Paired-domination on interval and circular-arc graphs*

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Abstract

We study the paired-domination problem on interval graphs and circular-arc graphs. Given an interval model with endpoints sorted, we give an $O(m+n)$ time algorithm to solve the paired-domination problem on interval graphs. The result is extended to solve the paired-domination problem on circular-arc graphs in $O(m(m+n))$ time.

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1 Introduction

Let $G=(V,E)$ be a graph without isolated vertices. Throughout this paper, n and m denote the number of vertices and edges of a graph, respectively. For a vertex $v \in V$, the *open neighborhood* of v is defined as $N(v)=\{u \in V | uv \in E\}$ and the *closed neighborhood* of v is defined as $N[v]=N(v)\cup\{v\}$. For $S \subseteq V$, the subgraph of G induced by the vertices in S is denoted by $\langle S\rangle$.

A set $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G is the minimum cardinality of a dominating set of G. A set $S \subseteq V$ is a *paired-dominating* set of G if S is a dominating set of G and the induced subgraph $\langle S\rangle$ has

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a perfect matching. If $v_jv_k = e_i \in M$, where M is a perfect matching of $\langle S\rangle$, we say that v_j and v_k are paired in S. The *paired-domination number* $\gamma_p(G)$ is defined as the minimum cardinality of a paired-dominating set S of G. Paired-domination was introduced by Haynes and Slater [6] with the following application in mind. If we think of each $s \in S \subseteq V$ as the location of a guard capable of protecting each vertex in $N[s]$, then "domination" requires every vertex to be protected. For paired-domination, we require the guards' locations to be selected as adjacent pairs of vertices so that each guard is assigned one other location and they are designed as backup for each other. Given a graph G and an integer K, the problem of determining whether G has a paired-dominating set whose cardinality is less than K is NP-complete [6, 7]. Qiao et al. [9] gave a linear algorithm to determine paired-dominating sets for trees.

A graph $G = (V, E)$ is called an *intersection graph* for a finite family $\mathcal{F}$ of a nonempty set if there is a one-to-one correspondence between $\mathcal{F}$ and V such that two sets in $\mathcal{F}$ have nonempty intersection if and only if their corresponding vertices in V are adjacent. We call $\mathcal{F}$ an *intersection* model of G. For an intersection model $\mathcal{F}$, we use $G(\mathcal{F})$ to denote the intersection graph for $\mathcal{F}$. If $\mathcal{F}$ is a family of intervals on a real line, then G is called an *interval graph* for $\mathcal{F}$ and $\mathcal{F}$ is called an *interval model* of G. If $\mathcal{F}$ is a family of arcs on a circle, then G is called a *circular-arc graph* for $\mathcal{F}$ and $\mathcal{F}$ is called a *circular-arc model* of G. For a family X of sets of vertices, Min(X) denotes a minimum cardinality vertex set in X.

Booth and Lueker [2] gave an $O(n + m)$-time algorithm for recognizing an interval graph and constructing an interval model using *PQ*-trees. Eschen and Spinrad [4] presented an $O(n^2)$-time algorithm for recognizing a circular-arc graph and constructing a circular-arc model. Interval graphs and circular-arc graphs have found applications in a wide range of fields such as scheduling and genetics, among others. Interval graphs and circular-arc graphs have been studied by many researchers [1, 5, 8, 10]. We only mention results pertinent to the class of domination problems studied in this paper. Chang [3] presented a unified approach to designing efficient $O(n)$ or $O(n \log\log n)$ algorithms for the weighted domination problem and the weighted independent, connected, and total domination problems on interval graphs, and extended the algorithms to solve the same problems on circular-arc graphs in $O(n + m)$ time.

2 Algorithms for the paired-domination problem on interval graphs

In this section we give a polynomial algorithm for the paired-domination problem on interval graphs. It is assumed that the input graph is given by an interval model I that is a set of n sorted intervals labelled by $1, 2, \ldots, n$ in increasing order of their right endpoints. The left endpoint of interval i is denoted by a_i and the right endpoint by b_i. By definition, $1 < a_i \leq b_i \leq 2n$ for $1 \leq i \leq n$. For convenience, we need the following notation.

(1) For a set S of intervals, the largest left (right) endpoint of the intervals in S is denoted by $\max a(S)$ ($\max b(S)$); the interval in S with the largest right endpoint is denoted by $last(S)$. We let $\max a(S) = 0$ ($\max b(S) = 0$) if S is empty. For endpoint e, we use $IFB(e)$ (interval finishing before endpoint e) to denote the set of all intervals whose right endpoint are less than e. Thus, $\max a(IFB(e))$ is the largest left endpoint of the intervals whose right endpoints are less than e. For any interval j, let l_j be the interval such that intervals l_j and j have nonempty intersection and $a(l_j)$ is minimum.

(2) For $j \in \{1, 2, \ldots, n\}$, we define $V_j = \{i : i \in \{1, 2, \ldots, n\}$ and $a_i \leq b_j\}$. Let $PD(j) = \{S : S \subseteq V_j, S$ is a paired-dominating set of $\langle V_j \rangle$ and $j \in S\}$. Let $PD(i, j) = \{S : S \subseteq V_j, S$ is a paired dominating set of $\langle V_j \rangle, i, j \in S$ and i, j are paired in $S\}$. Let $MPD(j) = \text{Min}\ (PD(j))$, $MPD(i, j) = \text{Min}\ (PD(i, j))$.

Following the above definitions, we have the following lemmas.

Lemma 2.1 *Let G be an interval graph with interval model I without isolated vertices, then $\langle V_j \rangle$ $(j = 1, 2, \ldots, n)$ has no isolated vertices.*

Lemma 2.2 *For $j \in \{1, 2, \ldots, n\}$, $|MPD(l_j, j)| = |MPD(j)|$.*

Proof. It is easily seen that $|MPD(j)| \leq |MPD(l_j, j)|$. Let S_j be an $MPD(j)$ and M be the perfect matching in $\langle S_j \rangle$ such that $ij \in M$. If $l_j \notin S_j$, then $S_j' = S_j \cup \{l_j\} - \{i\} \in PD(l_j, j)$. So, $|MPD(l_j, j)| \leq |S_j'| = |S_j| = |MPD(j)|$. Then, $|MPD(j)| = |MPD(l_j, j)|$. If $l_j \in S_j$ and $l_j p \in M, p \neq j$, we claim that $N_G(p) - S_j \neq \emptyset$. Otherwise, $S_j - \{p, i\} \in PD(j)$, which contradicts the minimality of S_j. Let $w \in N_G(p) - S_j$, then $S_j' = S_j \cup \{w\} - \{i\} \in PD(l_j, j)$. Hence, $|MPD(l_j, j)| \leq |S_j'| = |S_j| = |MPD(j)|$. Therefore, $|MPD(j)| = |MPD(l_j, j)|$. ■

From Lemma 2.2, clearly $MPD(l_j, j)$ is an $MPD(j)$.

Lemma 2.3 $|MPD(j)| \leq |MPD(j+1)|$ *for $j = 1, 2, \ldots, n-1$.*

Proof. Let M be a perfect matching in $\langle MPD(j+1) \rangle$. To prove the lemma, we consider four cases.

Case 1. $a_{j+1} > b_j$, $j \in MPD(j+1)$. If $(j+1)k \in M$ and $a_k > b_j$, then $MPD(j+1) - \{j+1, k\} \in PD(j)$. So, $|MPD(j)| < |MPD(j+1)|$. If $(j+1)k \in M$, $a_k < b_j$, and $N_{\langle V_j \rangle}(k) - MPD(j+1) = \emptyset$, then $MPD(j+1) - \{j+1, k\} \in PD(j)$. So, $|MPD(j)| < |MPD(j+1)|$. If $(j+1)k \in M$, $a_k < b_j$ and $N_{\langle V_j \rangle}(k) - MPD(j+1) \neq \emptyset$, let $k' \in N_{\langle V_j \rangle}(k) - MPD(j+1)$, then $MPD(j+1) \cup \{k'\} - \{j+1\} \in PD(j)$. Therefore, $|MPD(j)| \leq |MPD(j+1)|$.

Case 2. $a_{j+1} < b_j$, $j \in MPD(j+1)$. If $(j+1)k \in M$ and $a_k > b_j$, then $MPD(j+1) - \{j+1, k\} \in PD(j)$. So, $|MPD(j)| < |MPD(j+1)|$. If $(j+1)k \in M$ and $a_k < b_j$, then $MPD(j+1) \in PD(j)$. Therefore, $|MPD(j)| \leq |MPD(j+1)|$.

Case 3. $a_{j+1} > b_j, j \notin MPD(j+1)$. If $(j+1)k \in M$, $a_k > b_j$, and $N_G(j) - MPD(j+1) \neq \emptyset$, let $j' \in N_G(j) - MPD(j+1)$, then $MPD(j+1) \cup \{j, j'\} - \{j+1, k\} \in PD(j)$. So, $|MPD(j)| \leq |MPD(j+1)|$. If $(j+1)k \in M$, $a_k > b_j$, and $N_G(j) - MPD(j+1) = \emptyset$, let $p \in N_G(j)$ and $pp' \in M$, if $N_G(p') - MPD(j+1) = \emptyset$, then $MPD(j+1) \cup \{j\} - \{p', j+1, k\} \in PD(j)$; if $N_G(p') - MPD(j+1) \neq \emptyset$, let $p'' \in N_G(p') - MPD(j+1)$, then $MPD(j+1) \cup \{j, p''\} - \{j+1, k\} \in PD(j)$. So, $|MPD(j)| \leq |MPD(j+1)|$. If $(j+1)k \in M$ and $a_k < b_j$, then $MPD(j+1) \cup \{j\} - \{j+1\} \in PD(j)$. Consequently, $|MPD(j)| \leq |MPD(j+1)|$.

Case 4. $a_{j+1} < b_j, j \notin MPD(j+1)$. If $(j+1)k \in M$ and $a_k > b_j$, then $MPD(j+1) \cup \{j\} - \{k\} \in PD(j)$. So, $|MPD(j)| \leq |MPD(j+1)|$. If $(j+1)k \in M$ and $a_k < b_j$, then either $MPD(j+1) \cup \{j\} - \{j+1\} \in PD(j)$ or $MPD(j+1) \cup \{j\} - \{k\} \in PD(j)$. So, $|MPD(j)| \leq |MPD(j+1)|$.

Therefore, in all cases, we have shown that $|MPD(j)| \leq |MPD(j+1)|$. ■

Lemma 2.4 *For any $j \in \{1, 2, \ldots, n\}$, if $MPD(l_j, j) \neq \{l_j, j\}$, then there exists $k < j$ such that $MPD(l_j, j) = \{l_j, j\} \cup MPD(k)$ and $b_j > b_k > maxa(IFB(min(a_{l_j}, a_j)))$.*

Proof. Let $MPD(l_j, j)$ be $\{k_1, k_2, \ldots, k_t\}$ with $k_1 < k_2 < \ldots < k_t$ and M be the perfect matching in $\langle MPD(l_j, j) \rangle$ with $jl_j \in M$. To show the existence of such an $MPD(k)$, we consider the following four cases.

Case 1. $j = k_t$, $l_j < k_{t-1}$. It follows that $b_{l_j} < b_{k_{t-1}} < b_j$. Since $MPD(l_j, j)$ is a paired-dominating set of $\langle V_j \rangle$, there exists an interval k_l $(l < t-1)$ such that $k_l k_{t-1} \in M$. We claim that $a_{k_l} <$min(a_{l_j}, a_j). Otherwise, $MPD(l_j, j) - \{k_l, k_{t-1}\} \in PD(l_j, j)$. This contradicts the minimality of $MPD(l_j, j)$. We now claim that $MPD(l_j, j) - \{l_j, j\}$ is an $MPD(k_{t-1})$. First, it is easy to show that $MPD(l_j, j) - \{l_j, j\}$ dominates $V_{k_{t-1}}$. Next we will show that $|MPD(l_j, j) - \{l_j, j\}| = |MPD(k_{t-1})|$. Suppose there exists a paired-dominating set $S' \in PD(k_{t-1})$ such that

$|S'| < |MPD(l_j, j)| - 2$. We first claim that $l_j, j \notin S'$. Otherwise, if $j \in S'$, then $S' \in PD(j)$. Then $|MPD(l_j, j)| = |MPD(j)| \le |S'| < |MPD(l_j, j)| - 2$, a contradiction. If $l_j \in S'$ and $l_jp \in M$, then $N_G(p) - S' \neq \emptyset$. Otherwise, $S' \cup \{j\} - \{p\} \in PD(l_j, j)$, a contradiction to the minimality of $MPD(l_j, j)$. Let $p' \in N_G(p) - S'$, then $S = S' \cup \{p', j\} \in PD(l_j, j)$ and $|S| < |MPD(l_j, j)|$. This is also a contradiction. So, $l_j \notin S'$. Then, $S = S' \cup \{l_j, j\} \in PD(l_j, j)$ and $|S| < |MPD(l_j, j)|$. This is a contradiction to the minimality of $MPD(l_j, j)$. So, $MPD(l_j, j) - \{l_j, j\}$ is an $MPD(k_{t-1})$. Thus, $MPD(l_j, j) = \{l_j, j\} \cup MPD(k_{t-1})$ and $b_j > b_{k_{t-1}} >$max$a(IFB(\min(a_{l_j}, a_j)))$.

Case 2. $j = k_t, l_j = k_{t-1}$. Using a similar argument as that in Case 1, it is easy to show that $\{k_1, k_2, \ldots, k_{t-2}\}$ is an $MPD(k_{t-2})$. Thus, we have $MPD(j, l_j) = \{l_j, j\} \cup MPD(k_{t-2})$ and $b_j > b_{k_{t-2}} >$max$a(IFB(\min(a_{l_j}, a_j)))$.

Case 3. $j < k_t, l_j = k_t$. If $j = k_{t-1}$, then $b_{k_{t-2}} < b_j = b_{k_{t-1}}$. It is easy to show that $\{k_1, k_2, \ldots, k_{t-2}\}$ is an $MPD(k_{t-2})$. Thus,

$$MPD(j, l_j) = \{l_j, j\} \cup MPD(k_{t-2})$$

and

$$b_j > b_{k_{t-2}} > \text{max}a(IFB(\min(a_{l_j}, a_j))).$$

If $j < k_{t-1}$, then there exists an interval k_l $(l < t-1)$ such that $k_lk_{t-1} \in M$. We claim that $a_{k_l} <$min (a_{l_j}, a_j), and $a_{l_j} < b_{k_l} < a_j$. Otherwise, if $a_{k_l} >$min (a_{l_j}, a_j), then $MPD(l_j, j) - \{k_l, k_{t-1}\} \in PD(l_j, j)$. So, $a_{k_l} <$min(a_{l_j}, a_j). And if $b_{k_l} > a_j$, then intervals k_l and j have nonempty intersection, but $a_{k_l} < a_{l_j}$. This is a contraction to the choice of l_j. So, $b_{k_l} < a_j$. Since $a_{k_{t-1}} < b_{k_l} < a_j, b_j < b_{k_{t-1}}$, intervals j and k_{t-1} have nonempty intersection, and it follows that $a_{l_j} < a_{k_{t-1}}$. Combining this with $a_{k_{t-1}} < b_{k_l}$, we have $a_{l_j} < b_{k_l}$. Since $a_{k_{t-1}} < b_{k_l} < a_j$, $k_{t-1} \in V_{k_l}$ and $k_l < j$. Since $a_{l_j} < b_{k_l}$, $b_{k_l} >$max$a(IFB(\min(a_{l_j}, a_j)))$. As in Case 1, it is easy to see $MPD(l_j, j) - \{l_j, j\}$ is an $MPD(k_l)$. Thus, $MPD(l_j, j) = MPD(k_l) \cup \{l_j, j\}$ and $b_j > b_{k_l} >$max$a(IFB(\min(a_{l_j}, a_j)))$.

Case 4. $j < k_t, l_j < k_t$. Since $MPD(l_j, j)$ is a paired-dominating set of $\langle V_j \rangle$, then there exists an interval k_l $(l < t)$ such that $k_lk_t \in M$. $k_t \in V_j$ and $j < k_t$ imply that intervals j and k_t have nonempty intersection, so $a_{l_j} < a_{k_t}$. We claim that $a_{k_l} <$min$\{a_{l_j}, a_j\}$. Otherwise, $MPD(l_j, j) - \{k_l, k_t\} \in PD(l_j, j)$, which contradicts the minimality of $MPD(l_j, j)$. Using a similar argument as that in Case 3, we have $a_{l_j} < b_{k_l} < a_j$. So, $a_{k_t} < b_{k_l} < a_j < b_j$, and $k_t \in V_{k_l}$ and $k_l < j$. It is easy to see that $MPD(l_j, j) - \{l_j, j\}$ is an $MPD(k_l)$ and $b_j > b_{k_l} >$max$a(IFB(\min(a_{l_j}, a_j)))$. Thus, $MPD(l_j, j) = MPD(k_l) \cup \{l_j, j\}$.

Therefore, we always have an $MPD(k)$ $(k < j)$ such that $MPD(l_j, j) = \{l_j, j\} \cup MPD(k)$ and $b_j > b_k >$maxa$(IFB(\min(a_{l_j}, a_j)))$. The result follows. ■

Scan the endpoints of I to find the left endpoint sets $A_i = \{a_j : b_{i-1} < a_j < b_i\}$ for $i \in I$, where $b_0 = 0$.

Lemma 2.5 *Let b_K be the right endpoint of the interval K associated with the left endpoint set A_K containing maxa$(IFB(min(a_{l_j}, a_j)))$, $MPD(K) \cup \{l_j, j\} = MPD(l_j, j)$.*

Proof. We fist show that $MPD(K) \cup \{l_j, j\} \in PD(l_j, j)$. By the definition of $IFB(e)$, for any interval l in $V_j - V_K$, either intervals l_j, l have nonempty intersection or intervals j, l have nonempty intersection. Hence, $MPD(K) \cup \{l_j, j\}$ is a paired-dominating set of $\langle V_j \rangle$. Let S be an $MPD(l_j, j)$. From Lemma 2.4, there exists an $MPD(k)$ such that $S = MPD(k) \cup \{l_j, j\}$ and $b_j > b_k >$maxa$(IFB(\min(a_j, a_{l_j})))$. So, $b_k \geq b_K$. By Lemma 2.3, it follows that $|MPD(K)| \leq |MPD(k)|$. Hence, $|MPD(K) \cup \{l_j, j\}| \leq |MPD(k) \cup \{l_j, j\}| = |S|$. So, $MPD(K) \cup \{l_j, j\} = MPD(l_j, j)$. The lemma follows. ■

In the following we give an Algorithm MPD for computing MPD(j) for $j \in I$ in $O(m+n)$ time and space.

Introduce two intervals $n+1$ and $n+2$ with $a_{n+1} = 2n+1, a_{n+2} = 2n+2, b_{n+1} = 2n+3$, and $b_{n+2} = 2n+4$. Let I_p be the set of intervals obtained by augmenting I with the two intervals $n+1$ and $n+2$.

Algorithm MPD

Input. A set I_p of sorted intervals.

Output. A minimum cardinality paired-dominating set of $G(I_p)$.

1. Find maxa$(IFB(a_j))$ for all $j \in I_p$.

2. Find l_j for all $j \in I_p$.

3. Scan the endpoints of I_p to find the left endpoint sets $A_i = \{a_j : b_{i-1} < a_j < b_i\}$ for

   $i \in I_p$, where $b_0 = 0$.

4. $MPD(0) = \emptyset$.

5. for $j = 1$ to $n+2$ do

6. Find the left endpoint set A_k containing max$a(IFB(\min(a_j, a_{l_j})))$.

7. Let b_k be the right endpoint of the interval k associated with the left endpoint set A_k.

8. $MPD(j) = \{l_j, j\} \cup MPD(k)$.

9. end for

Output MPD(n+2).

The complexity of the above algorithm can be estimated as follows. Chang [3] gave a simple algorithm to find max$a(IFB(a_j))$ for every interval j in $O(n)$ time. So the time needed to perform Step 1 is clearly $O(n)$. The time taken in Step 2 is at most $O(m)$. The time taken in Step 6 is at most $O(n)$, so the time needed in the loop from Step 5 to Step 9 is at most $O(n)$. It follows that the total time needed to run the above algorithm is $O(m + n)$.

From Lemmas 2.2 and 2.5, it is easy to see the correctness of Algorithm MPD.

Lemma 2.6 *Given a set I of sorted intervals, we can compute $MPD(j)$ for all $j \in I$ in $O(m + n)$ time.*

We see that a subset S of I is a paired-dominating set of $G(I)$ if and only if $S \cup \{n+1, n+2\}$ is a paired-dominating set of $G(I_p)$. Thus, we can find a minimum cardinality paired-dominating set of $G(I_p)$ by using Algorithm MPD to compute $MPD(n + 2)$ of $G(I_p)$. Therefore, we have the following theorem.

Theorem 2.1 *Given a set I of sorted intervals, a minimum cardinality paired-dominating set of $G(I)$ can be found in $O(m + n)$ time.*

Given intervals x, y, where $a(x) = 1$ and x, y have nonempty intersection. For $\max(x, y) < j \leq n$, let $PD(j, x, y) = \{S : S \subseteq V_j, S$ is a paired-dominating set of $\langle V_j \rangle$, $j, x, y \in S$ and there exists a perfect matching M in S such that $xy \in M\}$, $PD(i, j, x, y) = \{S : S \subseteq V_j, S$ is a paired-dominating set of $\langle V_j \rangle$, $i, j, x, y \in S$ and there exists a perfect matching M in S such that $xy, ij \in M\}$. And let $MPD(i, j, x, y) = \min\ (PD(i, j, x, y))$, and $MPD(j, x, y) = \min\ (PD(j, x, y))$.

For $j >$max(x, y), let $l'_j \neq x, y$ be the interval such that l'_j, j have nonempty intersection and $a(l'_j)$ is minimum. Similar to Lemmas 2.2 and 2.3, we have the following lemmas.

Lemma 2.7 *For $j >$max(x, y), $|MPD(l_j, j', x, y)| = |MPD(j, x, y)|$.*

Lemma 2.8 *$|MPD(j, x, y)| \leq |MPD(j + 1, x, y)|$ for $j =$max$(x, y) + 1, \ldots, n - 1$.*

Lemma 2.9 *For $j >$max(x, y), either $MPD(j, l'_j, x, y) = \{j, l'_j, x, y\}$ or there exists an $MPD(k, x, y)$ $(j > k >$max$(x, y))$ such that $MPD(j, l'_j, x, y) = \{j, l'_j\} \cup MPD(k, x, y)$ and $b_j > b_k >$ maxa$(IFB($min$(a_{l'_j}, a_j)))$.*

Proof. It is easy to see that if $\min(a_j, a_{l'_j}) <$max(b_x, b_y), then $MPD(j, l'_j, x, y) = \{j, l'_j, x, y\}$. So, we may assume that $\min(a_j, a_{l'_j}) >$max(b_x, b_y). Let $MPD(j, l'_j, x, y)$ be $\{k_1, k_2, \ldots, k_t\}$ with $k_1 < k_2 < \ldots < k_t$ and M be the perfect matching of $\langle MPD(l_j, j', x, y)\rangle$ with $xy, jl'_j \in M$. To show the lemma, we distinguish the following four cases.

Case 1. $j = k_t, l'_j < k_{t-1}$. By the definition of $MPD(j, l'_j, x, y)$, there exists a k_l $(\neq x, y)$ such that $k_l k_{t-1} \in M$. We claim that $a_{k_l} <$min$(a_{l'_j}, a_j)$. Otherwise, $MPD(j, l'_j, x, y) - \{k_l, k_{t-1}\} \in PD(j, l'_j, x, y)$. This contradicts the minimality of $MPD(j, l'_j, x, y)$. Using a similar argument as that in Lemma 2.4, we claim that $MPD(j, l'_j, x, y) - \{j, l'_j\}$ is an $MPD(k_{t-1}, x, y)$, and $b_j > b_{k_{t-1}} >$max$(IFB($min$(a_{l'_j}, a_j)))$. Thus, $MPD(j, l'_j, x, y) = \{j, l'_j\} \cup MPD(k_{t-1}, x, y)$.

Case 2. $j = k_t, l'_j = k_{t-1}$. Using a similar argument as that in Case 1, it is easy to show that $\{k_1, k_2, \ldots, k_{t-2}\}$ is an $MPD(k_{t-2}, x, y)$. If $k_{t-2} =$max(x, y), then $MPD(j, l'_j, x, y)$ $= \{j, l'_j, x, y\}$. If $k_{t-2} >$max(x, y), then $MPD(j, l'_j, x, y) = \{j, l'_j\} \cup MPD(k_{t-2}, x, y)$ and $b_j > b_{k_{t-2}} >$ max$(IFB($min$(a_{l'_j}, a_j)))$.

Case 3. $j < k_t, l'_j = k_t$. If $j = k_{t-1}$, it is easy to show that $\{k_1, k_2, \ldots, k_{t-2}\}$ is an $MPD(k_{t-2}, x, y)$. As in Case 2, either $MPD(j, l'_j, x, y) = \{j, l'_j, x, y\}$ or $MPD(j, l'_j, x, y)$ $= \{j, l'_j\} \cup MPD(k_{t-2}, x, y)$ and $b_j > b_{k_{t-2}} >$max$(IFB($min$(a_{l'_j}, a_j)))$. If $k_{t-1} > j$, then there exists a k_l $(k_l \neq x, y)$ such that $k_l k_{t-1} \in M$. Using a similar argument as that in Lemma 2.4, we claim that $a_{k_l} <$min$(a_{l'_j}, a_j)$, and $a_{l'_j} < b_{k_l} < a_j$. So, $a_{k_{t-1}} < b_{k_l} < a_j < b_j$, then $k_{t-1} \in V_{k_l}$ and $k_l < j$. It is easy to see that $MPD(j, l'_j, x, y) - \{j, l'_j\}$ is an $MPD(k_l, x, y)$. We claim that $k_l >$max(x, y). Otherwise, since k_{t-1}, j have nonempty intersection, so $a_{l'_j} < a_{k_{t-1}}$. Then, $MPD(l_j, j', x, y) - \{k_l, k_{t-1}\} \in PD(j, l'_j, x, y)$, which contradicts the minimality of $MPD(l_j, j', x, y)$. So, $MPD(j, l'_j, x, y) = MPD(k_l, x, y) \cup \{l'_j, j\}$, and $b_j > b_{k_l} >$max$(IFB($ min$(a_{l'_j}, a_j)))$.

Case 4. $j < k_t, l'_j < k_t$. Since $MPD(l_j, j, x, y)$ is a paired-dominating set of $\langle V_j\rangle$, then there exists an interval k_l $(l < t)$ such that $k_l k_t \in M$. Intervals j and k_t have nonempty intersection, so $a_{l'_j} < a_{k_t}$. Using a similar argument as that in Case 3, we have $MPD(j, l'_j, x, y) = MPD(k_l, x, y) \cup \{l'_j, j\}$, and $b_j > b_{k_l}$ >max$(IFB(\min(a_{l'_j}, a_j)))$. ■

Using a similar argument as that in Lemma 2.5 and combining it with Lemmas 2.7, 2.8 and 2.9, we obtain the following lemma.

Lemma 2.10 *Let b_K be the right endpoint of the interval of K associated with the left endpoint set A_K containing maxa$(IFB(min(a_{l'_j}, a_j)))$, $MPD(j, x, y) = \{x, y, j, l'_j\}$ if $K \leq$max(x, y), and $MPD(j, x, y) = \{j, l'_j\} \cup MPD(K, x, y)$ if $K >$max(x, y).*

Following Lemma 2.10, we now design Algorithm MPD(x, y) for computing MPD(j, x, y) for all $j \in I$ in $O(m + n)$ time and space. Details of the algorithm are as follows.

Algorithm MPD(x, y)

Input. A set I of sorted intervals.

Output. $MPD(j, x, y)$ for $j >$max(x, y).

1. Find max$a(IFB(a_j))$ for all $j \in I$.

2. Find l'_j for all $j \in I$.

3. Scan the endpoints of I to find the left endpoint sets $A_i = \{a_j : b_{i-1} < a_j < b_i\}$ for

   $i \in I$, where $b_0 = 0$.

4. $MPD(\max(x, y), x, y) = \{x, y\}$.

5. for $j =$max$(x, y) + 1$ to n do

6. If $\min(a_j, a_{l'_j})$ <max(b_x, b_y), then $MPD(j, x, y) = \{x, y, j, l'_j\}$;

7. If $\min(a_j, a_{l'_j})$ >max(b_x, b_y), find the left endpoint set A_k containing max$a(IFB($ min

   $(a_{l'_j}, a_j)))$.

8. Let b_k be the right endpoint of interval k associated with the left endpoint set A_k.

9. $MPD(j, x, y) = \{j, l'_j\} \cup MPD(k, x, y)$ if $k >$max(x, y);

10. $MPD(j, x, y) = \{x, y, j, l'_j\}$ if $k \leq$max(x, y).

11. end for

Output $MPD(j, x, y)$ for $j >$max(x, y).

From Lemmas 2.7 and 2.10, we immediately obtain the following theorem, which ensures the correctness of the algorithm.

Theorem 2.2 *Given a set I of sorted intervals, we can compute $MPD(j, x, y)$ for all $j >max(x, y)$ in $O(m + n)$ time.*

3 Extension to circular-arc graphs

In this section we will extend the results of the previous section to solve the paired-domination problem on $G(A)$, given a set A of sorted arcs. An arc, starting from an endpoint h along the clockwise direction to the endpoint t, is denoted by $[h, t]$. We refer to endpoints h and t as the *head* and *tail* of arc $[h, t]$, respectively. We use "arc" to refer to a member of A and "segment $[c, d]$" to refer to the continuous part of the circle that begins with an endpoint c and ends with d in the clockwise direction. Arbitrarily choose an arc from A, starting from the head of this arc, label endpoints along the clockwise direction from 1 to $2n$. Arcs are numbered from 1 to n in increasing order of their tails. Denote the head and tail of arc i by h_i and t_i, respectively. Note that h_i can be larger than t_i, in which case arc $[h_i, t_i]$ extends $h_i, h_i + 1, \ldots, 2n, 1, \ldots, t_i$.

Lemma 3.1 *Suppose A is an arc model and x_0 is any arc of A. There exists a minimum cardinality paired-dominating set S of $G(A)$ such that S contains an arc x in $N[x_0]$ and S does not contain any other arc containing arc x.*

Proof. Let S be a paired-dominating set of $G(A)$ with minimum cardinality. Clearly, $S \cap N[x_0] \neq \emptyset$. There exists an arc $x \in S \cap N[x_0]$ such that x is not contained in any other arc of $S \cap N[x_0]$. Since every arc containing arc x is a neighbor of arc x, x is not contained in any other arc of S. ■

Following Lemma 3.1, we define the following:

$PRD(x) = \{S : S$ is a paired-dominating set of $G(A), x \in S$ and x is not contained in any other arc of $S\}$.

For $x \in A$, we define $\overline{N}(x)$ as the set of arcs of A that either contains arc x or is contained in arc x, and define $N_R(x)$ and $N_L(x)$ as the sets of arcs whose heads and tails are contained in arc x, respectively. Let $A_P(x) = A - \overline{N}(x)$, $A_R(x) = A_P(x) - N_L(x)$, and $A_L(x) = A_P(x) - N_R(x)$. It is straightforward to verify that $A_R(x)$ and $A_L(x)$ are interval graphs.

Lemma 3.2 *Suppose A is an arc model and x_0 is any arc of A. If there exists a minimum cardinality paired-dominating set S of $G(A)$ such that S contains an arc x in $N[x_0]$, S does not contain any other arc containing arc x, and $S \cap (N_L(x) \cup N_R(x)) \neq \emptyset$, then there exists a minimum cardinality paired-dominating set S' of $G(A)$ such that S' contains x and $S' \cap \overline{N}(x) = \emptyset$.*

Proof. Assume that S is a minimum cardinality paired-dominating set of $G(A)$ that contains x and does not contain any other arc containing arc x. Let M be a perfect matching in $\langle S \rangle$. If $S \cap \overline{N}(x) = \emptyset$, the result follows. If $S \cap \overline{N}(x) \neq \emptyset$, it is easy to prove that $|S \cap \overline{N}(x)| = 1$. Then there exists an arc y such that y is contained in x. If $xy \in M$, let $w \in S \cap (N_L(x) \cup N_R(x))$, $ww' \in M$, we claim that $N(w') - S \cup \overline{N}(x) \neq \emptyset$. Otherwise, $S' = S - \{w', y\}$ is a paired-dominating set of $G(A)$, a contradiction. Let $w'' \in N(w') - S \cup \overline{N}(x)$, then $S' = S \cup \{w''\} - \{y\}$ is a minimum cardinality paired-dominating set of $G(A)$. If $yw \in M$ $(w \neq x)$, we claim that $N(w) - S \cup \overline{N}(x) \neq \emptyset$. Otherwise, $S - \{w, y\}$ is a minimum paired-dominating set of $G(A)$, a contradiction. Let $w' \in N(w) - S \cup \overline{N}(x)$, so $S' = (S - \{y\}) \cup \{w'\}$ is also a minimum cardinality paired-dominating set of $G(A)$. Thus, we have a minimum cardinality paired-dominating set S' of $G(A)$ such that S' contains an arc x and $S' \cap \overline{N}(x) = \emptyset$. ■

Lemma 3.3 *If there exists a minimum cardinality paired-dominating set S of $G(A)$ such that S contains an arc x and $S \cap \overline{N}(x) = \emptyset$, then there exists a minimum cardinality paired-dominating set S' of $G(A)$ such that there exists $y \in S'$, x, y are paired in S', and $S' \cap \overline{N}(x) = S' \cap \overline{N}(y) = \emptyset$.*

Proof. Assume that S is a paired-dominating set of $G(A)$ with minimum cardinality that contains x, and $S \cap \overline{N}(x) = \emptyset$. Then there exists a perfect matching M in $\langle S \rangle$ such that $xw \in M$, where $w \in S$. If $S \cap \overline{N}(w) = \emptyset$, let $y = w$, then the result follows. If $S \cap \overline{N}(w) \neq \emptyset$, it is easy to show that $|S \cap \overline{N}(w)| = 1$. Otherwise, S is not a minimum cardinality paired-dominating set of $G(A)$. Let $w' \in S \cap \overline{N}(w)$. If w' is contained in w and $w'z \in M$, we claim that $N(z) - S \cup \overline{N}(w) \cup \overline{N}(x) \neq \emptyset$. Otherwise, $S - \{w', z\}$ is a paired-dominating set of $G(A)$, a contradiction. Let $z' \in N(z) - S \cup \overline{N}(w) \cup \overline{N}(x)$, so $S' = (S - \{w'\}) \cup \{z'\}$ is a minimum cardinality paired-dominating set of $G(A)$, and $S' \cap \overline{N}(x) = S' \cap \overline{N}(w) = \emptyset$. Let $y = w$, the result follows.

If w is contained in w' and $w'z \in M$, we claim that $N(z) - S \cup \overline{N}(w') \cup \overline{N}(x) \neq \emptyset$. Otherwise, $S - \{w, z\}$ is a paired-dominating set of $G(A)$, a contradiction. Let $z' \in N(z) - S \cup \overline{N}(w') \cup \overline{N}(x)$, so $S' = (S - \{w\}) \cup \{z'\}$ is a minimum cardinality paired-dominating set of $G(A)$, And x and w' are paired in S'. If $S' \cap \overline{N}(w') = \emptyset$, let $y = w'$, then the result follows. If $S' \cap \overline{N}(w') \neq \emptyset$, it is easy to show that $|S' \cap \overline{N}(w')| = 1$. Then there exists an arc w'' contained in arc w'; proceeding as above, let $y = w'$, the result follows. ∎

Furthermore, we define the following

$PRD_1(x) = \{S : S \in PRD(x), S \cap (N_L(x) \cup N_R(x)) = \emptyset\}$,

$PRD_2(x) = \{S : S \in PRD(x)$, there exists a vertex $y \in S$ such that x, y are paired in S, and $S \cap \overline{N}(x) = S \cap \overline{N}(y) = \emptyset\}$,

$MPRD_1(x) =$Min$(PRD_1(x)), MPRD_2(x) =$Min$(PRD_2(x))$.

$K(x) = \{y : y \in A, y \neq x,\ y$ is contained in $x\}$.

To find $MPRD_1(x)$, we need the following lemma.

Lemma 3.4 *The following two statements are true.*

(1) Suppose S is a paired-dominating set of $G(A - N[x])$ and y is an arc contained in arc x, $\{x, y\} \cup S \in PRD_1(x)$.

(2) Suppose $S \in PRD_1(x)$, $S - N[x]$ is a paired-dominating set of $G(A - N[x])$.

By Lemma 3.4, it is easy to see that $\{x, y\} \cup S$, where $y \in K(x)$ is an $MPRD_1(x)$ if S is a minimum paired-dominating set of $G(A - N[x])$. Since $G(A - N[x])$ is an interval graph, by Theorem 2.1, a minimum cardinality paired-dominating set of $G(A - N[x])$ can be computed in $O(m + n)$ time. So $MPD_1(x)$ can be computed in $O(m + n)$ time.

For $x \in N[x_0], y \in N_R(x)$, let $Z(x, y) = \{z : z$ is an arc contained in $[h_x, t_y], z \neq x, z \neq y\}$. For $x \in N[x_0], y \in N_L(x)$, let $Z(x, y) = \{z : z$ is an arc contained in $[h_y, t_x], z \neq x, z \neq y\}$. $PRD_2(x, y) = \{S : S \in PRD_2(x)$, there exists a perfect matching M in $\langle S \rangle$ such that $xy \in M$, and $S \cap \overline{N}(x) = S \cap \overline{N}(y) = \emptyset\}$, $MPRD_2(x, y) =$Min$(PRD_2(x, y))$.

Lemma 3.5 *For $y \in N_R(x)$, if $S \in PRD_2(x, y)$ is a minimum cardinality paired-dominating set of $G(A)$, then there exists a minimum cardinality paired-dominating set S' of $G(A)$ such* *that $S' \in PRD_2(x,y)$, $S' \cap Z(x,y) = \emptyset$, and there exists a perfect matching M in $\langle S' \rangle$ such that for any $w \in S' \cap N_L(x)$, there exists $w_1 \in S'$ with $ww_1 \in M$, and the intersection of arcs w, w_1 is not contained in arc x.*

Proof. We first prove that there exists a minimum cardinality paired-dominating set S' of $G(A)$ such that $S' \in PRD_2(x,y)$ and $S' \cap Z(x,y) = \emptyset$. If $Z(x,y) \cap S = \emptyset$, then the result follows. If $Z(x,y) \cap S \neq \emptyset$, then for any $w \in Z(x,y) \cap S$, there exists $w' \in S$ such that w, w' are paired in S. We claim that $N(w') - S \cup Z(x,y) \cup \overline{N}(x) \cup \overline{N}(y) \neq \emptyset$. Otherwise, $S - \{w, w'\}$ is a paired-dominating set of $G(A)$, a contradiction. Let $w'' \in N(w') - S \cup Z(x,y) \cup \overline{N}(x) \cup \overline{N}(y)$, so $S_1 = S \cup \{w''\} - \{w\}$ is a paired-dominating set of $G(A)$. Proceeding as above, we get a minimum cardinality paired-dominating set S' of $G(A)$ such that $Z(x,y) \cap S' = \emptyset$ and $S' \in PRD_2(x,y)$. Assume M is the perfect matching in $\langle S' \rangle$ such that $xy \in M$, then for any $w \in S' \cap N_L(x)$, there exists $w_1 \in S'$ such that $ww_1 \in M$. If the intersection of arcs w, w_1 is not contained in arc x, the result follows. Otherwise, $w_1 \in N_R(y), w_1 \notin Z(x,y)$ and the intersection of arcs w, w_1 is contained in arc x. Then $S - \{x,y\}$ is a paired-dominating set of $G(A)$, a contradiction to the minimality of S. The lemma follows. ■

Similar to Lemma 3.5, we can obtain the following result.

Lemma 3.6 *For $y \in N_L(x)$, if $S \in PRD_2(x,y)$ is a minimum cardinality paired-dominating set of $G(A)$, then there exists a minimum cardinality paired-dominating set S' of $G(A)$ such that $S' \in PRD_2(x,y), S' \cap Z(x,y) = \emptyset$, and there exists a perfect matching M in $\langle S' \rangle$ such that for any $w \in S' \cap N_L(y)$, there exists $w_1 \in S'$ such that $ww_1 \in M$, and the intersection of arcs w, w_1 is not contained in arc y.*

For $x \in N[x_0]$, we define the following:

$$PRD_{21}(x,y) = \begin{cases} \{S : S \in PRD_2(x,y), S \cap N_L(x) = \emptyset\} & if\ y \in N_R(x) \\ \{S : S \in PRD_2(x,y), S \cap N_L(y) = \emptyset\} & if\ y \in N_L(x) \end{cases}$$

$$PRD_{22}(x,y) = \begin{cases} \{S : S \in PRD_2(x,y), S \cap N_R(y) = \emptyset\} & if\ y \in N_R(x) \\ \{S : S \in PRD_2(x,y), S \cap N_R(x) = \emptyset\} & if\ y \in N_L(x) \end{cases}$$

$$PRD_{23}(x,y) = \begin{cases} \{S : S \in PRD_2(x,y),\ \text{covers the whole circle,} & \\ \text{and } S \text{ satisfies the properties of Lemma 3.5}\} & if\ y \in N_R(x) \\ \{S : S \in PRD_2(x,y),\ \text{covers the whole circle,} & \\ \text{and } S \text{ satisfies the properties of Lemma 3.6}\} & if\ y \in N_L(x) \end{cases}$$

$$PRD_{24}(x,y)=\begin{cases}\{S: S\in PRD_2(x,y),\ S\cap N_L(x)\neq\emptyset, S\cap N_R(y)\neq\emptyset, & \\ \text{S does not cover the whole circle, and satisfies the} & \\ \text{properties of Lemma 3.5}\} & if\ y\in N_R(x)\\ \{S: S\in PRD_2(x,y),\ S\cap N_L(y)\neq\emptyset, S\cap N_R(x)\neq\emptyset, & \\ \text{S does not cover the whole circle, and satisfies the} & \\ \text{properties of Lemma 3.6}\} & if\ y\in N_L(x)\end{cases}$$

$MPRD_{2j}(x,y)$ =Min$(PRD_{2,j}(x,y)), j=1,2,3,4$.

Without loss of generality, we consider the case $y\in N_R(x)$. We first compute $MPRD_{21}(x, y)$. It is easy to see that $S\subseteq A_R(x)$ if $S\in PRD_{21}(x,y)$. Clearly, $G(A_R(x)-\overline{N}(y))$ is an interval graph. For simplicity, arcs of $A_R(x)-\overline{N}(y)$ are considered as intervals in the following lemma, where the head and tail of an arc are considered as the left and right endpoint of its corresponding interval, respectively. We see that interval x is the first interval of $A_R(x)$.

Lemma 3.7 *Suppose $S\subseteq A$, $S\in PRD_{21}(x,y)$ if and only if $S\in PD(last(S),x,y)$ of $G(A_R(x)-\overline{N}(y)))$ and $b_{last(S)}>maxa(A_R(x)-\overline{N}(y))$.*

Proof. Suppose $S\in PRD_{21}(x,y)$, by the definition of $PRD_{21}(x,y)$, $S\subseteq A_R(x)-\overline{N}(y)$. Obviously, $S\in PD(last(S),x,y)$ of $G(A_R(x)-\overline{N}(y))$, and $b_{last}(S)>$max$a(A_R(x)-\overline{N}(y))$. On the other hand, suppose that $S\in PD(last(S),x,y)$ of $G(A_R(x)-\overline{N}(y))$, $b_{last(S)}>$max$a(A_R(x)-\overline{N}(y))$. Clearly, S is a paired-dominating set of $G(A_R(x)-\overline{N}(y))$, $S\subseteq A_R(x))-\overline{N}(y)$. Since x,y dominate $N[x]\cup N[y]$, S is a paired-dominating set of $G(A)$. Hence, $S\in PRD_{21}(x,y)$. ■

By Lemma 3.7, we can find $MPRD_{21}(x,y)$ by finding Min$(\{MPD(last(S),x,y): last(S)\in A_R(x)-\overline{N}(y), b_{last(S)}>maxa(A_R(x)-\overline{N}(y)\})$ from $G(A_R(x)-\overline{N}(y))$. By Theorem 2.2, it can be done in $O(m+n)$ time. Thus, $MPRD_{21}(x,y)$ can be found in $O(m+n)$ time.

By the symmetric property, $MPRD_{22}(x,y)$ can be found in $O(m+n)$ time in the same way.

In computing $MPRD_{23}(x,y)$, we first map $A_P(x)$ to a set of intervals. The endpoints of the arcs of $A_P(x)$ are numbered in the clockwise order from 1 to $2|A_P(x)|$, starting from the head of arc x. Then, for every arc $z\in A_R(x)$, we create an interval $I(z)=[h_z,t_z]$; for every arc $z\in N_L(x)$, we create an interval $I(z)=[h_z,t_z+2|A_P(x)|]$. For S, a subset of $A_P(x)$, let $I(S)$ denote $\{I(z): z\in S\}$.

The following two lemmas can be verified easily by the above procedure.

Lemma 3.8 ([3]) (1) *$I(x)$ is the first interval of $I(A_P(x))$.*

(2) *For two arcs w and z of $A_P(x)$, arc w overlaps arc z if $I(w)$ overlaps $I(z)$.*

(3) *For $w, z \in A_R(x)$, arc w overlaps z if and only if $I(w)$ overlaps $I(z)$.*

(4) *For $w \in A_P(x)$ and $z \in A - N[x]$, arcs w and z overlap if and only if $I(w)$ overlaps $I(z)$.*

Lemma 3.9 *For $w \in N_L(x)$ and the intersection of arcs w, z is not contained in arc x, arcs w and z overlap if and only if $I(w)$ overlaps $I(z)$.*

Lemma 3.10 *$S \in PRD_{23}(x, y)$ if and only if $I(S) \in PD(last(I(S)), x, y)$ of $G(I(A_P(x) - \overline{N}(y)))$ and $last(I(S)) \in I(N_L(x))$.*

Proof. Suppose $S \in PRD_{23}(x, y)$, by the definition of $PRD_{23}(x, y)$ and Lemmas 3.8, 3.9, clearly, $I(S) \in PD(last(I(S)), x, y)$ and $last(I(S)) \in I(N_L(x))$. On the other hand, suppose $I(S) \in PD(last(I(S)), x, y)$ of $G(I(A_P(x) - \overline{N}(y)))$ and $last(I(S)) \in I(N_L(x))$. For every arc $z \in A$, if $I(z)$ overlaps an interval in $I(S)$, then z overlaps an arc in S; if $I(z)$ does not overlap intervals in $I(S)$, $last(I(S)) \in I(N_L(x))$ implies that z overlaps x. So $S \in PRD_2(x, y)$, S covers the whole circle. Let M be the perfect matching of $\langle S \rangle$ corresponding to the perfect matching in $\langle I(S) \rangle$. It is clear that, for any $w \in S \cap N_L(x)$, there exists a $w' \in S$ such that $ww' \in M$ and the intersection of arcs w, w' is not contained in arc x. Therefore, $S \in PRD_{23}(x, y)$. ■

$MPRD_{23}(x, y)$ can be found by computing Min($\{MPD(last(I(S)), x, y) : last(I(S)) \in I(N_L(x))\}$) from $G(I(A_P(x) - \overline{N}(y)))$. By Theorem 2.2, it can be done in $O(m + n)$ time.

In the following, we show how to find $MPRD_{24}(x, y)$ by using the same technique in [3].

If $S \in PRD_{24}(x, y)$, then there exists an arc u of S such that h_u is not contained in any other arc of S. Apparently, $u \neq x$. Define $PRD_{24}(u, x, y) = \{S : S \in PRD_{24}(x, y), u \in S, h_u$ is not contained in any other arc of $S\}$, $MPRD_{24}(u, x, y) =$Min($PRD_{24}(u, x, y)$). Then, $MPRD_{24}(x, y) =$Min ($\{PRD_{24}(u, x, y) : u \in A_L(x) - \{x\}\}$). For arc $u \in A_L(x) - \{x\}$, define $LPRD(u, x, y)$ as the collection of all subsets S of $A_L(y) - Z(x, y) \cup \overline{N}(x)$ such that $x, y, u \in S$, $\langle S \rangle$ has a perfect matching M with $xy \in M$, all arcs of S are contained in segment $[h_u, t_y]$, and S dominates all arcs that overlap segment $[h_u, t_y]$. $MLPRD(u, x, y) =$Min($LPRD(u, x, y)$). Similarly, for arc $v \in A_R(y) - \{y\}$, define $RPRD(v, x, y)$ as the collection of all subsets S of $A_R(x) - Z(x, y) \cup \overline{N}(y)$ such that $x, y, v \in S$, $\langle S \rangle$ has a perfect matching M with $xy \in M$, all arcs of S are contained in segment $[h_x, t_v]$ and S dominates all arcs that overlap segment $[h_x, t_v]$. $MRPRD(u, x, y)$ =Min($RPRD(u, x, y)$).

Suppose $S \in PRD_{24}(x, y)$. Since S does not cover the whole circle, there exist two arcs u and v of S such that $u \in A_L(x) - \{x\}, v \in A_R(y) - \{y\}$, $h_u > t_v$, and all arcs of S are contained in segment $[h_u, t_v]$. Let $S_L(u, x, y)$ and $S_R(v, x, y)$ denote the set of arcs of S contained in segment $[h_u, t_y]$ and $[h_x, t_v]$, respectively. For arc $u \in A_L(x) - \{x\}$, define $RA(u)$ as the set of arcs of $A_R(x)$ that are contained in segment $[h_x, h_u]$. And define $\alpha(u)$ =max$\{h_w : w \in RA(u)\}$. Then, for $u \in A_L(x) - \{x\}$, $v \in A_R(y) - \{y\}$, and $t_v < h_u$, there does not exist any arc y contained in segment $[t_v, h_u]$ if and only if $t_v > \alpha(u)$. By the definition of $PRD_{24}(u, x, y)$, we observe that $S_L(u, x, y) \in LPRD(u, x, y)$ and $S_R(u, x, y) \in RPRD(v, x, y)$, $\alpha(u) < t_v < h_u$. If $u \in A_L(x) - \{x\}$, $S_1 \in LPRD(u, x, y)$, and $S_2 \in RPRD(v, x, y)$, where $v \in A_R(y) - \{y\}$ and $\alpha(u) < t_v < h(u)$, then $S_1 \cup S_2 \in PRD_{24}(u, x, y)$ since $S_1 \cup S_2$ dominates all arcs overlapping segment $[h_u, t_v]$ and there does not exist any arc z such that $t_v < h_z < t_z < h_u$.

Lemma 3.11 *$S \in PRD_{24}(u, x, y)$ if and only if there exists an arc v of S such that $S_L(u, x, y) \in LPRD(u, x, y)$, $S_R(v, x, y) \in RPRD(v, x, y)$ and $\alpha(u) < t_v < h(u)$.*

Following the above lemma, we immediately have $MPRD_{24}(u, x, y) = MLPRD(u, x, y) \cup$ Min($\{MRPRD(v, x, y) : v \in A_R(y) - \{y\}, \alpha(u) < t_v < h_u\}$). Min($\{MRPRD(v, x, y) : v \in A_R(y) - \{y\}, \alpha(u) < t_v < h_u\}$) and $MLPRD(u, x, y)$ can be found in $O(m + n)$ time by Algorithm $MPD(x, y)$. Thus, $MPRD_{24}(x, y)$ can be computed in $O(m + n)$ time.

Choosing a vertex x_0 of minimum degree and letting $N[x_0] = \{x_0, x_1, \ldots, x_d\}$, where d is the minimum degree of $G(A)$, we find $MPRD_1(x_k)(k = 1, 2, \ldots, d)$ and, for each $x \in N[x_0], y \in N(x)$, we find $MPRD_{21}(x, y), MPRD_{22}(x, y), MPRD_{23}(x, y)$, $MPRD_{24}(x, y)$. The one with minimum cardinality is a minimum cardinality paired-dominating set of $G(A)$. For each $x \in N[x_0]$, $MPRD_1(x)$ can be found in $O(m + n)$ time. And for each $x \in N[x_0], y \in N(x)$, $MPRD_{2i}(x, y)$ $(i = 1, 2, 3, 4)$ can be found in $O(m+n)$ time. So a minimum paired-dominating set of $G(A)$ can be found in $O(m(m + n))$ time.

Theorem 3.1 *Given a set of A of sorted arcs, the minimum paired-dominating set of $G(A)$ can be found in $O(m(m + n))$ time.*

4 Conclusion

We studied the paired-domination problem on interval graphs and circular-arc graphs. Given an interval model with endpoints sorted, we presented an $O(m + n)$ time algorithm to solve the paired-domination problem on interval graphs. We then extended the results to solve the paired-domination problem on circular-arc graphs in $O(m(m + n))$ time.

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References

[1] M. A. Bonuccelli, Dominating sets and domination number of circular-arc graphs, *Discrete Applied Mathematics* 12(1985), 203-213.

[2] K. S. Booth and G. S. Lueker, Testing for consecutive ones property, interval graphs and graph planarity using *PQ*-tree algorithms, *Journal of Computer System Sciences* 13(1976), 335-379.

[3] M. S. Chang, Efficient algorithms for the domination problems on interval and circular-arc graphs, *SIAM Journal on Computing* 27(1998), 1671-1694.

[4] E. M. Eschen and J. P. Spinrad, An $O(n^2)$ algorithm for circular-arc graph recognition, *in Proceedings of the 4th Annual ACM-SIAM Symposium on Discrete Algorithms*, Austin, Texas, 1993, 128-137.

[5] M. C. Golumbic, Interval graphs and related topics, *Discrete Mathematics* 55(1985), 113-121.

[6] T. W. Haynes, T. W. Slater, Paired-domination in graphs, *Networks* 32(1998), 199-206.

[7] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[8] R. Laskar, J. Pfaff, S. M. Hedetniemi, and S. T. Hedetniemi, On the algorithmic complexity of total domination, *SIAM Journal of Algebraic and Discrete Methods* 5(1984), 420-425.

[9] H. Qiao, L. Y. Kang, M. Cardei, D. Z. Du, Paired-domination of trees, *Journal of Global Optimization* 25(2003), 43-54.

[10] G. Ramalingan and C.P. Rangan, Total domination in interval graphs revisited, *Information Processing Letters* 27(1988), 17-21.