/N = 1$ (dashed line). Other parameters are $\omega_z = 2\pi\times 60$ Hz, $a_0 = 34.9a_B$, $a_2 = 45.8a_B$, $a_4 = 64.5a_B$ for ^{23}Na (a_B is the Bohr radius), $\eta = 1$, $N = 10\,000$.

$$i\hbar\frac{\partial\psi_{-1}}{\partial t} = \mathcal{L}\psi_{-1} + \sqrt{\frac{3}{2}}f_+\psi_0 + f_-\psi_{-2} - f_z\psi_{-1} - \psi_1^* s_- - b_x\left(\sqrt{\frac{3}{2}}\psi_0 + \psi_{-2}\right) + b_l(t)\,\psi_{-1}, \tag{26d}$$

$$i\hbar\frac{\partial\psi_{-2}}{\partial t} = \mathcal{L}\psi_2 + f_-\psi_{-1} - f_z\psi_{-2} + \psi_2^* s_- - b_x\psi_{-1} + 2b_l(t)\,\psi_{-2}, \tag{26e}$$

where $\mathcal{L} = -d^2/dz^2 + z^2/4 + \overline{c_0}N\eta(|\phi_2|^2 + |\phi_1|^2 + |\phi_0|^2 + |\phi_{-1}|^2 + |\phi_{-2}|^2)$ with transverse structure factor $\eta = \int|\phi_\perp(x,y)|^4 dxdy / \int|\phi_\perp(x,y)|^2 dxdy$, $f_+ = \overline{c_1}(\phi_2^*\phi_1 + \sqrt{3/2}\phi_1^*\phi_0 + \sqrt{3/2}\phi_0^*\phi_{-1} + \phi_{-1}^*\phi_{-2})$, $f_- = f_+^*$, $f_z = (\overline{c_1}/2)\Sigma_{i=-2}^{2}|\phi_i|^2$, and $s_- = (2\overline{c_2}/5)(\phi_0^2/2 - \phi_1\phi_{-1} + \phi_2\phi_{-2})$. The above equations have been written in dimensionless form and the units for length, energy, and time are $\sqrt{\hbar/(2m\omega_z)}$, $\hbar\omega_z$, and $1/\omega_z$, respectively, where ω_z is the axial trapping frequency. In the numerical simulation of the dynamic equations (26), the initial wave functions are taken to be ground state of the one-dimension GP equations $\mathcal{L}\phi_i(z,0) = \mu\phi_i(z,0)$, where $\phi_i(z,0)$ satisfies the normalization $\int|\phi_i(z,0)|^2 dz = N_i/N$ with N_i the initial particle number in spin-i component.

We first discuss the dynamics of the system in the absence of coupling magnetic fields. Figure 6 shows the time evolution of the population of spin-0 component for two kinds of initial population. The solid line in Fig. 6 corresponds to the initial state that five spin components are initially equally populated, whereas the dotted line corresponds to the case that the BEC initially populates spin-0 components. It shows (solid line) that during time evolution, the population of spin-0 component develops into a complex oscillatory structure, implying the excitation of high-energy modes. The excitation originates from the fact that the initial state is not a mean-field eigenstate of the SMA Hamiltonian (6), thus leading to a redistribution of the total energy E between the symmetric and unsymmetrical parts (denoted by E_s and E_a, respectively) of the total Hamiltonian. When E_a is sufficiently

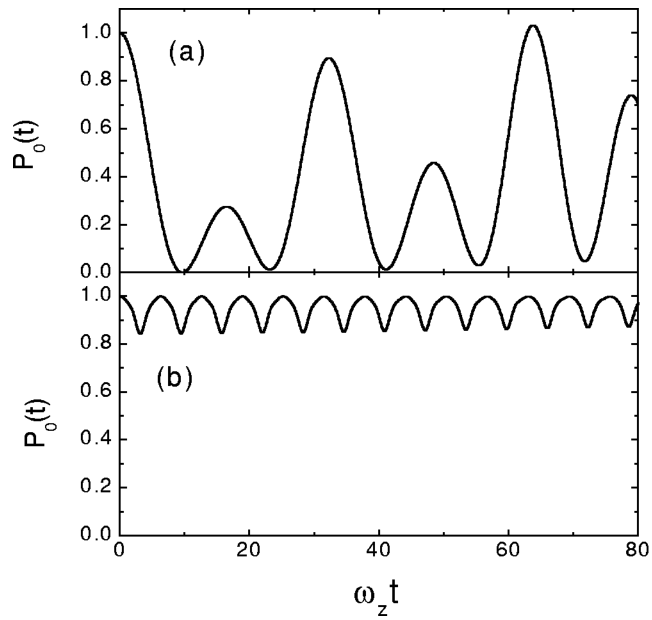

FIG. 7. Time evolution of $P_0(t)$ in the presence of a uniform field $b_x=1$. (a) The case without sinusoidal magnetic field, (b) the case with sinusoidal magnetic field $b_l(t)=b\cos(\omega t)$, $\omega=10$, $b=24$. Other parameters are the same as used in Fig. 6.

large, the higher modes are excited by the nonlinear spin-mixing processes and the SMA description breaks down for a large value of the particle number N. When the initial state is chosen to be the SMA eigenstate, the population of spin-0 state does not vary in time (dotted line). Thus, SMA treatment is appropriate for the spin-polarized state with the BEC localized in spin-0 component.

In the presence of external magnetic fields, the five spin components are coupled. The atomic population is featured by Rabi oscillations among the spin states. To see the validity of the SMA in studying the dynamic spin localization, we present in Fig. 7 the time evolution of the population of spin-0 state for several values of amplitude of oscillatory magnetic field. It shows that the weak transverse magnetic field couples the five spin components, inducing Rabi oscillation between the initial populated spin-0 state and the other spin states. The presence of an additional oscillatory magnetic field slows down the oscillation periods. Finally, when the field parameter is modulated to satisfy $b/\hbar\omega=2.4$, as shown in Fig. 7(b), the Rabi oscillation completely stops and the population of spin-0 state always remains close to unity during its time development. The good agreement between the SMA results and exact numerical solution suggests that *the phenomenon of dynamic spin localization is well-defined beyond the simple* SMA. This robustness of dynamic localization against nonlinear atomic interaction is in contrast to the case of two-component BEC system, where there are no energy-stable spin-polarized states.

VI. CONCLUSION

In summary, by the application of the external magnetic fields which may contain a random component, we have elaborated on the control of population transfer among the internal spin states for the $F=2$ spinor BEC. After taking the mean-field approximation, a general formula is obtained for the oscillation period to describe the quantum transition from the initial spin polar state to other spin states. In particular, when the frequency and reduced amplitude of the longitudinal magnetic field are related in a specific manner, the population of the initial spin-0 state is dynamically localized during the time evolution. In addition, the effects of external noise on the coherent population transfer dynamics of the spinor BEC are studied. Our findings are the following.

(i) The Rabi oscillations are sensitive to the external noise, and even weak coupling can destroy completely the Rabi oscillations in an earlier length of time.

(ii) When the strength of noise is weak, because of strong suppression caused by the systematic field, the static and dynamic localization may remain for experimentally relevant times.

(iii) When the strength of noise is strong, delocalization due to phase randomness occurs during time evolution and the initially localized BEC diffuses among five spin components.

(iv) The Rabi oscillation and dynamic spin localization are robust against the nonlinear spin exchange interaction.

Thus in order to observe the static or dynamic localization in spin space, one should produce stable magnetic fields, which have as little as possible the stochastic component. Although it is difficult to keep the external noise from intruding on dynamics, recent observation of the well-defined coherence between the two BECs [14,15] suggests that the external noise can be greatly suppressed in an experimental setup. Therefore, we expect such macroscopic coherent quantum controls as Rabi oscillation and dynamic localization may be realized in spinor BEC experiments.

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